General heavenly equation governs anti-self-dual gravity

A. A. Malykh[1] and M. B. Sheftel[2,3]

[1] Department of Numerical Modelling, Russian State Hydrometeorlogical University, Malookhtinsky pr. 98, 195196 St. Petersburg, Russia
[2] Department of Physics, Boğaziçi University 34342 Bebek, Istanbul, Turkey
[3] Department of Mathematics, North-West State Technical University Millionnaya St. 5, 191186, St. Petersburg, Russia

E-mail: andrei-malykh@mail.ru, mikhail.sheftel@boun.edu.tr

Abstract

We show that the general heavenly equation, suggested recently by Doubrov and Ferapontov [1], governs anti-self-dual (ASD) gravity. We derive ASD Ricci-flat vacuum metric governed by the general heavenly equation, null tetrad and basis of 1-forms for this metric. We present algebraic exact solutions of the general heavenly equation as a set of zeros of homogeneous polynomials in independent and dependent variables. A real solution is obtained for the case of neutral signature.

1 Introduction

There are several four-dimensional scalar equations of Monge-Ampère type that determine potentials of ASD Ricci-flat metrics: first and second heavenly equations of Plebañski [2], complex Monge-Ampère equation (CMA) as a real version of the first heavenly equation and Husain equation [3] together with the closely related mixed heavenly equation [4]. All these equations appear as canonical forms of the general second-order four-dimensional equation which admits partner symmetries and turns out to be of Monge-Ampère type with the additional constraints that it has a two-dimensional divergence form (instead of four-dimensional divergence form which it will have in general) and contains only second partial derivatives of the unknown [4]. In a recent paper [1], Doubrov and Ferapontov classified all integrable four-dimensional Monge-Ampère equations with no restriction of admitting a two-dimensional divergence form. Among resulting normal forms of these equations they presented only one equation that is not of a two-dimensional divergence form (it has a three-dimensional divergence form), which they called *general heavenly equation*

$$\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0, \qquad \alpha + \beta + \gamma = 0, \tag{1.1}$$

where α, β and γ are arbitrary constants with one linear dependence between them. Here and further on, subscripts mean partial derivatives with respect to variables z^1, z^2, z^3, z^4. All other normal forms are either well-known equations that govern self-dual gravity or the "modified heavenly equation" which is a particular case of our "asymmetric heavenly equation" [4]. Equation (1.1) also appeared in our recent paper [5], where it was shown that by using partner symmetries one can obtain its particular solutions, which also satisfy complex Monge-Ampère equation.

Our original motivation for this study of the general heavenly equation was just to make sure that all four-dimensional equations of the Monge-Ampère type can be used to describe anti-self-dual gravity. Then we realized the importance of a characteristic feature of the general heavenly equation that being homogeneous, in contrast to other heavenly equations mentioned above, it enables one to construct easily algebraic solutions, sets of zeros of polynomials in dependent and independent variables. By "homogeneous", we mean that the equation admits independent scaling transformations of each independent (and also dependent) variable. Any such algebraic solution, being a manifold of zeros of a homogeneous polynomial in a complex space, determines a compact complex smooth algebraic manifold. It is known for a long time that the complex Monge-Ampère equation has a solution which determines a compact $K3$ surface, which is called $K3$ instanton by physicists [6]. However, it is extraordinarily difficult to find algebraic solutions, and $K3$ in particular, by directly solving CMA because of its inhomogeneity. Here we construct a first example of algebraic solutions that determine Ricci-flat anti-self-dual metrics by solving the general heavenly equation.

In section 2, we modify the Lax pair of operators from [1] so that these operators commute on solutions of (1.1).

In section 3, we construct a null tetrad for anti-self-dual Ricci-flat metric governed by the general heavenly equation.

In section 4, we construct basis one-forms and ASD metric determined by solutions of the general heavenly equation.

In section 5, we explicitly obtain some examples of algebraic solutions of this equation and also show more general solutions.

