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Full nonlinear growing and decaying modes of superhorizon curvature perturbations

Yu-ichi Takamizu* and Jun'ichi Yokoyama†
Research Center for the Early Universe (RESCEU),
Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan
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We clarify the behavior of curvature perturbations in a nonlinear theory in case the inflaton temporarily stops during inflation. We focus on the evolution of curvature perturbation on superhorizon scales by adopting the spatial gradient expansion and show that the nonlinear theory, called the *beyond* δN-formalism for a general single scalar field as the next-leading order in the expansion. Both the leading-order in the expansion (δN-formalism) and our nonlinear theory include the solutions of full-nonlinear orders in the standard perturbative expansion. Additionally, in our formalism, we can deal with the time evolution in contrast to δN-formalism, where curvature perturbations remain just constant, and show decaying modes do not couple with growing modes as similar to the case with linear theory. We can conclude that although the decaying mode diverges when $\dot\phi$ vanishes, there appears no trouble for both the linear and nonlinear theory since these modes will vanish at late times.

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I. INTRODUCTION

Recent observations of the cosmic microwave background anisotropy [1] show very good agreement of the observational data with the prediction of standard inflationary cosmology, that is, adiabatic Gaussian random primordial fluctuations with an almost scale-invariant spectrum generated from quantum fluctuations of an inflaton field during inflation [2–8]. The amplitude of curvature perturbation on a comoving slicing, $\mathcal{R}_c$, is given by the formula $\mathcal{R}_c \approx H^2/|\dot\phi|$ evaluated at time of horizon crossing $t = t_k$ when the wavenumber k satisfying $k = aH$, where ϕ and H are the inflaton and the Hubble parameter during inflation, respectively. The reason why it gives an almost scale-invariant spectrum is that both H and $\dot\phi$ change very slowly during slow-roll inflaton.

The main purpose of this paper is to clarify what happens when the inflaton stops during that, namely, $\dot\phi \approx 0$, using a nonlinear perturbation theory. Such a situation naturally occurs in oscillating inflation or the chaotic new inflation models [9–11]. For example, it has been shown by Damour and Mukhanov that oscillating inflation is realized as the inflaton oscillates around a minimum of a nonconvex potential [9]. In another example of chaotic new inflation [11], it has been pointed that the inflaton changes its direction of motion if model parameters are approximately chosen.

In both examples, if we use the above formula for the amplitude of primordial curvature perturbation, it apparently diverges when $\dot\phi$ vanishes. However, Seto, Yokoyama and Kodama [12] have shown that even in the case that slow-roll conditions are violated, if a new formula is applied to this case, the amplitude has still finite value, where the time derivative of the scalar field is replaced by the potential gradient given as $\mathcal{R}_c \propto 3H^3/V'(\phi)$ at $t = t_k$. In this study, they have investigated the evolution of curvature perturbations in the linear theory and shown a decaying mode can diverge at and around temporary stopping of the inflaton [12, 13]. However, since the decaying mode vanishes at sufficient late times, there appears no trouble in the linear perturbation theory.

When we take nonlinear effects into account, the decaying modes can couple with the growing modes in general to convert into growing modes through such effects. Therefore the ill-behavior of the decaying mode may leave an observable trace if nonlinear perturbation is incorporated. The purpose of this paper is to clarify the behavior of curvature perturbations in a nonlinear theory in case the inflaton temporarily stops during inflation.

In order to incorporate nonlinearity of curvature perturbation, we focus on the evolution on superhorizon scales and consider a nonlinear cosmological perturbation theory by adopting a gradient expansion approach [14]. As for the leading-order in the expansion, δN-formalism [15–18] is a powerful tool to calculate the nonlinearity of primordial curvature perturbations (recently much attention to as their non-Gaussianity [19]) since it includes the solutions of full-nonlinear orders in the standard perturbative expansion, but this is just a lowest-order and in this formalism, we should ignore all decaying modes. Therefore, we have to use the next-leading order in the expansion, which was recently formulated by one of us, the so-called *beyond* δN-formalism [20]. In our formalism, which we will briefly review in the following section, there exists decaying and growing modes, having their time-dependences and they lead to time variations of superhorizon curvature perturbations. We will show such decaying mode also diverges, but they will vanish due to inflationary expansion in the same way as in the linear theory, when $\dot\phi$ vanishes.

