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Entanglement Entropy of Two Black Holes and Entanglement Entropic Force

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Abstract

We study the entanglement entropy, S_C, of a massless free scalar field on the outside region C of two black holes A and B whose radii are R_1 and R_2 and how it depends on the distance, $r(\gg R_1, R_2)$, between two black holes. If we can consider the entanglement entropy as thermodynamic entropy, we can see the entropic force acting on the two black holes from the r dependence of S_C. We develop the computational method based on that of Bombelli et al to obtain the r dependence of S_C of scalar fields whose Lagrangian is quadratic with respect to the scalar fields. First we study S_C in $d+1$ dimensional Minkowski spacetime. In this case the state of the massless free scalar field is the Minkowski vacuum state and we replace two black holes by two imaginary spheres, and we take the trace over the degrees of freedom residing in the imaginary spheres. We obtain the leading term of S_C with respect to $1/r$. The result is $S_C = S_A + S_B + \frac{1}{r^{2d-2}} G(R_1, R_2)$, where S_A and S_B are the entanglement entropy on the inside region of A and B, and $G(R_1, R_2) \leq 0$. We do not calculate $G(R_1, R_2)$ in detail, but we show how to calculate it. In the black hole case we use the method used in the Minkowski spacetime case with some modifications. We show that S_C can be expected to be the same form as that in the Minkowski spacetime case. But in the black hole case, S_A and S_B depend on r, so we do not fully obtain the r dependence of S_C. Finally we assume that the entanglement entropy can be regarded as thermodynamic entropy, and consider the entropic force acting on two black holes. We argue how to separate the entanglement entropic force from other force and how to cancel S_A and S_B whose r dependence are not obtained. Then we obtain the physical prediction which can be tested experimentally in principle.

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I. INTRODUCTION

Entanglement entropy in quantum field theory (QFT) was originally studied to explain the black hole entropy [1, 2]. Entanglement entropy is generally defined as the von Neumann entropy $S_A = -Tr\rho_A \ln \rho_A$ corresponding to the reduced density matrix ρ_A of a subsystem A. When we consider quantum field theory in $d+1$ dimensional spacetime $\mathbb{R} \times N$, where $\mathbb{R}$ and N denote the time direction and the d dimensional space-like manifold respectively, we define the subsystem by a d dimensional domain $A \subset N$ at fixed time $t = t_0$. (So this is also called geometric entropy.) Entanglement entropy naturally arises when we consider the black hole because we cannot obtain the information in the black hole. In fact, in the vacuum state the leading term of the entanglement entropy of A is proportional to the area of the boundary ∂A in many cases [1, 2]. This is similar to the black hole entropy, and extensive studies have been carried out [3–8].

In this paper we study the entanglement entropy, S_C, of a massless free scalar field on the outside region C of two black holes A and B whose radii are R_1 and R_2 and how it depends on the distance, $r(\gg R_1, R_2)$, between two black holes. We consider the case that the state of the massless free scalar field is the vacuum state which depends how to choose the time coordinate. We choose the coordinate system which covers whole space time and does not have the coordinate singularity on the horizons. If we can consider the entanglement entropy as thermodynamic entropy, we can see the entropic force (we call this entanglement entropic force) acting on the two black holes from the r dependence of S_C.

In Section II we obtain the general behavior of the entanglement entropy of two disjoint regions in a translational invariant vacuum in general QFT. In Section III we review the computational method of entanglement entropy in free scalar fields [1]. There are some computational methods of entanglement entropy [9–11] . See [12, 13] for reviews. That of Bombelli et al [1] is most straightforward and powerful enough to obtain the r dependence of S_C in free scalar field theory. In Section IV we study S_C in $d+1$ dimensional Minkowski spacetime. In this case the state of the massless free scalar field is the Minkowski vacuum state and we replace two black holes by two imaginary spheres, and we take the trace over the degrees of freedom residing in the imaginary spheres. We develop the method of Bombelli et al and obtain the leading term of S_C with respect to $1/r$. The result in this section agrees with the general behavior in Section II. This method can be used for any scalar fields in curved space time whose Lagrangian is quadratic with respect to the scalar fields (i.e higher derivative terms can exist). In Section V we consider the black hole case. We use the method used in Section IV with some modifications. We show that S_C can be expected to be the same form as that in the Minkowski spacetime case. But in the black hole case S_A and S_B depend on r, so we do not fully obtain the r dependence of S_C. In Section VI we assume that the entanglement entropy can be regarded as thermodynamic entropy, and consider the entanglement entropic force. We argue how to separate the entanglement entropic force from other force and how to cancel S_A and S_B whose r dependence are not obtained. Then we obtain the physical prediction which can be tested experimentally in principle, and discuss the possibility to measure the entanglement entropic force.

II. GENERAL BEHAVIOR

We consider entanglement entropy of two disjoint regions (A and B) in a translational invariant vacuum in general QFT in $d+1$ dimensional spacetime ($d \geq 2$). We will show that S_C reaches its maximum value when $r \to \infty$.

There are several useful properties which entanglement entropy enjoys generally.(See e.g. [14].) We summarize some of them for later use.

1. If a composite system AB is in a pure state, then $S_A = S_B$.
2. If $\rho_{AB} = \rho_A \otimes \rho_B$, then $S_{AB} = S_A + S_B$.
3. For any subsystem A and B, the following inequalities hold:

$$S_{AB} \leq S_A + S_B, \tag{1}$$

$$S_{AB} \geq \mid S_A - S_B \mid . \tag{2}$$

The first is the subadditivity inequality, and the second is the triangle inequality.

Because of translational invariance, S_A and S_B are independent of their positions, so,$\frac{\partial S_A}{\partial r} = 0$ and $\frac{\partial S_B}{\partial r} = 0$. And the total system is in a pure state, so we have $S_C = S_{AB}$. Moreover, in the vacuum state, $\lim_{r\to\infty} \rho_{AB} = \rho_A \otimes \rho_B$ because of the cluster decomposition principle [16]. So the property 2 suggests

$$\lim_{r\to\infty} S_C(r) = S_A + S_B. \tag{3}$$

We apply (1) and (2) to this system, then we obtain

$$\mid S_A - S_B \mid \leq S_C(r) \leq S_A + S_B. \tag{4}$$

Eqs. (3) and (4) show that S_C (as a function of r) reaches its maximum value when $r \to \infty$.

III. HOW TO COMPUTE ENTANGLEMENT ENTROPY

In this section we review the computational method developed by Bombelli et al [1].

A. Entanglement entropy of a collection of coupled harmonic oscillators

We model the scalar field on $\mathbb{R}^d$ as a collection of coupled oscillators on a lattice of space points, labeled by capital Latin indices, the displacement at each point giving the value of the scalar field there. In this case the Lagrangian can be given by

$$L = \frac{1}{2} G_{MN} \dot{q}^M \dot{q}^N - \frac{1}{2} V_{MN} q^M q^N, \tag{5}$$

where q^M gives the displacement of the Mth oscillator and $\dot{q}^M$ its generalized velocity. The symmetric matrix G_{MN} is positive definite, and therefore invertible, i.e, there exists the inverse matrix G^{MN} such that

$$G^{MP} G_{PN} = \delta^M{}_N. \tag{6}$$

The matrix V_{MN} is also symmetric and positive definite. Next, consider the positive definite symmetric matrix W_{MN} defined by

$$W_{MA} G^{AB} W_{BN} = V_{MN}. \tag{7}$$

The matrix W is the square root of V in the scalar product G.

Now consider a region Ω in $\mathbb{R}^d$. The oscillators in this region will be specified by Greek letters, and those in the complement of Ω will be specified by lowercase Latin letters. We will use the following notation

$$W_{AB} = \begin{pmatrix} W_{ab} & W_{a\beta} \\ W_{\alpha b} & W_{\alpha\beta} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \qquad W^{AB} = \begin{pmatrix} W^{ab} & W^{a\beta} \\ W^{\alpha b} & W^{\alpha\beta} \end{pmatrix} \equiv \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}, \tag{8}$$

where W^{AB} is the inverse matrix of W_{AB} (W^{AB} is *not* obtained by raising indices with G^{AB}). So we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} = \begin{pmatrix} AD + BE^T & AE + BF \\ B^T D + CE^T & B^T E + CF \end{pmatrix}. \tag{9}$$

If we consider the information on the displacement of the oscillators inside Ω as unavailable, we can obtain a reduced density matrix ρ_{red} for the outside Ω, integrating out over $\mathbf{R}$ for each of the oscillators in the region Ω, then we have

$$\rho_{red}(q^a, q'^b) = \int \prod_\alpha dq^\alpha \rho(q^a, q^\alpha, q'^b, q^\alpha), \tag{10}$$

where ρ is a density matrix of the total system.

We can obtain the density matrix for the ground state by standard method, and it is a Gaussian density matrix. Then, ρ_{red} is obtained by a Gaussian integral , and it is also a Gaussian density matrix. The entanglement entropy, $S = -tr \rho_{red} \ln \rho_{red}$, is given by [1]

$$S = \sum_n f(\lambda_n), \tag{11}$$

$$f(\lambda) \equiv \ln(\frac{1}{2}\lambda^{1/2}) + (1+\lambda)^{1/2} \ln[(1+\lambda^{-1})^{1/2} + \lambda^{-1/2}], \tag{12}$$

where λ_n are the eigenvalues of the matrix

$$\Lambda^a{}_b = -W^{a\alpha} W_{\alpha b} = -(EB^T)^a{}_b. \tag{13}$$

It can be shown that all of λ_n are nonnegative as follows. From (9) we have

$$A\Lambda = -AEB^T = BFB^T. \tag{14}$$

It is easy to show that A, C, D and F are positive definite matrices when W and W^{-1} are positive definite matrices. Then $A\Lambda$ is a positive semi definite matrix as can be seen from (14). So all eigenvalues of Λ are nonnegative. Finally, we can obtain the entanglement entropy by solving the eigenvalue problem of Λ.

