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The Spin-weighted Spheroidal Wave functions in the Case of $s = 1/2$

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Abstract

The spin-weighted spheroidal equations in the case $s = 1/2$ is thoroughly studied in the paper by means of the perturbation method in supersymmetry quantum mechanics. The first-five terms of the super-potential in the series of the parameter β are given. The general form of the nth term of the superpotential is also obtained, which could derived from the previous terms W_k, $k < n$. From the results, it is easy to give the ground eigenfunction of the equation. Furthermore, the shape-invariance property is investigated in the series form of the parameter β and is proven kept in this series form for the equations. This nice property guarantee one could obtain the excited eigenfunctions in the series form from the ground eigenfunctions by the method in supersymmetry quantum mechanics. This shows the perturbation method method in supersymmetry quantum mechanics could solve the spin-weight spheroidal wave equations completely in the series form of the small parameter β. PACS:11.30Pb; 04.25Nx; 04.70-s

1 Introduction

The spin-weighted spheroidal functions first appeared in the study of the stable problem of Kerr black hole. For the wave equation of the perturbation of φ of Kerr black hole[1-2]

$$\begin{aligned}
&\Big[\frac{(r^2+a^2)}{\Delta} - a^2\sin^2\theta\Big]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \Big[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\Big]\frac{\partial^2\psi}{\partial\phi^2}\\
&-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\Big[\frac{a\,(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\Big]\frac{\partial\psi}{\partial\phi}\\
&-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - 2s\Big[\frac{M\,(r^2-a^2)}{\Delta} - r - ia\cos\theta\Big]\frac{\partial\psi}{\partial t}\\
&+\left(s^2\cot^2\theta - s\right)\psi = 0
\end{aligned}\tag{1}$$

Teukolsky made nice separation for Eq.(1) and obtained radial equation for $R(r)$as

$$\begin{aligned}
&\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right)\\
+\quad&\Big[\frac{K^2 - 2is\,(r-M)\,K}{\Delta} + 4is\omega r - \bar{\lambda}\Big]R = 0,
\end{aligned}\tag{2}$$

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and the angular equations for $\Theta(\theta)$ as

$$\Big[\frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{d}{d\theta}\Big)+s+a^2\omega^2\cos^2\theta$$
$$-\quad 2as\omega\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+E\Big]\Theta=0 \tag{3}$$

where $\triangle = r^2-2Mr+a^2$, $K=(r^2+a^2)\omega-am$ and $\bar{\lambda}=E+a^2\omega^2-2am\omega$. The parameter s, the spin-weight of the perturbation fields could be $s=0,\pm\frac{1}{2},\pm1,\pm2$, and corresponds to the scalar, neutrino, electromagnetic or gravitational perturbations respectively. Eq.(3) is the spin-weighted spheroidal wave equation. This is kind of the Sturm-liuvelle problem, and the boundary conditions requires Θ is finite at $\theta=0,\ \pi$. Though it has been widely used in many fields, it has not been well studied due to the mathematical difficulty[1-5]. Until recent years, with the introduction of super-symmetric quantum mechanics to study the spheroidal equations (that is, s=0 case), it becomes possible to solve it[6-12]. This article would apply this method to study it in the case of $s=1/2$. When $s=0$, the problem is more complex than previous work. So more calculation is need in determining the eigenvalue.

2 Calculation of the first several terms of the super-potential

In order to employ the super-symmetry quantum mechanics (SUSYQM) to Eq.(3), we first make it into the Schrödinger form[6-12]

$$\frac{d^2\Psi}{d\theta^2}\quad+\quad\Big[\frac{1}{4}+s+\beta^2\cos^2\theta-2s\beta\cos\theta$$
$$-\quad\frac{(m+s\cos\theta)^2-\frac{1}{4}}{\sin^2\theta}+E\Big]\Psi=0 \tag{4}$$

by the transformation

$$\Theta(\theta)=\frac{\Psi(\theta)}{\sqrt{sin\theta}} \tag{5}$$

with $\beta=a\omega$. The corresponding boundary conditions now become $\Psi|_{\theta=0}=\Psi|_{\theta=\pi}=0$. For Eq.(4), we make a few instructions: Eq.(4) just has the Schrödinger form, it does not represent a particular quantum system. For convenience, we apply the terminology of quantum mechanics, such as potential energy, ground state, excited state, etc in the following. The potential in Eq.(4) is

$$V(\theta,\beta,s)\quad=\quad-\Big[\frac{1}{4}+s+\beta^2\cos^2\theta-2s\beta\cos\theta$$
$$+\frac{(m+s\cos\theta)^2-\frac{1}{4}}{\sin^2\theta}\Big], \tag{6}$$

Super potential W is a core concept in SUSYQM, and it is connected with the potential by[13]

$$W^2 - W' = V(\theta, \beta, s) - E_0. \tag{7}$$

According to the theory of the SUSYQM, the form of ground eigenfunction Ψ_0 is completely known through the super-potential W by the formula [13]

$$\Psi_0 = N \exp\Big[- \int W d\theta \Big]. \tag{8}$$

Hence, the key work in SUSYQM is solve Eq.(7). Practically, Eq.(7) is the same difficult to deal with as Eq.(4). Therefore, we rely on the perturbation methods to treat it. That is, we study it by expanding the super-potential W and the ground eigenvalue E_0 into series form of the parameter β [6-12]:

$$W = \sum_{n=0}^{\infty} \beta^n W_n \tag{9}$$

$$E_0 = \sum_{n=0}^{\infty} E_{0,n;m} \beta^n, \tag{10}$$

where the lower suffix m in $E_{0,n;m}$ is referred to the parameter m in Eq.(4) and the index 0 means belonging to the ground state energy and n refers to the nth-order item of the series expansion.

Substituting Eqs.(9), Eq.(10) into Eq.(7) gives the following equations[11]

$$\begin{aligned} W_0' - W_0^2 &= E_{0,0;m} + s + \frac{1}{4} \\ &\quad - \frac{(m + s\cos\theta)^2 - \frac{1}{4}}{\sin^2\theta} \equiv f_0(\theta) && (11) \\ W_1' - 2W_0W_1 &= -2s\cos\theta + E_{0,1;m} \\ &\equiv f_1(\theta) && (12) \\ W_2' - 2W_0W_2 &= \cos^2\theta + W_1^2 + E_{0,2;m} \\ &\equiv f_2(\theta) && (13) \\ W_n' - 2W_0W_n &= \sum_{k=1}^{n-1} W_k W_{n-k} + E_{0,n;m} \\ &\equiv f_n(\theta), \ (n \geq 3) && (14) \end{aligned}$$

by equating the coefficients of the same power of β in its two sides. Our main work is to find the solutions of the above equations in the case of $s = \frac{1}{2}$.