*Electronic address: takamizu˙at˙resceu.s.u-tokyo.ac.jp
†Electronic address: yokoyama˙at˙resceu.s.u-tokyo.ac.jp

The rest of the paper is organized as follows. In Sec. II, we review the full-nonlinear cosmological perturbation theory of superhorizon curvature perturbations. Then we discuss the growing and decaying modes in both the linear and nonlinear theories in Sec. III and discuss what happens on temporary stopping of the inflaton in Sec. IV. Section V is devoted to the conclusion.

II. BEYOND δN-FORMALISM

In this section, we will briefly review the nonlinear theory of cosmological perturbations valid up to $O(\epsilon^2)$ in the spatial gradient expansion and follow the previous works [20, 21], where ϵ is the ratio of the Hubble length scale $1/H$ to the characteristic length scale of perturbations L, used as a small expansion parameter, $\epsilon \equiv 1/(HL)$, of the superhorizon scales. First of all, we show the main result in our formula for the nonlinear curvature perturbation, $\mathcal{R}_c^{\rm NL}$,

$$\mathcal{R}_c^{\rm NL\,\prime\prime} + 2\frac{z'}{z}\mathcal{R}_c^{\rm NL\,\prime} + \frac{c_s^2}{4}K^{(2)}[\,\mathcal{R}_c^{\rm NL}\,] = O(\epsilon^4)\,, \qquad (2.1)$$

which shows two full-nonlinear effects;

1. Nonlinear variable: $\mathcal{R}_c^{\rm NL}$ including full-nonlinear curvature perturbation, δN
2. Source term: $K^{(2)}[\mathcal{R}_c^{\rm NL}]$ is a nonlinear function of curvature perturbations.

In (2.1), the prime denotes conformal time derivative and z is a well-known Mukhanov-Sasaki variable which will be seen later as (2.24). The explicit forms of both the definition of $\mathcal{R}_c^{\rm NL}$ and the source term $K^{(2)}[X]$, that is the Ricci scalar of the metric X, will be also seen later, in (2.23) and in (2.30), respectively. Of course, in the linear limit, it can be reduced to the well-known equation for the curvature perturbation on comoving hypersurfaces [22],

$$\mathcal{R}_c^{\rm Lin\,\prime\prime} + 2\frac{z'}{z}\mathcal{R}_c^{\rm Lin\,\prime} - c_s^2\,\Delta[\mathcal{R}_c^{\rm Lin}] = 0\,. \qquad (2.2)$$

We will briefly summarize our formula and show the above results in the following. Throughout this paper we consider a minimally-coupled single scalar field described by an action of the form

$$I = \int d^4x\sqrt{-g}P(X,\phi), \qquad (2.3)$$

where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Note that we do not assume the explicit forms of both kinetic term and its potential, that can be given as arbitrary function of $P(X,\phi)$.

We adopt the ADM decomposition and employ the gradient expansion. In the ADM decomposition, the metric is expressed as

$$ds^2 = -\alpha^2dt^2 + \gamma_{ij}(dx^i+\beta^idt)(dx^j+\beta^jdt), \qquad (2.4)$$

where α is the lapse function, β^i is the shift vector and Latin indices run over $1,2,3$. The equations of motion corresponding to α and β^i lead to constraint equations. Components of the spatial metric γ_{ij} are dynamical variables and the corresponding equations of motion are reduced to a set of first-order differential equations with respect to the time t. We introduce the extrinsic curvature K_{ij} defined by