B. The continuum limit

Next, we apply the above formalism to a massless free scalar field in $(d+1)$ dimensional Minkowski spacetime. We take the continuum limit in the above formalism. In this case the Lagrangian is given by

$$L = \int d^d x \frac{1}{2}[\dot{\phi}^2 - (\nabla\phi)^2]. \tag{15}$$

Then the potential term becomes

$$\frac{1}{2} V_{AB} q^A q^B \to \int d^d x \frac{1}{2}[(\nabla\phi)^2]. \tag{16}$$

The matrices V, W and W^{-1} are given in the momentum representation by,

$$V(x,y) = \int \frac{d^dk}{(2\pi)^d}(k^2)e^{ik\cdot(x-y)} \tag{17}$$

$$W(x,y) = \int \frac{d^dk}{(2\pi)^d}(k^2)^{1/2}e^{ik\cdot(x-y)} \tag{18}$$

$$W^{-1}(x,y) = \int \frac{d^dk}{(2\pi)^d}(k^2)^{-1/2}e^{ik\cdot(x-y)} \tag{19}$$

From (13), the matrix Λ is obtained as a sum over the oscillators in the region Ω,

$$\Lambda(x,y) = -\int_\Omega d^dz W^{-1}(x,z)W(z,y). \tag{20}$$

We now have to solve the eigenvalue equation

$$\int_{\Omega^c} d^dy \Lambda(x,y)f(y) = \lambda f(x), \tag{21}$$

where Ω^c is the complementary set of Ω, and then we use the eigenvalues in the expression for the entropy (11).

IV. ENTANGLEMENT ENTROPY OF TWO DISJOINT REGIONS IN A $d+1$ DIMENSIONAL MASSLESS FREE SCALAR FIELD

We consider two spheres A and B whose radii are R_1 and R_2, and define the outside region as C. (See Fig 1.) We derive the $r(\gg R_1, R_2)$ dependence of $S_C(r, R_1, R_2)$ by using the formalism of the preceding section. (In the later analysis we do not use the shapes of A and B, so all analysis in this section holds for A and B which have arbitrary shapes. In this case R_1 and R_2 are the characteristic sizes of A and B.)

We consider $S_{AB}(r, R_1, R_2)$ because $S_C = S_{AB}$ in a pure state and the r dependence of S_{AB} is clearer than that of S_C in the calculation. In this case the region Ω is C.

We obtain the r dependence of S_{AB} by following three steps:

(1) We obtain the r dependence of Λ by using the $\|x-y\|$ dependences of $W(x,y)$ and $W^{-1}(x,y)$. We decompose Λ into the non-perturbative part and the perturbative part as $\Lambda = \Lambda^0 + \delta\Lambda$, where $\Lambda^0 \equiv \lim_{r\to\infty}\Lambda$.

(2) We obtain $\lambda_m(r)$ which are the eigenvalues of Λ by perturbation theory. This is almost similar to the time-independent perturbation theory in quantum mechanics in the presence

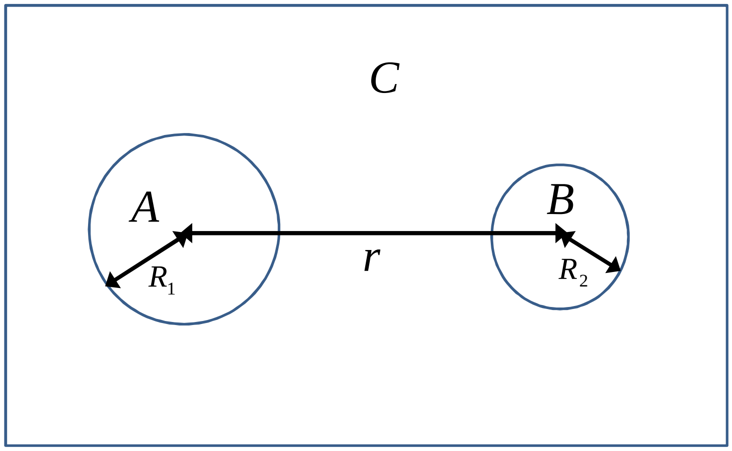

FIG. 1.

of degeneracy. We can regard Λ as Hamiltonian. Note that Λ is *not* a symmetric matrix. So we must slightly modify the perturbation theory in quantum mechanics.

(3) In Step (2), we had $\lambda_m(r)$ as $\lambda_m(r) = \lambda_m^0 + \delta\lambda_m(r)$, where λ_m^0 are the eigenvalues of Λ^0. We substitute these $\lambda_m(r)$ into (12), then we obtain $S_{AB}(r, R_1, R_2)$.

First we examine the $\|x-y\|$ dependences of $W(x,y)$ and $W^{-1}(x,y)$. Generally entanglement entropy has UV divergence as discussed in [1]. So we use a momentum cutoff l^{-1} in integrals (17)-(19), though these integrals are well defined as Fourier transforms of distributions. (The other regularization methods are discussed in [1].) When $d \geq 2$ and $l/\|x-y\| \to 0$, $W(x,y)$ and $W^{-1}(x,y)$ are

$$W(x,y) = \frac{A_d}{\|x-y\|^{d+1}} \ , \qquad W^{-1}(x,y) = \frac{B_d}{\|x-y\|^{d-1}}, \quad A_d, B_d \in \mathbb{R} \tag{22}$$

where A_d and B_d are nonzero dimensionless constants (see Appendix A). We *cannot* obtain (22) by only using a dimensional analysis because $l/\|x-y\|$ is dimensionless. Indeed $V(x,y) \to 0$ when $l/\|x-y\| \to 0$, i.e. $V(x,y)$ is zero when $\|x-y\|$ is finite. On the other hand W and W^{-1} have nonzero value for $\|x-y\| > 0$ because they are kernels of integral operators of nonlocal interaction (i.e Fourier transformations of $(\sqrt{k^2})^{\pm 1}$) . In Appendix A we explicitly show that W and W^{-1} have nonzero value for $\|x-y\| > 0$ and Eq. (22) holds.

Next, we obtain the r dependence of Λ by using (22). We represent the matrix elements of $W(W^{-1})$ diagrammatically in Fig 2. Instead of solid lines we use dotted lines for W^{-1}. The lines denote the matrix elements $W(x,y)$(or $W^{-1}(x,y)$) in (22). An initial point and an end point of an arrow denote a row and a column respectively. We can obtain products of

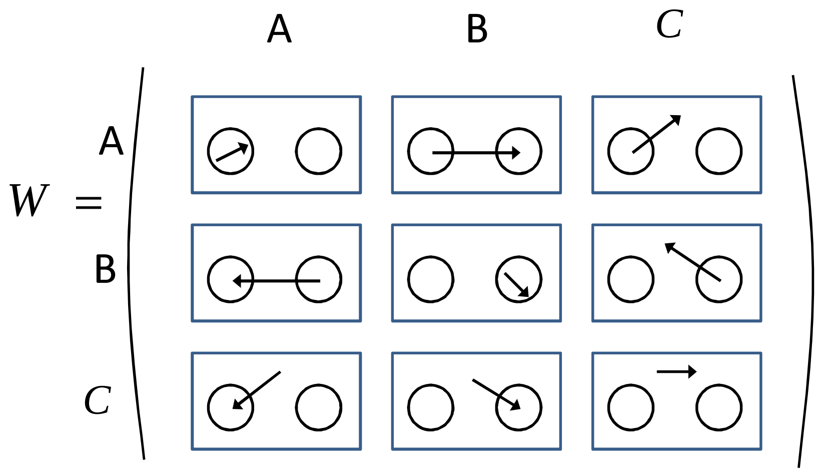

FIG. 2.

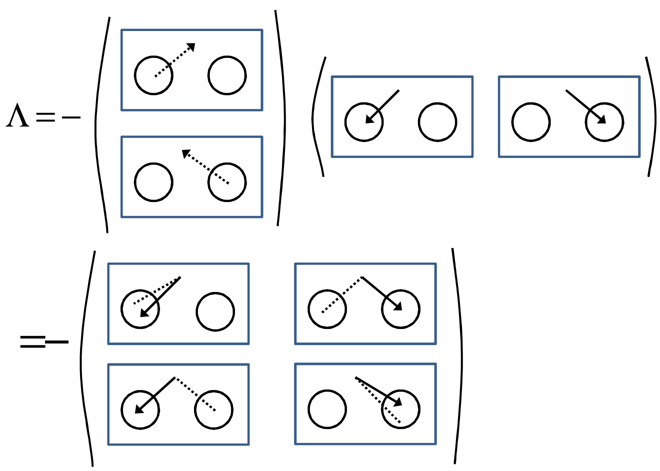

FIG. 3.

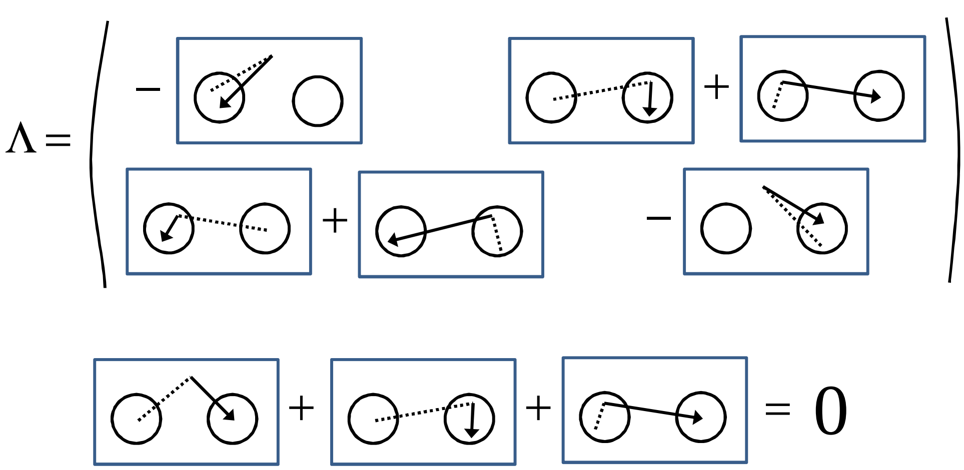

FIG. 4.

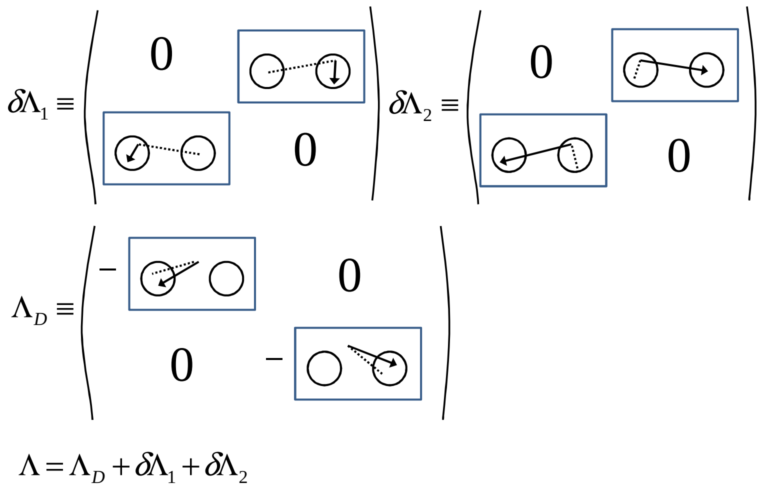

FIG. 5.