The solution of Eq.(11) is easy to find [11]

$$\begin{aligned} E_{0,0;m} &= m^2 + m - 3/4 \\ W_0 &= -\frac{1/2 + (m + \frac{1}{2})\cos\theta}{sin\theta} \end{aligned} \tag{15}$$

With W_0 known, it is easy to give W_n on according to the knowledge of differential equations,

$$\begin{aligned} W_n(\theta) &= e^{2\int W_0 d\theta} A_n(\theta) \\ &= \left[\tan\frac{\theta}{2}\sin^{2m+1}\theta\right]^{-1} A_n(\theta) \end{aligned} \tag{16}$$

where,

$$\begin{aligned} A_n(\theta) &= \int f_n(\theta) e^{-2\int W_0 d\theta} d\theta \\ &= \int f_n(\theta)\tan\frac{\theta}{2}\sin^{2m+1}\theta d\theta. \end{aligned} \tag{17}$$

In order to calculate Eqs.(16)-(17), we needs the following integral formulae [14]:

$$\begin{aligned} & P(2m,\theta) \\ =& \int sin^{2m}\theta d\theta \\ =& -\frac{\cot\theta}{2m+1}\Big[\sum_{k=0}^{m-1}\bar{I}(2m,k)sin^{2m-2k}\theta\Big] + \frac{(2m-1)!!}{(2m)!!}\theta \end{aligned} \tag{18}$$

where

$$\begin{aligned} \bar{I}(2m,k) &= \frac{(2m+1)(2m-1)\cdots(2m-2k+1)}{2m(2m-2)\cdots(2m-2k)}, \\ & \qquad (k \geq 0) \end{aligned} \tag{19}$$

After a simple derivation, we get

$$\begin{aligned} & P(2m+2,\theta) \\ =& \frac{2m+1}{2m+2}P(2m,\theta) - \frac{\cos\theta\sin^{2m+1}\theta}{2m+2}, \end{aligned} \tag{20}$$

$$\begin{aligned} & P(2m+4,\theta) \\ =& \frac{(2m+1)(2m+3)}{(2m+2)(2m+4)}P(2m,\theta) \\ & -\frac{\cos\theta\sin^{2m+3}}{2m+4} - \frac{(2m+3)\cos\theta\sin^{2m+1}\theta}{(2m+2)(2m+4)} \end{aligned} \tag{21}$$

$$\begin{aligned} & P(2m+6,\theta) \\ =& \frac{(2m+1)(2m+3)(2m+5)}{(2m+2)(2m+4)(2m+6)}P(2m,\theta) \\ & -\frac{\cos\theta\sin^{2m+5}}{2m+6} - \frac{(2m+5)\cos\theta\sin^{2m+3}\theta}{(2m+4)(2m+6)} \\ & -\frac{(2m+3)(2m+5)\cos\theta\sin^{2m+1}\theta}{(2m+2)(2m+4)(2m+6)} \end{aligned} \tag{22}$$

By the above equation,we can summarize the general formula

$$\begin{aligned} & P(2m+2n) \\ = & \frac{2m+1}{2m+2n+1}\bar{I}(2m+2n,n-1)P(2m,\theta) \\ & -\cos\theta\sum_{l=1}^{n}\frac{\bar{I}(2m+2n,n-l)}{2m+2n+1}\sin^{2m+2l-1}\theta \end{aligned} \quad (23)$$

This formula can be proved by mathematical induction. By the help of Eqs.(15), (23), $A_1(\theta)$ is now simplified as

$$\begin{aligned} A_1(\theta) &= \int(E_{0,1;m}-cos\theta)(1-cos\theta)sin^{2m}\theta d\theta \\ &= (E_{0,1;m}+1)P(2m,\theta)-P(2m+2,\theta) \\ &\quad -\frac{E_{0,1;m}+1}{2m+1}\sin^{2m+1}\theta \\ &= [E_{0,1;m}+1-\frac{2m+1}{2m+2}]P(2m,\theta) \\ &\quad +\Big[\frac{\cos\theta}{2m+2}-\frac{E_{0,1;m}+1}{2m+1}\Big]\sin^{2m+1}\theta \end{aligned} \quad (24)$$

Now we discuss the term $P(2m,\theta)$. According to Eq.(8) and Eq.(18), we can see that $\Psi_0(\theta)$ is ∞ at the boundaries $\theta = 0,\ \pi$. This result does not meet the boundary conditions that $\Psi(\theta)$ should finite at $\theta = 0,\ \pi$. So that the coefficient of the term $P(2m,\theta)$ must be zero. Thus,

$$E_{0,1} = -\frac{1}{2m+2} \quad (25)$$

Now we can simplify A_1

$$A_1(\theta) = -\frac{sin^{2m+1}\theta}{2m+2}+\frac{cos\theta sin^{2m+1}\theta}{2m+2} \quad (26)$$

With the help of Eq.(16) and Eq.(15), it is easy to obtain the first order $W_1(\theta)$

$$W_1(\theta) = -\frac{1}{2m+2}\sin\theta \quad (27)$$

By the same tedious calculation as that of $W_1(\theta)$, $E_{0,2;m}-E_{0,4;m}$ and $W_2(\theta)-W_4(\theta)$can also be obtained. The results are

$$E_{0,2;m} = -\frac{4m^2+10m-5}{(2m+2)^3} \quad (28)$$

$$E_{0,3;m} = -\frac{4(2m+1)^2(2m+3)}{(2m+2)^5(2m+4)} \quad (29)$$

$$E_{0,4;m} = -\frac{2(2m+1)^2(2m+3)(2m^2+9m+2)}{(2m+2)^7(2m+4)}$$

$$W_2(\theta) = b_{2,1} sin\theta + a_{2,1} sin\theta \cos\theta \tag{30}$$

$$W_3(\theta) = b_{3,1}\sin\theta + b_{3,2}\sin^3\theta + a_{3,1}\sin\theta\cos\theta \tag{31}$$

$$\begin{aligned} W_4(\theta) &= b_{4,1}\sin\theta + b_{4,1}\sin^3\theta \\ &\quad + a_{4,1}\sin\theta\cos\theta + a_{4,2}\sin^3\theta\cos\theta \end{aligned} \tag{32}$$

