The Hubble rate in averaged cosmology

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(Dated: November 23, 2010)

The calculation of the averaged Hubble expansion rate in an averaged perturbed Friedmann-Lemaître-Robertson-Walker cosmology leads to small corrections to the background value of the expansion rate, which could be important for measuring the Hubble constant from local observations. It also predicts an intrinsic variance associated with the finite scale of any measurement of H_0, the Hubble rate today. Both the mean Hubble rate and its variance depend on both the definition of the Hubble rate and the spatial surface on which the average is performed. We quantitatively study different definitions of the averaged Hubble rate encountered in the literature by consistently calculating the backreaction effect at second order in perturbation theory, and compare the results. We employ for the first time a recently developed gauge-invariant definition of an averaged scalar. We also give the value of the variance of the Hubble rate for the different definitions.

I. INTRODUCTION

The late time Universe is not perfectly homogeneous and isotropic, and the overdensities and voids that develop via gravitational collapse make it significantly inhomogeneous. As a result, the notion of a background, maximally symmetric, geometry, that is at the core of the standard concordance model needs to be addressed carefully. Specifically, one would like to construct such an averaged model, suitable to describe the Universe on sufficiently large scales, as a coarse-grained version of the actual distribution of matter and energy in the Universe. In the last decade, this issue has attracted considerable attention in cosmology, in particular through the so-called averaging problem (see [1] and references therein), mostly because it is sometimes believed that it could provide an answer to the Dark Energy problem (see e.g. [2–5]). Even though it has not been shown to be the case, the physics of averaging are still be worth investigating; the parameters for cosmological concordance are quite sensitive to the backreaction effect – important in the era of precision cosmology.

One method for evaluating this backreaction lies within the standard cosmological model. That is, one can evaluate the backreaction from the perturbations which describe structure formation. At second-order in perturbation theory, this gives rise to corrections to the local Hubble flow. This idea was first investigated in an Einstein-de Sitter model in [3], and followed up in more detail in [6–8]. This was extended to include the case of a cosmological constant in [9–12] . On the face of it there appears to be some discrepancy between these results: While [6–8] found an important effect from backreaction, [9–12] found much smaller changes to the value of H_0 . Our aim in this paper is to reconcile these results, and present them in a unified framework. Bearing this idea in mind, in this paper, we will be using the averaging formalism as developed in [12–14] in order to estimate the corrections to the averaged local Hubble flow induced by the small scale inhomogeneities in the matter distribution. Such studies have been performed in the past, [6–10, 12], with different definitions for the Hubble rate, different slicing for averaging, and/or different approaches to perturbation theory. Specifically, on the one hand, [9, 10] defined the averaged Hubble flow in the longitudinal gauge by following the expansion of the coordinate grid adapted to the gauge; this is the expansion associated with the gravitational potential [12]. In [10] they applied the same formalism, based on the expansion of the coordinated grids in various gauges. On the other hand, [6–8, 12] looked at the expansion of the matter fluid in the comoving synchronous gauge [6–8] and in the longitudinal gauge [12]. The results and the claims inferred from these results appear to differ from one analysis to the other approach. To clarify the issue and evaluate precisely the corrections to the concordance model due to the backreaction effect, we propose to compare quantitatively the different definitions and results. We will discuss how a consistent second order treatment of the backreaction effect in perturbation theory changes in the value of the Hubble rate. We will also analyse the intrinsic variance created by the fluctuations that could affect the measurements of H_0.

The paper is organized as follows: In Sections II and III, we briefly recall the averaging formalism we employ, and discuss the definition of the averaged Hubble rate in this context. We will see that there exists two class of definitions. In this section, we also use the gauge invariant formalism of [15, 16] to calculate in the (tilted) longitudinal gauge, the average Hubble rate defined in the fluid frame. To our knowledge, it is the first practical calculation making use of of this gauge invariant formalism. In Section IV we discuss our results and present a fitting formula for the variance of the Hubble rate. We also show that the two class of definitions can be clearly separated and we comment on their relevance. Finally, in Section V, we draw some conclusions and discuss future works. The Appendix presents the detailed expressions

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of the various Hubble rates considered in this paper, at second oder in perturbation theory.

II. EQUATIONS OF MOTION

Buchert's averaging formalism [13, 17] and its generalization to arbitrary coordinate systems [10, 11, 14, 18] rely on Einstein equations written in the Arnowitt-Deser-Misner form. Within this formalism, one considers a set of observers defined at each point of the spacetime manifold, and characterized by a unit 4-velocity field n^a that is everywhere timelike and future directed, i.e. $n^a n_a = -1$, with zero vorticity. This 4-velocity field induces a natural foliation of spacetime by a continuous set of spacelike hypersurfaces locally orthogonal to n^a. The projection tensor field onto these hypersurfaces is defined as $h_{ab} = g_{ab} + n_a n_b$. The line element can then be written, with respect to this foliation:

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j \ , \quad (1)$$

where we have introduced respectively the lapse function $N(x^a)$ and the shift 3-vector $N^i(x^a)$ such that the components of the 4-velocity read: $n^a = \frac{1}{N}(1, -N^i)$, $n_a = N(-1, 0, 0, 0)$ .The intrinsic curvature of the hypersurfaces orthogonal to n^a is given by $\mathcal{R} \equiv h^{ab}\mathcal{R}_{ab}$, where $\mathcal{R}_{ab}$ is the 3-Ricci curvature of the hypersurfaces and the extrinsic curvature (or second fundamental form): $K_{ab} \equiv -h^c_a h^d_b n_{c;d}$. Here we will consider only the Hamiltonian constraint (for the complete set of ADM decomposed Einstein equations see [14])

$$\left(\partial_t - \mathcal{L}_{\Sigma_t}\right) h_{ij} = -2N K_{ij} \quad (2)$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi\epsilon \quad (3)$$

where $\epsilon = n^a n^b T_{ab}$ and T_{ab} is the energy momentum tensor defined to include the cosmological constant as $T_{ab} = (\rho + p)u_a u_b + (p + \Lambda/(8\pi G))g_{ab}$. u^a is time-like 4-velocity $u^a u_a = -1$ for the matter field, it is related to the n^a through

$$u^a = \gamma(n^a + v^a) \text{ where } \gamma = \frac{1}{\sqrt{1 - v^a v_a}} \ , \quad (4)$$

v^a is spacelike and it is orthogonal to n^a ($v^a n_a = 0$). We have defined two different 4-velocities, which according to standard $1+3$ decomposition of a covariant derivative of 4-vector, will imply defining respectively two expansion rates.

II.1. Decomposition of velocities

The covariant derivatives of the two 4−velocities u^a and n^a, as well as the spacelike relative velocity v^a, may be invariantly decomposed with respect to the coordinate frame n^a (this corrects the decomposition presented in [14]; however the expression for the Hubble rate is not affected):

$$\begin{aligned}
\nabla_a n_b &= -n_a \dot{n}_b + \frac{1}{3}\xi h_{ab} + \Sigma_{ab} \\
\nabla_a u_b &= -\gamma v^c \left(\gamma^2 \dot{v}_c + \dot{n}_c\right) n_a n_b \\
&\quad -\gamma \left(\gamma^2 v^c \tilde{\nabla}_a v_c + \frac{1}{3}\xi v_a + \Sigma_{ac} v^c\right) n_b \\
&\quad +\gamma n_a \left(\gamma^2 v^c \dot{v}_c v_b + \dot{n}_{\langle b\rangle} + \dot{v}_{\langle b\rangle}\right) \\
&\quad +\frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \\
\nabla_a v_b &= -\dot{n}_c v^c n_a n_b - n_a \dot{v}_{\langle b\rangle} + \left(\frac{1}{3}\xi v_a + \Sigma_{ac} v^c\right) n_b \\
&\quad +\frac{1}{3}\kappa h_{ab} + \beta_{ab} + W_{ab}
\end{aligned} \quad (5)$$

where:

$$\xi \equiv h^{ab}\nabla_a n_b \qquad \Sigma_{ab} \equiv h^c_a h^d_b \nabla_{(c} n_{d)} - \frac{1}{3}\xi h_{ab} \ ,$$

$$\theta \equiv h^{ab}\nabla_a u_b \ , \qquad \sigma_{ab} \equiv h^c_a h^d_b \nabla_{(c} u_{d)} - \frac{1}{3}\theta h_{ab} \ ,$$

$$\omega_{ab} \equiv h^c_a h^d_b \nabla_{[c} u_{d]} \ , \qquad \kappa \equiv h^{ab}\nabla_a v_b \ ,$$

$$\beta_{ab} \equiv h^c_a h^d_b \nabla_{(c} v_{d)} - \frac{1}{3}\kappa h_{ab} \ , \qquad W_{ab} \equiv h^c_a h^d_b \nabla_{[c} v_{d]} \ .$$

Here we have used the notation $\dot{A}_{a\cdots b} = n^c \nabla_c A_{a\cdots b}$, angle brackets denote symmetric, trace free, and projected with respect to n_a. $\tilde{\nabla}$ denotes the spatial derivative with respect to n^a.