In section 6, we consider such a real cross-section of the general heavenly equation, which specifies the signature of corresponding real metric to be either Euclidean or neutral (ultrahyperbolic). By solving reality conditions imposed on the complex solutions obtained in section 5, we obtain real solutions as sets of zeros of homogeneous polynomials in dependent and independent variables. These solutions determine a real ASD Ricci-flat metric with neutral signature. More generally, we also have non-polynomial and functionally invariant solutions with the same property.

In section 7, we present all point symmetries of a real version of the general heavenly equation and find that our solutions, for any values of the parameters that they depend on, are invariant solutions with respect to a certain pair of symmetries of the equation. Hence, they determine ASD vacuum metrics which have two Killing vectors.

2 Lax pair

We start from the Lax pair for the equation (1.1) presented in [1]

$$\begin{aligned} X_1 &= u_{34}\partial_1 - u_{13}\partial_4 + \gamma\lambda(u_{34}\partial_1 - u_{14}\partial_3), \\ X_2 &= u_{23}\partial_4 - u_{34}\partial_2 + \beta\lambda(u_{34}\partial_2 - u_{24}\partial_3), \end{aligned} \tag{2.1}$$

where ∂_1 means ∂/∂_{z^1} and so on. The commutator of these operators does not vanish on solutions of equation (1.1):

$$\begin{aligned} u_{34}\,[X_1, X_2] &= \{u_{34}u_{234} - u_{23}u_{344} + \lambda\beta(u_{24}u_{334} - u_{34}u_{234})\}\,X_1 \\ &+ \{u_{34}u_{134} - u_{13}u_{344} + \lambda\gamma(u_{34}u_{134} - u_{14}u_{334})\}\,X_2. \end{aligned} \tag{2.2}$$

For our purposes, we need a Lax pair that commutes on solutions. It has the form

$$L_0 = \frac{1}{u_{34}}\,X_1, \quad M_0 = \frac{1}{u_{34}}\,X_2, \tag{2.3}$$

so that

$$[L_0, M_0] = \frac{\lambda}{u_{34}^2}\left\{\left(F_4 - \frac{u_{344}}{u_{34}}F\right)\partial_3 - \left(F_3 - \frac{u_{343}}{u_{34}}F\right)\partial_4\right\}, \tag{2.4}$$

where $F = \alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23}$ is the left-hand side of the general heavenly equation, F_3, F_4 are partial derivatives of F with respect to z^3, z^4 and $[L_0, M_0] = 0$ on solutions.

3 Null tetrad for anti-self-dual vacuum metric

In the following we use the notation and results from the book of Mason and Woodhouse [7]. Let Ω be a holomorphic function of z^1, z^2, z^3, z^4. We denote a null tetrad for the general heavenly equation by $W, Z, \tilde{W}, \tilde{Z}$ and set

$$L = W - \lambda\tilde{Z}, \quad M = Z - \lambda\tilde{W}. \tag{3.1}$$

Define Ω by the relations $L_0 = \Omega L$ and $M_0 = \Omega M$ with Ω yet unknown. Then

$$[L_0, M_0] = [\Omega L, \Omega M] = 0. \tag{3.2}$$

Let ν be a holomorphic holomorphic 4-form on a four-dimensional complex manifold with the coordinates $\{z^i\}$, which should satisfy the conditions

$$\mathscr{L}_L(\Omega^{-1}\nu) = \mathscr{L}_M(\Omega^{-1}\nu) = 0, \tag{3.3}$$

where $\mathscr{L}$ denotes Lie derivative.

