Asymptotic safety: a simple example

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We use the Gross-Neveu model in $2 < d < 4$ as a simple fermionic example for Weinberg's asymptotic safety scenario: despite being perturbatively nonrenormalizable, the model defines an interacting quantum field theory being valid to arbitrarily high momentum scales owing to the existence of a non-Gaußian fixed point. Using the functional renormalization group, we study the UV behavior of the model in both the purely fermionic as well as a partially bosonized language. We show that asymptotic safety is realized at non-Gaußian fixed points in both formulations, the universal critical exponents of which we determine quantitatively. The partially bosonized formulation allows to make contact to the large-N_{f} expansion where the model is known to be renormalizable to all-orders. In this limit, the fixed-point action as well as all universal critical exponents can be computed analytically. As asymptotic safety has become an important scenario for quantizing gravity, our description of a well-understood model is meant to provide for an easily accessible and controllable example of modern nonperturbative quantum field theory.

I. INTRODUCTION

Renormalizability is often described as a seemingly technical cornerstone for the construction of admissible models in particle physics. Renormalization fixes physical parameters of a model to measured values of observable quantities. A prime physical meaning of renormalizability is the capability of a model to provide an accurate description of a physical system over a wide range of scales at which measurements can be performed. The set of physical parameters, say, couplings or mass parameters etc., measured at different scales then define the renormalized trajectory in parameter space (theory space). If we demand for a specific model to provide a *fundamental* description of nature, the model must be valid on all scales, in particular down to arbitrarily short-distance scales. The renormalized trajectory then must exist on all scales without developing singularities.

The requirement of renormalizability can formally be verified and realized in perturbatively renormalizable theories in a weak coupling expansion. Here, all free parameters of a model can be fixed to physical values and the renormalized trajectory can be constructed order-by-order in a perturbative expansion. This strategy can successfully be applied to theories that exist at least over a wide range of scales, but still suffer from a maximum scale of UV extension, such as QED [1] or the standard model Higgs sector [2]. If a theory is asymptotically free, i.e., if the point of vanishing coupling (Gaußian fixed point) is a UV attractive fixed point, the perturbative construction can even be applied on all scales, as in QCD.

Renormalizability is by no means bound to a perturbative construction. Even though reliable nonperturbative information might be difficult to obtain, the concept of renormalizability and the existence of a renormalized trajectory on all scales can be formulated rather generally within Weinberg's asymptotic safety scenario [3]. Loosely speaking, asymptotic safety is the generalization of asymptotic freedom at the Gaußian fixed point to the case of a non-Gaußian fixed point. As a fixed point of the renormalization group (RG) by construction defines a point in parameter space where the system becomes scale invariant, RG trajectories that hit the fixed point towards the ultraviolet (UV) can be extended to arbitrarily high energy/momentum scales, thereby defining a fundamental theory, for reviews see [4, 5].

The asymptotic safety scenario has recently become an important ansatz for quantizing gravity. In contrast to other approaches, this scenario is based on the standard gravitational degrees of freedom and also the quantization procedure proceeds in a rather standard fashion. Here, significant progress was made with the aid of the functional RG, formulated in terms of a flow equation [6] for the effective average action for the metric field [7]. In simple truncations, the RG flow of gravity indeed reveals a non-Gaußian fixed point [8] – a necessary prerequisite for asymptotic safety. Most importantly, the fixed point has remained stable under extensions of the truncation, and its universal properties such as the critical exponents, in fact, exhibit a quantitative convergence under improvements of the approximations involved [9–11]. RG relevant directions in theory space have been identified and can be associated with a finite number of physical parameters to be fixed by experiment. Taken together, this provides for a rather strong evidence that a quantized version of Einstein gravity can consistently be formulated within the asymptotic safety scenario.

Still many questions are difficult to answer in the context of quantum gravity, mainly due to technical and computational limitations. For a confirmation of the asymptotic safety scenario, contact to other nonperturbative quantization schemes have to be made in a quantitative manner; first indications for a possible agreement, e.g., with dynamical triangulations have already been observed [12].

UV-complete scenarios for the matter sector of the standard model built on asymptotic safety have also been developed, for instance, for toy models of the Higgs sector [13–15] also involving non-linear sigma models [16], or for extra dimensional Yang-Mills theories [17] and gravity [10, 18].

As asymptotic safety is inherently linked with field theories in the nonperturbative domain, it appears highly worthwhile to identify and investigate other nontrivial examples. The present work is devoted to such a detailed study. Our benchmark model is given by the standard Gross-Neveu model [19] in $d = 3$ (or more generally in $2 < d < 4$) spacetime dimensions. This model is known to be perturbatively nonrenormalizable as the coupling constant carries a negative mass dimension. A naive loop expansion leads to a series in terms of diagrams with an increasing superficial degree of divergence. Proceeding in the standard fashion of perturbative renormalization would require infinitely many counterterms and thus infinitely many physical parameters to be fixed by experiment, implying that the theory has no predictive power at all. In fact, it has long been known that this conclusion is only an artifact of perturbative quantization. By means of a Hubbard-Stratonovich transformation, the fermionic theory can be partially bosonized such that an alternative expansion in terms of the inverse fermion flavor number N_{f} can conveniently be formulated. The large-N_{f} expansion turns out to be renormalizable to all orders rather similar to a small coupling expansion in a perturbatively renormalizable model [20].

Whereas this provides strong indications for the existence of an interacting Gross-Neveu model in $d = 3$, it remains an open question as to whether this conclusion holds for finite N_{f}. On the other hand, one may wonder whether this conclusion about nonperturbative renormalizability is indeed profoundly nontrivial: as the partially bosonized version of the Gross-Neveu model is identical to a Yukawa model. This seems to suggest that the renormalizability in the large-N_{f} expansion may simply reflect the super-renormalizability of the $d = 3$ Yukawa model. In fact, four-fermi models in $d = 4$ are known to be related to Yukawa models near the Gaußian fixed point [21, 22]. In this work, we wish to emphasize that this is, in fact, not the case in $2 < d < 4$. As we show below, also the bosonized Yukawa formulation is renormalized at a non-Gaußian fixed point within the asymptotic safety scenario. Similar observations have been made from a more pragmatic viewpoint by studying the scaling properties of corresponding lattice models towards the continuum limit [23, 24].

This work mainly has a pedagogical character. Our analysis is performed in a self-contained fashion within the modern formulation of the functional RG also to provide guidance to the recent literature on asymptotically safe quantum gravity. The fixed point analysis and the computation of universal properties is performed explicitly and contact is made to the large-N_{f} expansion where fully analytic results for the fixed-point potential and all universal critical exponents can be obtained.

As fermionic models occur in many circumstances of particle physics (effective models of QCD) and many-body physics (strongly correlated electron systems), our analysis of the microscopic completeness of the Gross-Neveu model also provides a lesson for the functional RG treatment of such systems. Of course, model studies of fermionic systems conventionally aim at long-range phenomena instead of short-distance behavior. As will be detailed below, the non-Gaußian fixed-point facilitating UV asymptotic safety in the Gross-Neveu model at the same time serves as a quantum critical point associated with a 2nd order quantum phase transition towards a phase with broken discrete chiral symmetry. This phase transition has been studied already with a variety of techniques [23–28] corresponding critical exponents are, of course, equivalent to those which we interpret as properties of the UV limit of the asymptotically safe model.

Our work is organized as follows: we begin with summarizing the essential details of the model in Sect. II. The basics of the asymptotic safety scenario are summarized in Sect. III. An RG analysis in the fermionic language is performed in Sect. IV. Section V contains the corresponding study in the partially bosonized formulation including a mean-field and large-N_{f} analysis and a numerical evaluation of the functional RG equations.

II. GROSS-NEVEU MODEL

The Gross-Neveu model describes the quantum field theory of N_{f} flavors of massless relativistic fermions in d space-time dimensions interacting via a four-fermion interaction term. It allows to study dynamical chiral symmetry breaking. The Euclidean action in d space-time dimensions reads

$$
\begin{aligned}
S[\bar{\psi},\psi] &= \int_x \left\{ \sum_{j=1}^{N_{\mathrm{f}}} \bar{\psi}_j \mathrm{i}\partial\!\!\!/\, \psi_j + \sum_{i,j=1}^{N_{\mathrm{f}}} \bar{\psi}_i\psi_i \frac{\bar{g}}{2N_{\mathrm{f}}} \bar{\psi}_j\psi_j \right\} \\
&\equiv \int_x \left\{ \bar{\psi}\mathrm{i}\partial\!\!\!/\, \psi + \frac{\bar{g}}{2N_{\mathrm{f}}} (\bar{\psi}\psi)^2 \right\} , \qquad (1)
\end{aligned}
$$

where $\int_x = \int d^dx$ is a shorthand for the integral over the d-dimensional Euclidean space-time. The model depends on a single parameter which is the coupling constant $\bar{g}$ with mass dimension $2 - d$.

In this work we restrict ourselves to $2 < d < 4$; in $d = 2$, the model is asymptotically free and perturbatively renormalizable, as the Gaußian fixed point is UV attractive. $d = 4$ will turn out to be a marginal case, where the asymptotic safety scenario no longer applies for integer N_{f} and the theory becomes "trivial", i.e., noninteracting in the continuum limit, see below and [14]. To be specific, we employ a four-component representation for the gamma matrices in the $d = 3$ case in this work, i.e., $d_\gamma = 4$ where d_γ denotes the dimension of the Dirac algebra. In $d = 3$, the explicit representation of our choice for the 4×4 representation of the Dirac algebra can be written as

$$
\gamma_0 = \tau_3 \otimes \tau_3 , \quad \gamma_1 = \tau_3 \otimes \tau_1 , \quad \gamma_2 = \tau_3 \otimes \tau_2 . \qquad (2)
$$

Here, the $\{\tau_i\}$'s denote the Pauli matrices which satisfy $\tau_i\tau_j = \delta_{ij}\tau_0 + \mathrm{i}\epsilon_{ijk}\tau_k$, with $i, j, k = 1, 2, 3$ and $\tau_0 = \mathbb{1}_2$

is a 2×2 unit matrix. The gamma matrices satisfy the anticommutation relation, i.e., the Dirac algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}_4 \,, \tag{3}$$

where $\mu, \nu = 0, 1, 2$ and $\mathbb{1}_4$ denotes the 4×4 unit matrix. Moreover we have two additional 4×4 matrices which anticommute with all γ_μ and with each other:

$$\gamma_3 = -\tau_1 \otimes \tau_0 \,, \quad \gamma_5 = \tau_2 \otimes \tau_0 \,, \quad \gamma_3^2 = \gamma_5^2 = \mathbb{1}_4 \tag{4}$$

