The Most General Theory for ELKOs

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Abstract

We consider the ELKO theory, and we study the question of what is their most general model; after having found for ELKOs the most general dynamical terms, and having recalled the already known most general potential terms, we outline consequences of this most general case: an outlook of the problems that this generalization may open is given.

Introduction

Quite recently, a new form of matter called ELKO field has been defined; this form of matter gets its name from the acronym of the German *Eigenspinoren des LadungsKonjugationsOperators* designating spinors that are eigenstates of the charge conjugation operator: mathematically they are spin-$\frac{1}{2}$ fermions indicated as λ verifying the conditions given by $\gamma^2\lambda^* = \pm\lambda$ respectively for self- and antiself-conjugated fields ([1]). The reason for which ELKO fields have been mathematically defined in this way is that in doing so these fields are meant to physically represent neutral fermions therefore displaying the physical property of non-locality ([2]). As a consequence it turned out that ELKOs are fermions of mass dimension 1 henceforth described by scalar-like field equations ([3, 4]).

That ELKOs are defined to be fermions which turn out to undergo second-order derivative field equations has many implications: first, it is known that fermions couple to torsion and in second-order derivative theories the spin of the fermion is given in terms of the derivative of the fermion beside the fact that in the field equations for the fermion there are derivatives of the torsion itself, and thus we have that derivatives of derivatives of the fermions given in terms of some matrix $C^{\alpha\beta}$ as $C^{\alpha\beta}\nabla_\alpha\nabla_\beta\lambda$ may arise beyond the second-order derivative of the fermion given as $\nabla^2\lambda$ hence threatening the causal propagation of the field; also, the same considerations can be used to see that as their field density tends to increase their energy tends to vanish; and finally, their richer dynamics gives rise to effects never seen for other forms of matter. Nevertheless, it has been proved that all torsional fermionic back-reactions actually cancel away leaving only the D'Alembertian operator acting on the field so that the causal propagation of the field itself is ensured ([5, 6]); then, it has been acknowledged that their gravitational asymptotic freedom is important because in this way it is even in terms of their gravitational pull that their non-local properties are preserved and also empowered ([7]); and eventually, it has been seen that their richer dynamics may be used to fit the exponential expansion of the inflationary universe and the curves of rotation of galaxies ([8, 16]). Some of these results have been extended and a comprehensive literature can be found ([17, 26]).

Thus said, ELKOs are a form of matter whose main feature is that their higher-order dynamical character gives rise to a peculiar behavior, which is very promising in solving several of the most important problems of the standard model of cosmology; the fact that these fields are so important for their applications to cosmology carries the question of what is the most general model of ELKO as another problem of interest for physics in general. In the present paper we will study ELKOs in terms of the most general theories that can be constructed and we will sketch a few consequences of these generalizations.

1 Most General Theory of ELKO Fields

In this paper, the Riemann-Cartan geometry is defined in terms of spacetime connections given by $\Gamma^{\mu}_{\alpha\sigma}$ and used to define the Riemann curvature tensor as

$$G^{\rho}{}_{\eta\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\eta\nu} - \partial_{\nu}\Gamma^{\rho}_{\eta\mu} + \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\eta\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\eta\mu} \tag{1}$$

which has one independent contraction given by $G^{\rho}{}_{\eta\rho\nu} = G_{\eta\nu}$ whose contraction is given by $G_{\eta\nu}g^{\eta\nu} = G$ as usual and then we define Cartan torsion tensor as

$$Q^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} \tag{2}$$

and contorsion tensor

$$K^{\rho}{}_{\mu\nu} - K^{\rho}{}_{\nu\mu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} \tag{3}$$

so that torsion and contorsion are linked by the relationship

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2}\left(Q^{\rho}{}_{\mu\nu} + Q_{\mu\nu}{}^{\rho} + Q_{\nu\mu}{}^{\rho}\right) \tag{4}$$