$$K_{ij} = -\frac{1}{2\alpha}\left(\partial_t\gamma_{ij} - D_i\beta_j - D_j\beta_i\right), \qquad (2.5)$$

where D is the covariant derivative compatible with the spatial metric γ_{ij}. As a result, the basic equations are reduced to the first-order equations for the dynamical variables (γ_{ij},K_{ij}), with the two constraint equations (the so-called Hamiltonian and Momentum constraint). We further decompose the spatial metric and the extrinsic curvature as

$$\begin{aligned}\gamma_{ij} &= a^2e^{2\zeta}\tilde{\gamma}_{ij},\\ K_{ij} &= a^2e^{2\zeta}\left(\frac{1}{3}K\tilde{\gamma}_{ij} + \tilde{A}_{ij}\right),\end{aligned} \qquad (2.6)$$

where $a(t)$ is the scale factor of the background FRW universe and $\det\tilde{\gamma}_{ij} = 1$.

Next, we will employ the gradient expansion. In this approach we introduce a flat FRW universe ($a(t)$, $\phi_0(t)$) as a background. As discussed in the first part of this section, we consider the perturbations on superhorizon scales, that is, L is longer than the Hubble length scale $1/H$ of the background, i.e. $HL \gg 1$. Therefore, we consider $\epsilon \equiv 1/(HL)$ as a small expansion parameter and systematically expand our equations by ϵ, considering a spatial derivative acted on perturbations is of order $O(\epsilon)$.

We assume the condition for the gradient expansion:

$$\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2). \qquad (2.7)$$

This corresponds to assuming the absence of any decaying modes at the leading-order in the expansion, namely, the absence of spatially homogeneous anisotropy. This is justified in most of the inflationary models in which the number of e-folds of inflation N is much larger than the number required to solve the horizon and flatness problem, $N \gg 60$. This assumption is sufficient to allow us discuss behavior of decaying modes when the inflaton stops, since all time dependent solutions at the leading order are reduced to just decaying modes and there exists no observable trace at late times.

When we focus on a contribution arising from the scalar-type perturbations, we may choose the gauge in which $\tilde{\gamma}_{ij}$ approaches the flat metric,

$$\tilde{\gamma}_{ij}\,(t\to\infty) = \delta_{ij}, \qquad (2.8)$$

where in reality the limit $t \to \infty$ may be reasonably interpreted as an epoch close to the end of inflation. We take the *comoving slicing, time-orthogonal* gauge:

$$\delta\phi_c(t,x^i) = \beta_c^i(t,x^i) = O(\epsilon^3), \qquad (2.9)$$

where $\delta\phi \equiv \phi - \phi_0$ denotes a fluctuation of a scalar field. The subscript c denotes this gauge throughout this paper.

Now we turn to the problem of properly defining a nonlinear curvature perturbation to $O(\epsilon^2)$ accuracy. Hereafter we will use the expression $\mathcal{R}_c$ on comoving slices to denote it. Let us consider the linear curvature perturbation which is given as

$$\mathcal{R}^{\rm Lin} = \left(H_L^{\rm Lin} + \frac{H_T^{\rm Lin}}{3}\right) Y, \tag{2.10}$$

where, following the notation in [23], the spatial metric in the linear limit is expressed as

$$\gamma_{ij} = a^2(\delta_{ij} + 2H_L^{\rm Lin} Y \delta_{ij} + 2H_T^{\rm Lin} Y_{ij})\,, \tag{2.11}$$

with Y being scalar harmonics with eigenvalue k^2 in Fourier space satisfying

$$(\Delta + k^2)Y = 0\,, \tag{2.12}$$

and

$$Y_{ij} = k^{-2}\left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right] Y\,. \tag{2.13}$$

These expressions in the linear theory correspond to the metric components in our notation as

$$\zeta = H_L^{\rm Lin} Y, \quad \tilde{\gamma}_{ij} = \delta_{ij} + 2H_T^{\rm Lin} Y_{ij}. \tag{2.14}$$

Notice that the variable ζ_c reduces to $\mathcal{R}_c^{\rm Lin}$ at leading-order in the gradient expansion, but not at second-order as (2.10) and it will be also similar to the nonlinear theory.