The matrix $\gamma_{35} \equiv \mathrm{i}\gamma_3\gamma_5$ on the other hand commutes with γ_μ and anticommutes with γ_3 and γ_5. The action of the Gross-Neveu model is invariant under global $\mathrm{U}(N_\mathrm{f})$ transformations of the fermion fields. This also implies invariance under $\mathrm{U}(1)^{\otimes N_\mathrm{f}}$ transformations, i.e., the associated U(1)-charge is conserved in each flavor-sector separately. The matrix γ_{35} further realizes a $\mathrm{U}^{35}(1)$ symmetry in each flavor sector:

$$\bar{\psi}_j \mapsto \bar{\psi}_j \mathrm{e}^{-\mathrm{i}\varphi\gamma_{35}}, \quad \psi_j \mapsto \mathrm{e}^{\mathrm{i}\varphi\gamma_{35}}\psi_j \,. \tag{5}$$

In addition to these two symmetries, the model is symmetric under discrete $\mathbb{Z}_2^5 = \{\mathbb{1}_4, \gamma_5\}$ chiral transformations acting on all flavors simultaneously:

$$\bar{\psi} \mapsto -\bar{\psi}\gamma_5 \,, \quad \psi \mapsto \gamma_5\psi \tag{6}$$

(A similar symmetry transformation involving γ_3 can be understood as a combination of $\mathbb{Z}_2^5$ and $\mathrm{U}^{35}(1)$ transformations.)

In continuous dimensions $2 < d < 4$, we assume that a suitable analytic continuation for the Dirac structure exists, such that traces over the algebraic structure yield analytic functions in d and d_γ. The chiral symmetry of the model can be associated with a $\mathbb{Z}_2$ symmetry for the order parameter. As we shall discuss below, the infrared regime of the theory is governed by dynamical chiral symmetry breaking, provided the only parameter of the model, namely $\bar{g}$, is adjusted accordingly.

III. ASYMPTOTIC SAFETY AND RG FLOW EQUATION

For a self-contained presentation, let us briefly summarize the essentials of Weinberg's asymptotic safety scenario which is based on the underlying general structure of the renormalization group (RG). In the space of parameters and couplings g_i, the RG provides a vector field $\boldsymbol{\beta}$, summarizing the RG β functions for these couplings $(\boldsymbol{\beta})_i = \beta_{g_i}(g_1, g_2, \dots) \equiv \partial_t g_i$. As the full content of a quantum system can be parameterized in terms of generating functionals for correlation functions, we can more generally study the RG behavior of a generating functional. Introducing an IR-regulated effective average action Γ_k, the RG flow of this action is determined by the Wetterich equation [6]

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{STr}\{[\Gamma_k^{(2)}[\Phi] + R_k]^{-1}(\partial_t R_k)\}, \;\; \partial_t = k\frac{d}{dk}. \tag{7}$$

Here, $\Gamma_k^{(2)}$ is the second functional derivative with respect to the field Φ, representing a collective field variable for all bosonic or fermionic degrees of freedom. The function R_k denotes a momentum-dependent regulator that suppresses IR modes below a momentum scale k. The solution to the Wetterich equation provides for an RG trajectory in the space of all action functionals, also known as *theory space*, interpolating between the bare action S_Λ to be quantized $\Gamma_{k\to\Lambda} \to S_\Lambda$ and the full quantum effective action $\Gamma = \Gamma_{k\to 0}$, being the generating functional of 1PI correlation functions; for reviews, see [5, 29, 30].

For a fundamental quantum field theory, the RG trajectory needs to be extendable over all scales which, in particular, requires that the UV cutoff Λ can be sent to infinity. This is, for instance, possible if the trajectory approaches a point in theory space which is a fixed point under the RG transformations, i.e., remains invariant under the variation of k. Parameterizing the effective average action Γ_k by a possibly infinite set of generalized dimensionless couplings g_i, the Wetterich equation provides us with the flow of these couplings $\partial_t g_i = \beta_{g_i}(g_1, g_2, \dots)$. A fixed point $g_{i,*}$ is defined by

$$\beta_i(g_{1,*}, g_{2,*}, \ldots) = 0 \;\forall\; i \,. \tag{8}$$

The fixed point is non-Gaußian, if at least one fixed-point coupling is nonzero $g_{j,*} \neq 0$. If the RG trajectory hits a fixed point in the UV, the UV cutoff can safely be taken to infinity and the system approaches a conformally invariant state for $k \to \Lambda \to \infty$.

For the theory to be predictive, the number of physical parameters required for specifying the RG trajectory needs to be finite. Considering the linearized flow in the fixed-point regime,

$$\partial_t g_i = B_i{}^j(g_{j,*} - g_j) + \dots, \quad B_i{}^j = \frac{\partial \beta_{g_i}}{\partial g_j}\Big|_{g=g_*}, \tag{9}$$

we encounter the stability matrix $B_i{}^j$, which we diagonalize

$$B_i{}^j\, V_j^I = -\Theta^I V_i^I, \tag{10}$$

in terms of right-eigenvectors V_i^I, enumerated by the index I. The resulting *critical exponents* Θ^I now provide for a classification of physical parameters. The solution of the coupling flow in the linearized fixed-point regime is given by

$$g_i = g_{i,*} + \sum_I C^I\, V_i^I \left(\frac{k_0}{k}\right)^{\Theta^I}, \tag{11}$$

where the integration constants C^I define the initial conditions at a reference scale k_0. All eigendirections with $\Theta^I < 0$ are suppressed towards the IR and thus are *irrelevant*. All *relevant* directions with $\Theta^I > 0$ increase towards the IR and thus determine the macroscopic physics. For the *marginal* directions $\Theta^I = 0$, higher-order terms in the expansion about the fixed point have to be regarded. Hence, the number of relevant and marginally-relevant directions determines the total number of physical parameters to be fixed. The theory is predictive if this number is finite.

For the flow towards the IR, the linearized fixed-point flow Eq. (9) is generally not sufficient and the full nonlinear β functions have to be taken into account. Even the parameterization of the effective action in terms of the same degrees of freedom in the UV and IR might be inappropriate, for instance, if bound states or condensates appear in the IR. This is precisely the case in fermionic models beyond criticality; below, we discuss asymptotic safety in the Gross-Neveu model therefore from both viewpoints, on the one hand in terms of microscopic fermionic degrees of freedom and on the other hand in terms of a mixed fermionic-bosonic description.

IV. FERMIONIC FIXED-POINT STRUCTURE

We begin with a discussion of the fixed-point structure of the Gross-Neveu model as it becomes apparent already in a very elementary approximation within a purely fermionic description. Let us consider only a point-like four-fermion interaction, such that our ansatz for the effective action reads

$$\Gamma_k[\bar{\psi},\psi] = \int_x \left\{ Z_\psi \bar{\psi} \mathrm{i}\partial\!\!\!/\, \psi + \frac{\bar{g}}{2N_\mathrm{f}}(\bar{\psi}\psi)^2 \right\} , \qquad (12)$$

where we allowed for a wave-function renormalization Z_ψ, and both Z_ψ and $\bar{g}$ are considered to be scale-dependent, i.e., a function of k. This simple ansatz can be viewed as a derivative expansion of the effective action, with the leading-order defined by Z_ψ =const. This expansion can, in fact, be associated with a potentially small expansion parameter in terms of the anomalous dimension $\eta_\psi = -\partial_t \ln Z_\psi$. Consequently, a running wave-function renormalization corresponds to a next-to-leading order derivative expansion. Aside from derivatives, further fermion channels and higher-order interactions compatible with the symmetries can be taken into account. We come back to the role of such interactions below.

Inserting the ansatz (12) into the flow equation (7), the flow of the dimensionless renormalized four-fermion coupling g,

$$g = Z_\psi^{-2} k^{2-d} \bar{g} \qquad (13)$$

is given by

$$\beta_g \equiv \partial_t g = (d-2+2\eta_\psi) g - 4 d_\gamma v_d\, l_1^F(0)\, g^2 , \qquad (14)$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$ and $\eta_\psi \sim \mathcal{O}(g)$. Here, we projected the full flow equation straightforwardly onto the pointlike limit of the Gross-Neveu coupling; contributions from further (possibly fluctuation-induced) interaction channels as well as dependencies on the Fierz basis [31–33] have been ignored for the sake of simplicity. The constant $l_1^F(0)$ depends on the choice of the regulator and parameterizes the regulator scheme dependence of the RG flow. For instance, for a linear regulator of the form [34–36]

$$R_k^\psi = Z_\psi p\!\!\!/\, r_\psi(p^2/k^2)\, ,\ r_\psi(x) = \left(\frac{1}{\sqrt{x}} - 1\right)\Theta(1-x)\, , \qquad (15)$$

where $p^2 = p_0^2 + \cdots + p_d^2$, we have $l_1^F(0) = 2/d$. Alternatively, for a sharp cutoff, we find $l_1^F(0) = 1$.