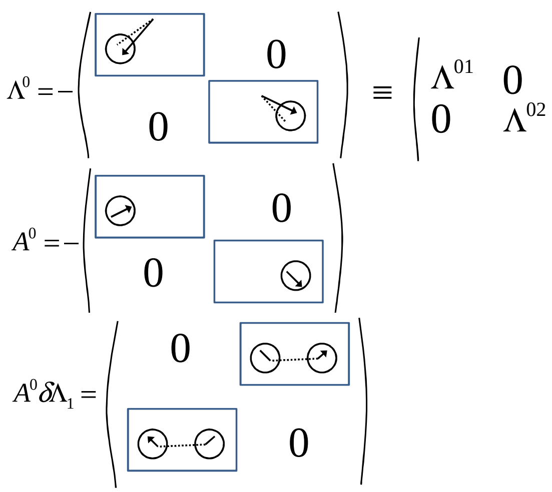

FIG. 6.

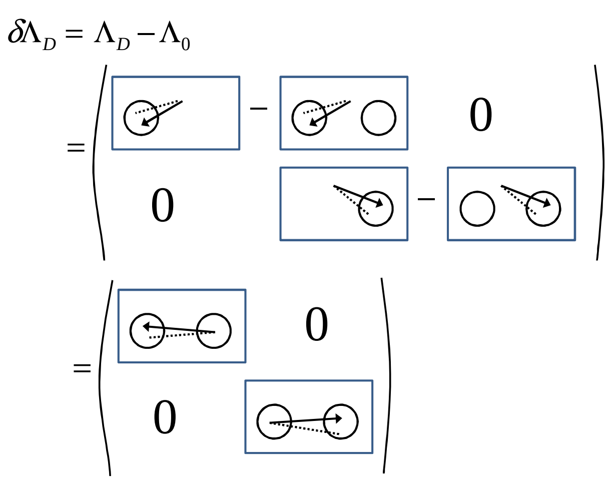

FIG. 7.

matrices by connecting arrows and integrating joint points on regions where the joint points exist. We label coordinates in A, B and C as x_a, x_b and x_c. Then, from Fig 3 we obtain $\Lambda = -EB^T$ as

$$\begin{aligned}\Lambda &= \begin{pmatrix} \Lambda(x_a, y_a) & \Lambda(x_a, y_b) \\ \Lambda(x_b, y_a) & \Lambda(x_b, y_b) \end{pmatrix} = -\begin{pmatrix} W^{-1}(x_a, z_c) \\ W^{-1}(x_b, z_c) \end{pmatrix} \begin{pmatrix} W(z_c, y_a) & W(z_c, y_b) \end{pmatrix} \\ &= -\begin{pmatrix} \int_C d^d z_c W^{-1}(x_a, z_c) W(z_c, y_a) & \int_C d^d z_c W^{-1}(x_a, z_c) W(z_c, y_b) \\ \int_C d^d z_c W^{-1}(x_b, z_c) W(z_c, y_a) & \int_C d^d z_c W^{-1}(x_b, z_c) W(z_c, y_b) \end{pmatrix}.\end{aligned} \tag{23}$$

To make the r dependence of the non-diagonal elements of Λ clear, we use the following identity,

$$\int_{A+B+C} d^d z W^{-1}(x_a, z) W(z, y_b) = \delta(x_a - y_b) = 0. \tag{24}$$

We represent this identity diagrammatically in Fig 4. From (23) and (24) we obtain (see Fig 4)

$$\begin{aligned}\Lambda(x_a, y_b) &= \int_A d^d z_a W^{-1}(x_a, z_a) W(z_a, y_b) + \int_B d^d z_b W^{-1}(x_a, z_b) W(z_b, y_b) \\ \Lambda(x_b, y_a) &= \int_A d^d z_a W^{-1}(x_b, z_a) W(z_a, y_a) + \int_B d^d z_b W^{-1}(x_b, z_b) W(z_b, y_a).\end{aligned} \tag{25}$$

Note that from (22) $W(x, y)$ and $W^{-1}(x, y)$ have the different $\|x - y\|$ dependence. So, from (23) and (25) we decompose Λ as

$$\Lambda = \Lambda_D + \delta\Lambda_1 + \delta\Lambda_2 \tag{26}$$

where we define (see Fig 5)

$$\Lambda_D \equiv \begin{pmatrix} \Lambda(x_a, y_a) & 0 \\ 0 & \Lambda(x_b, y_b) \end{pmatrix}, \tag{27}$$

$$\delta\Lambda_1 \equiv \begin{pmatrix} 0 & \int_B d^d z_b W^{-1}(x_a, z_b) W(z_b, y_b) \\ \int_A d^d z_a W^{-1}(x_b, z_a) W(z_a, y_a) & 0 \end{pmatrix}, \tag{28}$$

$$\delta\Lambda_2 \equiv \begin{pmatrix} 0 & \int_A d^d z_a W^{-1}(x_a, z_a) W(z_a, y_b) \\ \int_B d^d z_b W^{-1}(x_b, z_b) W(z_b, y_a) & 0 \end{pmatrix}. \tag{29}$$

We approximate $W(x_a, y_b) \approx \frac{A_d}{r^{d+1}}$ and $W^{-1}(x_a, y_b) \approx \frac{B_d}{r^{d-1}}$ because $r \gg R_1, R_2$. Then we have

$$\delta\Lambda_1 \approx \frac{B_d}{r^{d-1}} \begin{pmatrix} 0 & \int_B d^d z_b W(z_b, y_b) \\ \int_A d^d z_a W(z_a, y_a) & 0 \end{pmatrix}, \tag{30}$$

$$\delta\Lambda_2 \approx \frac{A_d}{r^{d+1}} \begin{pmatrix} 0 & \int_A d^d z_a W^{-1}(x_a, z_a) \\ \int_B d^d z_b W^{-1}(x_b, z_b) & 0 \end{pmatrix}. \tag{31}$$

Next we consider the non-perturbative part $\Lambda^0 = \lim_{r\to\infty} \Lambda$. From (30) and (31) we can see that $\delta\Lambda_1$ and $\delta\Lambda_2$ become 0 when $r \to \infty$. Note that the integral region of the integral in $\Lambda(x_a, y_a)$ ($\Lambda(x_b, y_b)$) become $A^c \equiv \mathbb{R}^d - A$ ($B^c \equiv \mathbb{R}^d - B$) when $r \to \infty$, then we obtain (see Fig 6)

$$\Lambda^0 = -\begin{pmatrix} \int_{A^c} d^d z W^{-1}(x_a, z) W(z, y_a) & 0 \\ 0 & \int_{B^c} d^d z W^{-1}(x_b, z) W(z, y_b) \end{pmatrix} \equiv \begin{pmatrix} \Lambda^{01}(x_a, y_a) & 0 \\ 0 & \Lambda^{02}(x_b, y_b) \end{pmatrix}. \tag{32}$$

From (32) we rewrite (26) as follows,

$$\Lambda = \Lambda^0 + \delta\Lambda_1 + \delta\Lambda_2 + \delta\Lambda_D, \tag{33}$$

where we define (see Fig 7)

$$\delta\Lambda_D \equiv \Lambda_D - \Lambda^0 = \begin{pmatrix} \int_B d^d z_b W^{-1}(x_a, z_b) W(z_b, y_a) & 0 \\ 0 & \int_A d^d z_a W^{-1}(x_b, z_a) W(z_a, y_b) \end{pmatrix}. \tag{34}$$

We use the same approximation as we used in (30) and (31), then we obtain

$$\delta\Lambda_D \approx \frac{A_d B_d}{r^{2d}} \begin{pmatrix} \int_B d^d z_b & 0 \\ 0 & \int_A d^d z_a \end{pmatrix}. \tag{35}$$

When we perform the perturbative calculation to obtain $\lambda_m(r)$ which is the eigenvalues of Λ, from (30), (31), (33) and (35) we can neglect $\delta\Lambda_D$ because it is higher order than $\delta\Lambda_1$ and $\delta\Lambda_2$ with respect to $1/r$. And we can neglect $\delta\Lambda_2$ because its nonzero matrix elements are in the same position as $\delta\Lambda_1$ and $\delta\Lambda_2$ is higher order than $\delta\Lambda_1$ with respect to $1/r$.

Because Λ is not a symmetric matrix, in the later perturbative calculation we need $A^0\delta\Lambda_1$ where A^0 is defined as (see Fig 6)

$$A^0 \equiv \lim_{r\to\infty} A = \begin{pmatrix} W(x_a, y_a) & 0 \\ 0 & W(x_b, y_b) \end{pmatrix}. \tag{36}$$

From (28), (30) and (36) we obtain (see Fig 6)

$$
\begin{aligned}
A^0\delta\Lambda_1 &= \begin{pmatrix} 0 & \int_A d^d z_a \int_B d^d z_b W(x_a,z_a)W^{-1}(z_a,z_b)W(z_b,y_b) \\ \int_B d^d z_b \int_A d^d z_a W(x_b,z_b)W^{-1}(z_b,z_a)W(z_a,y_a) & 0 \end{pmatrix} \\
&\approx \frac{B_d}{r^{d-1}} \begin{pmatrix} 0 & \int_A d^d z_a W(x_a,z_a) \int_B d^d z_b W(z_b,y_b) \\ \int_B d^d z_b W(x_b,z_b) \int_A d^d z_a W(z_a,y_a) & 0 \end{pmatrix}.
\end{aligned} \tag{37}
$$

We have finished the first step.

Next, we calculate $\lambda_m(r)$ by perturbation theory. This is almost similar to the time-independent perturbation theory with degeneracy of quantum mechanics. The only difference is that Λ is *not* a symmetric matrix and $A\Lambda$ is a symmetric matrix.