Where,

$$b_{2,1} = -\frac{2m+1}{(2m+2)^3} \quad a_{2,1} = \frac{2m+1}{(2m+2)^2} \tag{33}$$

$$b_{3,1} = \frac{4(2m+1)}{(2m+2)^5(2m+4)} \tag{34}$$

$$b_{3,2} = -\frac{2(2m+1)}{(2m+2)^3(2m+4)} \tag{35}$$

$$a_{3,1} = -\frac{4(2m+1)}{(2m+2)^4(2m+4)} \tag{36}$$

$$b_{4,1} = \frac{2(2m+1)(2m^2+9m+2)}{(2m+2)^7(2m+4)} \tag{37}$$

$$a_{4,1} = -\frac{2(2m+1)(2m^2+9m+2)}{(2m+2)^6(2m+4)} \tag{38}$$

$$b_{4,2} = -\frac{6m(2m+1)}{(2m+2)^5(2m+4)} \tag{39}$$

$$a_{4,2} = \frac{2m(2m+1)}{(2m+2)^4(2m+4)} \tag{40}$$

3 Calculation of the general n-th terms of the superpotential

From the four terms of $W_1 - W_4$, we hypothetically summarize a general formula for W_n as

$$W_n(\theta) = \sum_{k=1}^{[\frac{n}{2}]} a_{n,k}\sin^{2k-1}\theta\cos\theta + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k}\sin^{2k-1}\theta \tag{41}$$

Here we use mathematical induction to prove that the guess is true.

First it is easy to see the assumption (41) is the same as that of W_1 when $N = 1$. Under the condition that all W_N meet the requirement of (41) whenever $N \le n-1$, we will try to solve the differential equation for W_n to verify that it also can be written as that of (41). Back to Eqs.(14),(16),(17), one needs to simplify the term $\sum_{k=1}^{n-1} W_k W_{n-k}$ in order to calculate W_n. Whenever $1 \le k \le n-1$, one has $1 \le n-k \le n-1$ and $W_k(\theta)$, $W_{n-k}(\theta)$ could be written in the form of (41). That is,

$$W_k(\theta) = \sum_{i=1}^{[\frac{k}{2}]} a_{k,i}\sin^{2i-1}\theta\cos\theta + \sum_{i=1}^{[\frac{k+1}{2}]} b_{k,i}\sin^{2i-1}\theta$$

$$W_{n-k}(\theta) = \sum_{j=1}^{[\frac{n-k}{2}]} a_{n-k,j}\sin^{2j-1}\theta\cos\theta + \sum_{j=1}^{[\frac{n-k+1}{2}]} b_{n-k,j}\sin^{2j-1}\theta \tag{42}$$

For the sake of later use, the facts are true

$$a_{i,j} = 0, j < 1 \ or \ j > [\frac{i}{2}];$$
$$b_{i,j} = 0, \ j < 1 \ or \ j > [\frac{i+1}{2}] \tag{43}$$

for $i < n$. Thus

$$\sum_{k=1}^{n-1} W_k W_{n-k} = \bar{A}_1 + \bar{A}_2 + \bar{A}_3 + \bar{A}_4 \tag{44}$$

where

$$\bar{A}_1 = \sum_{k=1}^{n-1}\sum_{i=1}^{[\frac{k+1}{2}]}\sum_{j=1}^{[\frac{n-k+1}{2}]} b_{k,i}b_{n-k,j}\sin^{2i+2j-2}\theta$$
$$\bar{A}_2 = \sum_{k=1}^{n-1}\sum_{i=1}^{[\frac{k}{2}]}\sum_{j=1}^{[\frac{n-k}{2}]} a_{k,i}b_{n-k,j}\sin^{2i+2j-2}\theta\cos^2\theta$$
$$\bar{A}_3 = \cos\theta\sum_{k=1}^{n-1}\Big[\sum_{i=1}^{[\frac{k}{2}]}\sum_{j=1}^{[\frac{n-k+1}{2}]} a_{k,i}b_{n-k,j}\sin^{2i+2j-2}\theta\Big]$$
$$\bar{A}_4 = \cos\theta\sum_{k=1}^{n-1}\Big[\sum_{i=1}^{[\frac{k+1}{2}]}\sum_{j=1}^{[\frac{n-k}{2}]} b_{k,i}a_{n-k,j}\sin^{2i+2j-2}\theta\Big] \tag{45}$$

exchanging the order of the sums in all the above equations, they are simplified as

$$\bar{A}_1 = \sum_{p=2}^{\bar{c}_1}\sum_{k=1}^{n-1}\sum_{j=1}^{p-1} b_{k,p-j}b_{n-k,j}sin^{2p-2}\theta \tag{46}$$
$$\bar{A}_2 = \sum_{p=2}^{\bar{c}_2-1}\sum_{k=1}^{n-1}\sum_{j=1}^{p-1} a_{k,p-j}a_{n-k,j}sin^{2p-2}\theta - \sum_{p=3}^{\bar{c}_2}\sum_{k=1}^{n-1}\sum_{j=1}^{p-1} a_{k,p-1-j}a_{n-k,j}sin^{2p-2}\theta$$

$$
\begin{aligned}
\bar{A}_3 &= +\cos\theta \sum_{p=2}^{\bar{c}_3}\sum_{k=1}^{n-1}\sum_{j=1}^{p-1} b_{k,p-j}a_{n-k,j} sin^{2p+1}\theta \\
\bar{A}_4 &= \cos\theta \sum_{p=2}^{\bar{c}_4}\sum_{k=1}^{n-1}\sum_{j=1}^{p-1} a_{k,p-j}b_{n-k,j} sin^{2p+1}\theta
\end{aligned} \tag{47}
$$

where

$$
\begin{aligned}
\bar{c}_1 &= \left[\frac{k+1}{2}\right] + \left[\frac{n-k+1}{2}\right],\ \bar{c}_2 = \left[\frac{k}{2}\right] + \left[\frac{n-k}{2}\right] + 1 \\
\bar{c}_3 &= \left[\frac{k+1}{2}\right] + \left[\frac{n-k}{2}\right],\ \bar{c}_4 = \left[\frac{k}{2}\right] + \left[\frac{n-k+1}{2}\right]
\end{aligned} \tag{48}
$$

It is easy to see

$$
\bar{c}_1 = \bar{c}_2 = \frac{n}{2} + 1, \quad \bar{c}_3 = \bar{c}_4 = \frac{n}{2},\ when\ n\ is\ even \tag{49}
$$

$$
\bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \frac{n+1}{2}, \quad when\ n\ is\ odd \tag{50}
$$