Apart from the Gaußian fixed point we find a second non-trivial fixed point for the coupling g which is implicitly given by

$$g_* = \frac{d-2+2\,\eta_\psi(g_*)}{4 d_\gamma v_d\, l_1^F(0)} . \qquad (16)$$

At leading order of our derivative expansion we have $\eta_\psi \equiv 0$ and thus

$$g_* = \frac{d(d-2)}{8 d_\gamma v_d} \overset{(d=3)}{=} \frac{3\pi^2}{4}$$

for the linear regulator and

$$g_* = \frac{d-2}{4 d_\gamma v_d} \overset{(d=3)}{=} \frac{\pi^2}{2}$$

for the sharp cutoff. The regulator dependence of the fixed-point value exemplifies the non-universality of this quantity. Nevertheless, the existence of the fixed point is a universal statement, as the regulator-dependent constant $l_1^F(0)$ is a positive number for any regulator. Moreover, the value of the non-Gaußian fixed point does not depend on N_f. In Fig. 1, we show a sketch of $\beta_g = \partial_t g$. The arrows indicate the direction of the flow towards the infrared. The theory becomes trivial (non-interacting) in the infrared regime for initial values $g_\Lambda < g_*$. Choosing $g_\Lambda > g_*$, the four-fermion coupling increases rapidly towards the infrared and diverges eventually. The divergence of g at a finite RG scale actually is an artifact of the over-simplistic fermionic truncation, but can be associated with the onset of chiral symmetry breaking and the formation of a fermion condensate. This becomes more obvious in the bosonic formulation, see below. In fact, the scale for a given IR observable $\mathcal{O}$ is set by the scale k_cr at which $1/g \to 0$:

$$\mathcal{O} \sim k_\mathrm{cr}^{d_\mathcal{O}} , \qquad (17)$$

where $d_\mathcal{O}$ is the canonical mass dimension of the observable $\mathcal{O}$. For $g_\Lambda < g_*$, the coupling never diverges but approaches zero and thus $\mathcal{O} \equiv 0$. For $g_\Lambda > g_*$, we find

$$k_\mathrm{cr} = \Lambda \left(\frac{|g_\Lambda - g_*|}{g_\Lambda} \right)^{\frac{1}{|\Theta|}} , \qquad (18)$$

where the critical exponent Θ is given by

$$\Theta = -\frac{\partial \beta_g(g=g_*)}{\partial g} = d - 2 + 2\eta_{\psi,*} - 2 g_* \frac{\partial \eta_\psi}{\partial g}\Big|_{g_*} . \qquad (19)$$

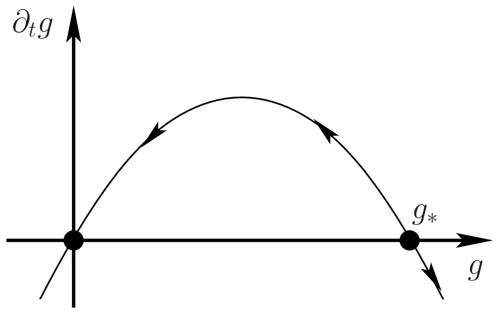

Figure 1: Sketch of the β function of the four-fermion coupling. For $g \geq g_*$ the infrared regime of the Gross-Neveu model is governed by chiral symmetry breaking (seemingly diverging fermionic interaction $g \to \infty$. For $g < g_*$ the model becomes a trivial (non-interacting) theory $g \to 0$ in the infrared.

Here, $\eta_{\psi,*}$ denotes the value of the fermionic anomalous dimension at the fixed point, $\eta_{\psi,*} = \eta_\psi(g = g_*)$. Relation (18) determines how a given IR observable scales when g_Λ is varied. As the initial condition g_Λ being larger or smaller than g_* distinguishes between two different phases in the long-range limit, the fixed point g_* can be viewed as a quantum critical point which divides the model into two physically different regimes.

For our main subject – asymptotic safety –, we ignore the infrared physics in the following and concentrate on the UV properties induced by the fixed point. In our simple fermionic truncation with only one coupling, the stability matrix boils down to a number, already represented by the critical exponent of the coupling direction in this one-dimensional theory space, see Eq. (19). At leading-order derivative expansion, where $\eta_\psi = 0$, the critical exponent is positive for all $d > 2$, such that the Gross-Neveu coupling corresponds to an RG relevant coupling being attracted by the non-Gaußian fixed point towards the UV. In this simple truncation, this suggests that the Gross-Neveu model can be renormalized and extended as a fundamental theory over all scales on RG trajectories that emanate from the non-Gaußian fixed point. As there is only one relevant direction, only one physical parameter has to be fixed (say the value of the coupling at a UV scale, g_Λ) in order to predict all physical quantities in the long-range limit.

It is instructive, to compare these conclusions within the asymptotic safety language with standard perturbation theory near the Gaußian fixed point $g_{*,\mathrm{Gauß}} = 0$. The corresponding critical exponent is

$$\Theta_{\mathrm{Gauß}} = -\frac{\partial \beta_g(g = g_{*,\mathrm{Gauß}})}{\partial g} = 2 - d, \qquad (20)$$

in agreement with (minus) the naive power-counting dimension of the coupling. At leading-order derivative expansion $\eta_\psi = 0$ and for $d > 2$, the critical exponent is negative and the fixed point thus infrared attractive. A UV limit $\Lambda \to \infty$ can only be taken if the RG trajectory emanates from the fixed point, but then the theory would be noninteracting on all scales and therefore "trivial". Within perturbation theory, the conclusion is that the Gross-Neveu model is perturbatively nonrenormalizable. Note that this conclusion remains unchanged also if the anomalous dimension is taken into account: within perturbation theory, $\eta_\psi = \mathcal{O}(g)$ (actually, accidentally $\mathcal{O}(g^2)$), such that $\eta_\psi = 0$ at the Gaußian fixed point, implying that standard power-counting can only be modified logarithmically.

Let us conclude with a word of caution on the derivative expansion in the fermionic truncation: in this simple approximation, the fixed-point seems to exist with similar properties in any dimension $d > 2$, in particular also in $d = 4$ and beyond. This conclusion will change, once composite bosonic degrees of freedom are taken into account. Fluctuations of the latter which are formed by fermionic interactions will remove the fixed point in the Gross-Neveu model for $d \geq 4$ such that no asymptotic safety scenario appears to exist for $d \geq 4$ in this model. In the fermionic language, the bosonic degrees of freedom correspond to specific nonlocal interactions or momentum-structures in the fermionic vertices. These are not properly resolved in a derivative expansion. As $d = 4$ is a marginal case, the conclusions for fermionic theories in $d = 4$ may depend on the details of the interaction and the algebraic structure of a given model; for instance, an asymptotic safety scenario in a standard-model-inspired $SU(N_{\mathrm{c}}) \times U(1)$ model has been discussed in [13].

A comparison with the asymptotic safety scenario for quantum gravity is also instructive: here, the upper critical dimension is $d = 2$ as in the fermionic models, and the non-Gaußian fixed point exist in simple truncations based on derivative expansions in all $d > 2$ [10, 18]. It is tempting to speculate whether strong-coupling phenomena such as bound-state formation may destabilize the fixed point above another so far unknown critical dimension. A similar phenomenon has been observed in extra dimensional Yang-Mills theories, where a non-Gaußian fixed point exists in $4 + \epsilon$ dimensions but is nonperturbatively destabilized for $\epsilon \gtrsim \mathcal{O}(1)$ [17].

V. PARTIALLY BOSONIZED GROSS-NEVEU MODEL

In this section, we study the (UV) fixed-point structure of the Gross-Neveu model by employing a (partially) bosonized version of the model. We relate our findings to the purely fermionic description, study analytically the large-N_{f} limit and show how corrections beyond the mean-field approximation can be systematically taken into account. Formulations of the Gross-Neveu model using partial bosonization are used for many aspects of the Gross-Neveu model such as the phase structure at zero and finite temperature and density [37–39].

Spontaneous symmetry-breaking and the formation of a fermion condensate in the Gross-Neveu model can conveniently be studied by introducing an auxiliary field σ into the functional integral. Formally, we introduce this auxiliary field by multiplying the path-integral by a suitable Gaußian factor. This is known as a Hubbard-Stratonovich transformation [40, 41]. The partially bosonized (PB) action of the Gross-Neveu model then reads

$$S_{\mathrm{PB}}[\bar{\psi},\psi,\sigma] = \int_x \left\{ \frac{N_{\mathrm{f}}}{2}\bar{m}^2\sigma^2 + \bar{\psi}\left(\mathrm{i}\partial\!\!\!/ + \mathrm{i}\bar{h}\sigma\right)\psi \right\}, \quad (21)$$

and the functional integral now includes integration measures for both fermionic and bosonic fields. As the auxiliary bosonic field only occurs quadratically in the action, it can be integrated out again; on the level of the action, this corresponds to replacing the bosonic field by its equation of motion, $N_{\mathrm{f}}\,\bar{m}^2\sigma = i\bar{h}\bar{\psi}\psi$. The action then reduces again to the Gross-Neveu action upon identifying $\bar{g} = \frac{\bar{h}^2}{\bar{m}^2}$. From the viewpoint of the Hubbard-Stratonovich transformation, the Yukawa coupling $\bar{h}$ is redundant as it can be scaled into the sigma field. Only the ratio $\bar{h}^2/\bar{m}^2$ has a physical meaning. In our formulation, the Yukawa coupling $\bar{h}$ is implicitly understood to carry mass dimension $[\bar{h}] = (4-d)/2$ in order to deal with an auxiliary field with canonical mass dimension $[\sigma] = (d-2)/2$. Moreover, the σ field has been scaled such that the first and second term in (21) are of the same order in N_{f}. Under a discrete chiral transformation, see Eq. (6), the σ-field transforms as $\sigma \mapsto -\sigma$. From a phenomenological point of view σ can be considered as a bound state of fermions, $\sigma \sim \bar{\psi}\psi$. Thus, the vacuum expectation value of σ is a proper order parameter for chiral symmetry breaking, as it is the case in the purely fermionic formulation of the model.

A. Mean-field analysis

Before we analyze the partially bosonized version of the Gross-Neveu model by means of the functional RG, let us start with a simple mean-field study, corresponding to the large-N_{f} limit, in order to rediscover aspects of the fermionic language of the preceding section in this standard textbook language. Due to the Hubbard-Stratonovich transformation, the N_{f} fermion flavors enter only bilinearly and can be integrated out from the corresponding functional integral. This yields a purely bosonic effective theory for the Gross-Neveu model:

$$S_{\mathrm{B}}[\sigma] = \int_x \frac{N_{\mathrm{f}}}{2}\bar{m}^2\sigma^2 - N_{\mathrm{f}}\mathrm{Tr}\ln\left[\mathrm{i}\partial\!\!\!/ + \mathrm{i}\bar{h}\sigma\right], \quad (22)$$

where Tr denotes a functional trace. Since σ depends on the space-time coordinates, the action S_{B} is highly non-local and therefore in general difficult to study. The ground state σ_0 can be obtained from the variational principle, i.e., the gap equation

$$\frac{\delta}{\delta\sigma(x)}S_{\mathrm{B}}[\sigma]\bigg|_{\sigma_0(x)} \stackrel{!}{=} 0. \quad (23)$$