Here ξ and θ are the expansion rates, while κ is the divergence of the 3-velocity v^a; Σ_{ab}, σ_{ab} and β_{ab} are the shear, while ω_{ab} and W_{ab} are the vorticity in the respective definitions. Note that:

$$\theta - \gamma\xi = \gamma\kappa \left(1 + \frac{1}{3}\gamma^2 v^2\right) + \gamma^3 v^a v^b \beta_{ab}. \quad (6)$$

The decomposition of the matter 4-velocity is quite unusual, since it is with respect to n^a. One can calculate directly the normal acceleration, vorticity and shear and so on; for us the intrinsic expansion rate is important:

$$\begin{aligned}
\Theta &= \nabla_a u^a \\
&= \theta + \gamma v^a \left(\gamma^2 \dot{v}_a + \dot{n}_a\right).
\end{aligned} \quad (7)$$

III. AVERAGED HUBBLE RATES

In general, the average of a scalar quantity $S(t, x)$ may be defined as:

$$\langle S(t,x)\rangle_{\mathcal{D}} \equiv \frac{\int d^3x J S(t,x)}{\int d^3x J}, \quad (8)$$

where $J = \sqrt{h}$ is the determinant of the metric on the hypersurface orthogonal to n^a. The time derivative of Eq. (8) leads to a commutation relation [14]

$$[\partial_t \cdot, \langle\cdot\rangle_{\mathcal{D}}] S(t, x^i) = \langle N\xi S\rangle_{\mathcal{D}} - \langle N\xi\rangle_{\mathcal{D}} \langle S\rangle_{\mathcal{D}} \ , \quad (9)$$

as is usual in the averaging context.

There have been different definitions of the averaged Hubble parameter $H_{\mathcal{D}}$ in the literature, and we would like to be able to compare them in the context of the standard model, up to second-order in perturbation theory. We shall employ the longitudinal gauge below in order to calculate averages in the concordance model, which fixes our coordinate frame n^a. In the longitudinal gauge the magnetic part of the Weyl tensor vanishes, and the electric part is a pure potential field in the absence of anisotropic stress [12] (see also Appendix A), making this the rest-frame of the gravitational field, or Newtonian frame. In this sense, both n^a and u^a are physically well defined reference frames.

There are different local expansion rates:

- ξ: the expansion of the coordinate observers. In the longitudinal gauge we employ below, this is the rest-frame of the gravitational field.
- Θ: The expansion of the fluid, as observed in the fluid rest-frame.
- θ: The expansion of the fluid, as observed in the gravitational rest-frame.

When performing averaging, there are two spatial hypersurfaces of interest:

- $\langle\ \rangle_{\mathcal{D}}$: Averaging in the gravitational frame.
- $\langle\ \rangle_{\mathcal{F}}$: Averaging in the rest frame of the fluid.

Finally, when averaging expansion rates associated with the gravitational field, there is the issue of the time coordinate to use: we can associate the time coordinate t with the proper time of the 'averaged observers', which, when using n^a requires an extra factor of N in the expansion rate [12].

Definitions based on

As argued in [9, 16], one can consider the evolution of the metric of the hypersurface:

$$\partial_t h_{ij} = \frac{2}{3} N h_{ij}\xi + 2N\Sigma_{ij} + D_i N_j + D_j N_i \tag{10}$$

and also assume that the dimensionless domain scale factor can be defined as $a_{\mathcal{D}} = \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}}}\right)^{1/3}$ where $V_{\mathcal{D}}$ is the volume of the domain. It is easy to show from equation (10), that

$$\begin{aligned} 3H_{\mathcal{D}} &= \frac{\partial_t V_{\mathcal{D}}}{V_{\mathcal{D}}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \left(N\xi + D_k N^k\right)\sqrt{h} d^3x \\ &= \left\langle N\xi + D_k N^k \right\rangle_{\mathcal{D}} . \end{aligned} \tag{11}$$

This definition describes the average expansion of the coordinate grid and says nothing about the matter field in general. In this sense it is not necessarily physical, but it has been used in the recent literature for calculations in the longitudinal gauge [9, 10, 16], in which case it can be interpreted as the expansion rate of the gravitational rest frame. We will find that these definitions exhibit an unnatural feature, namely a scale independence of backreaction effects, but they capture the correct behaviour when averaging on Hubble scales (as performed in [9, 10]).

With this definition, the averaged Hamiltonian constraint (3) becomes:

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G \left\langle N^2\gamma^2\left(\rho + v^2 p\right)\right\rangle_{\mathcal{D}} + 2\left\langle N^2\Lambda\right\rangle_{\mathcal{D}} \\ &\quad - \left\langle N^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q} + \mathcal{P} \end{aligned} \tag{12}$$

$$\mathcal{Q} \equiv \frac{2}{3}\left\langle N^2\xi^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle N\xi\right\rangle_{\mathcal{D}}^2 - 2\left\langle N^2\Sigma^2\right\rangle_{\mathcal{D}}$$

$$\mathcal{P} \equiv \frac{4}{3}\left\langle N\xi\right\rangle_{\mathcal{D}}\left\langle D_k N^k\right\rangle_{\mathcal{D}} + \frac{2}{3}\left\langle D_k N^k\right\rangle_{\mathcal{D}} \tag{13}$$

This definition was used in [9, 10, 16]. One can also choose to define the Hubble factor without the lapse function as $3H_{\mathcal{D}} = \langle\xi\rangle_{\mathcal{D}}$and the corresponding averaged Hamiltonian constraint becomes

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G \left\langle \gamma^2\left(\rho + v^2 p\right)\right\rangle_{\mathcal{D}} + 2\left\langle\Lambda\right\rangle_{\mathcal{D}} - \left\langle\mathcal{R}\right\rangle_{\mathcal{D}} \\ &\quad - \mathcal{Q}_{\mathcal{D}} \end{aligned} \tag{14}$$

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left\langle\xi^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle\xi\right\rangle_{\mathcal{D}}^2 - 2\left\langle\Sigma^2\right\rangle_{\mathcal{D}} . \tag{15}$$

Definitions based on Θ

Assuming all types of matter follow the same 4-velocity, the local expansion of the matter is given by Θ. If we average this on spatial surfaces orthogonal to u^a, we have $3H_{\mathcal{D}} = \langle\Theta\rangle_{\mathcal{F}}$. This definition is equivalent to that studied in [7, 8, 19], and is the same as the expansion of the coordinates if we choose the synchronous gauge. The equations in that case are well known and presented in [1].