Hence

$$
\sum_{k=1}^{n-1} W_k W_{n-k} = \sum_{p=2}^{[\frac{n}{2}]+1} \Big[h_{n,p} + g_{n,p}\cos\theta\Big] sin^{2p-2}\theta \tag{51}
$$

where $h_{n,p}$ and $h_{n,p}$ are constant coefficients:

$$
h_{n,p} = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}\Big[b_{k,p-j}b_{n-k,j} + a_{k,p-j}a_{n-k,j} - a_{k,p-1-j}a_{n-k,j}\Big] \tag{52}
$$

$$
g_{n,p} = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}\Big[b_{k,p-j}a_{n-k,j} + a_{k,p-j}b_{n-k,j}\Big] \tag{53}
$$

Hence, one has

$$
\begin{aligned}
&when\ n\ is\ even \\
g_{n,p} &= 0, \quad p < 2\ or\ p > \frac{n}{2}; \\
h_{n,p} &= 0, \quad p < 2\ or\ p > \frac{n}{2} + 1, \\
&when\ n\ is\ odd \\
g_{n,p} &= h_{n,p} = 0, \quad p < 2\ or\ p > \frac{n+1}{2},.
\end{aligned} \tag{54}
$$

We have used the fact (43) and substituted the quantities $\bar{c}_1,\ \bar{c}_2,\ \bar{c}_3,\ \bar{c}_4$ by the maximum $[\frac{n}{2}]+1$ of them in last line in the above equation for convenience in the following calculation process. On the final conclusion, we will take Eq.(54) into consideration.

Substituting $f_n(z) = \sum_{k=1}^{n-1} W_k W_{n-k} + E_{0,n;m}$ into Eqs.(17) and by the use of Eq.(18), we can have

$$\begin{aligned}
& A_n(\theta) \\
= & \int \sum_{p=2}^{[\frac{n}{2}]+1} \Big[h_{n,p} + g_{n,p}\cos\theta\Big] sin^{2p-2}\theta(1-cos\theta)sin^{2m}\theta d\theta \\
& + \int E_{0,n;m} * (1-cos\theta)sin^{2m}\theta d\theta \\
= & \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p}-h_{n,p}}{2m+2p-1} sin^{2m+2p-1}\theta - \frac{E_{0,n;m}}{2m+1}\sin^{2m+1}\theta \\
& + \sum_{p=2}^{[\frac{n}{2}]+1} \Big(h_{n,p} - g_{n,p}\Big) P(2m+2p-2,\theta) \\
& + \sum_{p=2}^{[\frac{n}{2}]+1} g_{n,p} P(2m+2p,\theta) + E_{0,n;m} P(2m,\theta)
\end{aligned} \tag{55}$$

According to Eq.(23), one has

$$\begin{aligned}
P(2m+2p-2) &= \frac{2m+2p}{2m+2p-1} P(2m+2p,\theta) \\
& - \cos\theta \frac{\sin^{2m+2p-1}\theta}{2m+2p-1}
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
& P(2m+2p) \\
= & \frac{(2m+1)\bar{I}(2m+2p,p-1)}{2m+2p+1} P(2m,\theta) \\
& - \cos\theta \sum_{l=1}^{p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta
\end{aligned} \tag{57}$$

Hence,

$$\begin{aligned}
& A_n(\theta) \\
= & \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p}-h_{n,p}}{2m+2p-1} sin^{2m+2p-1}\theta - \frac{E_{0,n;m}}{2m+1}\sin^{2m+1}\theta \\
& - \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p}-h_{n,p}}{2m+2p-1} sin^{2m+2p-1}\theta \\
& - \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} * \sum_{l=1}^{p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta \\
& + b * P(2m,\theta)
\end{aligned} \tag{58}$$

where the quantities $c_{n,p}$, and b as the coefficient of $P(2m,\theta)$ are

$$e_{n,p} = \frac{2m+2p}{2m+2p-1}(h_{n,p} - \frac{g_{n,p}}{2m+2p}), \tag{59}$$

$$b = E_{0,n;m} + \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} * \frac{(2m+1)\bar{I}(2m+2p,p-1)}{2m+2p+1} \tag{60}$$

respectively. As stated before, the coefficient b must be zero to make the eigenfunction finite at the boundary, then

$$E_{0,n;m} \quad = \quad -\sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} * \frac{(2m+1)\bar{I}(2m+2p,p-1)}{2m+2p+1} \tag{61}$$

Now, we first simplify some terms in Eq.(58)

$$\begin{aligned}
& \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} * \sum_{l=1}^{p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta \\
= \;& \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} * \sum_{l=2}^{p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta \\
& + \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} \frac{\bar{I}(2m+2p,p-1)}{2m+2p+1} \sin^{2m+1}\theta \\
= \;& \cos\theta \sum_{l=2}^{[\frac{n}{2}]+1} \sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta \\
& + \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} \frac{\bar{I}(2m+2p,p-1)}{2m+2p+1} \sin^{2m+1}\theta
\end{aligned} \tag{62}$$

Taking Eqs.(61), (62) into Eq.(58),we can obtain

$$\begin{aligned}
& A_n(\theta) \\
= \;& \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p} - h_{n,p}}{2m+2p-1} sin^{2m+2p-1}\theta - \frac{E_{0,n;m}}{2m+1} \sin^{2m+1}\theta \\
& - \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p} - h_{n,p}}{2m+2p-1} sin^{2m+2p-1}\theta \\
& - \cos\theta \sum_{l=2}^{[\frac{n}{2}]+1} \sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta
\end{aligned}$$

$$
\begin{aligned}
& -\cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-1)}{2m+2p+1}\sin^{2m+1}\theta \\
= \quad & F_1+\cos\theta F_2
\end{aligned} \tag{63}
$$

where $F_1,\ F_2$ are

$$
F_1=\sum_{p=2}^{[\frac{n}{2}]+1}\frac{g_{n,p}-h_{n,p}}{2m+2p-1}sin^{2m+2p-1}\theta-\frac{E_{0,n;m}}{2m+1}\sin^{2m+1}\theta \tag{64}
$$

$$
\begin{aligned}
F_2 \quad = \quad & -\sum_{p=2}^{[\frac{n}{2}]+1}\frac{g_{n,p}-h_{n,p}}{2m+2p-1}sin^{2m+2p-1}\theta \\
& -\sum_{l=1}^{[\frac{n}{2}]+1}\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta \\
= \quad & -\sum_{p=2}^{[\frac{n}{2}]+1}\frac{g_{n,p}-h_{n,p}}{2m+2p-1}sin^{2m+2p-1}\theta \\
& -\sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-1)}{2m+2p+1}\sin^{2m+1}\theta \\
& -\sum_{l=2}^{[\frac{n}{2}]+1}\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta
\end{aligned}
$$