As we are interested in the UV properties of the model, we assume that the ground state is homogeneous. (In $d=2$, inhomogeneous condensates have been identified in some parts of the phase diagram at finite temperature and large values of the chemical potential [38]; the status for higher-dimensional fermionic models is subject to ongoing work, see e. g. [42, 43]) Such a homogeneous ground state σ_0 =const. is then implicitly given by the solution of the following equation:

$$\sigma_0 = 4\frac{\bar{h}^2}{\bar{m}^2}\int_p \frac{\sigma_0}{p^2+\bar{h}^2\sigma_0^2}\,, \quad (24)$$

with $\int_p = \int \frac{d^dp}{(2\pi)^d}$. Apparently this equation has a trivial solution, $\sigma_0 = 0$. Moreover, non-trivial solutions for σ_0 can be found for suitably adjusted values of the four-fermion coupling $\bar{g} = \bar{h}^2/\bar{m}^2$. Since in $d=3$ ($d=4$) the right-hand side of Eq. (24) is linearly (quadratically) divergent, we impose a sharp UV cutoff $\Lambda \gg m_{\mathrm{f}} = \bar{h}^2\sigma_0^2$. For $d=3$, we find

$$m_{\mathrm{f}} = \frac{2}{\pi}\left(\Lambda - \frac{\pi^2}{2}\frac{\bar{m}^2}{\bar{h}^2}\right). \quad (25)$$

Thus, the fermions acquire a non-zero mass due to the spontaneous breakdown of chiral symmetry, provided we choose $\bar{m}^2/\bar{h}^2 < 2\Lambda/\pi^2$. In terms of the four-fermion coupling, we can read off a critical value for the dimensionless coupling $g = \Lambda\bar{g} = \Lambda\bar{h}^2/\bar{m}^2$ above which the IR physics is governed by spontaneous symmetry breaking:

$$g_{\mathrm{cr}} = \frac{\pi^2}{2}. \quad (26)$$

This critical value g_{cr} can be identified with the sharp-cutoff value of the non-trivial fixed point g_* which we found in our study of the fermionic fixed-point structure. The role of the critical value as a fixed point becomes obvious from the fact that if $g = g_{\mathrm{cr}}$ the theory is interacting but remains massless and ungapped on all scales. From the viewpoint of the partially bosonized theory, we find that the Yukawa coupling and the boson mass are not independent parameters. For a fixed ratio $\bar{h}^2/\bar{m}^2 > g_{\mathrm{cr}}$ the IR physics remains unchanged. Thus, the purely bosonic description of the theory in this approximation depends only on a single parameter as in the fermionic formulation.

B. RG flow of the partially bosonized theory

Let us now discuss the fixed-point structure of the partially bosonized theory. A partially bosonized description of the theory is appealing from a field-theoretical point as it also forms the basis for the expansion in $1/N_\mathrm{f}$ for a large number of flavors. In addition, it allows us to systematically resolve parts of the momentum dependence of the vertices by means of a derivative expansion. As we shall see below, these two expansion schemes are not identical and should therefore not be confused with each other. For our study we employ the following ansatz for the effective action:

$$\Gamma[\bar{\psi},\psi,\sigma] \;=\; \int_x \Big\{ \frac{N_\mathrm{f}}{2} Z_\sigma (\partial_\mu \sigma)^2 + \bar{\psi}\left(Z_\psi \mathrm{i}\partial\!\!\!/ + \mathrm{i}\bar{h}\sigma\right)\psi + N_\mathrm{f}\, U(\sigma^2)\Big\}\,, \tag{27}$$

where we allow all couplings and wave function renormalizations $Z_{\sigma,\psi}$ to be scale dependent. The kinetic term of the boson field adds a new aspect: on the one hand, it goes beyond the local approximation of simple mean-field theory; in terms of the fermionic language, this kinetic term corresponds to a specific momentum dependence in the scalar s channel of the four-fermion coupling on the other hand. As we shall see below, this term and the associated wave function renormalization receive contributions to leading order in the large-N_f approximation (in order to simplify the large-N_f counting of orders, the purely bosonic sector is multiplied by an N_f factor in Eq. (27) similar to Eq. (21)). The large-N_f flow corresponds to the choice

$$\begin{aligned} Z_\sigma\big|_{k\to\Lambda} &\to 0\,, \qquad \partial_t Z_\sigma \neq 0\,,\\ Z_\psi\big|_{k\to\Lambda} &\equiv 1\,, \qquad \partial_t Z_\psi \equiv 0\,. \end{aligned} \tag{28}$$

This exemplifies the difference between large-N_f and derivative expansion, as with this choice we include next-to-leading order corrections in terms of a derivative expansion in the bosonic sector but treat the fermionic sector in the leading-order approximation.

From our ansatz (27) we see that a nonzero homogeneous expectation value for $\mathrm{i}\bar{h}\sigma$ plays the role of a mass term for the fermions. By expanding the effective action about the homogeneous background σ we anticipate that condensation occurs only in the homogeneous channel. The bosonized Yukawa model is fixed to the fermionic Gross-Neveu model by a suitable choice of initial conditions for Γ at the UV scale Λ. This correspondence is established by the choice

$$Z_\sigma\big|_{k\to\Lambda} \ll 1\,,\quad Z_\psi\big|_{k\to\Lambda} \to 1\,,\quad U_\Lambda = \frac{1}{2}\bar{m}_\Lambda^2\,. \tag{29}$$

Thus, the renormalized boson mass $m = \bar{m}/\sqrt{Z_\sigma}$ at the UV cutoff Λ becomes much larger than Λ and renders the boson propagator essentially momentum independent. This compositeness condition for the bosonic formulation can be considered as a locality condition at the UV scale for the four-fermion coupling g in the purely fermionic formulation of the model.

Since we are interested in the UV fixed-point structure of the partially bosonized (chirally-symmetric) Gross-Neveu model, we anticipate that a possible non-Gaußian fixed point occurs in the symmetric regime. Therefore, we only need to study the RG flow in the symmetric regime with vanishing vacuum expectation value for the σ field. We then find the following flow equation for the dimensionless effective potential:

$$\begin{aligned} \partial_t u \;&=\; -du + (d-2+\eta_\sigma)u'\rho - 2d_\gamma v_d\, l_0^{(F)d}(2h^2\rho;\eta_\psi)\\ &\quad + \frac{1}{N_\mathrm{f}} 2v_d\, l_0^d(u' + 2\,\rho u'';\eta_\sigma)\,, \end{aligned} \tag{30}$$

where

$$u(\rho) = k^{-d}U(\sigma)\,,\qquad \rho = \frac{1}{2}Z_\sigma k^{2-d}\sigma^2, \tag{31}$$

and the dimensionless renormalized Yukawa coupling is given by

$$h^2 = Z_\sigma^{-1} Z_\psi^{-2} k^{d-4}\bar{h}^2\,. \tag{32}$$

The flow equation for the potential can be solved either directly or by an expansion around $\rho = 0$,

$$u(\rho) = \sum_{n=0}^{\infty} \frac{\lambda_{2n}}{n!}\rho^n\,. \tag{33}$$

In this paper, we shall mainly focus on the latter one which allows us to directly project onto the (dimensionless) bosonic mass parameter λ_2 and the higher-order bosonic couplings λ_{2n}:

$$\lambda_2 = \frac{\bar{m}^2}{Z_\sigma k^2}\,,\qquad \lambda_{2n} = Z_\sigma^{-n} k^{(n-1)d-2n} U^{(n)}\big|_{\sigma=0}\,. \tag{34}$$

The flow equations for the Yukawa coupling as well as the anomalous dimensions $\eta_\sigma = -\partial_t \ln Z_\sigma$ and $\eta_\psi = -\partial_t \ln Z_\psi$ read[1]

$$\begin{aligned} \partial_t h^2 \;&=\; (d-4+2\eta_\psi+\eta_\sigma)h^2\\ &\quad + \frac{1}{N_\mathrm{f}} 8 v_d\, h^4\, l_{1,1}^{(FB)d}(0,\lambda_2;\eta_\psi,\eta_\sigma)\,, \end{aligned} \tag{35}$$

$$\eta_\sigma \;=\; 8\frac{d_\gamma v_d}{d} h^2\, m_4^{(F)d}(0;\eta_\psi)\,, \tag{36}$$

$$\eta_\psi \;=\; \frac{1}{N_\mathrm{f}} 8 \frac{v_d}{d} h^2 m_{1,2}^{(FB)d}(0,\lambda_2;\eta_\psi,\eta_\sigma)\,, \tag{37}$$

The threshold functions l and m in Eqs. (31) and (35)-(37) depend on the details of the regulator. For practical computations, we use an optimized regulator [34–36] for the fermionic fields (15) and the bosonic fields:

$$R_k = Z_\sigma r(p^2/k^2)\,,\quad r(x) = \left(\frac{1}{x} - 1\right)\Theta(1-x)\,. \tag{38}$$

The corresponding threshold functions are listed in App. A. These functions essentially describe the threshold behavior of regularized 1PI diagrams.

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[1] These flow equations agree with those derived in [14, 28] in the symmetric phase, upon a rescaling of the wave function renormalization Z_σ by a factor of N_f, cf. Eq. (27).

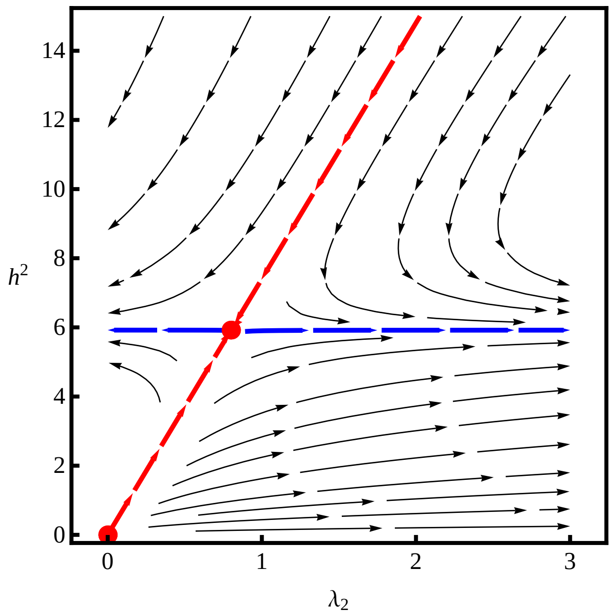

Figure 2: Leading-order RG flow in the $1/N_\mathrm{f}$ expansion for the partially bosonized Gross-Neveu model in the (h^2, λ_2) plane. The red line denotes the critical manifold of points drawn into the fixed point towards the infrared. The blue line depicts the critical surface attracting the flow towards the IR. The critical surface contains all trajectories that emanate from the non-Gaußian fixed point in the UV; its dimensionality equals the number of relevant directions and thus the number of physical parameters to be fixed. The red dots denote the Yukawa Gaußian fixed point and the non-Gaußian fixed point, respectively. The arrows indicate the direction of flow towards the infrared.