We can approximate $\Lambda \approx \Lambda^0 + \delta\Lambda_1$ and regard $\delta\Lambda_1$ as the perturbative part. Then, from (30) we expand $\lambda_m(r)$ with respect to $1/r^{d-1}$. We expand λ_m around $\lambda_m^0 \equiv \lambda_m(r=\infty)$,

$$\lambda_m = \lambda_m^0 + \delta\lambda_m^1 + \delta\lambda_m^2, \tag{38}$$

where $\delta\lambda_m^1$ and $\delta\lambda_m^2$ are the first and the second order perturbations.

Next we substitute (38) into (12),

$$
\begin{aligned}
S_{AB}(r,R_1,R_2) &= \sum_m f(\lambda_m) \\
&= S_A(R_1) + S_B(R_2) + \sum_m \left[\delta\lambda_m \left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda_m^0} + \frac{1}{2}(\delta\lambda_m)^2 \left.\frac{d^2 f}{d\lambda_m^2}\right|_{\lambda_m=\lambda_m^0} \right],
\end{aligned} \tag{39}
$$

where $\delta\lambda_m \equiv \delta\lambda_m^1 + \delta\lambda_m^2$. We will show that the first order perturbation in (39) (i.e $\sum_m \delta\lambda_m^1 \left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda_m^0}$) is zero, so we must calculate the second order perturbations.

We label the λ_m^0's as $\lambda_m^0 > \lambda_n^0$ when $m > n$. And we define the eigenvectors of Λ^0,

$$f_{m1\alpha}^0 = \begin{pmatrix} f_{m1\alpha}^0(x_a) \\ 0 \end{pmatrix}, f_{n2\beta}^0 = \begin{pmatrix} 0 \\ f_{n2\beta}^0(x_b) \end{pmatrix}, \Lambda^0 f_{m1\alpha}^0 = \lambda_m^0 f_{m1\alpha}^0, \Lambda^0 f_{m2\beta}^0 = \lambda_m^0 f_{m2\beta}^0, \tag{40}$$

where $\alpha = 1, \cdots, M_m$ and $\beta = 1, \cdots, N_n$ are the labels of the degeneracy, and $f_{m1\alpha}^0(x_a)$ and $f_{n2\beta}^0(x_b)$ are the eigenvectors of Λ^{01} and Λ^{02} which are defined in (32). And we normalize $f_{mi\alpha}^0$ $(i=1,2)$ as follow,

$$f_{mi\alpha}^{0T} A^0 f_{nj\beta}^0 = \delta_{mn}\delta_{ij}\delta_{\alpha\beta}. \tag{41}$$

This normalization is always possible because A^0 is a positive definite symmetric matrix. For general R_1 and R_2, Λ^{01} and Λ^{02} have different eigenvalues, so there are two groups of

λ_m^0; one is the group of the common eigenvalues of Λ^{01} and Λ^{02}, the other is not. We will see that $\delta\lambda_m^1$ of the latter group are zero. We expand $f_{m\gamma}$ which is the eigenvector of Λ in the following way,

$$f_{m\gamma} = \sum_\alpha a_{\gamma\alpha} f^0_{m1\alpha} + \sum_\beta b_{\gamma\beta} f^0_{m2\beta} + f^1_{m\gamma} + f^2_{m\gamma} \equiv \xi^0_{m\gamma} + f^1_{m\gamma} + f^2_{m\gamma} \tag{42}$$

where $f^1_{m\gamma}$ and $f^2_{m\gamma}$ are the first and the second order perturbations. Note that when λ_m^0 is an eigenvalue of Λ^{01} (Λ^{02}) and is not an eigenvalue of Λ^{02} (Λ^{01}), then the coefficients $b_{\gamma\beta}$ ($a_{\gamma\alpha}$) are zero ; because the zeroth order eigenvectors $f^0_{m2\beta}$ ($f^0_{m1\alpha}$) do not exist. So either the coefficients $a_{\gamma\alpha}$ or $b_{\gamma\beta}$ are zero when λ_m^0 is not a common eigenvalue of Λ^{01} and Λ^{02}. We substitute (42) into the eigenvalue equation (we approximate $\Lambda \approx \Lambda^0 + \delta\Lambda_1$) , then we have

$$(\Lambda^0 + \delta\Lambda_1) f_{m\gamma} = (\lambda^0_m + \delta\lambda^1_{m\gamma} + \delta\lambda^2_{m\gamma}) f_{m\gamma}. \tag{43}$$

We obtain equations of the first and the second order perturbation.

$$\Lambda^0 f^1_{m\gamma} + \delta\Lambda_1 \xi^0_{m\gamma} = \lambda^0_m f^1_{m\gamma} + \delta\lambda^1_{m\gamma}\xi^0_{m\gamma}, \tag{44}$$

$$\Lambda^0 f^2_{m\gamma} + \delta\Lambda_1 f^1_{m\gamma} = \lambda^0_m f^2_{m\gamma} + \delta\lambda^1_{m\gamma} f^1_{m\gamma} + \delta\lambda^2_{m\gamma}\xi^0_{m\gamma}. \tag{45}$$

We multiply (44) by $f^{0T}_{mj\gamma'}A^0$ from the left . The first term of the left hand side of (44) cancel the first term of the right hand side of (44) because $A^0\Lambda^0$ is a symmetric matrix, then we obtain

$$\sum_\alpha a_{\gamma\alpha} V^{j1}_{m\gamma' m\alpha} + \sum_\beta b_{\gamma\beta} V^{j2}_{m\gamma' m\beta} = \delta\lambda^1_{m\gamma}(a_{\gamma\gamma'}\delta^{j1} + b_{\gamma\gamma'}\delta^{j2}), \tag{46}$$

where

$$V^{ij}_{m\alpha n\beta} \equiv f^{0T}_{mi\alpha} A^0 \delta\Lambda_1 f^0_{nj\beta}. \tag{47}$$

From (37) we obtain $V^{11}_{m\alpha n\beta} = V^{22}_{m\alpha n\beta} = 0$ and

$$\begin{aligned} V^{12}_{m\alpha n\beta} &= \begin{pmatrix} f^0_{m1\alpha}(x_a) & 0 \end{pmatrix} \begin{pmatrix} A^0\delta\Lambda_1(x_a,y_a) & A^0\delta\Lambda_1(x_a,y_b) \\ A^0\delta\Lambda_1(x_b,y_a) & A^0\delta\Lambda_1(x_b,y_b) \end{pmatrix} \begin{pmatrix} 0 \\ f^0_{n2\beta}(y_b) \end{pmatrix} \\ &= \frac{B_d}{r^{d-1}} \int_A d^d x_a \int_A d^d z_a W(x_a,z_a) f^0_{m1\alpha}(x_a) \int_B d^d y_b \int_B d^d z_b W(y_b,z_b) f^0_{n2\beta}(y_b) \equiv \frac{B_d}{r^{d-1}} C_{m\alpha n\beta} \end{aligned} \tag{48}$$

and $V^{12}_{m\alpha n\beta} = V^{21}_{n\beta m\alpha}$. We define an $M_m \times N_n$ matrix C_{mn} as $(C_{mn})_{\alpha\beta} = C_{m\alpha n\beta}$ and write (46) as follows,

$$\frac{B_d}{r^{d-1}} \begin{pmatrix} 0 & C_{mm} \\ C^T_{mm} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_\gamma \\ \mathbf{b}_\gamma \end{pmatrix} = \delta\lambda^1_{m\gamma} \begin{pmatrix} \mathbf{a}_\gamma \\ \mathbf{b}_\gamma \end{pmatrix} \tag{49}$$

where $(\mathbf{a}_\gamma)_\alpha = a_{\gamma\alpha}$ and $(\mathbf{b}_\gamma)_\beta = b_{\gamma\beta}$. From (49), if λ^0_m is not a common eigenvalue of Λ^{01} and Λ^{02}, $\delta\lambda^1_{m\gamma}$ is zero; because either $a_{\gamma\alpha}$ or $b_{\gamma\beta}$ are zero when λ^0_m is not a common eigenvalue of Λ^{01} and Λ^{02}. We first consider the case that $M_m \geq N_m$. In this case we obtain the following eigenvalue equation [17].

$$\begin{aligned} \det \begin{vmatrix} x1_{M_m\times M_m} & -C_{mm} \\ -C^T_{mm} & x1_{N_m\times N_m} \end{vmatrix} &= \det(x1_{M_m\times M_m})\det(x1_{N_m\times N_m} - x^{-1}C^T_{mm}C_{mm}) \\ &= x^{M_m-N_m}\det(x^2 1_{N_m\times N_m} - C^T_{mm}C_{mm}) = 0. \end{aligned} \tag{50}$$

We define the eigenvalues of $C^T_{mm}C_{mm}$ as $c_{m\alpha}$ $\quad(\alpha = 1, \cdots M_m)$. $C^T_{mm}C_{mm}$ is a positive semidefinite matrix because C_{mm} is a real matrix, so $c_{m\alpha} \geq 0$. Then we obtain $\delta\lambda^1_{m\gamma}$ from (49) and (50) .

$$\delta\lambda^1_{m\gamma} = \begin{cases} 0 \\ \pm\dfrac{B_d}{r^{d-1}}\sqrt{c_{m\alpha}} \quad (\alpha = 1, \cdots M_m) \end{cases}. \tag{51}$$

When $M_m < N_m$, we can obtain $\delta\lambda^1_{m\gamma}$ in the same way. We define the eigenvalues of $C_{mm}C^T_{mm}$ as $d_{m\alpha}(\geq 0)$ $\quad(\alpha = 1, \cdots N_m)$. Then we obtain

$$\delta\lambda^1_{m\gamma} = \begin{cases} 0 \\ \pm\dfrac{B_d}{r^{d-1}}\sqrt{d_{m\alpha}} \quad (\alpha = 1, \cdots N_m) \end{cases}. \tag{52}$$

Then $\sum_{m,\gamma}\delta\lambda^1_{m\gamma}\left.\dfrac{df}{d\lambda_m}\right|_{\lambda_m=\lambda^0_m} = 0$, because we have $\sum_\gamma \delta\lambda^1_{m\gamma} = 0$ from (51) and (52).