We note that

$$\mathscr{L}_{L_0}(u_{34}dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = \mathscr{L}_{M_0}(u_{34}dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = 0,$$

equivalent to

$$\mathscr{L}_L(\Omega u_{34}dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = \mathscr{L}_M(\Omega u_{34}dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4) = 0. \tag{3.4}$$

Comparing (3.3) and (3.4) we deduce that

$$\nu = \Omega^2 u_{34} dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \tag{3.5}$$

satisfies condition (3.3). According to Proposition 13.4.8 in [7], if L and M satisfy conditions (3.2), (3.3) and the normalization condition

$$24\nu(W, Z, \tilde{W}, \tilde{Z}) = 1, \tag{3.6}$$

then $W, Z, \tilde{W}, \tilde{Z}$, defined in (3.1), is a null tetrad for an ASD vacuum metric. Substituting ν from (3.5) into (3.6), we obtain $\Omega^2 = \beta\gamma\Delta/u_{34}$, where $\Delta = u_{13}u_{24} - u_{14}u_{23}$, which determines Ω

$$\Omega = \sqrt{\frac{\beta\gamma\Delta}{u_{34}}}. \tag{3.7}$$

The 4-form ν becomes

$$\nu = \beta\gamma\Delta dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4. \tag{3.8}$$

Since $L = \Omega^{-1}L_0$, $M = \Omega^{-1}M_0$ with L_0 and M_0 defined by (2.3), from the definition of $W, Z, \tilde{W}, \tilde{Z}$ in (3.1) we obtain the explicit form of an ASD tetrad frame

$$\begin{aligned} W &= \frac{u_{34}\partial_1 - u_{13}\partial_4}{\sqrt{\beta\gamma u_{34}\Delta}}, \qquad & Z &= \frac{u_{23}\partial_4 - u_{34}\partial_2}{\sqrt{\beta\gamma u_{34}\Delta}}, \\ \tilde{W} &= \frac{\sqrt{\beta}(u_{24}\partial_3 - u_{34}\partial_2)}{\sqrt{\gamma u_{34}\Delta}}, & \tilde{Z} &= \frac{\sqrt{\gamma}(u_{14}\partial_3 - u_{34}\partial_1)}{\sqrt{\beta u_{34}\Delta}}, \end{aligned} \tag{3.9}$$

which is governed by solutions of the general heavenly equation (1.1) expressed solely in terms of the independent parameters β and γ

$$(\beta + \gamma)u_{12}u_{34} = \beta u_{13}u_{24} + \gamma u_{14}u_{23}. \tag{3.10}$$

4 Basis one-forms and ASD metric governed by the general heavenly equation

The corresponding coframe consists of four 1-forms $\omega^i = \omega^i_j dz^j$ which satisfy the following normalization conditions

$$\omega^1(W) = \omega^2(Z) = \omega^3(\tilde{W}) = \omega^4(\tilde{Z}) = 1 \tag{4.1}$$

with all other $\omega^i(W), \omega^i(Z), \omega^i(\tilde{W}), \omega^i(\tilde{Z})$ vanishing. By solving these biorthogonality relations, we obtain the following coframe 1-forms

$$\begin{aligned} \omega^1 &= \sqrt{\frac{\beta\gamma}{u_{34}\Delta}}\left\{u_{23}(u_{14}dz^1 + u_{24}dz^2) + u_{34}(u_{23}dz^3 + u_{24}dz^4)\right\} \\ \omega^2 &= \sqrt{\frac{\beta\gamma}{u_{34}\Delta}}\left\{u_{13}(u_{14}dz^1 + u_{24}dz^2) + u_{34}(u_{13}dz^3 + u_{14}dz^4)\right\} \\ \omega^3 &= -\sqrt{\frac{\gamma}{\beta u_{34}\Delta}}\left\{u_{14}(u_{13}dz^1 + u_{23}dz^2) + u_{34}(u_{13}dz^3 + u_{14}dz^4)\right\} \\ \omega^4 &= \sqrt{\frac{\beta}{\gamma u_{34}\Delta}}\left\{u_{24}(u_{13}dz^1 + u_{23}dz^2) + u_{34}(u_{23}dz^3 + u_{24}dz^4)\right\}. \end{aligned} \tag{4.2}$$