C. RG flow at large N_f

As the nonperturbative renormalizability of the Gross-Neveu model beyond $d = 2$ has been proved to all orders in the $1/N_\mathrm{f}$ expansion, it is worthwhile to study asymptotic safety on the basis of the RG flow in the same limit. Here, our set of flow equations for the partially bosonized Gross-Neveu model reduces to

$$\begin{aligned}\partial_t u &= -du + (d-2+\eta_\sigma)u'\rho \\ &\quad -2d_\gamma v_d\, l_0^{(F)d}(2h^2\rho;\eta_\psi), \qquad (39)\\ \partial_t h^2 &= (d-4+2\eta_\psi+\eta_\sigma)h^2, \qquad (40)\\ \eta_\sigma &= 8\frac{d_\gamma v_d}{d}h^2\, m_4^{(F)d}(0;\eta_\psi)\,, \qquad (41)\\ \eta_\psi &= 0\,. \qquad (42)\end{aligned}$$

As the bosonic fluctuations carry no flavor number, we observe that 1PI diagrams with at least one inner bosonic line decouple completely from the large-N_f RG flow. As a consequence, η_σ is non-vanishing in leading order in the $1/N_\mathrm{f}$ expansion, whereas the fermionic anomalous dimension is zero.[2] The flow equations for the bosonic couplings are essentially driven by the fermion loop,

$$\begin{aligned}\partial_t\lambda_{2n} &= \big(n(d-2+\eta_\sigma)-d\big)\lambda_{2n} \\ &\quad -(-1)^n n!\,2^{n+2}\left(\frac{d_\gamma v_d}{d}\right)\left(h^2\right)^n. \qquad (43)\end{aligned}$$

Fixed point values h_*^2, $\eta_{\sigma,*}$, $\lambda_{2,*}$, $\lambda_{4,*}$, $\lambda_{6,*}$, ... can be identified as the zeroes of the corresponding β functions, $\partial_t h^2 \stackrel{!}{=} 0\,,\ \partial_t\lambda_2 \stackrel{!}{=} 0, \ldots$ Of course, the Gaußian fixed point with all couplings vanishing solves these fixed point equations. As the RG flows for the bosonic couplings decouple in the large-N_f limit, a non-trivial fixed point requires $h_* \neq 0$. This immediately requires

$$\eta_{\sigma,*} = 4-d. \qquad (44)$$

Whereas this tight relation between the dimensionality and the bosonic anomalous dimension here is an artifact of the large-N_f expansion, a similar relation exists in gravity for the graviton anomalous dimension at the fixed point as a consequence of background gauge invariance. Similar sum rules are known for Yukawa theories with chiral symmetries [33]. Such a sum rule for a corresponding fixed point is also responsible for the universality of the BCS-BEC crossover in the broad resonance limit of ultracold fermi gases [46]. In the present case, this fixes the value of the Yukawa fixed-point coupling,

$$h_*^2 = \left(\frac{d}{d_\gamma v_d}\right)\frac{(d-4)(d-2)}{(8-6d)}. \qquad (45)$$

The fixed point is interacting for $2 < d < 4$ and merges with the Gaußian fixed point in $d = 2$ and $d = 4$. Also the fixed-point values for the bosonic mass parameter and couplings can be given analytically

$$\lambda_{2n,*} = \frac{(-1)^n n!\,2^{n+2}}{(2n-d)}\left(\frac{d}{d_\gamma v_d}\right)^{n-1}\left(\frac{(d-4)(d-2)}{(8-6d)}\right)^n. \qquad (46)$$

Thus, the fixed-point values for all bosonic vertices λ_{2n} are non-vanishing. In a purely fermionic formulation of the model, these higher bosonic self-interactions correspond to higher (non-local) fermionic self-interactions. In any case, we find that the UV fixed-point theory in $2 < d < 4$ for $N_\mathrm{f} \to \infty$ is not identical to the action S_PB but involves infinitely many operators.

It turns out that the alternating series for the full renormalized effective potential can be resummed and yields a representation of the Gaußian Hypergeometric

[2] It is worthwhile to emphasize, that the large-N_f counting is very different from scalar O(N) models, where the the anomalous dimensions are zero to leading order at large N_f [44]. Also the structure of the potential equation is very different, such that also the search for an exact solution requires a different strategy from that of O(N) models [45], see below.

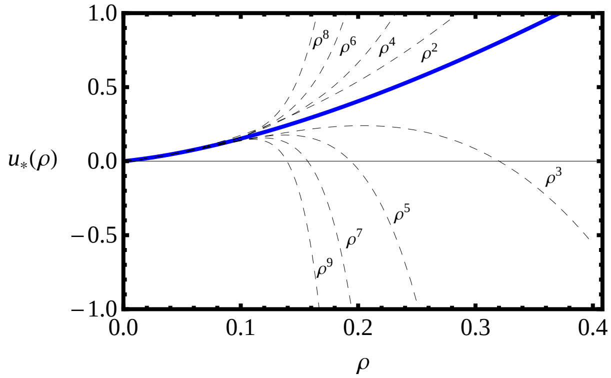

Figure 3: (Resummed) Fixed-point potential $u_*(\rho)$ in the large-N_f limit (blue/solid), see Eq. (47), and low-order polynomial approximations thereof ranging from ρ^2 to ρ^9 for small values of ρ (dashed lines).

function ${}_2F_1(a,b;c;z)$ (see e.g. [47]). In terms of renormalized fields, the scale invariant fixed-point action then describes massless fermions coupled to a scalar boson with a potential

$$u_*(\rho) = -\frac{2d-8}{3d-4}\rho \times \\ {}_2F_1\left(1-\frac{d}{2}, 1; 2-\frac{d}{2}; \frac{(d-4)(d-2)}{6d-8}\frac{d}{d_\gamma v_d}\rho\right). \tag{47}$$

This potential has a large-field asymptotic behavior $\propto \rho^{\frac{3}{2}}$. The small-field region is depicted in Fig. 3.

The theory has predictive power, as the number of physical parameters is determined by the number of RG relevant directions corresponding to the number of positive critical exponents. At the non-Gaußian fixed point in the large-N_f limit, the stability matrix assumes a particularly simple form where only a single column and the main diagonal are non-vanishing,

$$B = \begin{pmatrix} b_{h^2,h^2} & 0 & \cdots & & \\ b_{h^2,\lambda_2} & b_{\lambda_2,\lambda_2} & 0 & \cdots & \\ b_{h^2,\lambda_4} & 0 & b_{\lambda_4,\lambda_4} & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \end{pmatrix}. \tag{48}$$

Therefore only the main diagonal of B enters into the stability analysis around the non-Gaußian fixed point. The important non-vanishing entries b_{h^2,h^2} and $b_{\lambda_{2n},\lambda_{2n}}$ turn out to be completely universal and are given by

$$b_{h^2,h^2} \equiv \frac{\partial \beta_{h^2}}{\partial h^2}\Big|_{g=g_*} = \eta_{\sigma,*} = 4-d \tag{49}$$

and

$$b_{\lambda_{2n},\lambda_{2n}} \equiv \frac{\partial \beta_{\lambda_{2n}}}{\partial \lambda_{2n}}\Big|_{g=g_*} = -d + n(d-2+\eta_{\sigma,*}) = 2n-d. \tag{50}$$

The characteristic polynomial $\det(B+\Theta\mathbb{1})$ of the matrix B yielding the eigenvalues $-\Theta^I$ via its zeroes is then easily found to be

$$(-\Theta-(4-d))\prod_{n=1}^{\infty}(-\Theta-(2n-d)). \tag{51}$$

All large-N_f critical exponents are thus given by $d-4$ and $d-2n$. In $d=3$ this boils down to one positive critical exponent with value 1, i.e., one relevant RG direction, and infinitely many negative critical exponents $-1,-1,-3,-5,-7,\ldots$, corresponding to irrelevant RG directions. For the case of the Gaußian Yukawa fixed point, the characteristic polynomial of the stability matrix B is changed to

$$(-\Theta-(d-4))\prod_{n=1}^{\infty}(-\Theta-(n(d-2)-d)). \tag{52}$$

As expected, the critical exponents coincide with the mass dimension of the Yukawa coupling and the bosonic couplings, reproducing simple perturbative power counting. In total, this yields three relevant RG directions and one marginal RG direction. Note that the Gaußian fixed point found in the purely fermionic flow in Sect. IV translates into a diverging dimensionless renormalized boson mass $\lambda_2^{1/2} \sim g^{-1/2}$.

Returning to the non-Gaußian fixed point, we can, of course, make contact with the purely fermionic description and deduce the fixed point of the four-fermion coupling from the fixed point values of the Yukawa coupling and the bosonic mass parameter. We find

$$g_* = \frac{h_*^2}{\lambda_{2,*}} = \left(\frac{d}{d_\gamma v_d}\right)\frac{d-2}{8} \overset{(d=3)}{=} \frac{3\pi^2}{4}, \tag{53}$$

for the linear regulator which agrees with our findings in Sect. IV (using the same regulator). In fact, the flow equation of the four-fermion interaction g can be reconstructed from the flow of h^2/λ_2,

$$\partial_t\left(\frac{h^2}{\lambda_2}\right) = (d-2)\left(\frac{h^2}{\lambda_2}\right) - \frac{8d_\gamma v_d}{d}\left(\frac{h^2}{\lambda_2}\right)^2 + \mathcal{O}\left(\frac{1}{N_\mathrm{f}}\right). \tag{54}$$

Using the linear regulator in Eq. (14), we observe that the flow equation for g and h^2/λ_2 are identical in the large-N_f limit. Recall that $\eta_\psi = 0$ in this limit.

Due to the equivalence of g and h^2/λ_2, the quantum critical point found in the purely fermionic formulation is also present in our study of the partially bosonized theory for $N_\mathrm{f}\to\infty$, as it should be the case. As it can be seen from the scaling law (18), this quantum critical point is associated with a vanishing boson mass, i.e., a diverging correlation length, in the long-range limit.