Next we consider $\delta\lambda^2_{m\gamma}$. We skip the detailed calculation because it is also almost similar to the time-independent perturbation theory with degeneracy of quantum mechanics. Then we can write $\delta\lambda^2_{m\gamma}$ as follows

$$\begin{aligned} \delta\lambda^2_{m\gamma} &= \sum_{n(\neq m),i,\beta} \frac{1}{\lambda^0_m - \lambda^0_n}(f^{0T}_{ni\beta}A^0\delta\Lambda_1\xi^0_{m\gamma})(\xi^{0T}_{m\gamma}A^0\delta\Lambda_1 f^0_{ni\beta}) \\ &= \sum_{n(\neq m)} \frac{1}{\lambda^0_m - \lambda^0_n}\xi^{0T}_{m\gamma}A^0\delta\Lambda_1\hat{\phi}_n\delta\Lambda_1\xi^0_{m\gamma} \end{aligned} \tag{53}$$

where

$$\hat{\phi}_n \equiv \sum_{i,\beta} f^0_{ni\beta} f^{0T}_{ni\beta} A^0. \tag{54}$$

$\hat{\phi}_n$ is a projection operator on the eigenspace of λ^0_n. To obtain $\delta\lambda^2_{m\gamma}$ we must obtain $\xi^0_{m\gamma}$ by solving the eigenvalue problem, but it is not necessary for our purpose because we want to know only $\sum_{m,\gamma} \delta\lambda^2_{m\gamma} \left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda^0_m}$. From (53) we obtain

$$\begin{aligned}
\sum_{m,\gamma} \delta\lambda^2_{m\gamma} \left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda^0_m} &= \sum_{m,\gamma} \sum_{n(\neq m)} \frac{1}{\lambda^0_m - \lambda^0_n} \xi^{0T}_{m\gamma} A^0 \delta\Lambda_1 \hat{\phi}_n \delta\Lambda_1 \xi^0_{m\gamma} \left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda^0_m} \\
&= \sum_{m,n(m\neq n)} \frac{1}{\lambda^0_m - \lambda^0_n} Tr(\hat{\phi}_m \delta\Lambda_1 \hat{\phi}_n \delta\Lambda_1) \left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda^0_m} \\
&= \sum_{m,n(m>n)} \frac{1}{\lambda^0_m - \lambda^0_n} Tr(\hat{\phi}_m \delta\Lambda_1 \hat{\phi}_n \delta\Lambda_1) \left(\left.\frac{df}{d\lambda_m}\right|_{\lambda_m=\lambda^0_m} - \left.\frac{df}{d\lambda_n}\right|_{\lambda_n=\lambda^0_n} \right).
\end{aligned} \tag{55}$$

In the second line we have used

$$\sum_{\gamma} \xi^0_{m\gamma} \xi^{0T}_{m\gamma} A^0 = \hat{\phi}_m, \tag{56}$$

and in the third line we have used cyclic property of trace. Next we examine the sign of (55). Its trace term is positive because

$$\begin{aligned}
Tr(\hat{\phi}_m \delta\Lambda_1 \hat{\phi}_n \delta\Lambda_1) &= \sum_{i,\alpha,j,\beta} (f^{0T}_{ni\alpha} A^0 \delta\Lambda_1 f^0_{mj\beta})(f^{0T}_{mj\beta} A^0 \delta\Lambda_1 f^0_{ni\alpha}) \\
&= \sum_{i,\alpha,j,\beta} V^{ij}_{n\alpha m\beta} V^{ji}_{m\beta n\alpha} = \sum_{i,\alpha,j,\beta} (V^{ij}_{n\alpha m\beta})^2 = 2\left(\frac{B_d}{r^{d-1}}\right)^2 \sum_{\alpha,\beta} (C_{m\alpha n\beta})^2 \geq 0.
\end{aligned} \tag{57}$$

And from (12) we obtain

$$\frac{df}{d\lambda} = \frac{1}{2\sqrt{1+\lambda}} \ln\left[\sqrt{1+\frac{1}{\lambda}} + \frac{1}{\sqrt{\lambda}}\right] > 0 \quad \text{for} \quad \lambda > 0, \tag{58}$$

$$\frac{d^2 f}{d\lambda^2} = -\frac{1}{4\sqrt{1+\lambda}} \left[\frac{1}{1+\lambda} \ln\left[\sqrt{1+\frac{1}{\lambda}} + \frac{1}{\sqrt{\lambda}}\right] + \frac{1}{\lambda\sqrt{1+\lambda}}\right] < 0 \quad \text{for} \quad \lambda > 0. \tag{59}$$

From (57), (59) and $\lambda^0_m > \lambda^0_n \ \ (m > n)$, (55) is negative. And from (59) we obtain

$$\sum_{m,\gamma} (\delta\lambda^1_{m\gamma})^2 \left.\frac{d^2 f}{d\lambda^2_m}\right|_{\lambda_m=\lambda^0_m} \leq 0. \tag{60}$$

Finally, from (39), (51), (52), (55) and (57) we obtain

$$
\begin{aligned}
&S_{AB}(r,R_1,R_2) - S_A(R_1) - S_B(R_2) = \sum_{m,\gamma}\left[\delta\lambda^2_{m\gamma}\,\frac{df}{d\lambda_m}\bigg|_{\lambda_m=\lambda^0_m} + \frac{1}{2}(\delta\lambda^1_m)^2\,\frac{d^2 f}{d\lambda^2_m}\bigg|_{\lambda_m=\lambda^0_m}\right] \\
&= \left(\frac{B_d}{r^{d-1}}\right)^2 \Bigg[\sum_{m,n(m>n)} \frac{2}{\lambda^0_m-\lambda^0_n}\sum_{\alpha,\beta}(C_{m\alpha n\beta})^2\left(\frac{df}{d\lambda_m}\bigg|_{\lambda_m=\lambda^0_m} - \frac{df}{d\lambda_n}\bigg|_{\lambda_n=\lambda^0_n}\right) \\
&+ \sum_{m',\alpha} c_{m'\alpha}\,\frac{d^2 f}{d\lambda^2_m}\bigg|_{\lambda_m=\lambda^0_{m'}} + \sum_{m'',\alpha} d_{m''\alpha}\,\frac{d^2 f}{d\lambda^2_m}\bigg|_{\lambda_m=\lambda^0_{m''}}\Bigg] \equiv \frac{1}{r^{2d-2}}G(R_1,R_2) \leq 0
\end{aligned}
\tag{61}
$$

where $\sum_{m'}$ denotes the summation taken over the common eigenvalues of Λ^{01} and Λ^{02}, whose degeneracy is $M_m \geq N_m$, and $\sum_{m''}$ denotes the summation taken over the common eigenvalues of Λ^{01} and Λ^{02}, whose degeneracy is $M_m < N_m$.

We have obtained the r dependence of $S_C(r,R_1,R_2) = S_{AB}(r,R_1,R_2)$ in (61), then we next consider $G(R_1,R_2)$. To calculate $G(R_1,R_2)$ we need to know λ^0_m and $f^0_{mi\alpha}$ which we do not examine in this paper. But from $C_{m\alpha n\beta}(R_1=0,R_2) = C_{m\alpha n\beta}(R_1,R_2=0) = 0$ we obtain a trivial property of $G(R_1,R_2)$,

$$
G(R_1=0,R_2) = G(R_1,R_2=0) = 0. \tag{62}
$$

And $G(R_1,R_2)$ depends on the cutoff length l because λ^0_m , $f^0_{mi\alpha}$ and $C_{m\alpha n\beta}$ depend on l. (λ^0_m are dimensionless, so they depend on R_1/l or R_2/l. And in (48) $\int_A d^d z_a W(x_a,z_a)$ and $\int_B d^d z_b W(y_b,z_b)$ depend on l because $W(x,y)$ depend on l for $x \approx y$, so $C_{m\alpha n\beta}$ depends on l.) Probably $G(R_1,R_2)$ diverges when $l \to 0$, as $S_A(R_1)$ and $S_B(R_2)$ have $1/l^{d-1}$ divergence [1, 2]. And $G(R_1,R_2)$ most likely diverges more weakly than $S_A(R_1)$ and $S_B(R_2)$. Then, by dimensional analysis, when $R_1 = R_2 \equiv R$ we can assume

$$
G(R_1=R,R_2=R) = gR^{2d-2}\left(\frac{R}{l}\right)^m\left(\ln\left(\frac{R}{l}\right)\right)^n \qquad d-1 \geq m \geq 0, n \geq 0, g < 0 \tag{63}
$$

where g is a dimensionless constant.

Finally we consider the condition under which the approximations are good. When $r \gg R_1, R_2, \quad \delta\Lambda \approx \delta\Lambda_1$ is a good approximation. When $|\frac{B_d}{r^{d-1}}C_{m\alpha n\beta}| \ll |\lambda^0_m - \lambda^0_n|$, the perturbation theory is a good approximation. From the l dependences of λ^0_m and $C_{m\alpha n\beta}$ in the latter condition, we might need the condition $R/r \ll (l/R)^a$, where $a \geq 0$.

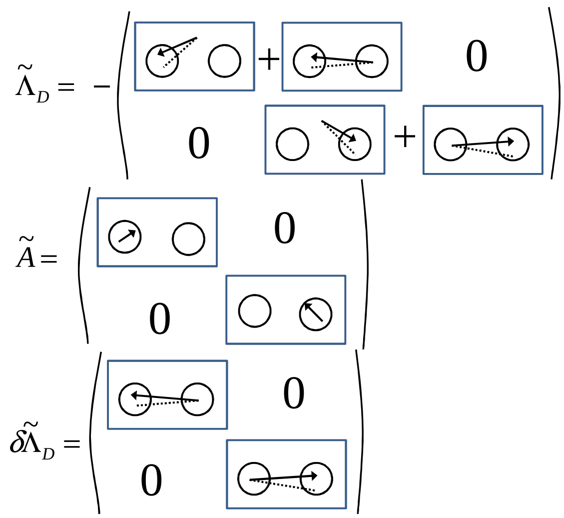

FIG. 8.

V. ENTANGLEMENT ENTROPY OF TWO BLACK HOLES IN A $d+1$ DIMENSIONAL MASSLESS FREE SCALAR FIELD

In this section we consider the entanglement entropy of a massless free scalar field on the outside region C of two black holes A and B whose radii are R_1 and R_2. The action of the massless free scalar field is given by

$$S = -\frac{1}{2}\int d^d x\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi. \tag{64}$$

First we specify the vacuum state of the scalar field. The vacuum state is specified by specifying the time coordinate t. We use the coordinate system which have following properties: this coordinate system covers the inside and the outside regions of two black holes and does not have the coordinate singularity on the horizons and becomes the orthogonal coordinate system of Minkowski spacetime in the region far from the two black holes. To construct this coordinate system, we use the coordinates which is similar to the Kruskal coordinates in the inside regions and the neighborhood of black holes, and similar to the Schwarzschild coordinates in the other region. In this coordinate system g^{tt} is positive everywhere, then from (64) G_{MN} and V_{MN} in (5) are positive definite. So we can use the formalism in the Section III.