On solutions of (3.10) the corresponding ASD vacuum metric reads

$$
\begin{aligned}
& ds^2 = 2(\omega^2\omega^4 - \omega^1\omega^3) = \frac{2(\beta+\gamma)}{\Delta} \times \\
& \big\{u_{12}\left[u_{13}u_{14}(dz^1)^2 + u_{23}u_{24}(dz^2)^2\right] + u_{34}\left[u_{13}u_{23}(dz^3)^2 + u_{14}u_{24}(dz^4)^2\right] \\
& \ + (u_{13}u_{24} + u_{14}u_{23})\left(u_{12}dz^1dz^2 + u_{34}dz^3dz^4\right) \\
& \ + (u_{12}u_{34} + u_{14}u_{23})\left(u_{13}dz^1dz^3 + u_{24}dz^2dz^4\right) \\
& \ + (u_{12}u_{34} + u_{13}u_{24})\left(u_{23}dz^2dz^3 + u_{14}dz^1dz^4\right)\big\}\,. \qquad (4.3)
\end{aligned}
$$

Using the program EXCALC run by REDUCE, we have computed the Riemann curvature 2-forms and checked vanishing of the Ricci tensor on solutions of the general heavenly equation in the form (3.10), so our metric is indeed Ricci-flat. The expressions for the Riemann curvature 2-forms are too lengthy to be presented here.

5 Algebraic solutions of the general heavenly equation

We look for solutions of the general heavenly equation (1.1), which are algebraic surfaces of even order $2m$, $m = 1, 2, 3, \dots$ of the special form

$$
\begin{aligned}
& u^{2m} - \big\{a_1(z^1)^{2m} + a_2(z^2)^{2m} + a_3(z^3)^{2m} + a_4(z^4)^{2m} + a_5(z^1)^m(z^2)^m \\
& \ + a_6(z^1)^m(z^4)^m + a_7(z^2)^m(z^3)^m) + a_8(z^3)^m(z^4)^m \\
& \ + a_9(z^1)^m(z^3)^m + a_{10}(z^2)^m(z^4)^m + C\big\} = 0, \qquad (5.1)
\end{aligned}
$$

where C is an arbitrary constant and the constant coefficients a_i are determined by equation (1.1). There are two solutions for the set of coefficients in (5.1), such that $\Delta = u_{13}u_{24} - u_{14}u_{23} \neq 0$ in the denominators of coframe 1-forms (4.2) and metric (4.3). The first solution reads

$$
\begin{aligned}
& a_1 = \frac{a_6(\alpha a_9 a_{10} + \beta a_5 a_8) + 2\gamma a_4 a_5 a_9}{2\gamma(2a_4a_7 - a_8a_{10})}\,, \quad a_7 = -\frac{\alpha a_5 a_8 + \beta a_9 a_{10}}{\gamma a_6}\,, \\
& a_2 = \frac{\alpha a_{10}(a_5a_8 - a_9a_{10}) + \gamma a_5(2a_4a_7 - a_8a_{10})}{2\gamma(2a_4a_9 - a_6a_8)}\,, \\
& a_3 = \frac{\beta a_8(a_5a_8 - a_9a_{10}) - \gamma a_9(2a_4a_7 - a_8a_{10})}{2\gamma(a_6a_{10} - 2a_4a_5)}\,, \qquad (5.2)
\end{aligned}
$$

where seven coefficients $a_4, a_5, a_6, a_8, a_9, a_{10}, C$ are free parameters. It is remarkable that these coefficients do not depend on the choice of power m.

Let $z = \{z^1, z^2, z^3, z^4\}$. The proof of the following statement is staright-forward.