Let us conclude our large-N_f analysis with a word of caution on the widely used so-called local potential approximation (LPA) in which the running of the wave-function renormalizations are neglected. If we ignored the running of the wave-function renormalization of the bosonic field in the present case, the model would artificially depend on more than one physical parameter. To be more specific, let us consider the mass spectrum of the theory in the regime with broken chiral symmetry and assume that we have already fixed the mass of the fermions. Using the definition of the masses and the flow equations of the couplings, we find that the (dimensionless) renormalized boson mass in the broken regime can be written in terms of the (renormalized) fermion mass m_{f}:

$$m^2 = 2\lambda_4 \rho_0 \sim Z_\sigma^{-1} \bar{h}^2 (\bar{h}^2 \bar{\rho}_0) \sim Z_\sigma^{-1} \bar{h}^2 m_{\mathrm{f}}^2 \,, \qquad (55)$$

where $\rho_0 = (1/2)\sigma_0^2$. Neglecting the running of Z_σ, i. e. $Z_\sigma =$ const. as is done in the LPA, we observe that the boson mass does not depend on a single physical parameter, as it should be, but on two parameters independently, namely the fermion mass and the (bare) Yukawa coupling. By contrast, taking the running of $Z_\sigma \sim \bar{h}^2$ into account, the value of the boson mass is fixed solely in terms of the fermion mass, in agreement with our fixed-point analysis. While this argument might be altered in $d = 4$ space-time dimensions where the Yukawa coupling is marginal, it is true for the Gross-Neveu model (as well as the Nambu-Jona-Lasinio model) in any dimension d in which the flow equation for Z_σ is non-vanishing even at leading order in an expansion in $1/N_{\mathrm{f}}$. Therefore the flow of Z_σ has to be taken into account in a systematic and consistent expansion of the flow equations in powers of $1/N_{\mathrm{f}}$. To be specific, the flow of the order parameter potential (30) incorporates already fluctuations at next-to-leading order in $1/N_{\mathrm{f}}$ due to the presence of the bosonic loop[3]. However, for a systematic and consistent study of the effects of corrections beyond the large-N_{f} expansion the flow of Z_σ, Z_ψ as well as of the Yukawa coupling needs to be taken into account.

D. RG flow for general flavor number N_{f}

Beyond the limit of large N_{f}, bosonic fluctuations begin to play a role. An immediate consequence is that a new fixed point for $h = 0$ arises for the flow of the effective potential for $2 < d < 4$. This fixed point of the purely bosonic theory is nothing but the Wilson-Fisher fixed point which describes critical phenomena in the Ising universality class.

[3] We stress that our parameter N_{f} plays the role of the number of colors N_{c} in QCD. The number of flavors N_{f} in, e. g., QCD low-energy models is related to the number of involved mesonic scalar fields $N = N_{\mathrm{f}}^2$. Thus, a large-N_{c} expansion in QCD models corresponds to a large-N_{f} expansion in the Gross-Neveu model. For an analysis of the role of corrections beyond the large-N_{c} approximation for the thermodynamics of QCD low-energy models we refer to Ref. [48].

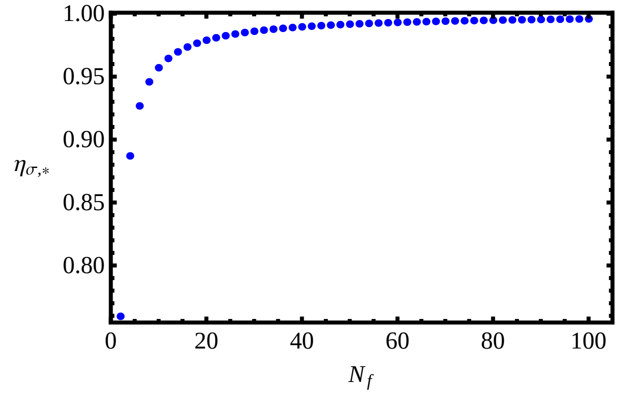

Figure 4: Scalar anomalous dimension η_σ for $N_{\mathrm{f}} = 2, 4, 6, \ldots, 100$ in $d = 3$.

The non-Gaußian fixed point of the full Gross-Neveu system can now be understood as being sourced from the leading large-N_{f} terms discussed above and the bosonic fluctuations inducing a Wilson-Fisher fixed point. Depending on the value of N_{f} the non-Gaußian Gross-Neveu fixed point interpolates between the large-N_{f} fixed point for $N_{\mathrm{f}} \to \infty$ and the Wilson-Fisher fixed point in the formal limit of $N_{\mathrm{f}} \to 0$. For the latter, the functional RG has already proven to be a useful quantitative tool for describing nonperturbative critical phenomena, see e. g. Refs. [30, 49–54].

Let us repeat the preceding large-N_{f} analysis, now using the full flow equations at next-to-leading order derivative expansion, i.e., Eqs. (30), (35), (37), and (36). For all quantities of interest, such as critical exponents and fixed point values of couplings, a solution of the potential flow in a polynomial expansion is sufficient. Confining our numerical studies to $d = 3$, all figures are produced within a truncated expansion up to 22nd order in σ; quantitative results are derived from an expansion to the same order in σ. In the symmetric regime, a nontrivial fixed point in the Yukawa coupling requires the following inequality to be satisfied,

$$d - 4 + 2\eta_{\psi,*} + \eta_{\sigma,*} < 0, \quad \text{for } N_{\mathrm{f}} < \infty. \qquad (56)$$

This is because the second term of the Yukawa flow Eq. (35) is strictly positive for admissible values of the anomalous dimensions $\eta_\sigma, \eta_\psi \lesssim \mathcal{O}(1)$. For instance, in $d = 3$, the sum of the anomalous-dimension terms is always slightly smaller than 1, see Figs. 4 and 5. The inequality becomes an equality in the large-N_{f} limit, see Eq. (44).

The resulting fixed point values for the Yukawa coupling in $d = 3$ are depicted in Fig. 6. For increasing N_{f}, the fixed point quickly approaches its large-N_{f} limit (45), whereas it tends to zero for small N_{f} leaving us with the pure Wilson-Fisher fixed point of a pure scalar model. As the latter is known to exhibit a fixed-point potential in the broken regime, i.e., with a non-vanishing expectation value of the scalar field σ, we expect such

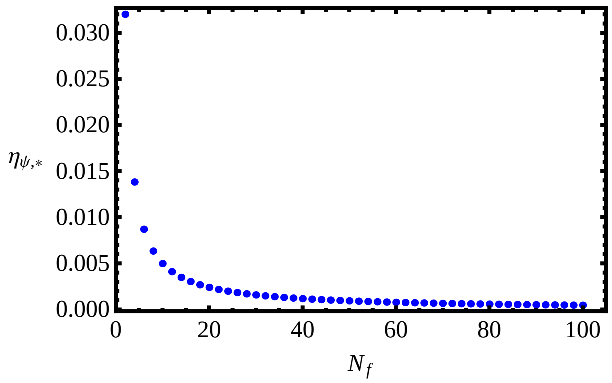

Figure 5: Scalar anomalous dimension η_ψ for $N_\mathrm{f} = 2, 4, 6, \ldots, 100$ in $d = 3$.

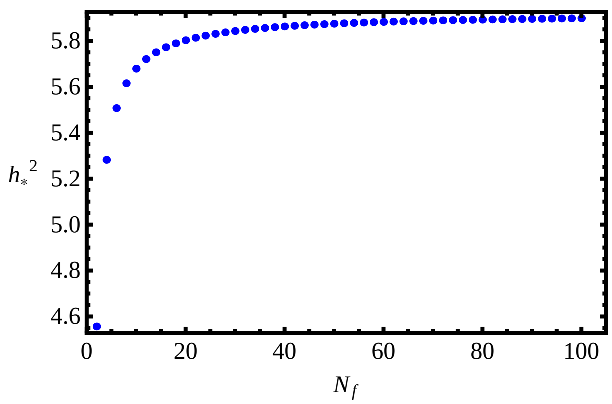

Figure 6: Yukawa fixed-point value over $N_\mathrm{f} = 2, 4, 6, \ldots, 100$ in $d = 3$.

a fixed-point potential featuring a nontrivial minimum to occur for small N_f. For all integer values of $N_\mathrm{f} \geq 1$ Dirac (four-component) fermions, we still observe fixed-point potentials in the symmetric regime, in agreement with [28]. Nevertheless, the fixed point seems to occur in the broken regime for the model with one two-component fermion (corresponding to $N_\mathrm{f} = 1/2$ in our language) [28].

The fixed-point potential in $d = 3$ for various values of N_f is plotted in Fig. 7 in a 22nd-order approximation. Also the potential converges rapidly to the large-N_f result for increasing values of N_f.

Let us now turn to the leading universal critical exponents $\Theta^{1,2}$. The convergence of the polynomial expansion of the potential is demonstrated in Tabs. I II, where the leading critical exponents for $N_\mathrm{f} = 2$ and $N_\mathrm{f} = 12$ are listed for increasing truncation order. In each case, the leading exponents converge to a stable value. We observe a more rapid convergence for larger values of N_f. The somewhat slower convergence for more scalar dominated models is familiar from pure O(N) models. The leading two critical exponents are plotted as a function of N_f in Figs. 8 and 9. Table III lists the leading exponents for increasing number of N_f, illustrating the approach to the analytical large-N_f results.

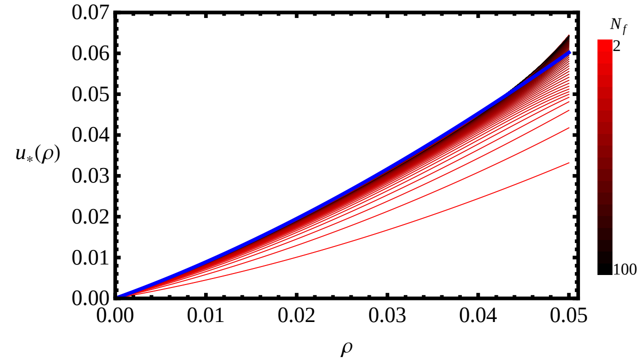

Figure 7: Fixed-point potentials $u_*(\rho)$ for $N_\mathrm{f} = 2, 4, 6, \ldots, 100$ in $d = 3$. The large-N_f result is shown as a thick/blue line. Incidentally, for $N_\mathrm{f} = 0$ the potential approaches the Wilson-Fisher fixed-point potential with a nontrivial minimum near $\rho \simeq 0.3$ (not shown).