We can use the method of the last section with some modifications. In the black hole case $W(x,y)$ and $W^{-1}(x,y)$ depend on r, so we write them as $W(x,y;r)$ and $W^{-1}(x,y;r)$. Exactly in the same way as in Minkowski spacetime, Eqs. (26)-(29) hold because (24) holds.

On the other hand $\Lambda_0 (= \lim_{r\to\infty}\Lambda)$ changes because $W(x,y;r)$ and $W^{-1}(x,y;r)$ depend on r. We define $W_A(x,y)$ and $W_A^{-1}(x,y)(W_B(x,y)$ and $W_B^{-1}(x,y))$ as $W(x,y)$ and $W^{-1}(x,y)$ in the case that the only one black hole $A(B)$ exists. Then we have

$$\Lambda^0 = -\begin{pmatrix} \int_{A^c} d^dz W_A^{-1}(x_a,z)W_A(z,y_a) & 0 \\ 0 & \int_{B^c} d^dz W_B^{-1}(x_b,z)W_B(z,y_b) \end{pmatrix}. \tag{65}$$

It is difficult to evaluate the r dependence of $\delta\Lambda_D = \Lambda_D - \Lambda^0$ because it is difficult to evaluate $W(x,y;r) - W_{A(B)}(x,y)$ and $W^{-1}(x,y;r) - W^{-1}_{A(B)}(x,y)$. So, in the black hole case we *do not* consider Λ_0 as the non-perturbative part. Instead we define $\tilde{\Lambda}_D(r)$ and $\tilde{A}(r)$ as (see Fig 8)

$$\begin{aligned} \tilde{\Lambda}_D(r) &\equiv \begin{pmatrix} \Lambda_A(x_a,y_a;r) & 0 \\ 0 & \Lambda_B(x_b,y_b;r) \end{pmatrix} \\ &\equiv \begin{pmatrix} -\int_{A^c} d^dz W^{-1}(x_a,z;r)W(z,y_a;r) & 0 \\ 0 & -\int_{B^c} d^dz W^{-1}(x_b,z;r)W(z,y_b;r) \end{pmatrix}, \end{aligned} \tag{66}$$

$$\tilde{A}(r) \equiv \begin{pmatrix} W(x_a,y_a;r) & 0 \\ 0 & W(x_b,y_b;r) \end{pmatrix}, \tag{67}$$

and we consider $\tilde{\Lambda}_D$ as the non-perturbative part. Note that Λ_A and Λ_B are the matrices Λ corresponding to $S_A(r,R_1,R_2)$ and $S_B(r,R_1,R_2)$. So we will obtain S_{AB} as the following form, $S_{AB}(r,R_1,R_2) = S_A(r,R_1,R_2) + S_B(r.R_1,R_2) + \delta S_{AB}(r,R_1,R_2)$. We calculate the leading term of $\delta S_{AB}(r,R_1,R_2)$ with respect to $1/r$.

We define $\delta\tilde{\Lambda}_D \equiv \Lambda_D - \tilde{\Lambda}_D$ (see Fig 8), then we have $\Lambda = \tilde{\Lambda}_D + \delta\Lambda_1 + \delta\Lambda_2 + \delta\tilde{\Lambda}_D$. To evaluate $\delta\Lambda_1$, $\delta\Lambda_2$ and $\delta\tilde{\Lambda}_D$, we evaluate $W(x_a,y_b;r)$ and $W^{-1}(x_a,y_b;r)$. When $r \gg R_1, R_2$, by dimensional analysis we obtain $W(x_a,y_b;r) \approx \frac{A_d}{r^{d+1}}L_1(R_1/r,R_2/r)$ and $W^{-1}(x_a,y_b;r) \approx \frac{B_d}{r^{d-1}}L_2(R_1/r,R_2/r)$, where L_1 and L_2 are dimensionless functions of R_1/r and R_2/r. The space time becomes Minkowski space time when $R_1 \to 0$ and $R_2 \to 0$, so in this limit probably we have $L_1 \to 1$ and $L_2 \to 1$. This limit is equivalent to $r \to \infty$, so we have $\lim_{r\to\infty} L_1 = \lim_{r\to\infty} L_2 = 1$. Then we obtain $\delta\Lambda_1 = O(1/r^{d-1})$, $\delta\Lambda_2 = O(1/r^{d+1})$ and $\delta\tilde{\Lambda}_D = O(1/r^{2d})$ as well as the Minkowski spacetime case. We can neglect $\delta\Lambda_2$ and $\delta\tilde{\Lambda}_D$ for the same reason as in the Minkowski spacetime case (see below Eq.(35)). So we can approximate $\Lambda \approx \tilde{\Lambda}_D + \delta\Lambda_1$. Then we change the perturbative calculation in the last section as follows

$$\Lambda^0 \to \tilde{\Lambda}_D(r) \quad A^0 \to \tilde{A}(r) \quad \lambda^0_m \to \tilde{\lambda}^0_m(r) \quad f^0_{mi\alpha} \to \tilde{f}^0_{mi\alpha}(r) \tag{68}$$

where $\tilde{\lambda}^0_m(r)$ and $\tilde{f}^0_{mi\alpha}(r)$ are the eigenvalues and the eigenvectors of $\tilde{\Lambda}_D(r)$. The perturbative calculation is the same as that in the last section. In this case $\tilde{A}(r)$, $\tilde{\lambda}^0_m(r)$ and $\tilde{f}^0_{mi\alpha}(r)$ depend on r, but we can remove their r dependence as follow. Because we want to calculate the leading term of $S_{AB}(r,R_1,R_2) - S_A(r,R_1,R_2) - S_B(r,R_1,R_2)$ with respect to $1/r$, we can approximate

$$\begin{aligned}
\tilde{A}\delta\Lambda_1 &\approx \frac{B_d L_2(\frac{R_1}{r},\frac{R_2}{r})}{r^{d-1}} \begin{pmatrix} 0 & \int_A d^d z_a W(x_a,z_a;r)\int_B d^d z_b W(z_b,y_b;r) \\ \int_B d^d z_b W(x_b,z_b;r)\int_A d^d z_a W(z_a,y_a;r) & 0 \end{pmatrix}. \\
&\approx \frac{B_d}{r^{d-1}} \begin{pmatrix} 0 & \int_A d^d z_a W_A(x_a,z_a)\int_B d^d z_b W_B(z_b,y_b) \\ \int_B d^d z_b W_B(x_b,z_b)\int_A d^d z_a W_A(z_a,y_a) & 0 \end{pmatrix}
\end{aligned} \tag{69}$$

In the second line we have approximated $L_2(\frac{R_1}{r},\frac{R_2}{r}) \approx 1$, $W(x_a,z_a;r) \approx W_A(x_a,z_a)$ and $W(z_b,y_b;r) \approx W_B(z_b,y_b)$. And we can approximate $\tilde{\lambda}^0_m(r) \approx \tilde{\lambda}^0_m(r=\infty) \equiv \lambda^0_m$ and $\tilde{f}^0_{mi\alpha}(r) \approx \tilde{f}^0_{mi\alpha}(r=\infty) \equiv f^0_{mi\alpha}$. Note that λ^0_m and $f^0_{mi\alpha}$ are the eigenvalues and the eigenvectors of Λ^0, i.e. (Λ^0 is in (65))

$$f^0_{m1\alpha} = \begin{pmatrix} f^0_{m1\alpha}(x_a) \\ 0 \end{pmatrix}, f^0_{n2\beta} = \begin{pmatrix} 0 \\ f^0_{n2\beta}(x_b) \end{pmatrix}, \Lambda^0 f^0_{m1\alpha} = \lambda^0_m f^0_{m1\alpha}, \Lambda^0 f^0_{m2\beta} = \lambda^0_m f^0_{m2\beta}, \tag{70}$$

where $\alpha = 1,\cdots,M_m$ and $\beta = 1,\cdots,N_n$ are the labels of the degeneracy.

Finally we obtain

$$S_{AB}(r,R_1,R_2) = S_A(r,R_1,R_2) + S_B(r,R_1,R_2) + \frac{1}{r^{2d-2}}G(R_1,R_2) \tag{71}$$

where $G(R_1,R_2)$ is the same function as that in (61). Note that in this case from (69) $C_{m\alpha n\beta}$ in $G(R_1,R_2)$ is

$$C_{m\alpha n\beta} = \int_A d^d x_a \int_A d^d z_a W_A(x_a,z_a) f^0_{m1\alpha}(x_a) \int_B d^d y_b \int_B d^d z_b W_B(y_b,z_b) f^0_{n2\beta}(y_b). \tag{72}$$

As in the Minkowski spacetime case, we obtain $G(R_1=0,R_2) = G(R_1,R_2=0) = 0$ from $C_{m\alpha n\beta}(R_1=0,R_2) = C_{m\alpha n\beta}(R_1,R_2=0) = 0$. The $1/l$ dependence of $G(R_1,R_2)$ is most likely the same as that in the Minkowski spacetime, where l is the cutoff length. Then we obtain

$$G(R_1=R,R_2=R) = g_{BH}R^{2d-2}\left(\frac{R}{l}\right)^m \left(\ln\left(\frac{R}{l}\right)\right)^n \quad d-1 \geq m \geq 0, n \geq 0, g_{BH} < 0 \tag{73}$$

where g_{BH} is a dimensionless constant, and m and n are the same numbers as those in the Minkowski spacetime.

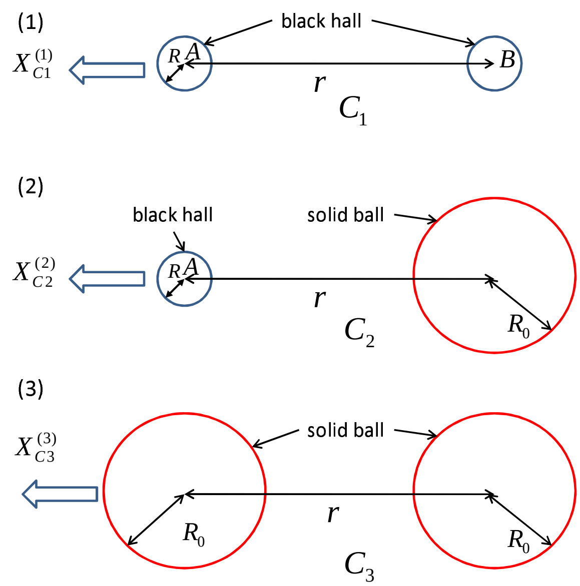

FIG. 9.