Proposition 1 *If $P(z)$ satisfies equation (1.1) and the differential constraint*

$$\alpha(P_1P_2P_{34} + P_3P_4P_{12}) + \beta(P_1P_3P_{24} + P_2P_4P_{13}) + \gamma(P_1P_4P_{23} + P_2P_3P_{14}) = 0, \tag{5.3}$$

then the function $u(z)$, implicitly determined by the equation $G(u, P) = 0$ where G is an arbitrary smooth function, is also a solution of (1.1).

Thus, we have obtained a functionally invariant solution. An example of $P(z)$, that satisfies both conditions of this theorem, is given in curly braces in (5.1) with coefficients determined by (5.2). We also note that solutions of the form (5.1) are valid more generally for an arbitrary real or complex parameter m but then they are not algebraic manifolds.

The second solution for the coefficients of (5.1) has the form

$$a_1 = \frac{\beta a_4 a_5 a_9^2}{\alpha a_5 a_8^2 + 2\beta a_4 a_7 a_9}, \quad a_2 = \frac{a_5(2\beta^2 a_4 a_7 a_9 - \alpha\gamma a_5 a_8^2)}{4\beta^2 a_4 a_9^2}, \\ a_3 = \frac{2\beta a_4 a_7 a_9 - \gamma a_5 a_8^2}{4\beta a_4 a_5}, , \quad a_6 = 0, \quad a_{10} = -\frac{\alpha a_5 a_8}{\beta a_9} \tag{5.4}$$

with five free parameters a_4, a_5, a_7, a_8, a_9.

6 Real cross-section of the general heavenly equation and its real algebraic solutions

For applications to self-dual gravity we need real cross-sections of the general heavenly equation and its solutions. We specify the real cross-section by the requirement that the corresponding real metric should have a certain signature. Then we have to make the following identifications: $z^2 = \bar{z}^1$ and $z^4 = \bar{z}^3$ ($z^2 = -\bar{z}^1$ would also do) and then replace everywhere index 3 by 2. Here the bar means complex conjugation.

The real general heavenly equation takes the form

$$\alpha u_{1\bar{1}} u_{2\bar{2}} + \beta u_{12} u_{\bar{1}\bar{2}} + \gamma u_{1\bar{2}} u_{2\bar{1}} = 0, \tag{6.1}$$

while $\Delta = u_{12}u_{\bar{1}\bar{2}} - u_{1\bar{2}}u_{2\bar{1}}$. In the following we assume that $\gamma \neq 0$.

The signature of the metric depends on the sign of β/γ. If $\beta/\gamma > 0$, we set $\beta = \gamma\delta^2$ with $\delta > 0$. Then basis 1-forms (4.2) become

$$\omega^1 = \frac{|\gamma|\delta}{\sqrt{u_{2\bar{2}}\Delta}}\,\bar{l}_1, \qquad \omega^2 = \frac{|\gamma|\delta}{\sqrt{u_{2\bar{2}}\Delta}}\,l_2,$$
$$\omega^3 = -\frac{1}{\delta\sqrt{u_{2\bar{2}}\Delta}}\,l_1, \qquad \omega^4 = \frac{\delta}{\sqrt{u_{2\bar{2}}\Delta}}\,\bar{l}_2, \tag{6.2}$$

where 1-forms l_1 and l_2 are defined as

$$l_1 = u_{1\bar{2}}(u_{12}dz^1 + u_{\bar{1}\bar{2}}d\bar{z}^1) + u_{2\bar{2}}(u_{12}dz^2 + u_{1\bar{2}}d\bar{z}^2), \tag{6.3}$$
$$l_2 = u_{12}(u_{1\bar{2}}dz^1 + u_{\bar{1}\bar{2}}d\bar{z}^1) + u_{2\bar{2}}(u_{12}dz^2 + u_{1\bar{2}}d\bar{z}^2). \tag{6.4}$$