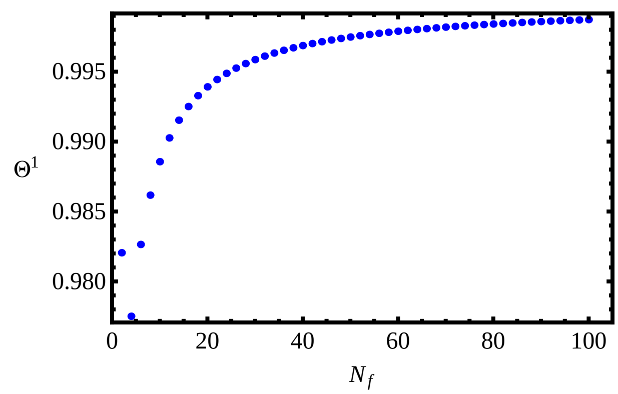

Figure 8: Relevant critical exponent Θ^1 for $N_\mathrm{f} = 2, 4, 6, \ldots, 100$ in $d = 3$. The non-monotonic behavior for small N_f is expected, since Θ^1 has to approach $\Theta^1 \simeq 1.6$ for $N_\mathrm{f} = 0$ corresponding to a correlation length exponent $\nu = 1/\Theta^1 \simeq 0.63$ of the Ising universality class.

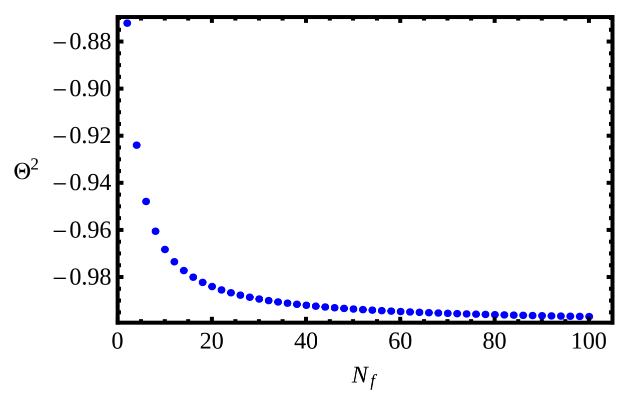

Figure 9: First subleading irrelevant critical exponent Θ^2 for $N_\mathrm{f} = 2, 4, 6, \ldots, 100$ in $d = 3$.

$2n$	Θ^1	Θ^2	Θ^3	Θ^4	Θ^5	Θ^6
4	0.9928	−0.8687	−1.5743	–	–	–
6	0.9766	−0.8743	−1.0624	−5.4313	–	–
8	0.9831	−0.8721	−1.0790	−3.5622	−10.5959	–
10	0.9821	−0.8720	−1.0999	−3.4194	−6.8111	−17.4807
12	0.9819	−0.8723	−1.0897	−3.5628	−5.7949	−11.2156
14	0.9821	−0.8722	−1.0911	−3.5104	−6.0610	−8.6408
16	0.9820	−0.8722	−1.0920	−3.5062	−6.1190	−8.3598
18	0.9820	−0.8722	−1.0914	−3.5202	−5.9849	−8.9972
20	0.9821	−0.8722	−1.0915	−3.5132	−6.0516	−8.5869
22	0.9821	−0.8722	−1.0916	−3.5135	−6.0514	−8.5820

Table I: Non-Gaußian critical exponents in $d = 3$ for increasing polynomial truncations for $N_{\rm f} = 2$. The results for the critical exponent Θ^1 agree within the error bars with the result from Monte-Carlo (MC) simulations [24], $1/\Theta^1_{\rm MC} = \nu_{\rm MC} \approx 1.00(4)$.

$2n$	Θ^1	Θ^2	Θ^3	Θ^4	Θ^5	Θ^6
4	0.9898	−0.9735	−1.0701	–	–	–
6	0.9903	−0.9735	−1.0489	−3.2583	–	–
8	0.9903	−0.9735	−1.0507	−3.1714	−5.5889	–
10	0.9903	−0.9735	−1.0505	−3.1821	−5.3368	−8.1011

Table II: Non-Gaußian critical exponents in $d = 3$ for increasing polynomial truncations for $N_{\rm f} = 12$.

$N_{\rm f}$	Θ^1	Θ^2	Θ^3	Θ^4	Θ^5	Θ^6
2	0.9821	−0.8722	−1.0916	−3.5135	−6.0514	−8.5820
4	0.9775	−0.9240	−1.1010	−3.3910	−5.7739	−8.2429
12	0.9903	−0.9735	−1.0506	−3.1810	−5.3665	−7.6004
50	0.9975	−0.9936	−1.0143	−3.0510	−5.1062	−7.1789
100	0.9987	−0.9968	−1.0073	−3.0263	−5.0550	−7.0934
∞	1	−1	−1	−3	−5	−7

Table III: Non-Gaußian critical exponents for various flavor numbers $N_{\rm f}$ in $d = 3$ in the ρ^{11} truncation.

This approach is also visible in the anomalous dimension and the fixed point values for the coupling, see Tab. IV. Whereas the fixed-point couplings are non-universal (holding for the linear regulator in this case), the anomalous dimensions are universal and illustrate the inequality (56).

Quantitatively, our results for the leading critical exponent can be compared to those studies of the quantum critical phase transition aiming at the long-range physics, as Θ^1 is related to the correlation length exponent ν by $\nu = 1/\Theta^1$. Together with the scalar anomalous dimension and corresponding scaling and hyperscaling relation, all thermodynamic exponents of the phase transition are determined. Wherever comparable, our results agree quantitatively with the functional RG study of [28] where both a polynomial expansion as well as a grid solution of the potential was used (note that our $N_{\rm f}$ counts four-component fermions, whereas [28] uses two-component fermions). Also the agreement with results from other methods such as $1/N_{\rm f}$ expansions [26] and Monte Carlo simulations [23, 24] is satisfactory. Discrepancies are mainly visible only in the anomalous dimensions for small $N_{\rm f}$, a feature familiar from scalar models. Also the polynomial expansion of the potential converges somewhat slower for the special case $N_{\rm f} = 1$, where our results for the leading exponents are compatible with those of [28], but subleading exponents seem to require a precise solution of the potential flow.

To summarize, the nonperturbative features of the Gross-Neveu model near criticality can well be described by the functional renormalization group, as the model interpolates between two well-accessible limits: the large-$N_{\rm f}$ limit for $N_{\rm f} \to \infty$ and the Wilson-Fisher fixed point in the Ising universality class for $N_{\rm f} \to 0$. Our results suggest that the Gross-Neveu model is asymptotically safe for all $N_{\rm f} > 0$ and that the model depends on only one parameter, even when we take into account corrections beyond the large-$N_{\rm f}$ limit. This might provide helpful information, e.g., for a systematic study of the finite-temperature phase diagram of the Gross-Neveu model beyond the large-$N_{\rm f}$ approximation.

$N_{\rm f}$	$\eta_{\sigma,*}$	$\eta_{\psi,*}$	h_*^2	$\lambda_{2,*}$
2	0.7596	0.0320	4.5565	0.3956
4	0.8870	0.0138	5.2820	0.5846
12	0.9644	0.0041	5.7206	0.7263
50	0.9917	0.0009	5.8746	0.7822
100	0.9958	0.0005	5.8983	0.7911
∞	1	0	5.9218	0.8

Table IV: Non-Gaußian fixed-point values of the universal anomalous dimensions for various flavor numbers $N_{\rm f}$ in $d = 3$. The (non-universal) fixed-point couplings hold for the linear regulator. In Monte-Carlo (MC) simulations [24] $\eta_{\sigma,*}^{\rm MC} = 0.754(8)$ has been found for $N_{\rm f} = 4$ two-component fermions (corresponding to $N_{\rm f} = 2$ in our language).

VI. CONCLUSIONS AND OUTLOOK

We have used the functional RG to describe the UV behavior of the Gross-Neveu model in $2 < d < 4$ dimensions. In agreement with many earlier results in the literature, the model is nonperturbatively renormalizable by means of a non-Gaußian fixed point, providing a simple example of asymptotic safety in Weinberg's sense. The perturbative conclusion about nonrenormalizability is a mere artifact of naive power-counting which is only justified near the Gaußian fixed point.

In this work, we have summarized these conclusions in the functional RG language as it is also extensively used recently to explore the possibility of quantizing gravity within pure quantum field theory. This pedagogic character of our work is amended by new results in the large-$N_{\rm f}$ limit, where the fixed-point potential as well as all critical exponents can be computed analytically. Moreover, we have also provided finite $N_{\rm f}$ results, demonstrating that the model interpolates between the large $N_{\rm f}$ limit and a purely bosonic model in the Ising universality class.

Some final comments are in order: the simple fermionic Gross-Neveu action is minimalistic in the sense that it suffices to put the system into the right "Gross-Neveu universality class". Comparing the fermionic action with the fixed-point action in the partially bosonized version, we have to conclude that the fixed-point action in the fermionic language is far more complicated than the simple ansatz (12). In general, it will contain higher-order as well as non-local interaction terms. The deviations from the simple Gross-Neveu structure, however, are irrelevant operators which do not modify the predictive power of the Gross-Neveu model.

The simple Gross-Neveu action is also incomplete in the sense that it does not exhibit all possible four-fermi terms compatible with the defining symmetries of the model. For instance, a Thirring-like interaction $(\bar\psi\gamma_\mu\psi)^2$ is also invariant under the symmetries of the Gross-Neveu model and will thus generically be generated by the RG flow. A more complete RG analysis thus has to include these terms facilitating the appearance of further fixed-points, as is known for the Thirring model [55]. If so, this implies the possible existence of further UV completions of fermionic models with the same symmetries as the Gross-Neveu model.

Finally, let as mention once more that the property of asymptotic safety in the Gross-Neveu model is tightly related to the occurrence of a quantum phase transition of 2nd order separating a disordered phase from a phase with broken (discrete) chiral symmetry. More generally, models with such 2nd-order quantum phase transitions are guaranteed to be asymptotically safe. Whether the converse is true, i.e., whether asymptotically safe models always exhibit a physically relevant order-disorder quantum phase transition, is an interesting question for future studies.