With the help of Eq.(16), it is easy to obtain

$$
\begin{aligned}
& W_n \\
= \quad & \frac{\cos\theta\Big[F_1+F_2\Big]+\Big[F_1+F_2-\sin^2\theta F_2\Big]}{\sin^{2m+2}\theta} \\
= \quad & \frac{\cos\theta\Big[F_1+F_2\Big]}{\sin^{2m+2}\theta} \\
& +\frac{F_1+F_2-\sin^2\theta(F_1+F_2)+F_1\sin^2\theta}{\sin^{2m+2}\theta}.
\end{aligned} \tag{65}
$$

With the help of Eq.(61), it is easy to have

$$
F_1+F_2=-\sum_{l=2}^{[\frac{n}{2}]+1}\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta. \tag{66}
$$

one achieves

$$W_n = \cos\theta \sum_{l=2}^{[\frac{n}{2}]+1} a_{n,l-1} \sin^{2l-3}\theta + \sum_{l=2}^{[\frac{n}{2}]+1} b_{n,l-1} \sin^{2l-3}\theta. \tag{67}$$

Where,

$$\begin{aligned} a_{n,l-1} &= -\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p} \frac{\bar{I}(2m+2p,p-l)}{2m+2p+1},\ l \geq 2 \\ &= -\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p} \frac{(2m+2l-2)\bar{I}(2m+2p,p-l+1)}{(2m+2l-1)(2m+2p+1)} \end{aligned} \tag{68}$$

$$b_{n,1} = a_{n,1} - \frac{E_{0,n;m}}{2m+1} \tag{69}$$

$$b_{n,l-1} = a_{n,l-1} - a_{n,l-2} + \frac{g_{n,l-1} - h_{n,l-1}}{2m+2l-3},\ l \geq 3, \tag{70}$$

Eq.(67) could be written as

$$W_n = \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l} \sin^{2l-1}\theta + \sum_{l=1}^{[\frac{n}{2}]+1} b_{n,l} \sin^{2l-1}\theta \tag{71}$$

where for $l \geq 1$

$$a_{n,l} = -\sum_{p=l+1}^{[\frac{n}{2}]+1} e_{n,p} \frac{(2m+2l)\bar{I}(2m+2p,p-l)}{(2m+2l+1)(2m+2p+1)}, \tag{72}$$

$$b_{n,1} = a_{n,1} - \frac{E_{0,n;m}}{2m+1} \tag{73}$$

$$b_{n,l} = a_{n,l} - a_{n,l-1} + \frac{g_{n,l} - h_{n,l}}{2m+2l-1},\ l \geq 2. \tag{74}$$

Compare Eqs.(71), (41), their difference lies in the upper limit of the second sum in W_n Whenever n is odd, $[\frac{n+1}{2}] = [\frac{n}{2}] + 1$ means Eq. (41) the same as Eq.(71). While n is even, we could use Eq. (43) and Eq. (54) to obtain

$$a_{n,\frac{n}{2}+1} = 0 \tag{75}$$

$$\begin{aligned} b_{n,[\frac{n}{2}]+1} &= b_{n,\frac{n}{2}+1} \\ &= -a_{n,\frac{n}{2}} - \frac{h_{n,\frac{n}{2}+1}}{2m+n+1} = 0 \end{aligned} \tag{76}$$

Hence the upper limit for $b_{n,p}$ becomes $\frac{n}{2} = [\frac{n+1}{2}]$ when n is even. The correctness of our induction about the general formula with W_n is completed. According to Eq.(72),Eq.(74) and Eq.(61),we can get some interesting results:

$$b_{n,l} = \frac{1}{2m+2l}(g_{n,l} - a_{n,l}),\ l \geq 2 \tag{77}$$

$$a_{n,1} = \frac{2m+2}{(2m+1)(2m+3)} E_{0,n;m} \quad (78)$$

We can compare these interesting results with $W_3(\theta)$ and $W_4(\theta)$ to verify the correctness of the general formula of $W_n(\theta)$.The validation results are satisfactory. So that,we can say that the general formula of $W_n(\theta)$ is accurate.

4 The ground state eigenfunctions

Based on the above conclusions, the super-potential W could be written as

$$W = W_0 + \sum_{n=1}^{\infty} \beta^n W_n \quad (79)$$

The ground eigenfunction becomes

$$\begin{aligned} \Psi_0 &= N(1-\cos\theta)^{\frac{1}{2}} \sin^m \theta \\ &* \exp\Big[-\sum_{n=1}^{\infty} \beta^n \Big(\sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \frac{\sin^{2k}\theta}{2k} \\ &+ \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} P(2k-1,\theta) \Big) \Big] \end{aligned} \quad (80)$$

Back to the Θ, the above ground eigenfunction becomes

$$\begin{aligned} \Theta_0 &= N(1-\cos\theta)^{\frac{1}{2}} \sin^{m-\frac{1}{2}} \theta \\ &* \exp\Big[-\sum_{n=1}^{\infty} \beta^n \Big(\sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \frac{\sin^{2k}\theta}{2k} \\ &+ \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} P(2k-1,\theta) bigg) \Big] \end{aligned} \quad (81)$$

and the ground energy is

$$E_{0;m} = m(m+1) - \frac{3}{4} + \sum_{n=1}^{\infty} E_{0,n;m} \beta^n \quad (82)$$

with $E_{0,n;m}$ determined by Eq.(61).

5 The excited eigenfunctions

In the following, we will compute the excited eigenfunctions. As done in Ref.[7], we hope to extend the study of the recurrence relations by the means of super-symmetric quantum mechanics to Eq. (4).

The super-potential W connects the two partner potential $V_{\mp}$ by

$$V^{\mp}(\theta) = W^2(\theta) \mp W'(\theta). \tag{83}$$

The shape-invariance properties mean that the pair of partner potentials $V^{\pm}(x)$ are similar in shape and differ only in the parameters, that is

$$V^{+}(\theta; a_1) = V^{-}(\theta; a_2) + R(a_1), \tag{84}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of θ.