The metric becomes

$$ds^2 = 2(\omega^2\omega^4 - \omega^1\omega^3) = \frac{2|\gamma|}{|u_{2\bar{2}}\Delta|}\left\{|l_1|^2 + \delta^2|l_2|^2\right\}, \tag{6.5}$$

which has obviously Euclidean signature. It is determined by solutions of the real version (6.1) of the general heavenly equation in the form

$$(\delta^2 + 1)u_{1\bar{1}}u_{2\bar{2}} = \delta^2 u_{12}u_{\bar{1}\bar{2}} + u_{1\bar{2}}u_{2\bar{1}}. \tag{6.6}$$

If $\beta/\gamma < 0$, we set $\beta = -\gamma\delta^2$ and then the real cross-section of the general heavenly equation (6.1) becomes

$$(\delta^2 - 1)u_{1\bar{1}}u_{2\bar{2}} = \delta^2 u_{12}u_{\bar{1}\bar{2}} - u_{1\bar{2}}u_{2\bar{1}}. \tag{6.7}$$

Basis 1-forms (4.2) become

$$\omega^1 = \frac{i|\gamma|\delta}{\sqrt{u_{2\bar{2}}\Delta}}\,\bar{l}_1, \qquad \omega^2 = \frac{i|\gamma|\delta}{\sqrt{u_{2\bar{2}}\Delta}}\,l_2,$$
$$\omega^3 = -\frac{1}{i\delta\sqrt{u_{2\bar{2}}\Delta}}\,l_1, \qquad \omega^4 = \frac{i\delta}{\sqrt{u_{2\bar{2}}\Delta}}\,\bar{l}_2, \tag{6.8}$$

The metric (4.3) takes the form

$$ds^2 = 2(\omega^2\omega^4 - \omega^1\omega^3) = \frac{2|\gamma|}{|u_{2\bar{2}}\Delta|}\left\{|l_1|^2 - \delta^2|l_2|^2\right\}, \tag{6.9}$$

which obviously has neutral signature. This metric is determined by solutions of equation (6.7).

So far, we were able to obtain a real solution only in the case of neutral signature. This solution of equation (6.7) is obtained by solving reality conditions imposed on the complex algebraic solution (5.1) with coefficients (5.2). A relatively simple particular solution has the following form:

$$\begin{aligned} & u^{2m} - \Big\{ \frac{A}{2} \frac{(F-\delta R)}{(\delta F - R)} \left[\exp(2i\theta)(z^1)^{2m} + \exp(-2i\theta)(\bar{z}^1)^{2m}\right] \\ & + \frac{1}{2A}\left[\delta(AB+F^2) - FR\right]\left[(z^2)^{2m} + (\bar{z}^2)^{2m}\right] - A(z^1)^m(\bar{z}^1)^m \\ & + R\left[\exp(i\theta)(z^1)^m(\bar{z}^2)^m + \exp(-i\theta)(z^2)^m(\bar{z}^1)^m\right] + B(z^2)^m(\bar{z}^2)^m \\ & + F\left[\exp(i\theta)(z^1)^m(z^2)^m + \exp(-i\theta)(\bar{z}^1)^m(\bar{z}^2)^m\right] + C\Big\} = 0, \end{aligned} \tag{6.10}$$

where $R = \sqrt{(\delta^2-1)AB + \delta^2 F^2}$ and the constants m, A, B, F, θ, C are free real parameters. Parameter θ is inessential, since it can be scaled out by using scaling symmetries of the general heavenly equation (6.7), so that we can set $\theta = 0$ in (6.10) and our solution depends only on five essential parameters. For $m = 1, 2, 3, \ldots$ this solution determines an algebraic hypersurface in the real five-dimensional space.

The real cross-section of the second complex solution (5.1), (5.4) can be transformed to a particular case of solution (6.10) by using scaling symmetries of the equation (6.7) and therefore does not yield an essentially new real solution.