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Appendix A: Threshold functions

The regulator dependence of the flow equations is encoded in dimensionless threshold functions which arise from 1PI diagrams incorporating bosonic and/or fermionic fields. In this work we have employed a linear regulator [34–36], see Eqs. (15) and (38). For the threshold functions appearing in the flow equations in Sect. V, we then find

$$l_0^d(\omega;\eta_\sigma) = \frac{2}{d}\left(1-\frac{\eta_\sigma}{d+2}\right)\frac{1}{1+\omega}\,,$$

$$l_0^{(F)d}(\omega;\eta_\psi) = \frac{2}{d}\left(1-\frac{\eta_\psi}{d+1}\right)\frac{1}{1+\omega}\,,$$

$$l_{1,1}^{(FB)d}(\omega_1,\omega_2;\eta_\psi,\eta_\sigma) = \frac{2}{d}\frac{1}{(1+\omega_1)(1+\omega_2)}\times$$
$$\times\left\{\left(1-\frac{\eta_\psi}{d+1}\right)\frac{1}{1+\omega_1}+\left(1-\frac{\eta_\sigma}{d+2}\right)\frac{1}{1+\omega_2}\right\}\,,$$

$$m_4^{(F)d}(\omega;\eta_\psi) = \frac{1}{(1+\omega)^4} + \frac{1-\eta_\psi}{d-2}\frac{1}{(1+\omega)^3} - \left(\frac{1-\eta_\psi}{2d-4} + \frac{1}{4}\right)\frac{1}{(1+\omega)^2},$$

$$m_{1,2}^{(FB)d}(\omega_1,\omega_2;\eta_\psi,\eta_\sigma) = \left(1 - \frac{\eta_\sigma}{d+1}\right)\frac{1}{(1+\omega_1)(1+\omega_2)^2}.$$

Note that the linear regulators (15) and (38) render $m_{1,2}^{(FB)d}$ independent of η_ψ.

[1] L. L.D., in Niels Bohr and the Development of Physics edited by W. Pauli (Pergamon Press, London) (1955); M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954); M. Gockeler et al., Phys. Rev. Lett. **80**, 4119 (1998), hep-th/9712244; H. Gies and J. Jaeckel, Phys. Rev. Lett. **93**, 110405 (2004), hep-ph/0405183.

[2] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin (1976), in *Phase Transitions and Critical Phenomena, Vol.6*, London 1976, 125-247; M. Luscher and P. Weisz, Nucl. Phys. **B290**, 25 (1987); U. Wolff, Phys. Rev. **D79**, 105002 (2009), 0902.3100;

[3] S. Weinberg (1976), lectures presented at Int. School of Subnuclear Physics, Ettore Majorana, Erice, Sicily, Jul 23 - Aug 8, 1976.

[4] S. Weinberg (1996), hep-th/9702027; S. Weinberg (2009), 0903.0568; M. Niedermaier and M. Reuter, Living Rev. Rel. **9**, 5 (2006); R. Percacci (2007), 0709.3851.

[5] O. J. Rosten (2010), 1003.1366.

[6] C. Wetterich, Phys. Lett. **B301**, 90 (1993).

[7] M. Reuter, Phys. Rev. **D57**, 971 (1998), hep-th/9605030.

[8] D. Dou and R. Percacci, Class. Quant. Grav. **15**, 3449 (1998), hep-th/9707239; W. Souma, Prog. Theor. Phys. **102**, 181 (1999), hep-th/9907027; M. Reuter and F. Saueressig, Phys. Rev. **D65**, 065016 (2002), hep-th/0110054.

[9] O. Lauscher and M. Reuter, Phys. Rev. **D65**, 025013 (2002), hep-th/0108040.

[10] D. F. Litim, Phys. Rev. Lett. **92**, 201301 (2004), hep-th/0312114.

[11] A. Codello and R. Percacci, Phys. Rev. Lett. **97**, 221301 (2006), hep-th/0607128; A. Codello, R. Percacci, and C. Rahmede, Int. J. Mod. Phys. **A23**, 143 (2008), 0705.1769; P. F. Machado and F. Saueressig, Phys. Rev. **D77**, 124045 (2008), 0712.0445; A. Eichhorn, H. Gies, and M. M. Scherer, Phys. Rev. **D80**, 104003 (2009), 0907.1828; K. Groh and F. Saueressig, J. Phys. **A43**, 365403 (2010), 1001.5032; A. Eichhorn and H. Gies, Phys. Rev. **D81**, 104010 (2010), 1001.5033.

[12] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. **D72**, 064014 (2005), hep-th/0505154.

[13] H. Gies, J. Jaeckel, and C. Wetterich, Phys. Rev. **D69**, 105008 (2004), hep-ph/0312034.

[14] H. Gies and M. M. Scherer, Eur. Phys. J. **C66**, 387 (2010), 0901.2459.

[15] H. Gies, S. Rechenberger, and M. M. Scherer, Eur. Phys. J. **C66**, 403 (2010), 0907.0327.

[16] R. Percacci and O. Zanusso, Phys. Rev. **D81**, 065012 (2010), 0910.0851; M. Fabbrichesi, R. Percacci, A. Tonero, and O. Zanusso (2010), 1010.0912.

[17] H. Gies, Phys. Rev. **D68**, 085015 (2003), hep-th/0305208.

[18] P. Fischer and D. F. Litim, AIP Conf. Proc. **861**, 336 (2006), hep-th/0606135; P. Fischer and D. F. Litim, Phys. Lett. **B638**, 497 (2006), hep-th/0602203.

[19] D. J. Gross and A. Neveu, Phys. Rev. **D10**, 3235 (1974).

[20] B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rev. Lett. **62**, 1433 (1989); K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **55**, 363 (1985); C. de Calan, P. A. Faria da Veiga, J. Magnen, and R. Seneor, Phys. Rev. Lett. **66**, 3233 (1991).

[21] J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991).

[22] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti, and Y. Shen, Nucl. Phys. **B365**, 79 (1991).

[23] S. Hands, A. Kocic, and J. B. Kogut, Ann. Phys. **224**, 29 (1993), hep-lat/9208022.

[24] L. Karkkainen, R. Lacaze, P. Lacock, and B. Petersson, Nucl. Phys. **B415**, 781 (1994), hep-lat/9310020.

[25] G. Gat, A. Kovner, and B. Rosenstein, Nucl. Phys. **B385**, 76 (1992).

[26] J. A. Gracey, Int. J. Mod. Phys. **A9**, 727 (1994), hep-th/9306107; A. N. Vasiliev, S. E. Derkachov, N. A. Kivel, and A. S. Stepanenko, Theor. Math. Phys. **94**, 127 (1993).

[27] E. Focht, J. Jersak, and J. Paul, Phys. Rev. **D53**, 4616 (1996), hep-lat/9511005; T. Reisz (1997), hep-lat/9712017; E. Babaev, Phys. Lett. **B497**, 323 (2001), hep-th/9907089; L. Rosa, P. Vitale, and C. Wetterich, Phys. Rev. Lett. **86**, 958 (2001), hep-th/0007093.

[28] F. Hofling, C. Nowak, and C. Wetterich, Phys. Rev. **B66**, 205111 (2002), cond-mat/0203588.

[29] D. F. Litim and J. M. Pawlowski (1998), hep-th/9901063; C. Bagnuls and C. Bervillier, Phys. Rept. **348**, 91 (2001), hep-th/0002034; J. Polonyi, Central Eur. J. Phys. **1**, 1 (2003), hep-th/0110026; B. Delamotte, D. Mouhanna, and M. Tissier, Phys. Rev. **B69**, 134413 (2004), cond-mat/0309101; J. M. Pawlowski, Annals Phys. **322**, 2831 (2007), hep-th/0512261; H. Gies (2006), hep-ph/0611146; B. Delamotte (2007), cond-mat/0702365.

[30] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rept. **363**, 223 (2002), hep-ph/0005122.

[31] J. Jaeckel and C. Wetterich, Phys. Rev. **D68**, 025020 (2003), hep-ph/0207094.

[32] J. Jaeckel (2003), hep-ph/0309090.

[33] H. Gies, L. Janssen, S. Rechenberger, and M. M. Scherer,

Phys. Rev. **D81**, 025009 (2010), 0910.0764.
[34] D. F. Litim, Phys. Lett. **B486**, 92 (2000), hep-th/0005245.
[35] D. F. Litim, Phys. Rev. **D64**, 105007 (2001), hep-th/0103195.
[36] D. F. Litim, Int. J. Mod. Phys. **A16**, 2081 (2001), hep-th/0104221.
[37] S. Hands, S. Kim, and J. B. Kogut, Nucl. Phys. **B442**, 364 (1995), hep-lat/9501037; V. Schon and M. Thies, Phys. Rev. **D62**, 096002 (2000), hep-th/0003195.
[38] M. Thies and K. Urlichs, Phys. Rev. **D67**, 125015 (2003), hep-th/0302092; M. Thies, Phys. Rev. **D69**, 067703 (2004), hep-th/0308164; G. Basar and G. V. Dunne, Phys. Rev. Lett. **100**, 200404 (2008), 0803.1501; G. Basar and G. V. Dunne, Phys. Rev. **D78**, 065022 (2008), 0806.2659.
[39] P. Castorina, M. Mazza, and D. Zappala, Phys. Lett. **B567**, 31 (2003), hep-th/0305162.
[40] R. Stratonovich, Dokl. Akad. Nauk. **115**, 1097 (1957).
[41] J. Hubbard, Phys. Rev. Lett. **3**, 77 (1959).
[42] D. Nickel, Phys. Rev. **D80**, 074025 (2009), 0906.5295.
[43] T. Kojo, Y. Hidaka, L. McLerran, and R. D. Pisarski, Nucl. Phys. **A843**, 37 (2010), 0912.3800.
[44] J. Zinn-Justin, Int. Ser. Monogr. Phys. **113**, 1 (2002).
[45] N. Tetradis and D. F. Litim, Nucl. Phys. **B464**, 492 (1996), hep-th/9512073.
[46] S. Diehl, H. Gies, J. M. Pawlowski, and C. Wetterich, Phys. Rev. **A76**, 021602 (2007), cond-mat/0701198; Phys. Rev. **A76**, 053627 (2007), cond-mat/0703366.
[47] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products* (Jeffrey, Alan (ed.), Academic Press, 2000).
[48] J. Braun, Phys. Rev. **D81**, 016008 (2010), 0908.1543.
[49] N. Tetradis and C. Wetterich, Nucl. Phys. **B422**, 541 (1994), hep-ph/9308214.
[50] D. F. Litim and J. M. Pawlowski, Phys. Lett. **B516**, 197 (2001), hep-th/0107020.
[51] D. F. Litim, Nucl. Phys. **B631**, 128 (2002), hep-th/0203006.
[52] C. Bervillier, A. Juttner, and D. F. Litim, Nucl. Phys. **B783**, 213 (2007), hep-th/0701172.
[53] F. Benitez et al., Phys. Rev. **E80**, 030103 (2009), 0901.0128.
[54] D. F. Litim and D. Zappala (2010), 1009.1948.
[55] H. Gies and L. Janssen, Phys. Rev. **D82**, 085018 (2010), 1006.3747.