VI. ENTANGLEMENT ENTROPIC FORCE AND THE PHYSICAL PREDICTION

We *assume* that we can consider the entanglement entropy of two black holes as thermodynamic entropy. If this assumption is correct, the entropic force acts on two black holes. We consider the force of the scalar field which acts on two black holes. We consider two black holes which have same radius $R_1 = R_2 \equiv R$, then we can consider the temperature T to be the Hawking temperature. We define the energy and the free energy of the field on the region C as $E_C(r, R)$ and $F_C(r, R)$,

$$F_C(r, R) = E_C(r, R) - TS_C(r, R) = E_C(r, R) - T\left(2S_A(r, R) + \frac{1}{r^{2d-2}}G(R)\right). \quad (74)$$

where $G(R) \equiv G(R_1 = R, R_2 = R)$ and we have used (71). We define the force of the field on the region C which acts on one black hole in the direction of increasing r as X_C. We obtain X_C by partially differentiating F_C with R fixed,

$$X_C(r, R) = -\frac{\partial F_C}{\partial r} = -\frac{\partial E_C(r, R)}{\partial r} + T\left(2\frac{\partial S_A(r, R)}{\partial r} - (2d-2)\frac{1}{r^{2d-1}}G(R)\right). \quad (75)$$

In (75) the second term is the entropic force.

We cannot see the effect of the entropic force only from (75) because we do not know $S_A(r, R)$. To see the effect of the entropic force we consider three situations . (See Fig 9) (1) There are two black holes which have the same radius R and the distance between them is r.(This is the situation we have considered.) (2) There are one black hole whose radius is R and one solid ball whose radius is $R_0 \approx R(R_0 > R)$, and the distance between them is r. This ball has mass M which is the same as that of a black hole whose radius is R. And the scalar field does not exist in this ball. The boundary condition on the scalar field on the surface of this ball is not so important in the later calculation that we do not specify the boundary condition in detail. We only require that the scalar field on the outside region of this ball is not so different from that in the situation (1). (3) There are two solid balls which have the same radius R_0 and the distance between them is r. These balls have the same properties as those in the situation (2).

We define the force of the field which acts on one black hole or on one ball in the direction of increasing r as $X^{(1)}_{C_1}$, $X^{(2)}_{C_2}$ and $X^{(3)}_{C_3}$. We illustrate in Fig 9 the directions of force and the names of the regions.

In the situation (2) the state of the field is $|0\rangle^{(2)}_{A+C_2}$, where $|0\rangle^{(2)}_{A+C_2}$ is the vacuum state on $A + C_2$. Because $|0\rangle^{(2)}_{A+C_2}$ is a pure state, then $S^{(2)}_{C_2} = S^{(2)}_A$. We define $\Lambda^{(1)}_A(\Lambda^{(2)}_A)$ as Λ corresponding to $S^{(1)}_A(S^{(2)}_A)$. Because the scalar field does not exist in the ball, then we obtain

$$\Lambda^{(2)}_A - \Lambda^{(1)}_A \approx \int_B d^d z_b W^{-1}(x_a, z_b) W(z_b, y_a) = O\left(\frac{1}{r^{2d}}\right). \tag{76}$$

Then we can approximate $S^{(2)}_A = S^{(1)}_A(r, R) + O(\frac{1}{r^{2d}}) \approx S^{(1)}_A(r, R)$. Then we obtain

$$X^{(2)}_{C_2}(r, R) = -\frac{\partial F^{(2)}_{C_2}}{\partial r} = -\frac{\partial E^{(2)}_{C_2}(r, R)}{\partial r} + T\frac{\partial S^{(2)}_A(r, R)}{\partial r} \approx -\frac{\partial E^{(2)}_{C_2}(r, R)}{\partial r} + T\frac{\partial S^{(1)}_A(r, R)}{\partial r}. \tag{77}$$

In the situation (3) the state of the field on the region C_3 is a pure state, so $S^{(3)}_{C_3} = 0$. Then we obtain

$$X^{(3)}_{C_3}(r, R) = -\frac{\partial F^{(3)}_{C_3}}{\partial r} = -\frac{\partial E^{(3)}_{C_3}(r, R)}{\partial r}. \tag{78}$$

From (75) (77) and (78) we obtain

$$X^{(1)}_{C_1} - 2X^{(2)}_{C_2} + X^{(3)}_{C_3} \approx -\frac{\partial}{\partial r}[E^{(1)}_{C_1} - 2E^{(2)}_{C_2} + E^{(3)}_{C_3}] - (2d-2)T\frac{1}{r^{2d-1}}G(R). \tag{79}$$

$E^{(1)}_{C_1} - 2E^{(2)}_{C_2} + E^{(3)}_{C_3}$ is Casimir energy.

We have not considered the force of gravity. But we can include them in (79) easily. We define total force acting on one black hole or on one ball in the direction of increasing r as $\mathcal{F}^{(1)}_A$, $\mathcal{F}^{(2)}_A$ and $\mathcal{F}^{(3)}_{ball}$. Then we obtain

$$\mathcal{F}^{(1)}_A - 2\mathcal{F}^{(2)}_A + \mathcal{F}^{(3)}_{ball} = X^{(1)}_{C_1} - 2X^{(2)}_{C_2} - X^{(3)}_{C_3} \approx -\frac{\partial}{\partial r}[E^{(1)}_{C_1} - 2E^{(2)}_{C_2} + E^{(3)}_{C_3}] - (2d-2)T\frac{1}{r^{2d-1}}G(R). \tag{80}$$

The force of gravity is canceled in (80). The first and the second terms in the left hand side are the Casimir force and the effect of entropic force, respectively.

Finally we consider the case $d = 3$. In this case the Hawking temperature is $T = \frac{1}{8\pi G_N M} = \frac{1}{4\pi R}$. From (73) and (80) we obtain

$$\mathcal{F}_A^{(1)} - 2\mathcal{F}_A^{(2)} + \mathcal{F}_{ball}^{(3)} \approx -\frac{\partial}{\partial r}[E_{C_1}^{(1)} - 2E_{C_2}^{(2)} + E_{C_3}^{(3)}] - \frac{g_{BH}}{\pi}\frac{R^3}{r^5}\left(\frac{R}{l}\right)^m \left(\ln\left(\frac{R}{l}\right)\right)^n \quad 2 \geq m \geq 0, n \geq 0, g < 0. \tag{81}$$

We roughly estimate the Casimir force by analogy with that of electromagnetic field between two dielectric spheres with center-to-center distance r in Minkowski spacetime. The Casimir force between the two sphere was calculated in [15], and it is $O(1/r^8)$. So, in our case we can probably neglect $-\frac{\partial}{\partial r}[E_{C_1}^{(1)} - 2E_{C_2}^{(2)} + E_{C_3}^{(3)}]$ in (81). The left hand side of (81) can be measured experimentally, so (81) is the physical prediction. From (81) the effect of the entropic force becomes significant when R is large. We can probably use heavy stars as the balls in the situation (2) and (3). So we can possibly confirm the effect of the entropic force by the cosmic observation.

VII. CONCLUSION AND DISCUSSION

In Section II we showed that the entanglement entropy ($S_C = S_{AB}$) of two disjoint regions in a translational invariant vacuum in general QFT reaches its maximum value when $r \to \infty$. And we obtained the inequality (4). In Section IV we developed the method to obtain the r dependence of S_C and obtained the r dependence of S_C (61) of a free massless scalar field in $(d+1)$ dimensional Minkowski spacetime. We can use this method in curved space time and for scalar field theory whose Lagrangian is quadratic. To know only the r dependence we need only the $\|x-y\|$ dependence of $W(x,y)$ and $W^{-1}(x,y)$ when $\|x-y\|$ is large. To know the R_1 and R_2 dependence we must solve the zeroth order eigenvalue equation and obtain λ_m^0 and $f_{mi\alpha}^0$. It is difficult to solve the zeroth order eigenvalue equation analytically, so we will need to perform numerical calculation. But we assumed the R_1 and R_2 dependence (63) by using dimensional analysis and the cutoff dependence of S_A and S_B.

In Section V we showed that S_C can be expected to be the form (71) in the black hole case. In this case the only assumption we made is the r dependence of $W(x_a, y_b)$ and $W^{-1}(x_a, y_b)$. We did not explicitly calculate $W(x_a, y_b)$ and $W^{-1}(x_a, y_b)$, but assumed the r dependence of $W(x_a, y_b)$ and $W^{-1}(x_a, y_b)$ by dimensional analysis.

In Section VI we assumed that we can consider the entanglement entropy of two black holes as thermodynamic entropy, and investigated its entropic force. We considered three situations (1), (2) and (3) and obtained the relationship (81) between the force acting on one black hole or on one ball and the sum of the Casimir force and the effect of the entanglement entropic force. Because we can probably neglect the Casimir force, we can confirm (81) experimentally in principle. And we can possibly confirm the effect of the entropic force by the cosmic observation because it is significant for large black holes.

Next we discuss the entanglement entropic force in different systems. In the black hole case, black holes act as *"walls" which hide inside regions but hold the entanglement between inside and outside regions*. So if there are walls of this type, the entanglement entropic force will exist between regions surrounded by these walls. Then we will be able to confirm the entanglement entropic force by experiments in a laboratory if we make this wall. And if entanglement entropy depends on some external parameter, entanglement entropic force probably appears also in quantum mechanical (i.e. not quantum field theoretical) systems.

Finally we mention our assumption that we can consider the entanglement entropy of two black holes as thermodynamic entropy. Entanglement entropy has properties which are different from those of thermodynamic entropy. For example entanglement entropy is not a extensive variable in general. So we must reconsider statistical mechanics from a fundamental level to judge whether our assumption is correct or not. We can also use (81) to judge the correctness of our assumption by experiments.

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Appendix A: The calculation of W and W^{-1}

In this appendix we calculate $W(x, y)$ and $W^{-1}(x, y)$ ((18) and (19)) explicitly. We regularize them by including convergence factor $e^{-l|k|}$ in them, where l is the cutoff length.