A general solution of reality conditions for (5.1) with coefficients (5.2) reads

$$\begin{aligned} & u^{2m} - \big\{a_1(z^1)^{2m} + \bar{a}_1(\bar{z}^1)^{2m} + E\left[(z^2)^{2m} + (\bar{z}^2)^{2m}\right] - A(z^1)^m(\bar{z}^1)^m \\ & + R\left[(z^1)^m(\bar{z}^2)^m + (z^2)^m(\bar{z}^1)^m)\right] + B(z^2)^m(\bar{z}^2)^m \\ & + F\left[e^{i\phi}(z^1)^m(z^2)^m + e^{-i\phi}(\bar{z}^1)^m(\bar{z}^2)^m\right] + C\big\} = 0, \end{aligned} \tag{6.11}$$

where all inessential parameters are scaled out by scaling symmetries of the general heavenly equation (6.7), the coefficients satisfy the relations

$$a_1 = \frac{2AEFe^{i\phi} - R(R^2 + AB - F^2)}{2(BFe^{-i\phi} - 2ER)}\,, \tag{6.12}$$

$$\cos\phi = \frac{B(R^2 + AB + F^2) - 4AE^2}{4EFR}, \tag{6.13}$$

R is defined above and $\bar{a}_1$ is complex conjugate to (6.12). This solution at $m = 1, 2, 3, \dots$ and fixed δ depends on five real parameters, four of which, A, B, E, F, should satisfy the relation

$$|B(R^2 + F^2 + AB) - 4AE^2| \leqslant 4|EF|R \tag{6.14}$$

to ensure $|\cos\phi| \leqslant 1$. It is not difficult to check that inequality (6.14) can indeed be satisfied for the considered case of neutral signature, when $\beta/\gamma = -\delta^2 < 0$, while for $\beta/\gamma = \delta^2 > 0$ in the case of Euclidean signature it cannot be satisfied for real solutions.

The particular solution (6.10) with $\theta = 0$ is obtained from (6.11) at $\phi = 0$, which implies the relation $E = \dfrac{\pm\delta|AB + F^2| - FR}{2A}$.

Using our solutions (6.10), (6.11) in metric (6.9) together with definitions (6.4) of 1-forms l_1, l_2, we obtain real ASD Ricci-flat metric with neutral signature, which depends on five real parameters. We have checked that $\Delta \not\equiv 0$ as far as $A\cdot(AB+F^2)\cdot(\delta^2-1) \neq 0$ (for $m = 1$ also $C \neq 0$) while $u_{2\bar{2}} \neq 0$ obviously implies $B \neq 0$. Under these conditions, metric (6.9) has no identically vanishing denominators.

7 Symmetries of general heavenly equation

We have determined all point symmetries of the general heavenly equation. For its real version (6.7), which governs the ASD metrics with the neutral signature, symmetry generators have the form

$$\begin{aligned} &X_1 = a(z^1)\partial_1,\ \bar{X}_1 = \bar{a}(\bar{z}^1)\partial_{\bar{1}}, \quad X_2 = b(z^2)\partial_2,\ \bar{X}_2 = \bar{b}(\bar{z}^2)\partial_{\bar{2}},\ X_3 = u\partial_u, \\ &X_4 = c(z^1)\partial_u,\ \bar{X}_4 = \bar{c}(\bar{z}^1)\partial_u, \quad X_5 = d(z^2)\partial_u,\ \bar{X}_5 = \bar{d}(\bar{z}^2)\partial_u, \end{aligned} \tag{7.1}$$

where a, b, c, d and their complex conjugates are arbitrary functions of one variable. Finite symmetry transformations generated by X_3, X_4, X_5 and $\bar{X}_4, \bar{X}_5$ in (7.1) incorporate scaling $\tilde{u} = \lambda u$ and translations $\tilde{u} = u + \varepsilon c(z^i)$ for $i = 1, 2$, together with their complex conjugates. Symmetry transformations generated by X_1, X_2 have the form $\hat{a}(\tilde{z}^i) = \hat{a}(z^i) + \varepsilon$, where we have introduced the notation $\hat{a}(z) = \int \dfrac{dz}{a(z)}$, plus complex conjugate equations. In particular, if either $a(z) = 1$ or $a(z) = z$ and similarly for their complex conjugates, we obtain translations and scaling in each variable $z^i, \bar{z}^i$ respectively. We note that our solutions (6.10) and (6.11) with $C \neq 0$