with one independent contraction given by $K_{\nu\rho}{}^{\rho} = Q^{\rho}{}_{\rho\nu} = Q_{\nu} = K_{\nu}$ as convention: when in the connection the torsion or contorsion tensors are separated away we are left with the symmetric connection in terms of which we define the Riemann metric curvature tensor $R^{\rho}{}_{\eta\mu\nu}$ with one contraction $R_{\eta\nu}$ whose contraction is R as usual. The contraction of the Riemann curvature tensor will be called Ricci curvature tensors: then the contractions of the Riemann metric curvature tensor will be called Ricci metric curvature tensors. By using these tensors it is possible to build the covariant derivatives, the commutator of covariant derivatives and the cyclic permutations of covariant derivatives known as Jacobi-Bianchi identities useful for what we are going to do next.

In fact by employing these tensors and their identities we can postulate the coupling between the curvature and the energy density $T^{\mu\nu}$ and between the contorsion and the spin density $S^{\rho\mu\nu}$ as

$$G^{\mu\nu} - \frac{1}{2}g^{\mu\nu}G = \frac{1}{2}T^{\mu\nu} \tag{5}$$

and

$$K_{\mu\alpha\beta} - K_{\mu\beta\alpha} + K_{\alpha}g_{\beta\mu} - K_{\beta}g_{\alpha\mu} = -S_{\mu\alpha\beta} \tag{6}$$

where the gravitational constant has been normalized away, known as Einstein-Sciama-Kibble field equations: again it is possible to separate contorsion everywhere in the equations for the energy (11) and to invert contorsion in terms of the spin in the equations for the spin (12) in order to substitute it into the field equations for the energy (11) obtaining the field equations that relate the metric curvature tensor to the energy density.

In the scheme provided by the Riemann-Cartan geometry the Jacobi-Bianchi identities can be used to see that the Einstein-Sciama-Kibble field equations are transformed into conservation laws given by

$$D_\rho S^{\rho\mu\nu} + Q_\rho S^{\rho\mu\nu} + \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right) = 0 \tag{7}$$

and also

$$D_\mu T^{\mu\rho} + Q_\mu T^{\mu\rho} - T_{\mu\beta}Q^{\beta\mu\rho} - S_{\beta\mu\kappa}G^{\mu\kappa\beta\rho} = 0 \tag{8}$$

which must be satisfied once the matter field undergoes specific field equations.

The matter fields we will employ will be spin-$\frac{1}{2}$ fermion fields for which the derivatives D_μ are defined by the contorsionless derivatives ∇_μ according to the decomposition

$$D_\mu\lambda = \nabla_\mu\lambda + \frac{1}{2}K^{ij}{}_{\mu}\sigma_{ij}\lambda \tag{9}$$

in terms of the σ_{ij} matrices; the commutator of the derivatives is given by

$$[D_\mu, D_\nu]\lambda = Q^\rho{}_{\mu\nu}D_\rho\lambda + \frac{1}{2}G^{ij}{}_{\mu\nu}\sigma_{ij}\lambda \tag{10}$$

in terms of the σ_{ij} matrices given as $\sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$ where the gamma matrices satisfy the anticommutation relationships represented by the Clifford algebra.

Now in order to write the most general matter field theory of ELKO we define for the ELKO the conserved quantities given by the energy

$$\begin{aligned} T_{\mu\nu} = &\left(D_\mu \overset{\neg}{\lambda} D_\nu\lambda + D_\nu \overset{\neg}{\lambda} D_\mu\lambda - g_{\mu\nu}D_\rho \overset{\neg}{\lambda} D^\rho\lambda\right) + \\ &+a\left(D_\nu \overset{\neg}{\lambda} \sigma_{\mu\rho}D^\rho\lambda + D^\rho \overset{\neg}{\lambda} \sigma_{\rho\mu}D_\nu\lambda - g_{\mu\nu}D_\rho \overset{\neg}{\lambda} \sigma^{\rho\sigma}D_\sigma\lambda\right) \end{aligned} \tag{11}$$

and the spin

$$\begin{aligned} S_{\mu\alpha\beta} = &\tfrac{1}{2}\left(D_\mu \overset{\neg}{\lambda} \sigma_{\alpha\beta}\lambda - \overset{\neg}{\lambda} \sigma_{\alpha\beta}D_\mu\lambda\right) + \\ &+\tfrac{a}{2}\left(D^\rho \overset{\neg}{\lambda} \sigma_{\rho\mu}\sigma_{\alpha\beta}\lambda - \overset{\neg}{\lambda} \sigma_{\alpha\beta}\sigma_{\mu\rho}D^\rho\lambda\right) \end{aligned} \tag{12}$$

and as it is clear they must satisfy the coupled conservation laws given above.