Definitions based on θ

A final definition of the expansion we consider is given by θ: the derivative of the matter observers worldline projected into the rest-space of the gravitational frame. This was introduced in [12, 14] as a way of recognising the fact the rest-frame of the matter before and after averaging are not the same. Hence, a useful definition of the average Hubble factor is: $3H_{\mathcal{D}} = \langle N\theta\rangle_{\mathcal{D}}$ .This will lead to the following averaged Friedmann's equation:

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G \left(\left\langle\gamma^4 N^2\rho\right\rangle_{\mathcal{D}} + \left\langle\gamma^2(\gamma^2-1)N^2 p\right\rangle_{\mathcal{D}}\right) \\ &\quad + 2\Lambda\left\langle N^2\gamma^2\right\rangle_{\mathcal{D}} - \left\langle\gamma^2 N^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}} \ , \end{aligned} \tag{16}$$

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\left\langle(N\theta)^2\right\rangle_{\mathcal{D}} - \left\langle N\theta\right\rangle_{\mathcal{D}}^2\right) - 2\left\langle N^2\sigma^2\right\rangle_{\mathcal{D}} \ ,$$

$$\mathcal{L}_{\mathcal{D}} \equiv 2\left\langle N^2\sigma_B^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle(N\theta_B)^2\right\rangle_{\mathcal{D}} - \frac{4}{3}\left\langle N^2\theta\theta_B\right\rangle_{\mathcal{D}}$$

In the same vein, we can also consider a definition of average Hubble factor without scaling with lapse function as $3H_{\mathcal{D}} = \langle\theta\rangle_{\mathcal{D}}$, in this case the averaged Friedmann's equation becomes

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G \left(\left\langle\gamma^4\rho\right\rangle_{\mathcal{D}} + \left\langle\gamma^2(\gamma^2-1)p\right\rangle_{\mathcal{D}}\right) + 2\Lambda\left\langle\gamma^2\right\rangle_{\mathcal{D}} \\ &\quad - \left\langle\gamma^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}} \end{aligned} \tag{17}$$

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\left\langle(\theta)^2\right\rangle_{\mathcal{D}} - \left\langle\theta\right\rangle_{\mathcal{D}}^2\right) - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}} \ ,$$

$$\mathcal{L}_{\mathcal{D}} \equiv 2\left\langle\sigma_B^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle(\theta_B)^2\right\rangle_{\mathcal{D}} - \frac{4}{3}\left\langle\theta\theta_B\right\rangle_{\mathcal{D}}$$

Notice that the Friedman part of the Buchert equations averaged on the comoving hypersurface may be recovered from the last two definitions in the limit where $v_i \to 0$, $\gamma \to 1$ and $\langle\theta\rangle_{\mathcal{D}} \to \langle\Theta\rangle_{\mathcal{F}}$ [1].

III.1. Spatial averaging of a perturbed FLRW model

The equations derived in Sec. II are not closed, but physical information can be extracted from them if we suppose that the Universe is well described by a perturbed FLRW background. We shall consider perturbations in the longitudinal (Poisson) gauge, where the metric may be written as

$$ds^2 = -\left(1 + 2\Phi + \Phi_2\right) dt^2 + a^2 \left(1 - 2\Phi - \Psi_2\right) \delta_{ij} dx^i dx^j \ . \tag{18}$$

Here, the coordinates are chosen to coincide with the n^a frame such that $n_a = -N\partial_a t$ where the lapse function is $N = \left(1 + \Phi + \frac{1}{2}\Phi_2 - \frac{1}{2}\Phi^2\right)$. We have used the trace-free part of the momentum constraint to set: $\Psi_1 = \Phi_1 = \Phi$ (that is, there is no anisotropic stress at first-order). The peculiar velocity v^i can be expanded to second order and is given by:

$$v_i = \frac{1}{2a}\partial_i(2v_1 + v_2).$$

As usual, the background Friedmann's equation and the deceleration parameter are given by:

$$H(z)^2 = H_0^2\left[\Omega_0(1+z)^3 + 1 - \Omega_0\right],$$
$$q(z) = -\frac{1}{H^2}\frac{\ddot{a}}{a} = -1 + \frac{1+z}{H(z)}\frac{dH}{dz} = -1 + \frac{3}{2}\Omega_m(z)$$

respectively, where

$$\Omega_m(z) = \frac{\Omega_0(1+z)^3}{\left[\Omega_0(1+z)^3 + 1 - \Omega_0\right]^{1/2}}$$

and the first order peculiar velocity given in terms Φ reads:

$$v^{(1)} = -\frac{2}{3aH^2\Omega_m}\left(\dot{\Phi} + H\Phi\right). \tag{19}$$

For details about the solution to the first and the second order equations used in this work, see Appendix A.

In this framework, the average quantities on the hypersurface orthogonal to n^a can easily be expanded to second order in perturbation theory, so that one would rather evaluate Euclidean integrals instead of a Riemann integrals [3]:

$$\langle S\rangle_{\mathcal{D}} = S_0 + \langle S_1\rangle + \langle S_2\rangle + \frac{1}{J_0}\langle S_1 J_1\rangle - \frac{1}{J_0}\langle S_1\rangle\langle J_1\rangle \ . \tag{20}$$

where J_0 and J_1 respectively stand for the background and the first order piece of the determinant of the metric $\sqrt{h}$. while S_0, S_1 and S_2 is the background, first order and the second order component of any perturbed scalar on the hypersurface orthogonal to n^a. Note the important terms of the form $\langle\ \rangle\langle\ \rangle$ which appear due to the Riemann average – such terms do not appear if we average perturbations on the background only.

III.1.1. Frame switching

In other to perform spatial averages on the hypersurface comoving with the matter fluid, i.e. on the hypersurface orthogonal to u^a, while using the coordinate system of the longitudinal gauge presented in Eq. (18) we employ the technique developed in [15]. This will allow us to perform an average in a frame which is titled with respect to the coordinates. We do this because the longitudinal gauge is well defined at second-order, and the solutions up to second-order are known in the case where the cosmological constant is non-zero.

Before applying the formalism of [15] to the particular case of interest here, we summarise it and generalise it for our purposes. When defining the average of a spacetime scalar there is considerable freedom in the definition, and this freedom can be used to switch from an average defined in one frame to that in another ([15] used it to define gauge-invariant averages). Consider defining the average of a quantity using a spacetime window function W_Ω:

$$\langle S\rangle_{A_0,r_0} = \frac{\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, S\, W_\Omega(x)}{\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, W_\Omega(x)} \ , \tag{21}$$

where a a suitable window function might be:

$$W_\Omega(x) = \delta(A(x) - A_0)H(r_0 - B(x)) \ . \tag{22}$$

In this definition of the window function, H is the Heaviside step function and $B(x)$ is a positive function of the coordinates with space-like gradient, $\nabla_a B$, and A is a suitable scalar field with time-like gradient, $\nabla_a A$, such that it takes on a constant value A_0 on the hyper-surface of interest. The scalar field A then defines the foliation of spacetime for averaging. The range of integration across the hyper-surface is specified by inserting a step-like definition of the spatial boundary using the function $B(x)$, which is then bounded by a constant positive value $r_0 > 0$.

It was argued in [15, 16] that one can consistently integrate out the coordinate time to define an average of the scalar field S on the hypersurface of constant A by performing a suitable change of coordinates that transforms the integration variable from $t \mapsto \tilde{t}$. This can be achieved by defining $t = f(\tilde{t}, x)$, where the function f is chosen to ensure that the scalar field S transforms as $S(f(\tilde{t}, x), x) = \tilde{S}(\tilde{t}, x)$. By the use of the Jacobian factor $\partial t/\partial\tilde{t}$, the 3-metric is also transformed from h_{ij} into another metric $\tilde{h}_{ij}$. The function f ensures that the scalar field $A(x,t)$ is homogeneous in the tilde frame: $A(f(\tilde{t},x),x) = \tilde{A}(\tilde{t},x) \equiv A^{(0)}(\tilde{t})$ (see [15] for details). Inserting this into Eq. (21), one finds:

$$\langle S\rangle_{A_0} = \frac{\int_{\Sigma_{A_0}} d^3x\tilde{J}\ \ \tilde{S}(t_0,x)}{\int_{\Sigma_{A_0}} d^3x\tilde{J}} \ , \tag{23}$$

where the tilde quantities are evaluated in the new coordinate system. According to [15], this represents a gauge invariant prescription for the average of a scalar object S on the hypersurface Σ_{A_0} of constant $A = A_0$.