Thus to define a nonlinear generalization of the linear curvature perturbation (2.10), we need nonlinear generalizations of $H_L Y$ and $H_T Y$. Our nonlinear ζ is an apparent natural generalization of $H_L^{\rm Lin} Y$,

$$H_L Y = \zeta\,. \tag{2.15}$$

As for $H_T Y$, however, the generalization is non-trivial. It corresponds to the $O(\epsilon^2)$ part of $\tilde{\gamma}_{ij}$ and we have obtained a general solution of the dynamical equation for $\tilde{\gamma}_{ij}$ as a first-order differential equation in [20, 21] and the time-dependent part includes the following solution;

$$\tilde{\gamma}_{ij}(t) \ni C_{ij}^{(2)} \int \frac{dt'}{a^3(t')}, \tag{2.16}$$

with the Momentum constraint:

$$e^{3\ell^{(0)}}\partial_i C^{(2)} \;=\; 6 f_{(0)}^{jk}\partial_j\left[e^{3\ell^{(0)}} C_{ki}^{(2)}\right]. \tag{2.17}$$

The explicit forms of solutions can be seen in [21]. Here we attach the superscript (m) to a quantity of $O(\epsilon^m)$, and both $\ell^{(0)}$ and $f_{ij}^{(0)}$ will be denoted as the leading-order metric in (2.19) and (2.20). Our aim is to derive the scalar-type solution $C^{(2)}$ from the tensor $C_{ij}^{(2)}$ in (2.16) by using (2.17). As shown in [20], it can be done by introducing the inverse Laplacian operator Δ^{-1} on the flat background and we defined the nonlinear generalization of $H_T Y$ as

$$H_T Y = E \equiv -\frac{3}{4}\Delta^{-1}\left[\partial^i e^{-3\ell^{(0)}}\partial^j e^{3\ell^{(0)}}(\ln\tilde{\gamma})_{ij}\right]. \tag{2.18}$$

It is easy to see that $E \ni C^{(2)}$ which we expected.

At leading-order, the only non-trivial quantities for the spatial metric, ζ and $\tilde{\gamma}_{ij}$, are given by

$$\zeta = \ell^{(0)}(x^k) + O(\epsilon^2), \tag{2.19}$$

and

$$\tilde{\gamma}_{ij} = f_{ij}^{(0)}(x^k) + O(\epsilon^2), \tag{2.20}$$

where $\ell^{(0)}(x^k)$ is an arbitrary function of the spatial coordinates $\{x^k\}$ $(k = 1, 2, 3)$ and $f_{ij}^{(0)}(x^k)$ is a (3×3)-matrix function of the spatial coordinates with a unit determinant, respectively. Throughout this paper, this leading-order of spatial metric can be chosen as

$$f_{ij}^{(0)} = \delta_{ij}, \tag{2.21}$$

consistent with the gauge condition of (2.8). On the other hand, $\ell^{(0)}$ represents a conserved comoving curvature perturbation, equivalent to a fluctuation of the number of e-folds, which is denoted by the so-called δN term from some final uniform density (or comoving) hypersurface to the initial flat hypersurface at $t = t_*$,

$$\ell^{(0)} = \delta N(t_*, x^i)\,. \tag{2.22}$$

With these definitions of $H_L Y$ and $H_T Y$, we can define the nonlinear curvature perturbation valid up through $O(\epsilon^2)$ as

$$\mathcal{R}_c^{\rm NL} \;\equiv\; \zeta_c \;+\; \frac{E_c}{3}\,. \tag{2.23}$$

It is easy to show that this nonlinear quantity can be reduced to (2.10) in the linear limit. As clear from (2.18), finding $H_T Y$ generally requires a spatially non-local operation, however, in the comoving slicing, time-orthogonal gauge with the asymptotic condition on the spatial coordinates (2.8), we find it is possible to obtain the explicit form of $H_T Y$ without any non-local operation as seen in [20].