are noninvariant under these particular cases of symmetry transformations. However, consider the invariance condition of our solution (6.11) under the symmetry generator $X = X_1 + X_2 + \bar{X}_1 + \bar{X}_2$ in (7.1), with the choice $a(z^1) = c_1/(z^1)^{m-1}, b(z^2) = c_2/(z^2)^{m-1}$ and their complex conjugates:

$$\begin{aligned}
&2a_1c_1 - A\bar{c}_1 + Fe^{i\phi}c_2 + R\bar{c}_2 = 0,\\
&-Ac_1 + 2\bar{a}_1\bar{c}_1 + Rc_2 + Fe^{-i\phi}\bar{c}_2 = 0,\\
&Fe^{i\phi}c_1 + R\bar{c}_1 + 2Ec_2 + B\bar{c}_2 = 0,\\
&Rc_1 + Fe^{-i\phi}\bar{c}_1 + Bc_2 + 2E\bar{c}_2 = 0.
\end{aligned} \tag{7.2}$$

This system of linear algebraic equations admits nonzero solution for the coefficients $c_i, \bar{c}_i$ since the determinant of this system turns out to be zero. Moreover, the rank of this system equals 2. This proves the existence of two independent symmetries of our equation, such that our real solution of the general form is invariant with respect to these symmetries for any choice of parameters in the solution and hence it is an invariant solution. Therefore, the corresponding metric (6.9) will have two Killing vectors.

8 Conclusion

We have shown that the general heavenly equation, introduced by Doubrov and Ferapontov [1], governs anti-self-dual gravity, similar to heavenly equations of Plebañski and Husain. We have derived the ASD Ricci-flat vacuum metric, determined by solutions of the general heavenly equation, together with the corresponding null tetrad and basis 1-forms. Unlike other heavenly equations that describe anti-self-dual gravity, the general heavenly equation is homogeneous, i.e. admit independent scaling transformations of each independent (and also dependent) variable. This property allows us to obtain algebraic solutions to this equation in the form of homogeneous polynomials in independent and dependent variables, which can be modified by adding an arbitrary constant C. In this respect, the homogeneity seems to be most important property of the considered equation, so that we would suggest to call it *homogeneous heavenly equation.* Though for $C \neq 0$, these solutions do not admit scaling or any other obvious symmetries of the equation, we have proved that they are invariant solutions with respect to certain two symmetries of this equation, so that the corresponding metric will have two Killing vectors. The work on noninvariant algebraic solutions to heavenly equation, such that their real form will determine metrics with Euclidean signature, is currently in progress. Such a solution may be relevant to the search of the famous $K3$ instanton [6].

References

[1] Doubrov B and Ferapontov E V 2010 arXiv:0910.3407v2 [math.DG]

[2] Plebañski J F 1975 *J. Math. Phys.* **16** 2395–402

[3] Husain V 1994 *Class. Quantum Grav.* **11** 927–937 (arXiv:gr-qc/9310003)

[4] Sheftel M B and Malykh A A 2009 *J. Phys. A: Math. Gen.* **42** (2009) 395202; Preprint arXiv:0904.2909v3

[5] Malykh A A and Sheftel M B 2010 arXiv:1005.0153v1 [math-ph]

[6] Atiyah M F, Hitchin N J and Singer I M 1978 *Proc. Roy. Soc. A* **362** 452

[7] Mason L J and Woodhouse N M J 1996 *Integrability, Self-Duality, and Twistor Theory*, London Mathematical Society Monographs New Series. Clarendon Press, Oxford