This can actually be achieved as soon as the ELKO is governed by the field equations given by

$$\left(D^2\lambda + K^\mu D_\mu\lambda\right) + a\left(\sigma^{\rho\mu}D_\rho D_\mu\lambda + K_\rho\sigma^{\rho\mu}D_\mu\lambda\right) = 0 \tag{13}$$

as it can be checked by performing direct calculations in a straightforward way.

Notice that the terms that carries the constant a are precisely those that generalizes the simplest ELKO theory known by now.

It is possible to see that this set of field equations can also be derived from a variational principle starting from the action given by

$$S = \int \left[D_i \overset{\neg}{\lambda} (\eta^{ij} + a\sigma^{ij})D_j\lambda - G\right] d\Omega \tag{14}$$

and varying with respect to each of the independent fields involved.

Notice that the two terms containing ELKO fields combined in terms of the a parameter form what is actually the most general second-order derivative action showing that this generalization of the theory of ELKO is actually the most general theory of ELKO possible.

2 Extra Effects in the Dynamical Terms

We have now defined the ELKO in their most general theory, in the sense that it is described by the most general second-order derivative action (14); as a consequence the subsequent field equations for the matter field (13) with the geometry-matter coupling (6) and (5) with conserved quantities (12) and (11) given by

$$D^2\lambda + K^\mu D_\mu\lambda + a\sigma^{\rho\mu}D_\rho D_\mu\lambda + aK_\rho\sigma^{\rho\mu}D_\mu\lambda = 0 \tag{15}$$

together with

$$\begin{aligned}2\left(K_{\mu\beta\alpha} - K_\alpha g_{\beta\mu} - K_{\mu\alpha\beta} + K_\beta g_{\alpha\mu}\right) = D_\mu \overset{\neg}{\lambda}\sigma_{\alpha\beta}\lambda - \overset{\neg}{\lambda}\sigma_{\alpha\beta}D_\mu\lambda +\\ aD^\rho \overset{\neg}{\lambda}\sigma_{\rho\mu}\sigma_{\alpha\beta}\lambda - a\overset{\neg}{\lambda}\sigma_{\alpha\beta}\sigma_{\mu\rho}D^\rho\lambda\end{aligned} \tag{16}$$

and

$$\begin{aligned}2G^{\mu\nu} - g^{\mu\nu}G = D_\mu \overset{\neg}{\lambda} D_\nu\lambda + D_\nu \overset{\neg}{\lambda} D_\mu\lambda - g_{\mu\nu}D_\rho \overset{\neg}{\lambda} D^\rho\lambda +\\ +aD_\nu \overset{\neg}{\lambda}\sigma_{\mu\rho}D^\rho\lambda + aD^\rho \overset{\neg}{\lambda}\sigma_{\rho\mu}D_\nu\lambda - ag_{\mu\nu}D_\rho \overset{\neg}{\lambda}\sigma^{\rho\sigma}D_\sigma\lambda\end{aligned} \tag{17}$$

are the most general satisfying the conservation laws for which the Jacobi-Bianchi geometrical identities are verified: however this theory is more general than the simplest theory because of a single set of terms with a one constant a alone. In the following we shall explore some of the effects due to the dynamics that comes from this extra set of terms with the a parameter.