We must introduce the parameters $A_{i,j},\ B_{i,j}$ into the super-potential W in order to study the shape-invariance properties of the spin-weighted spheroidal equations as:

$$\begin{aligned} W(A_{n,j}, B_{n,j}, \theta) &= -A_{0,0}(m+\frac{1}{2})\cot\theta - \frac{1}{2}B_{0,0}\csc\theta \\ &\quad + \sum_{n=1}^{\infty}\beta^n W_n(A_{n,j}, B_{n,j}, \theta), \end{aligned} \tag{85}$$

where

$$\begin{aligned} &W_n(A_{n,j}, B_{n,j}, \theta) \\ &= \sum_{j=1}^{[\frac{n+1}{2}]} \bar{b}_{n,j}\sin^{2j-1}\theta + \cos\theta \sum_{j=1}^{[\frac{n}{2}]} \bar{a}_{n,j}\sin^{2j-1}\theta \end{aligned} \tag{86}$$

with

$$\bar{a}_{n,j} = A_{n,j}a_{n,j},\ \ \bar{b}_{n,j} = B_{n,j}b_{n,j} \tag{87}$$

Then, $V^{\pm}(A_{n,j}, B_{n,j}, \theta)$ are $V^{\pm}(A_{n,j}, B_{n,j}, \theta)$ are defined as

$$\begin{aligned} V^{\pm}(A_{n,j}, B_{n,j}, \theta) &= W^2(A_{n,j}, B_{n,j}, \theta) \pm W' \\ &= \sum_{n=0}^{\infty}\beta^n V_n^{\pm}(A_{i,j}, B_{n,j}, \theta). \end{aligned} \tag{88}$$

The key point is to try to find some quantities $C_{i,j}, D_{i,j}$ to make the relations

$$V_n^{+}(A_{i,j}, B_{n,j}, \theta) = V_n^{-}(C_{i,j}, D_{n,j}, \theta) + R_{n;m}(A_{i,j}, B_{n,j}) \tag{89}$$

retain with $R_{n;m}(A_{i,j}, B_{n,j}) = R_{n;m}$ pure quantities. Now, we will prove that it is true for the special cases $n = 0,\ 1, 2$. It is easy to obtain

$$V_0^{+}(A_{0,0}, B_{0,0}, \theta) = V_0^{-}(C_{0,0}, D_{0,0}, \theta) + R_{0;m} \tag{90}$$

$$V_1^{+}(A_{1,1}, B_{1,1}, \theta) = V_1^{-}(C_{1,1}, D_{1,1}, \theta) + R_{1;m} \tag{91}$$

$$V_2^{+}(B_{2,1}, A_{2,1}, \theta) = V_2^{-}(C_{2,1}, D_{2,1}, \theta) + R_{2;m}. \tag{92}$$

In order to retain the above equations, one must have

$$
\begin{aligned}
C_{0,0} &= A_{0,0} + \frac{2}{2m+1}, && (93)\\
D_{0,0} &= B_{0,0} && (94)\\
D_{1,1} &= \frac{(2m+1)A_{0,0}-1}{(2m+1)A_{0,0}+3}B_{1,1} && (95)
\end{aligned}
$$

and

$$
\begin{aligned}
D_{2,1} &= \frac{(2m+1)A_{0,0}-1}{(2m+1)A_{0,0}+3}B_{2,1}\\
&+\frac{6B_{0,0}B_{2,1}}{[(2m+1)A_{0,0}+3][(2m+1)A_{0,0}+4]}\\
&-\frac{8[(2m+1)A_{0,0}+1]B_{0,0}B_{1,1}^2}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]} && (96)\\
C_{2,1} &= \frac{8[(2m+1)A_{0,0}+1]B_{1,1}^2}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]}\\
&+\frac{(2m+1)A_{0,0}-2}{(2m+1)A_{0,0}+4}A_{2,1} && (97)
\end{aligned}
$$

with

$$
\begin{aligned}
& R_{0;m}(A_{0,0}) = (2m+1)A_{0,0}+1, && (98)\\
& R_{1;m}(A_{0,0},B_{0,0},B_{1,1}) = -\frac{4B_{0,0}B_{1,1}}{(2m+1)A_{0,0}+3} && (99)\\
& R_{2;m}(A_{0,0},B_{0,0},B_{1,1},B_{2,1},A_{2,1})\\
= & [-\frac{4B_{0,0}B_{2,1}}{(2m+1)A_{0,0}+3} + AB_{1,1}^2 + BA_{2,1}] && (100)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{8B_{0,0}^2 - 8[(2m+1)A_{0,0}-1][(2m+1)A_{0,0}+3]}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]}\\
B &= \frac{6B_{0,0}^2 - 2[(2m+1)A_{0,0}-1][(2m+1)A_{0,0}+3]}{[(2m+1)A_{0,0}+3][(2m+1)A_{0,0}+4]} && (101)
\end{aligned}
$$

For the general proof of n, one uses the induction methods to proceed. The above result shows that the formula (89) is true for $N = 0,\ 1, 2$. Suppose it is true for $N \leq n-1$, we need to prove it is also true for $N = n$. First, one simplifies the expressions of $V_n^{\pm}(A_{i,j}, B_{i,j}, \theta)$. With the help of

$$
W^2 = \sum_{n=0}^{\infty} \beta^n \sum_{k=0}^{n} W_k(A_{i,j}, B_{n,j}, \theta) W_{n-k}(A_{i,j}, B_{n,j}, \theta) \qquad (102)
$$

we have the formulae for $V_n^{\pm}$ in the case $n \geq 1$ as following

$$V_n^-(A_{i,j}, B_{n,j}, \theta) = P_n^- + \bar{P}_n \tag{103}$$

$$V_n^+(A_{i,j}, B_{n,j}, \theta) = P_n^+ + \bar{P}_n \tag{104}$$

There are three parts in the above equations and we simplify them separately.

$$P_n^- = 2W_0 W_n(A_{n,j}, B_{n,j}, \theta) - W_n'(A_{i,j}, B_{n,j}, \theta) \tag{105}$$

$$= \cos\theta \sum_{p=1}^{[\frac{n+1}{2}]} P_{n,p}^-(A_{i,j}, B_{n,j}) \sin^{2p-2}\theta + \sum_{p=1}^{[\frac{n+2}{2}]} Q_{n,p}^-(A_{i,j}, B_{n,j}) \sin^{2p-2}\theta \tag{106}$$

where

$$P_{n,p}^-(A_{n,j}, B_{n,j}) = -B_{0,0}A_{n,p}a_{n,p} + \Big[(1-2p) - (2m+1)A_{0,0}\Big] B_{n,p}b_{n,p} \tag{107}$$

$$Q_{n,p}^-(A_{n,j}, B_{n,j}) = -B_{0,0}B_{n,p}b_{n,p} + \Big[(1-2p) - (2m+1)A_{0,0}\Big] A_{n,p}a_{n,p} + \Big[(2p-2) + (2m+1)A_{0,0}\Big] A_{n,p-1}a_{n,p-1} \tag{108}$$

$$P_n^+ = 2W_0 W_n(A_{n,j}, B_{n,j}, \theta) + W_n'(A_{i,j}, B_{n,j}, \theta) \tag{109}$$