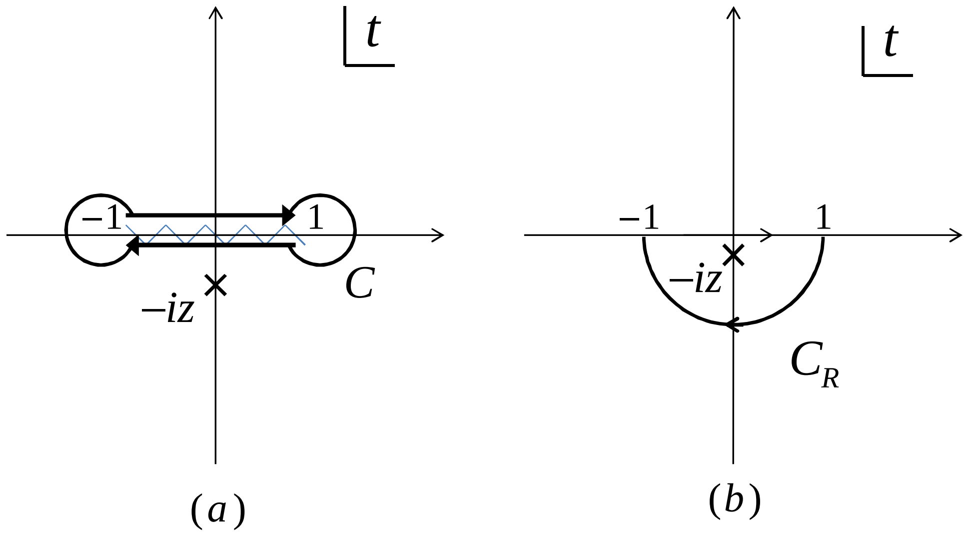

FIG. 10.

We define W_α as

$$W_\alpha(x,y) = \int \frac{d^dk}{(2\pi)^d}(k^2)^{(1-\alpha)/2} e^{ik\cdot(x-y)} e^{-l|k|}. \tag{A1}$$

Then we have $W_0 = W$ and $W_2 = W^{-1}$. First we consider the case $d \geq 3$.

(1) $d \geq 3$

We perform the integrals of angular coordinates which do not enter the inner product,

$$W_\alpha(x,y) = \frac{1}{(2\pi)^d} \prod_{m-2}^{d-2} \left(\sqrt{\pi} \frac{\Gamma\left(\dfrac{d-m}{2}\right)}{\Gamma\left(\dfrac{d-m+1}{2}\right)} \right) \int_0^\infty dk \int_{-1}^1 dt [1-t^2]^{\frac{d-3}{2}} e^{ikrt} e^{-lk} k^{d-\alpha} \tag{A2}$$

where $r \equiv \|x-y\|$ and we change the variable as $t = \cos\theta$. Next we perform the k integral

$$\begin{aligned} &\int_0^\infty dk \int_{-1}^1 dt [1-t^2]^{\frac{d-3}{2}} e^{ikrt} e^{-lk} k^{d-\alpha} = \int_{-1}^1 dt [1-t^2]^{\frac{d-3}{2}} \left(\frac{1}{it}\frac{d}{dr}\right)^{d-\alpha} \int_0^\infty dk e^{ikrt} e^{-lk} \\ &= (-i)^{d-\alpha-1}(d-\alpha)! \frac{1}{r^{d-\alpha+1}} \int_{-1}^1 dt [1-t^2]^{\frac{d-3}{2}} \frac{1}{(t+iz)^{d-\alpha+1}} \end{aligned} \tag{A3}$$

where $z \equiv l/r$. We define

$$g(t) \equiv [1-t^2]^{\frac{d-3}{2}} \frac{1}{(t+iz)^{d-\alpha+1}}. \tag{A4}$$

We want to show $W_\alpha \neq 0$ when $z \to 0$.

(i) $d = 2m+2 \quad (m \geq 1)$

In this case $g(t)$ has branch cut on the real axis from -1 to 1. We perform the integration along the contour shown in Fig 10 (a), and obtain

$$\int_{-1}^{1} dt g(t) = \pi i \mathrm{Res}_{t=-iz} g(t) = \pi i \frac{1}{(d-\alpha)!} \left(\frac{d^{d-\alpha}}{dt^{d-\alpha}} [1-t^2]^{\frac{d-3}{2}} \right) \Bigg|_{t=-iz}. \tag{A5}$$

The derivative in (A5) can be calculated by the derivative of a composite function,

$$\begin{aligned} \left(\frac{d^{d-\alpha}}{dt^{d-\alpha}} [1-t^2]^{\frac{d-3}{2}} \right) = & \sum_{r=0}^{\left[\frac{1}{2}(d-\alpha)\right]} \frac{(d-\alpha)!}{r!(d-\alpha-2r)!} (2t)^{d-\alpha-2r} (-1)^{d-\alpha-r} \\ & (\frac{d-3}{2})(\frac{d-3}{2}-1)\cdots(\frac{d-3}{2}-(d-\alpha-r-1))(1-t^2)^{\frac{d-3}{2}-(d-\alpha-r)} \end{aligned} \tag{A6}$$

where $\left[\frac{1}{2}(d-\alpha)\right]$ is the Gauss' symbol which is the greatest integer that is less than or equal to $\frac{1}{2}(d-\alpha)$.

Then, when $z \to 0$, we obtain

$$\begin{aligned} \int_{-1}^{1} dt g(t) &= \pi i \frac{1}{(\frac{d-\alpha}{2})!} (-1)^{\frac{d-\alpha}{2}} (\frac{d-3}{2})(\frac{d-3}{2}-1)\cdots(\frac{d-3}{2}-(\frac{d-\alpha}{2}-1)) \\ &= \pi i \frac{1}{(\frac{d-\alpha}{2})!} \left(-\frac{1}{2}\right)^{\frac{d-\alpha}{2}} (d-3)(d-1)\cdots(\alpha-1). \end{aligned} \tag{A7}$$

Then, from (A2), (A3) and (A7), for $\alpha = 2l \le d (l \in \mathbb{Z})$ W_α is nonzero and W_α has the form of (22) when $z \to 0$. (When $\alpha = d$, we obtain $\int_{-1}^{1} dt g(t) = \pi i$ from (A5).)

(ii) $d = 2m+1 \quad (m \ge 1)$

We perform the integration along the contour shown in Fig 10 (b), and obtain

$$\int_{-1}^{1} dt g(t) = -2\pi i \mathrm{Res}_{t=-iz} g(t) - \int_{C_R} dt g(t). \tag{A8}$$

For $\alpha = 2l (l \in \mathbb{Z})$, $d-\alpha$ is odd, so we obtain $\lim_{z\to 0} \mathrm{Res}_{t=-iz} g(t) = 0$ from (A6). Then, for $\alpha = 2l$ and $z \to 0$ we obtain

$$\begin{aligned} \int_{-1}^{1} dt g(t) &= -\int_{C_R} dt g(t) = i(-1)^{d-\alpha} \int_{0}^{\pi} d\theta e^{-i(d-\alpha)\theta} [1-e^{2i\theta}]^{\frac{d-3}{2}} \\ &= i(-1)^{d-\alpha}(-2i)^{m-1} \int_0^\pi d\theta e^{-i(2m+1-\alpha)\theta} e^{i(m-1)\theta} (\sin\theta)^{m-1} \\ &= (-1)^m 2^{m-1} i^m (I_{sc}[m-1, m+2-\alpha] - i I_{ss}[m-1, m+2-\alpha]) \\ &= \begin{cases} 2^{m-1} i^{m+1} I_{ss}[m-1, m+2-\alpha] \neq 0 & \text{for odd } \ m, \\ 2^{m-1} i^m I_{sc}[m-1, m+2-\alpha] \neq 0 & \text{for even } \ m, \end{cases} \end{aligned} \tag{A9}$$

where

$$I_{sc}[m,n] \equiv \int_0^\pi d\theta (\sin\theta)^m \cos(n\theta) \ , \quad I_{ss}[m,n] \equiv \int_0^\pi d\theta (\sin\theta)^m \sin(n\theta). \tag{A10}$$

Then, from (A2) (A7) and (A9) , for $\alpha = 2l(l \in \mathbb{Z})$ W_α is nonzero and W_α has the form of (22) when $z \to 0$.

From (i) and (ii) we showed (22) for $d \geq 3$. Next we consider $d = 2$.

(2) $d = 2$

In this case we can perform the angular integral first,

$$\begin{aligned} W_\alpha(x,y) &= \frac{1}{(2\pi)^2}\int_0^\infty kdk \int_0^{2\pi} d\theta k^{1-\alpha} e^{ikr\cos\theta - lk} \\ &= \frac{1}{2\pi}\int_0^\infty dk k^{2-\alpha} e^{-lk} J_0(kr) = \frac{1}{2\pi r^{3-\alpha}}\int_0^\infty dx x^{2-\alpha} e^{-zx} J_0(x) \end{aligned} \tag{A11}$$

where J_0 is the Bessel function of zeroth order. We perform the integral for $\alpha = 2$ and $\alpha = 0$.

(i) $\alpha = 2$

In this case we have

$$\int_0^\infty dx e^{-zx} J_0(x) = \frac{1}{\sqrt{z^2+1}}. \tag{A12}$$

Then, when $z \to 0$ we obtain

$$W_{\alpha=2}(x,y) = W^{-1}(x,y) = \frac{1}{2\pi r}. \tag{A13}$$

(i) $\alpha = 0$

In this case we have

$$\int_0^\infty dx x^2 e^{-zx} J_0(x) = \frac{\Gamma(3)}{z^3} F\left(\frac{3}{2}, 2, 1; -\frac{1}{z^2}\right) \to -1 \quad (z \to 0) \tag{A14}$$

where F is the Gaussian hypergeometric function. Then, when $z \to 0$ we obtain

$$W_{\alpha=0}(x,y) = W(x,y) = \frac{-1}{2\pi r^3}. \tag{A15}$$

Finally we have showed (22) for $d \geq 2$.

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[16] When d=1, this is not correct because generally the correlation function $\langle 0|\,\phi(t,x)\phi(t,y)\,|0\rangle$ does not become zero when $|x-y|\to\infty$. For example the correlation function of massless free scalar fields does not become zero when $|x-y|\to\infty$.

[17] We use the following easily verifiable identity,

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \begin{pmatrix} A - CB^{-1}D & CB^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ D & 1 \end{pmatrix} \begin{pmatrix} 1 & A^{-1}C \\ 0 & B - DA^{-1}C \end{pmatrix}.$$