In cosmological perturbation theory, the determinant of the metric $\tilde{J}$ and the scalar fields $\tilde{S}$ can be expanded to second order in perturbation theory to give the average of a scalar field in the new coordinate system:

$$\langle S \rangle_{A_0} = S_0 + \left\langle \tilde{S_1} \right\rangle + \left\langle \tilde{S_2} \right\rangle + \frac{1}{\tilde{J_0}} \left\langle \tilde{S_1} \right\rangle - \frac{1}{\tilde{J_0}} \left\langle \tilde{S_1} \right\rangle \left\langle \tilde{J_1} \right\rangle . \tag{24}$$

where $\tilde{J}_0 = \sqrt{\tilde{h}_0}$ and $\tilde{J}_1 = \sqrt{\tilde{h}_1}$ are the background and the first order piece of the determinant of the metric $\tilde{h}$ respectively. By making a gauge transformation [20] back to the original coordinates, we obtain:

$$\begin{aligned}
\langle S \rangle_{A_0} = {} & S_0 + \langle S_1 \rangle + \langle S_2 \rangle + \frac{1}{J_0} \langle J_1 S_1 \rangle - \frac{1}{J_0} \langle S_1 \rangle \langle J_1 \rangle \\
& - \frac{\dot{S}_0}{\dot{A}_0} \left[\langle A_1 \rangle + \frac{1}{J_0} \langle J_1 A_1 \rangle + \langle A_2 \rangle \right] \\
& + 2 \frac{\dot{S}_0}{\dot{A}_0^2} \left\langle A_1 \dot{A}_1 \right\rangle - \frac{1}{\dot{A}_0} \left[\left\langle A_1 \dot{S}_1 \right\rangle + \left\langle S_1 \dot{A}_1 \right\rangle \right] \\
& + \frac{1}{2} \left[\frac{\ddot{S}_0}{\dot{A}_0^2} - 3 \frac{\ddot{A}_0 \dot{S}_0}{\dot{A}_0^3} + 2 \frac{\partial_t \left(\ln J_0 \right) \dot{S}_0}{\dot{A}_0^2} \dot{S}_0 \right] \langle A_1^2 \rangle \\
& + \left[\frac{\ddot{A}_0}{\dot{A}_0^2} - \frac{\partial_t \left(\ln J_0 \right)}{\dot{A}_0} \right] \langle S_1 A_1 \rangle + 2 \frac{\dot{S}_0}{J_0 \dot{A}_0} \langle A_1 \rangle \langle J_1 \rangle \\
& - \left[\frac{\ddot{A}_0}{\dot{A}_0^2} - \frac{\partial_t \left(\ln J_0 \right)}{\dot{A}_0} \right] \langle S_1 \rangle \langle A_1 \rangle \\
& - \left[\frac{\dot{S}_0 \ddot{A}_0}{\dot{A}_0^3} + \frac{\partial_t \left(\ln J_0 \right) \dot{S}_0}{\dot{A}_0^2} \right] \langle A_1 \rangle^2 \\
& - \frac{\dot{S}_0}{\dot{A}_0^2} \langle A_1 \rangle \left\langle \dot{A}_1 \right\rangle + \frac{1}{\dot{A}_0} \langle S_1 \rangle \left\langle \dot{A}_1 \right\rangle
\end{aligned} \tag{25}$$

Once the scalar variable A is chosen to specify the averaging hypersurface, the above averaging prescription can easily be applied. Eq. (25) was first derived in [15], but the authors set the spatial average of a first order scalar quantity $\langle S_1 \rangle$ to zero (see Eq. (3.10) in [15]) before performing the gauge transformation, thereby neglecting the terms of the form $\langle S_1 \rangle \langle A_1 \rangle$, $\langle S_1 \rangle \left\langle \dot{A}_1 \right\rangle$, etc, which are non-zero and are explicitly scale dependent at second order [12]. We have inserted them as they play an important role in the average of the Hubble rate.

To fix the definition of A in terms of the quantities of the perturbed FLRW background and at the same time fix the foliation of interest, we employ the technique used in [3]. This involves relating the scalar field A to the time, τ, measured by the average observers with 4-velocity u^a comoving with the fluid: $u^0 \partial_0 + u^i \partial_i = \partial_\tau$.The scalar field A can be expanded to second order in perturbation theory, subject to the condition $\tilde{A}(t,x) = A_0(t) + A_1(t,x) + A_2(t,x) \equiv \tau$ [15] to give (using $u^a \nabla_a \tau$):

$$\begin{aligned}
& (1 - \Phi - \frac{1}{2}\Phi_2 + \frac{3}{2}\Phi^2 + v_1^k v_{1k}) \partial_t \tilde{A}(t,x) \\
& \quad + \frac{1}{a^2}(v_1^i + v_2^i) \partial_i \tilde{A}(t,x) = 1 \ .
\end{aligned} \tag{26}$$

We can now calculate the higher order A in terms of Φ_1 and Φ_2 of the perturbed FLRW background. This gives:

$$A_0(t) = t \tag{27}$$

$$A_1(t,x) = \int_0^t \Phi_1 dt \tag{28}$$

$$\begin{aligned}
A_2(t,x) = {} & \frac{1}{2} \int_0^t \Phi_2 dt - \frac{1}{2} \int_0^t \Phi_1^2 dt - \int_0^t v_1^k v_{1k} dt \\
& - \int_0^t \frac{1}{a^2} v_1^i \partial_i A_1 dt \ .
\end{aligned} \tag{29}$$

The average Hubble factor calculated using this prescription is given in the Appendix B.

III.2. The ensemble average and the variance

With the tools developed in Sec. III, we have performed a consistent second order perturbative expansion of the Riemann average defined in Sec. III to obtain a corresponding Euclidean average. Given a specific realisation of a cosmology, we could now calculate spatial averages directly. Alternatively, we can calculate the ensemble average of a given spatial average which will tell us the expectation values of spatially averaged quantities. The ensemble-variance tell us how much we expect that to vary from one domain to another.

The ensemble average of a spatial average may be written as:

$$\overline{\langle X(x) \rangle} = \frac{1}{V} \int d^3x W(x/R_{\mathcal{D}}) \overline{X(x)} \tag{30}$$

where the overbar denotes an ensemble average. We have specified the domain though the window function W. The Euclidean volume of the spatial domain of averaging $\mathcal{D}$ is then given by: $V = \int d^3x W(x/R_{\mathcal{D}})$ which in the case of a Gaussian window function which we mostly employ is $V = 4\pi R_{\mathcal{D}}^3 \int_0^\infty y^2 W(y) dy = (2\pi)^{3/2} R_{\mathcal{D}}^3$ for any $R_{\mathcal{D}}$. The inverse Fourier transform of this window function reads: $W(kR_{\mathcal{D}}) = \frac{1}{V} \int d^3x W(x/R_{\mathcal{D}}) e^{-ik \cdot x}$. The Fourier and the inverse Fourier transforms of any scalar quantity Φ are given as

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \Phi(k) e^{ik \cdot x}, \tag{31}$$

$$\Phi(k) = \frac{1}{(2\pi)^{3/2}} \int d^3x \Phi(x) e^{-ik \cdot x}. \tag{32}$$

For statistically homogenous Gaussian random variables, we have: $\overline{\Phi(k)} = 0$, and the power spectrum of Φ is defined by

$$\overline{\Phi(k)\Phi(k')} = \frac{2\pi^2}{k^3} \mathcal{P}_\Phi(k) \delta(k + k'). \tag{33}$$

Assuming scale-invariant initial conditions from inflation, this is given by

$$\mathcal{P}_\Phi(z,k) = \left(\frac{3\Delta_{\mathcal{R}}}{5 g_\infty} \right)^2 g(z)^2 T(k)^2 \tag{34}$$

where $T(k)$ is the normalised transfer function, $\Delta^2_{\mathcal{R}}$ is the primordial power of the curvature perturbation [21], with $\Delta^2_{\mathcal{R}} \approx 2.41 \times 10^{-9}$ at a scale $k_{CMB} = 0.002\text{Mpc}^{-1}$.

It is not difficult to notice from the equations displayed in the appendix that most of the terms we are dealing with are scalars which schematically appear in the form $\partial^m \Phi(x) \partial^n \Phi(x)$ where m and n represent the number of derivatives (not indices), such that $m+n$ is even so that there are no free indices. (For example, $\partial_i \Phi \partial^2 \partial^i \Phi$ has $m = 1$ and $n = 3$.) Then from the results of [12], the ensemble average of these kind of terms, if a Gaussian window function is assumed, is given by:

$$\overline{\langle \partial^m \Phi(x) \partial^n \Phi(x) \rangle} = \frac{(-1)^{(m+3n)/2}}{2\pi^2} \int dk\, k^{m+n-1} k^3 \mathcal{P}_\Phi(k). \tag{35}$$

Using $\Phi = g(t)\Phi_0(x)$, $g(t)$ being the growth suppression factor and $\Phi_0(x)$ the spatial dependent initial condition (see the Appendix), the terms that appear with a time derivative of the gravitational potential can be rewritten to pull out the time component before evaluating the ensemble average:

$$\dot{\Phi}(t, x) = -(1+z)\, H \frac{d \ln g}{dz} \Phi(t, x). \tag{36}$$

For the details of the calculation of the ensemble average of the inverse laplacian appearing the second order Bardeen potential refer to [12].