$$= \cos\theta \sum_{p=1}^{[\frac{n+1}{2}]} P_{n,p}^+(A_{i,j}, B_{n,j}) \sin^{2p-2}\theta + \sum_{p=1}^{[\frac{n+2}{2}]} Q_{n,p}^+(A_{i,j}, B_{n,j}) \sin^{2p-2}\theta \tag{110}$$

where

$$P_{n,p}^+(A_{n,j}, B_{n,j}) = -B_{0,0}A_{n,p}a_{n,p} + \Big[(2p-1) - (2m+1)A_{0,0}\Big] B_{n,p}b_{n,p} \tag{111}$$

$$Q_{n,p}^+(A_{n,j}, B_{n,j}) = -B_{0,0}B_{n,p}b_{n,p} + \Big[-(2m+1)A_{0,0} + (2p-1)\Big] A_{n,p}a_{n,p} + \Big[(2-2p) + (2m+1)A_{0,0}\Big] A_{n,p-1}a_{n,p-1} \tag{112}$$

Notice the fact

$$P^{-}_{n,p}(A_{n,j},B_{n,j}) = P^{+}_{n,p}(A_{n,j},B_{n,j})$$
$$= Q^{-}_{n,p}(A_{n,j},B_{n,j}) = Q^{+}_{n,p}(A_{n,j},B_{n,j}) = 0,$$
$$p < 1 \ or \ p > [\frac{n+1}{2}]. \tag{113}$$

It is easy to obtain [?]

$$\begin{aligned}\bar{P}_n &= \sum_{k=1}^{n-1}\Big[W_k(A_{i,j},B_{n,j},\theta)W_{n-k}(A_{i,j},B_{n,j},\theta)\Big] \\ &= \cos\theta\sum_{p=1}^{[\frac{n+1}{2}]}G_{n,p}(A_{i,j},B_{i,j})\sin^{2p-2}\theta \\ &\quad+\sum_{p=1}^{[\frac{n+2}{2}]}H_{n,p}(A_{i,j},B_{i,j})\sin^{2p-2}\theta\end{aligned} \tag{114}$$

where

$$G_{n,p}(A_{i,j},B_{i,j}) = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}\Big[\bar{b}_{k,p-j}\bar{a}_{n-k,j}+\bar{a}_{k,p-j}\bar{b}_{n-k,j}\Big] \tag{115}$$

$$\begin{aligned}H_{n,p}(A_{i,j},B_{i,j}) &= \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}\Big[\bar{b}_{k,p-j}\bar{b}_{n-k,j}+\bar{a}_{k,p-j}\bar{a}_{n-k,j} \\ &\quad -\bar{a}_{k,p-1-j}\bar{a}_{n-k,j}\Big]\end{aligned} \tag{116}$$

where

$$\bar{a}_{i,j} = A_{i,j}a_{i,j}, \ \bar{b}_{i,j} = B_{i,j}b_{i,j}. \tag{117}$$

When n is even,

$$\begin{aligned}G_{n,p}(A_{i,j},B_{i,j}) &= 0, \quad p < 2 \ or \ p > \frac{n}{2}; \\ H_{n,p}(A_{i,j},B_{i,j}) &= 0, \quad p < 2 \ or \ p > \frac{n}{2}+1,\end{aligned} \tag{118}$$

and when n is odd,

$$G_{n,p}(A_{i,j},B_{i,j}) = H_{n,p}(A_{i,j},B_{i,j}) = 0,. \tag{119}$$

Thus,

$$V^{-}_{n}(A_{i,j},B_{n,j})$$

$$
\begin{aligned}
&= \cos\theta \sum_{k=2}^{[\frac{n+1}{2}]} [G_{n,p}(A_{i,j}, B_{i,j}) + P^{-}_{n,p}(A_{n,j}, B_{n,j})] sin^{2p-2}\theta \\
&\quad + \sum_{k=2}^{[\frac{n+2}{2}]} [H_{n,p}(A_{i,j}, B_{i,j}) + Q^{-}_{n,p}(A_{n,j}, B_{n,j})] sin^{2p-2}\theta \\
&\quad + P^{-}_{n,1}(A_{n,j}, B_{n,j}) + Q^{-}_{n,1}(A_{n,j}, B_{n,j}) \qquad (120)
\end{aligned}
$$

$$
\begin{aligned}
&V^{+}_{n}(A_{i,j}, B_{n,j}) \\
&= \cos\theta \sum_{k=2}^{[\frac{n+1}{2}]} [G_{n,p}(A_{i,j}, B_{i,j}) + P^{+}_{n,p}(A_{n,j}, B_{n,j})] sin^{2p-2}\theta \\
&\quad + \sum_{k=2}^{[\frac{n+2}{2}]} [H_{n,p}(A_{i,j}, B_{i,j}) + Q^{+}_{n,p}(A_{n,j}, B_{n,j})] sin^{2p-2}\theta \\
&\quad + P^{+}_{n,1}(A_{n,j}, B_{n,j}) + Q^{+}_{n,1}(A_{n,j}, B_{n,j}) \qquad (121)
\end{aligned}
$$

one could rewrite $V^{+}_{n}(A_{i,j})$ as

$$
\begin{aligned}
&V^{+}_{n}(A_{i,j}, B_{i,j}) = V^{-}_{n}(C_{i,j}, D_{i,j}) + R_{n;m}(A_{i,j}, B_{i,j}) \\
&= \cos\theta \sum_{k=2}^{[\frac{n+1}{2}]} [G_{n,p}(C_{i,j}, D_{i,j}) + P^{-}_{n,p}(C_{n,j}, D_{n,j})] sin^{2p-2}\theta \\
&\quad + \sum_{k=2}^{[\frac{n+2}{2}]} [H_{n,p}(C_{i,j}, D_{i,j}) + Q^{-}_{n,p}(C_{n,j}, D_{n,j})] sin^{2p-2}\theta \\
&\quad + P^{-}_{n,1}(C_{n,j}, D_{n,j}) + Q^{-}_{n,1}(C_{n,j}, D_{n,j}) + R_{n;m}(A_{i,j}, B_{i,j}) \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
R_{n;m}(A_{i,j}, B_{i,j}) &= P^{+}_{n,1}(A_{n,j}, B_{n,j}) + Q^{+}_{n,1}(A_{n,j}, B_{n,j}) \\
&\quad - P^{-}_{n,1}(C_{n,j}, D_{n,j}) - Q^{-}_{n,1}(C_{n,j}, D_{n,j}) \qquad (123)
\end{aligned}
$$