The first problem that need be solved is the problem of the inversion of contorsion: as it is clear from equation (16), the contorsion tensor is given by the spin, which is written in terms of derivatives containing contorsion; thus equation (16), once the derivative are written in terms of the contorsionless derivatives, only gives contorsion as an implicit function, and it needs to be inverted. The problem of the inversion of contorsion for the simplest theory where $a = 0$ has been solved in [5, 6] and so it is possible to employ the same strategy to get a solution also in this most general case having a of general values, and indeed by following that procedure it is possible to see that the contorsion has completely antisymmetric and trace irreducible decompositions that are given by the expressions

$$K_{\alpha\beta\mu}\varepsilon^{\alpha\beta\mu\rho}\left[(8+(1+a)\overset{\neg}{\lambda}\lambda)^{2}+(1+a)^{2}(i\overset{\neg}{\lambda}\gamma\lambda)^{2}\right]=$$
$$=4(1+a)^{2}(i\overset{\neg}{\lambda}\gamma\lambda)\left(\nabla_{\mu}\overset{\neg}{\lambda}\sigma^{\mu\rho}\lambda-\overset{\neg}{\lambda}\sigma^{\mu\rho}\nabla_{\mu}\lambda\right)-$$
$$-4i(1+a)(8+(1+a)\overset{\neg}{\lambda}\lambda)\left(\nabla_{\mu}\overset{\neg}{\lambda}\gamma\sigma^{\mu\rho}\lambda-\overset{\neg}{\lambda}\sigma^{\mu\rho}\gamma\nabla_{\mu}\lambda\right)+$$
$$+3a(1+a)(i\overset{\neg}{\lambda}\gamma\lambda)\nabla^{\rho}(\overset{\neg}{\lambda}\lambda)-3a(8+(1+a)\overset{\neg}{\lambda}\lambda)\nabla^{\rho}(i\overset{\neg}{\lambda}\gamma\lambda) \tag{18}$$
$$K^{\rho}\left[(8+(1+a)\overset{\neg}{\lambda}\lambda)^{2}+(1+a)^{2}(i\overset{\neg}{\lambda}\gamma\lambda)^{2}\right]=$$
$$=2(1+a)(8+(1+a)\overset{\neg}{\lambda}\lambda)(i\overset{\neg}{\lambda}\gamma\lambda)\left(\nabla_{\mu}\overset{\neg}{\lambda}\sigma^{\mu\rho}\lambda-\overset{\neg}{\lambda}\sigma^{\mu\rho}\nabla_{\mu}\lambda\right)+$$
$$+2i(1+a)^{2}(i\overset{\neg}{\lambda}\gamma\lambda)\left(\nabla_{\mu}\overset{\neg}{\lambda}\gamma\sigma^{\mu\rho}\lambda-\overset{\neg}{\lambda}\sigma^{\mu\rho}\gamma\nabla_{\mu}\lambda\right)+$$
$$+\tfrac{3a}{2}(1+a)(i\overset{\neg}{\lambda}\gamma\lambda)\nabla^{\rho}(i\overset{\neg}{\lambda}\gamma\lambda)+\tfrac{3a}{2}(8+(1+a)\overset{\neg}{\lambda}\lambda)\nabla^{\rho}(\overset{\neg}{\lambda}\lambda) \tag{19}$$

and we know that with these expression contorsion can be inverted; however having contorsion inverted explicitly is a long work beyond the purpose of the present article. It is worth noticing however that since the relationship that implicitly defines contorsion is linear in the contorsion itself then this problem always admits solution, although finding it may be rather difficult under a technical perspective. In the following we will try to remain in the case in which contorsion is not separated away performing calculations not in terms of the contorsionless but in terms of the contorsional covariant derivatives.