The ensemble variance in the Hubble factor is given by

$$\text{Var}[H_{\mathcal{X}}] = \overline{H^2_{\mathcal{X}}} - \overline{H}^2_{\mathcal{X}}\ , \tag{37}$$

where $H_{\mathcal{X}}$ can be any definition of averaged expansion rate we are studying. With this definition, it is easy to see that pure second order contributions drop out of the variance, so that only terms that are quadratic in first order quantities remain.

IV. RESULTS AND DISCUSSION

We shall now investigate the expectation values of the different average Hubble rates, along with their variances. For this we will consider an Einstein-de Sitter model, and a standard concordance model. We shall use length scales intrinsic to the model as reference points for averaging: the equality scale, k^{-1}_{eq}, and the Hubble scale, k^{-1}_{H}:

$$\begin{aligned} k_{\text{eq}} &\approx 7.46 \times 10^{-2} \Omega_0 h^2 \text{Mpc}^{-1}, \\ k_{\text{H}} &\approx \frac{h}{3000} \text{Mpc}^{-1}, \end{aligned} \tag{38}$$

where Ω_b and Ω_0 are the baryon and total matter contributions today and $H_0 = 100\, h\, \text{kms}^{-1}\text{Mpc}^{-1}$. We shall use two models for comparison: Einstein-de Sitter with $h = 0.7$ and 5% baryon fraction (WMAP [21] estimates $\Omega_b \approx 0.046$). This has $k^{-1}_{\text{eq}} \simeq 27.9$Mpc. The other model we shall use is a concordance model with $\Omega_0 = 0.26, h = 0.7, f_{\text{baryon}} = 0.175$ (this is the WMAP best fit [22]). The key length scales in this model is $k^{-1}_{\text{eq}} \simeq 107.2$Mpc. Both models have the Hubble scale $k^{-1}_{\text{H}} \simeq 4.3$Gpc. To calculate the integrals we use transfer functions presented in [23]. All lengths shown are in Mpc. As some of the integrals have a logarithmic IR divergence, all k-integrals have an IR cut-off set at ten times the Hubble scale, it did not appear explicitly in any of our calculations.

We show the ensemble averages of some of the second-order terms which appear in the Hubble rates in Fig. 1. Note that we also show the result of $\langle \partial^2 \Phi \partial^2 \Phi \rangle$ for comparison.

IV.1. Comparison between the different definitions

We can now turn to estimating and comparing the Hubble rates as well as their intrinsic variances as defined above consistently up to second order in perturbation theory. When determining the ensemble average of the Hubble rate, we shall consider two alternatives: a kinematical ensemble average given by $\overline{H}_{\mathcal{D}}$, and a dynamical one, which arrises from taking the ensemble average of the Friedman equation: $\sqrt{\overline{H^2_{\mathcal{D}}}}$. We shall find that the difference between these two is large because the variance is large.

Fig. 2 presents the evolution as functions of redshift of the different definitions for the Hubble rate in a ΛCDM and an EdS scenarios. Fig. 3 depicts the values of the same Hubbles rates at $z = 0$ as functions of the averaging scale R_D, and Fig. 4 shows the scaling of their variances with the averaging scale R_D.

It is clear that the two types of Hubble rates defined in this paper, i.e. those of the gravitational frame, and the ones defined in terms of the physical matter flow can be distinguished as far as their value and their variance are concerned.

First, the ones defined through the local expansion of the observers' worldlines, $\langle \xi \rangle$ and $\langle N\xi \rangle$ present a correction to the FLRW background Hubble rate that is very small, of order 10^{-5} for ΛCDM and 10^{-4} for an EdS scenario. Moreover, they don't exhibit any scale dependence. Finally, their variance is also scale independent. Let us emphasize that this scale independence is very unrealistic, since one would expect the dispersion around the average value of the expansion rate to depend on the scale, illustrating at least the finite volume effect arising in any sampling of a random process.

Second, the Hubble rates defined through the local expansion of the matter worldlines systematically present corrections to the background Hubble rate two orders of magnitude bigger than the previous ones, and are indistinguishable from each other, except when the averaging scale is much larger than the equality scale. Their values as well as their variances are scale dependent. This scale dependence can be traced back to the presence of significant non-connected terms like $\langle \Phi \rangle \langle \partial^2 \Phi \rangle$ in their development. It is interesting to note that both the values of these Hubble rates and their variances are indistinguishable up to scales of averaging of order >100 Mpc,

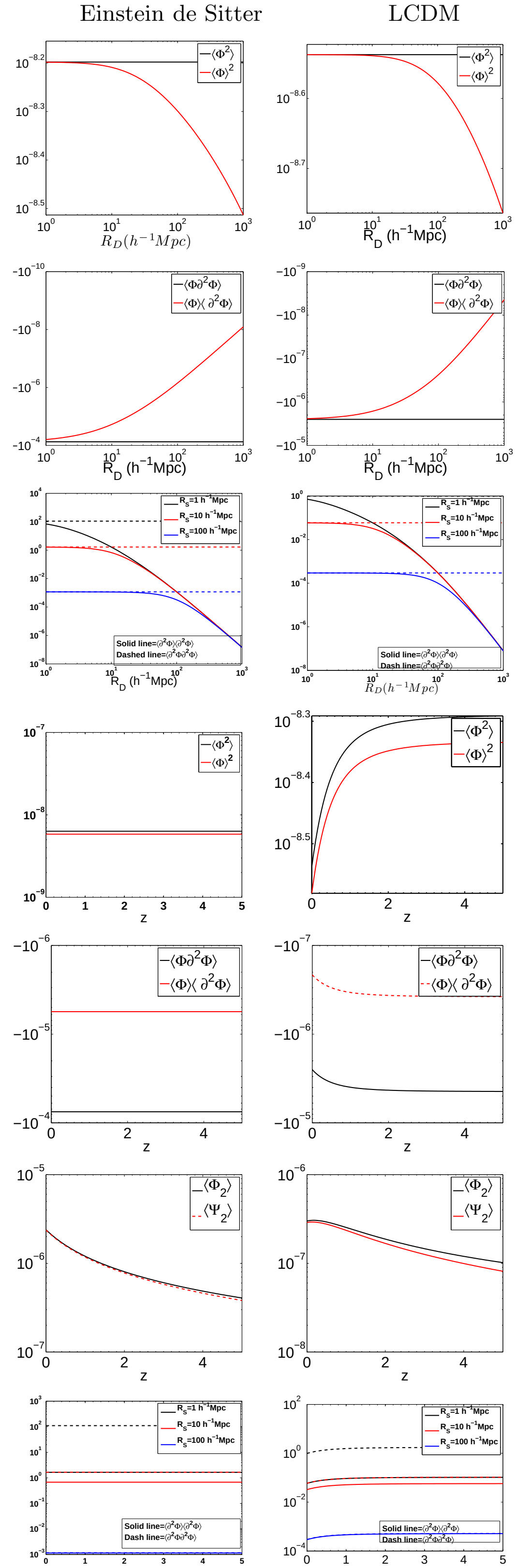

FIG. 1. The amplitude of the ensemble averages of various second-order terms which appear in the Hubble rate.