where

$$P^{-}_{n,1}(C_{n,j}, D_{n,j}) = -[(2m+1)C_{0,0} + 1]C_{n,1}b_{n,1} \qquad (124)$$

$$Q^{-}_{n,1}(C_{n,j}, D_{n,j}) = -D_{0,0}C_{n,1}b_{n,1} \qquad (125)$$

and

$$P^{+}_{n,1}(A_{n,j}, B_{n,j}) = -[(2m+1)A_{0,0} - 1]B_{n,1}b_{n,1} \qquad (126)$$

$$Q^{+}_{n,1}(A_{n,j}, B_{n,j}) = -B_{0,0}B_{n,1}b_{n,1} \qquad (127)$$

by the use of

$$b_{n,0} = 0 \qquad a_{n,0} = 0 \qquad n = 1, 2, \ldots \qquad (128)$$

In order to maintain the shape-invariance property for the nth term, the following equations must be satisfied

$$\begin{aligned}
&G_{n,p}(C_{i,j}, D_{i,j}) + P^-_{n,p}(C_{n,j}, D_{n,j})\\
&= G_{n,p}(A_{i,j}, B_{i,j}) + P^+_{n,p}(A_{n,j}, B_{n,j})\\
&H_{n,p}(C_{i,j}, D_{i,j}) + Q^-_{n,p}(C_{n,j}, D_{n,j})\\
&= H_{n,p}(A_{i,j}, B_{i,j}) + Q^+_{n,p}(A_{n,j}, B_{n,j})\\
&\qquad p = 2, 3, \ldots .[\frac{n+2}{2}]
\end{aligned}$$

Define

$$\begin{aligned}
&G_{n,p}(A_{i,j}, B_{i,j}) - G_{n,p}(C_{i,j}, D_{i,j}) + P^+_{n,p}(A_{n,j}, B_{n,j})\\
&\quad \equiv U_{n,p}, \qquad (129)\\
&H_{n,p}(A_{i,j}, B_{i,j}) - H_{n,p}(C_{i,j}, D_{i,j}) + Q^+_{n,p}(A_{n,j}, B_{n,j})\\
&\quad \equiv \check{U}_{n,p}, \qquad (130)
\end{aligned}$$

then we have

$$\begin{aligned}
P^-_{n,p}(C_{n,j}, D_{n,j}) &= -\alpha_p D_{n,p} b_{n,p} - D_{0,0} C_{n,p} a_{n,p}\\
&= U_{n,p}\\
Q^-_{n,p}(C_{n,j}, D_{n,j}) &= -\alpha_p C_{n,p} a_{n,p} - D_{0,0} D_{n,p} b_{n,p}\\
&\quad +(\alpha_p - 1) C_{n,p-1} a_{n,p-1}\\
&= \check{U}_{n,p}, \qquad (131)
\end{aligned}$$

where $\alpha_p = \Big[(2m+1)C_{0,0} + (2p-1)\Big]$.

In the above equations, the quantities $C_{i,j}, i < n,\ j < [\frac{i}{2}]$ and $D_{i,j}, i < n,\ j < [\frac{i+1}{2}]$ are known as the functions of the variables $A_{i,j}, B_{i,j},\ i < n,\ j < n$. The only unknown quantities are the n quantities $C_{n,p},\ p = 1, 2, \ldots,\ [\frac{n}{2}],\ D_{n,p},\ p = 1, 2, \ldots,\ [\frac{n+1}{2}]$. Therefore,

From the above equations, one obtains

$$\begin{aligned}
D_{n,p} &= \frac{D_{0,0} a_{n,p}}{\alpha_p b_{n,p}} C_{n,p} - \frac{U_{n,p}}{\alpha_p b_{n,p}} \qquad (132)\\
C_{n,p-1} &= \frac{\Big(\alpha_p + \frac{D^2_{0,0}}{\alpha_p}\Big) a_{n,p}}{(\alpha_p - 1) a_{n,p-1}} C_{n,p} + \frac{\check{U}_{n,p} - \frac{D_{0,0}}{\alpha_p} U_{n,p}}{(\alpha_p - 1) a_{n,p-1}}, \qquad (133)\\
&\qquad p = 2, 3, \ldots, [\frac{n+2}{2}] \qquad (134)
\end{aligned}$$

with $D_{n,[\frac{n+1}{2}]+1} = C_{n,[\frac{n}{2}]+1} = 0$. Here we give some notes: (1) when n is odd, one could obtain first $D_{n,[\frac{n+1}{2}]}$ from Eq.(132) and $C_{n,[\frac{n}{2}]}$ from Eq.(133) under the condition $p = [\frac{n+1}{2}],\ C_{n,[\frac{n+1}{2}]} = 0$. Then, it is easy to calculate subsequently. (2) when n is even, one needs to calculate $C_{n,\frac{n}{2}}$ from Eq.(133) under the condition $p = [\frac{n+1}{2}],\ C_{n,[\frac{n+1}{2}]} = 0$. Then, it is easy to calculate subsequently.

Then, the excited eigen-values $E_{l;m}$and eigenfunctions Ψ_l is obtained by the recurrence relation :

$$E^{-}_{l;m} = E_{0;m} + \sum_{k=1}^{l} R(a_k, b_k), \tag{135}$$

$$E_{0;m} = m(m+1) - \frac{3}{4} + \sum_{n=1}^{\infty} E_{0,n;m}\beta^n \tag{136}$$

$$R(a_k, b_k) = R_{0;m} + \sum_{n=1}^{\infty} \beta^n R_{n;m}(a_k, b_k), \tag{137}$$

$$a_1 = (A_{i,j}, B_{i,j}),\ a_2 = (C_{i,j}, D_{i,j}), \ldots, \tag{138}$$

$$\Psi_0 \propto \exp\Big[- \int_{\theta_0}^{\theta} W(A_{n,j}, B_{n,j}, \theta) d\theta \Big], \tag{139}$$

$$\mathcal{A}^{\dagger} = -\frac{d}{d\theta} + W(A_{n,j}, B_{n,j}, \theta) \tag{140}$$

$$\Psi^{-}_{n} = \mathcal{A}^{\dagger}\,(A_{n,j}, B_{n,j}, \theta)\Psi^{-}_{n-1}(C_{n,j}, D_{n,j}, \theta), \tag{141}$$

$$n = 1, 2, 3, \ldots \tag{142}$$

In conclusion, we have proved the shape-invariance properties for the spin-weighted equations in the case of $s = \frac{1}{2}$ and obtain the recurrence relations for them. By these results we can get the exited eigenvalue and eigenfunction. Similar process could also extends to the case $s = 1, 2, \frac{3}{2}$ for the spin-weighted functions.

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