To see what is a special behaviour the field may follow we will look for solutions of a peculiar form, and following [7] we will look for solutions given by

$$D_{\mu}\lambda=-iP_{\mu}\gamma\lambda \tag{20}$$
$$D_{\mu}\overset{\neg}{\lambda}=iP_{\mu}\overset{\neg}{\lambda}\gamma \tag{21}$$

in terms of the $\gamma=i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$ matrix; it is easy to see that in the moment in which the vector P_{μ} is the momentum of the free particle and thus covariantly constant $\nabla_{\alpha}P_{\mu}=0$ then these are actually solutions for the field equations with torsion given by the expression

$$Q_{\alpha\mu\nu}=\tfrac{1}{2}\left(i\overset{\neg}{\lambda}\gamma\sigma_{\rho\nu}\lambda g_{\alpha\mu}-i\overset{\neg}{\lambda}\gamma\sigma_{\rho\mu}\lambda g_{\alpha\nu}-2i\overset{\neg}{\lambda}\gamma\sigma_{\mu\nu}\lambda g_{\alpha\rho}\right)P^{\rho}+$$
$$+\tfrac{a}{2}(i\overset{\neg}{\lambda}\gamma\sigma_{\rho\nu}\lambda g_{\alpha\mu}-i\overset{\neg}{\lambda}\gamma\sigma_{\rho\mu}\lambda g_{\alpha\nu}-i\overset{\neg}{\lambda}\gamma[\sigma_{\rho\alpha},\sigma_{\mu\nu}]\lambda)P^{\rho} \tag{22}$$

as it can be checked directly, and for which the curvature is given by

$$G_{\mu\nu}=P_{\mu}P_{\nu}\overset{\neg}{\lambda}\lambda \tag{23}$$

showing that the curvature is independent on the a parameter and so the same we have in the simplest theory. Also notice that it is not reducible in the sense that in this case all contributions to the curvature scalar vanish identically.

3 Further Effects for the Potential Terms

We remark that in talking about the ELKO in their most general theory, the utmost generality concerned the dynamical terms and not eventual potential terms; such generalizations could easily be achieved by means of a quartic potential term, a mass term and a cosmological term as it is usually done. The point we would like to stress is that when the two generalizations, both dynamical and potential, are taken into account, then there are extra effects in the dynamics and in the potentials, and there may even be effects coming from dynamical-potential coupled together in interaction: for instance, if within this general theory with one parameter a we were to include also the cosmological constant term Λ then a mass term proportional to $2m^2 = a\Lambda$ would appear in the field equations; by setting the sign of a to be the same of Λ this term would be a real mass term for the field equations. This fact is important in the context of mass generation because it provides such a mechanism without an additional scalar field with suitable potential, although here like in the usual mass generation mechanism the mass of the particle is determined in terms of two fundamental constants still unknown and so no prediction can be made.

Conclusion

In this paper we have discussed the ELKO fields in what is their most general theory: by this we mean most general in the sense of the terms that determine the dynamics; terms giving general potentials were already known. We have seen that when all extra contributions are taken into account there are four fundamental constants in the model: an a parameter accounting for the extra dynamical terms; a coupling constant delineating the strength of the extra potential of autointeraction, the mass constant and the cosmological constant.

We have then discussed what are the main problems that could be faced in this most general context: the problem of the inversion of contorsion, of which we have indicated the way that have to be followed in order to solve it; the problem of finding solutions of the field equations and checking the behaviour of the curvature-energy couplings, for which we have found a solution of special form for which the curvature-energy coupling has been calculated. A consequence for the mass generation mechanism has been discussed. Now it would be interesting to know more about the possible implications of this most general theory, for instance: in the moment in which the contorsion is actually inverted, it will be possible to calculate all second-order derivative terms in the field equations, to study the propagation of the particle; further the metric curvature could be explicitly calculated to check whether for high-densities the energy conditions that may give rise to gravitational asymptotic freedom are still valid. Of course the most important of these problems concerns the applications to physical situations, where all extra contribution are likely to change the behaviour of inflationary scenarios or curves of rotation.

The ELKO theory is certainly one of the best candidates to solve many problems in the standard model of cosmology, and having ELKO fields described in terms of the most general theory is the simplest way we have to increase our explanatory power so to embrace into a unique theory as many of the problems of the standard model of cosmology as we can. The fact that what we have presented here for ELKOs is their most general theory actually provides this possibility; and in the case in which some effects should be found that cannot be described within this scheme then this would mean that something even more general than ELKO must be found necessary.

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