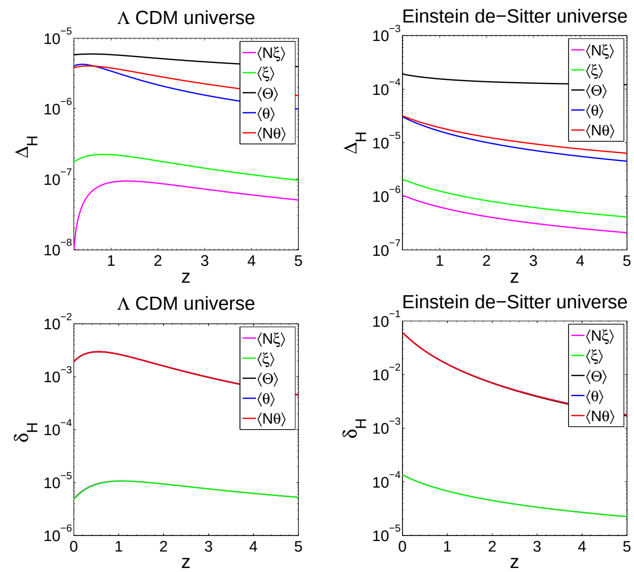

FIG. 2. Fractional change to the background Hubble rate as a function of redshift for the different definitions of averaged Hubble rates under study. Here we have averaged at the equality scale. Here, $\Delta_H = (\overline{H}_{\mathcal{D}} - H_0)/H_0$, and $\delta_H = (\sqrt{\overline{H^2_{\mathcal{D}}}} - H_0)/H_0$.

after which they start differing. This scale have been interpreted in a previous work [12] as naturally defining the scale of statistical homogeneity of the universe (note that this is the case even for EdS; so it is not simply the equality scale). Around the same scale the expansion rate of the gravitational frame becomes comparable with the others because the peculiar velocity tends to zero.

Finally, let us note that the results are consistent, for a pure CDM Universe, with those found on small scales in [7, 8]. This can be seen on Fig. 5.

This analysis shows that the averaged Hubble rates defined through the expansion of the Newtonian-like or gravitational frame, as in [9, 10], is not a good tracer of the expansion of the cosmic fluid, except beyond the homogeneity scale. The fluid frame is more relevant for local measurements since it is attached to the matter component of the Universe. The 'gravitational frame', as we have referred to it here, seems useful on much larger scales, which is the situation in [9, 10] in which it was first evaluated – their domain was the Hubble scale.

IV.2. Fluctuations in the measurement of H_0

We would like to finish this paper by addressing the following questions:

- What is the physical relevance of the averaged Hubble rate and its variance?
- Can there be any signature of backreaction in the observations leading to the measurement of H_0?

First, let us note that on sufficiently small scales, such as scales smaller than ~ 100 Mpc, which are the stan-

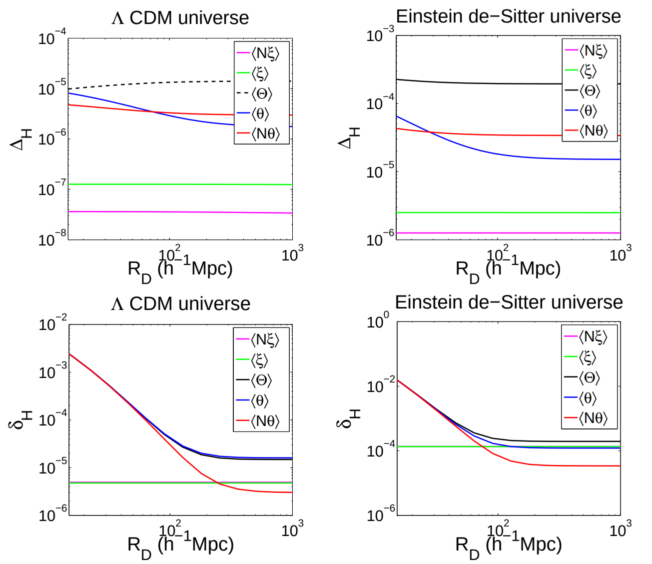

FIG. 3. Fractional change to the background Hubble rate as a function of the averaging scale for the different definitions of averaged Hubble rates under study. In both models, $\langle \xi \rangle_{\mathcal{D}}$ and $\langle N\xi \rangle_{\mathcal{D}}$ almost coincide and are indistinguishable in the figure.

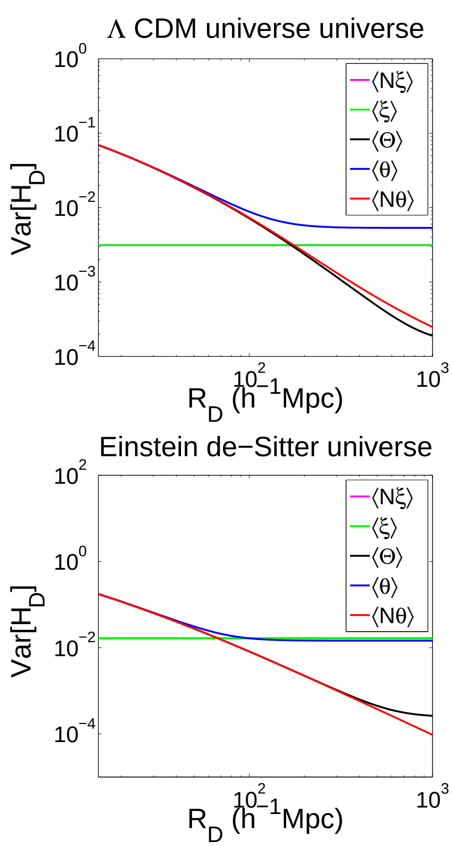

FIG. 4. Variance in the fractional change to the background Hubble rate as a function of redshift for the different definitions of averaged Hubble rates under study.

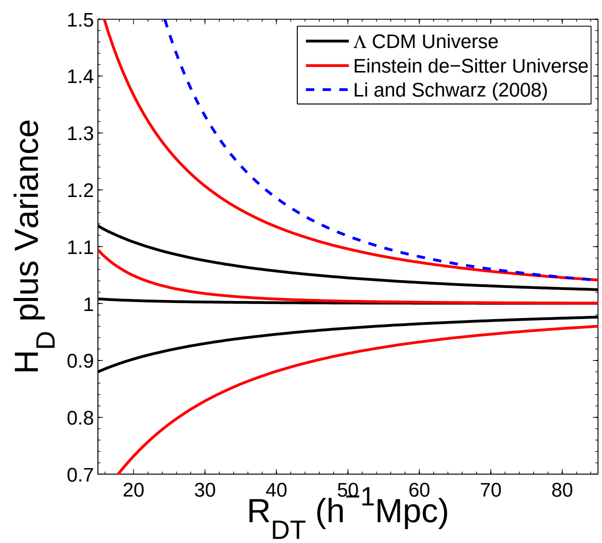

FIG. 5. Dynamical Hubble rate $\langle \Theta \rangle_{\mathcal{F}}$ today, plus/minus the variance as a function of the averaging scale (normalized to the background Hubble rate), where we have used a top-hat window function to define our domain for comparison with [7, 8]. The blue curve represents the variance calculated in [7, 8]; differences for small domains are a consequence of the different transfer functions used here.

dard scales at which the Hubble rate is evaluated, and in a statistically homogeneous and isotropic Universe, spatial averages are expected to be a good approximation of what happens along the past lightcone on which observations are made. Along the past lightcone the monopole contribution to the Hubble rate, which is the one that remains once a full sky average has been performed, is exactly the covariant quantity $\Theta = \nabla_a u^a$ [24].

Hence, our estimate of $H_{\mathcal{D}}$ on a scale $R_{\mathcal{D}}$ can be interpreted as the average Hubble rate in a patch of the local Universe of size $R_{\mathcal{D}}$ as long as this size remains sufficiently small compared with the Hubble scale. Moreover the variance we calculated is the intrinsic dispersion on the measurement of H_0 that comes from the fluctuations in the peculiar velocity of the sources and gravitational potential. In a concordance cosmology, this dispersion appears small, of order 1% at a scale of 100 Mpc, and even less on larger scales, as can be seen on Fig. 5.

This is consistent with previous estimates that were based on an estimate of the first order velocity power spectrum [19, 25]. It is due to the fact that the pure second order terms cancel out consistently at second order in our expression of the variance, allowing only contributions of squares of first order quantities. As noted before, this is a similar effect to that found in [7, 8], where the calculations were made in the comoving synchronous gauge, for a pure CDM Universe. Our calculation of $\langle \Theta \rangle_{\mathcal{F}}$ using the gauge-invariant approach of [15] corresponds to a gauge-invariant version of the average expansion rate in the synchronous gauge.

To quantify the backreaction effect on the variance for a large class of cosmological models, we provide a fitting formula for the variance of the Hubble rate (defined via the flow of matter), $Var[H]$, that is accurate to a few percents across the scales of interest:

$$\ln Var[H] = -43.61 + 46.0\Omega_m^{0.0293} - 0.7969 f_b^{0.0347} + \lambda R^\alpha + \gamma \exp(-\beta R^2) \quad (39)$$

where

$$\begin{aligned} \lambda &= 10.32 - 9.084\Omega_m^{0.0469} - 3.611/f_b^{0.00497} \quad (40)\\ \gamma &= 1.309 - 2.355\Omega_m^{0.055} - 1.073 f_b^{2.1778}\\ \beta &= -1.805 + 3.260\Omega_m^{0.0279} - 0.7180 f_b^{0.665}\\ \alpha &= 1.222 + 0.0334\Omega_m^{3.635} + 0.0591 f_b^{0.3944}. \end{aligned}$$

This formula gives the variance on the measurement of H_0, normalised to the value of H_0: Ω_m is the CDM density parameter, f_b the baryon fraction, and R the length characteristic of the survey, i.e. the distance to the farthest object (in units of h^{-1}Mpc). Note that this fitting formula is valid for a Gaussian window function. Top-hat window functions generically lead to a slight increase of the variance.

V. CONCLUSION

In this work, we have first compared the different definitions of the averaged Hubble rate that can be found in the literature, by calculating their behaviour in perturbation theory consistently to second order. For the first time we have calculated average of the expansion rate using the formalism of [15]. We found that the physical definitions that involve the flow of the dust matter component are consistent with each other at second order, but differ significantly on small scales from the quantities defined only in terms of the specific coordinate system used to perform the calculations. In particular, we find that, in terms of the ensemble averaged Hubble rate:

- On small scales all definitions which involve the matter flow agree, and give a small sub-percent change to the background Hubble rate.
- The ensemble average of the expansion rate of the gravitational frame exhibits no scale dependence.
- On large scales (much larger than the equality scale) all definitions become scale invariant once their ensemble average is evaluated.
- The hypersurface used for averaging is important only on large scales, and only makes a noticeable difference in EdS models.
- Including N in the definition of the averaged expansion again only leaves a residual effect on large scales, and tends to reduce the backreaction effect.

We have also derived the dispersion affecting the Hubble rate and arising from the peculiar velocities of the matter flow. We found a effect consistent with previous estimates from backreaction in the literature [7, 8], and our results are consistent with effects evaluated previously [12, 19].

We close with a comment on the origin of the scale dependence of the various quantities. The scale dependence we have found appearing after ensemble averaging comes only from 'non-connected' terms such as $\langle\Phi\rangle\left\langle\partial^2\Phi\right\rangle$ since the domain size factors out of all other terms. Non-connected terms only arise when we perform averages in the spacetime itself, which many authors on backreaction have stressed as important. It is interesting to note that these terms do not appear if we treat perturbations as fields propagating on the background, and calculate average quantities only with respect to the background geometry (i.e., if we perform a Euclidean average and not a Riemannian one) [26].

ACKNOWLEDGMENTS

We thank George Ellis, Syksy Räsänen, Cyril Pitrou and Dominik Schwarz for discussions. JL is supported by the Claude Leon Foundation (South Africa). OU is supported by National Institute for Theoretical Physics (NITheP), South Africa and the University of Cape Town. CC is funded by the NRF (South Africa)

Appendix A: Second-order perturbation theory

The Poisson gauge is particularly elegant for scalar perturbations because with n^a defined orthogonal to the spatial metric h_{ij}, the Weyl tensor becomes

$$E_{ij}^{(n)} = \frac{1}{2}\left(h_i{}^a h_j{}^b - \frac{1}{3}h_{ij}h^{ab}\right)\left\{\tilde\nabla_a\tilde\nabla_b\left[\Phi + \Psi - \Phi^2 - \Psi^2 + \frac{1}{2}\left(\Phi^{(2)} + \Psi^{(2)}\right)\right] + \tilde\nabla_a\Phi\tilde\nabla_b\Phi - \tilde\nabla_a\Psi\tilde\nabla_b\Psi\right\} \quad (A1)$$

$$H_{ij}^{(n)} = 0. \quad (A2)$$

In the rest frame n^a, then, the gravitational field is silent, and, with $\Psi = \Phi$ is a pure potential field. Hence, n^a may be considered as the rest-frame of the gravitational field, or the Newtonian-like frame, and so defines natural hypersurfaces with which to perform our averages. By contrast, in the frame u^a the Weyl tensor has non-zero H_{ab} [12].

The Einstein Equation for a single fluid with zero pressure and no anisotropic stress $\Psi = \Phi$, and Φ obeys the

Bardeen equation

$$\Phi'' + 3\mathcal{H}\Phi' + a^2\Lambda\Phi = 0 = \ddot{\Phi} + 4H\dot{\Phi} + \Lambda\Phi\,. \tag{A3}$$

and $' = d/d\eta$, and $\mathcal{H} = a'/a$ is the conformal Hubble rate. All first-order quantities can be derived from Φ. The solution to the growing mode of the Bardeen equation may be written as

$$\Phi(\eta, x) = g(\eta)\Phi_0(x) \tag{A4}$$

where $\Phi_0(x)$ is the Bardeen potential today ($\eta = \eta_0,\ z = 0$) and $g(\eta)$ is the growth suppression factor, which may be approximated, in terms of redshift, as [27, 28]

$$g(z) = \frac{5}{2}g_\infty\Omega_m(z)\left\{\Omega_m(z)^{4/7} - \Omega_\Lambda(z) + \left[1 + \frac{1}{2}\Omega_m(z)\right]\left[1 + \frac{1}{70}\Omega_\Lambda(z)\right]\right\}^{-1}. \tag{A5}$$

and g_∞ is chosen so that $g(z = 0) = 1$.

The second-order solutions for $\Psi^{(2)}$ and $\Phi^{(2)}$ are given by [29]. We quote their results directly:

$$\Psi^{(2)}(\eta, x) = \left(B_1(\eta) - 2g(\eta)g_m - \frac{10}{3}(a_{\rm nl} - 1)g(\eta)g_m\right)\Phi_0^2 + \left(B_2(\eta) - \frac{4}{3}g(\eta)g_m\right)\Big[\nabla^{-2}\left(\partial^i\Phi_0\partial_i\Phi_0\right)$$
$$- 3\nabla^{-4}\partial_i\partial^j\left(\partial^i\Phi_0\partial_j\Phi_0\right)\Big] + B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\Phi_0\partial_j\Phi_0) + B_4(\eta)\partial^i\Phi_0\partial_i\Phi_0\,, \tag{A6}$$

$$\Phi^{(2)}(\eta, x) = \left(B_1(\eta) + 4g^2(\eta) - 2g(\eta)g_m - \frac{10}{3}(a_{\rm nl} - 1)g(\eta)g_m\right)\Phi_0^2 + \left[B_2(\eta) + \frac{4}{3}g^2(\eta)\left(e(\eta) + \frac{3}{2}\right) - \frac{4}{3}g(\eta)g_m\right]$$
$$\times\left[\nabla^{-2}\left(\partial^i\Phi_0\partial_i\Phi_0\right) - 3\nabla^{-4}\partial_i\partial^j\left(\partial^i\Phi_0\partial_j\Phi_0\right)\right] + B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\Phi_0\partial_j\Phi_0) + B_4(\eta)\partial^i\Phi_0\partial_i\Phi_0\,, \tag{A7}$$

where $B_i(\eta) = \mathcal{H}_0^{-2}\left(f_0 + 3\Omega_0/2\right)^{-1}\tilde{B}_i(\eta)$ with the following definitions

$$\tilde{B}_1(\eta) = \int_{\eta_m}^{\eta} {\rm d}\tilde{\eta}\,\mathcal{H}^2(\tilde{\eta})(f(\tilde{\eta}) - 1)^2 C(\eta, \tilde{\eta}), \qquad \tilde{B}_2(\eta) = 2\int_{\eta_m}^{\eta} {\rm d}\tilde{\eta}\,\mathcal{H}^2(\tilde{\eta})\Big[2(f(\tilde{\eta}) - 1)^2 - 3 + 3\Omega_m(\tilde{\eta})\Big]C(\eta, \tilde{\eta})\,, \tag{A8}$$

$$\tilde{B}_3(\eta) = \frac{4}{3}\int_{\eta_m}^{\eta} {\rm d}\tilde{\eta}\left(e(\tilde{\eta}) + \frac{3}{2}\right)C(\eta, \tilde{\eta}), \qquad \tilde{B}_4(\eta) = -\int_{\eta_m}^{\eta} {\rm d}\tilde{\eta}\,C(\eta, \tilde{\eta},) \tag{A9}$$

and

$$C(\eta, \tilde{\eta}) = g^2(\tilde{\eta})a(\tilde{\eta})\Big[g(\eta)\mathcal{H}(\tilde{\eta}) - g(\tilde{\eta})\frac{a^2(\tilde{\eta})}{a^2(\eta)}\mathcal{H}(\eta)\Big]\,, \tag{A10}$$

with $e(\eta) = f^2(\eta)/\Omega_m(\eta)$ and

$$f(\eta) = 1 + \frac{g'(\eta)}{\mathcal{H}g(\eta)}\,. \tag{A11}$$

g_m denotes the value of $g(\eta_m)$, deep in the matter era before the cosmological constant was important. We also have $a_{\rm nl}$ which denotes any primordial non-Gaussianity present. We set this to unity, representing a single field slow-roll inflationary model. For details on how the spatial average of the second order Bardeen Potential may be evaluated see [12]

Appendix B: Hubble rates

In this appendix, we present the different Hubble rates, consistently at second order. The superscript determines the quantity that has been averaged to define the average Hubble rate.

$$H_{\mathcal{D}}^{N\xi} = H - \left\langle\dot{\Phi}\right\rangle - 3\left\langle\dot{\Phi}\right\rangle\langle\Phi\rangle + \left\langle\Phi\dot{\Phi}\right\rangle - \frac{1}{2}\langle\Psi_2\rangle \tag{B1}$$

$$H_{\mathcal{D}}^{\xi} = H - \left\langle\dot{\Phi}\right\rangle - H\langle\Phi\rangle - 3\left\langle\dot{\Phi}\right\rangle\langle\Phi\rangle + 2\left\langle\Phi\dot{\Phi}\right\rangle - 3H\langle\Phi\rangle^2 + \frac{9}{2}\left\langle\Phi^2\right\rangle - \frac{1}{2}\left\langle\dot{\Psi}_2\right\rangle - \frac{1}{2}H\langle\Phi_2\rangle \tag{B2}$$

$$
\begin{aligned}
H_{\mathcal{D}}^{N\theta} = {} & H - \left\langle \dot{\Phi} \right\rangle - 3 \left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle + 2 \left\langle \dot{\Phi}\Phi \right\rangle - \frac{1}{2} \left\langle \dot{\Psi}_2 \right\rangle + \frac{(1+z)}{6} \partial_k v_2^k - \frac{2\,(1+z)^2}{9\Omega_m H^2} \left[\left\langle \partial^2 \dot{\Phi} \right\rangle + H \left\langle \partial^2 \Phi \right\rangle \right] \\
& + \frac{(1+z)^2}{\Omega_m^2 H^3} \left[\frac{8}{9} H \left(1 + \frac{\Omega_m}{2} \right) \left\langle \partial_k \dot{\Phi} \partial^k \Phi \right\rangle + \frac{4}{9} H^2 \left(1 + \Omega_m \right) \left\langle \partial_k \Phi \partial^k \Phi \right\rangle - \frac{4}{9} \left\langle \partial_k \dot{\Phi} \partial^k \dot{\Phi} \right\rangle \right] \\
& + \frac{2\,(1+z)^2}{3\Omega_m H^2} \left[\frac{2}{3} \left\langle \Phi \partial^2 \dot{\Phi} \right\rangle + \frac{2}{3} H \left\langle \Phi \partial^2 \Phi \right\rangle - \left\langle \partial^2 \dot{\Phi} \right\rangle \langle \Phi \rangle - H \left\langle \partial^2 \Phi \right\rangle \langle \Phi \rangle \right]
\end{aligned} \tag{B3}
$$

$$
\begin{aligned}
H_{\mathcal{D}}^{\theta} = {} & H - \left\langle \dot{\Phi} \right\rangle - H \langle \Phi \rangle - 3 \left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle + 2 \left\langle \Phi \dot{\Phi} \right\rangle - 3H \langle \Phi \rangle^2 + \frac{9}{2} H \left\langle \Phi^2 \right\rangle - \frac{1}{2} \left[\left\langle \dot{\Psi}_2 \right\rangle + 2H \left\langle \Phi_2 \right\rangle \right] \\
& + \frac{(1+z)}{6} \left\langle \partial_k v^k \right\rangle - \frac{2\,(1+z)^2}{9\Omega_m H^2} \left[\left\langle \partial^2 \dot{\Phi} \right\rangle + H \left\langle \partial^2 \Phi \right\rangle \right] + \frac{2\,(1+z)^2}{9\Omega_m^2 H^3} \left[\left\langle \partial_k \dot{\Phi} \partial^k \dot{\Phi} \right\rangle + 2H \left\langle \partial_k \dot{\Phi} \partial^k \Phi \right\rangle \right. \\
& \left. \times \left(1 + \frac{3}{2} \Omega_m \right) + H^2 \left\langle \partial_k \Phi \partial^k \Phi \right\rangle \left(1 + 3\Omega_m \right) \right] + \frac{2\,(1+z)^2}{3\Omega_m H^2} \left[\left\langle \Phi \partial^2 \dot{\Phi} \right\rangle - \left\langle \partial^2 \dot{\Phi} \right\rangle \langle \Phi \rangle \right. \\
& \left. + H \left\langle \Phi \partial^2 \Phi \right\rangle - H \left\langle \partial^2 \Phi \right\rangle \langle \Phi \rangle \right]
\end{aligned} \tag{B4}
$$

$$
\begin{aligned}
H_{\mathcal{F}}^{\Theta} = {} & H - \left\langle \dot{\Phi} \right\rangle + 3 \left\langle \Phi \dot{\Phi} \right\rangle - \frac{1}{2} \left[\left\langle \dot{\Psi}_2 \right\rangle + H \left\langle \Phi_2 \right\rangle + 3\Omega_m H^2 \left\langle A_2 \right\rangle \right] - \langle \Phi \rangle H \left[1 - \frac{3}{2} \Omega_m H g_I \right] \\
& - 4 \left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle \left[1 + \frac{3}{4} H g_I \right] + \frac{\left\langle \Phi^2 \right\rangle H}{4} \left[22 - 9H\Omega_m g_I \left(1 + \frac{1}{2} g_I H \right) \right] - \frac{7}{2} H \langle \Phi \rangle^2 \left[\frac{8}{7} \left(1 + \frac{3}{4} H g_I \right) \right. \\
& \left. - 3\Omega_m H g_I \left(1 + \frac{3}{7} H g_I \right) \right] + \frac{1}{6} (1+z) \left\langle \partial_k v_2^k \right\rangle - \frac{2(1+z)^2}{9H^2} \left[\left\langle \partial^2 \dot{\Phi} \right\rangle + H \left\langle \partial^2 \Phi \right\rangle \right] \\
& + \frac{2(1+z)^2}{27H^3\Omega_m^2} \left[\left\langle \partial_k \dot{\Phi} \partial^k \dot{\Phi} \right\rangle + 2H \left\langle \partial_k \dot{\Phi} \partial^k \Phi \right\rangle \left(1 + \frac{9}{2} \Omega_m \right) + H^2 \left\langle \partial_k \Phi \partial^k \Phi \right\rangle \left(1 + 9\Omega_m \right) \right] \\
& + \frac{2(1+z)^2}{3\Omega_m H^2} \left[\left\langle \partial^2 \dot{\Phi} \right\rangle \langle \Phi \rangle \left(H g_I - \frac{4}{3} \right) + H \left(H g_I - \frac{2}{3} \right) \left(\left\langle \partial^2 \Phi \right\rangle \langle \Phi \rangle - \left\langle \Phi \partial^2 \Phi \right\rangle \right) + \frac{2}{3} \left\langle \Phi \partial^2 \dot{\Phi} \right\rangle \right]
\end{aligned} \tag{B5}
$$

where $g_I = \frac{1}{g(t)} \int_0^t g(t') dt'$.

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