Next, we can derive a nonlinear second-order differential equation that $\mathcal{R}_c^{\rm NL}$ (2.23) satisfies at $O(\epsilon^2)$ accuracy by introducing the conformal time η, defined by $d\eta = dt/a(t)$ and the Mukhanov-Sasaki variable [22],

$$z = \frac{a}{H}\left(\frac{\rho + P}{c_s^2}\right)^{\frac{1}{2}}, \tag{2.24}$$

where notice that c_s is the speed of sound for the gauge invariant scalar perturbation in the linear theory [24], given by

$$c_s^2 = \frac{P_X}{P_X + 2P_{XX}X}, \tag{2.25}$$

where the subscript X represents derivative with respect to X. The result can be reduced to a simple equation of the form (2.1) as a natural extension of the linear version (2.2). We also obtain the solution of the nonlinear equation (2.1) as

$$\mathcal{R}_c^{\mathrm{NL}}(\eta) = \ell^{(0)} + \frac{1}{4}\big[F(\eta) - F_*\big]K^{(2)} + \big[D(\eta) - D_*\big]C^{(2)} + O(\epsilon^4), \quad (2.26)$$

where

$$D(\eta) = 3\mathcal{H}_* \int_\eta^0 \frac{z^2(\eta_*)}{z^2(\eta')} d\eta' \,,$$
$$F(\eta) = \int_\eta^0 \frac{d\eta'}{z^2(\eta')} \int_{\eta_*}^{\eta'} z^2 c_s^2(\eta'') d\eta'' \,. \quad (2.27)$$

Here $D_* = D(\eta_*)$, $F_* = F(\eta_*)$ and $\mathcal{H}_*$ denotes the conformal Hubble parameter $\mathcal{H} = d\ln a/d\eta$ at $\eta = \eta_*$ which we take the time as some after the horizon crossing. Note that $t \to \infty$ corresponds to $\eta \to 0$ in the conformal time. Thus the functions D and F vanish asymptotically at late times, $D(0) = F(0) = 0$. Deviation of the solution (2.26) can be easily understood as follows. The second-order differential equation (2.1) contains two solutions (even though its independent relation appears only for the linear theory), i.e. decaying mode and growing mode. We can find that the function $D(\eta)$ satisfies

$$D'' + 2\frac{z'}{z}D' = 0 \,, \quad (2.28)$$

in the long-wavelength limit, i.e. no source term in (2.1). It will be seen that it corresponds to the decaying mode in the linear theory in the next section. On the other hand, the function $F(\eta)$ corresponds to the source term in (2.1), satisfying

$$F'' + 2\frac{z'}{z}F' + c_s^2 = 0 \,, \quad (2.29)$$

as the $O(\epsilon^2)$ correction to a constant mode at the leading-order, i.e. as the growing mode in the linear theory, which is taken the form $1 + F(\eta)K^{(2)} + O(\epsilon^4)$.

Moreover the equation (2.1) includes two 'constants' of integration, or arbitrary spatial functions, which in general appear as the initial conditions, namely, the initial value and its time derivative. Let us consider the spatial functions, which we have introduced as $\ell^{(0)}, C^{(2)}$ and $K^{(2)}$. Here the last one is related to the Ricci scalar of the 0th-order spatial metric as

$$K^{(2)}[\ell^{(0)}] = R\left[e^{2\ell^{(0)}}\delta_{ij}\right] = -2(2\Delta\ell^{(0)} + \delta^{ij}\partial_i\ell^{(0)}\partial_j\ell^{(0)})e^{-2\ell^{(0)}}, \quad (2.30)$$

where we have used $f_{ij}^{(0)} = \delta_{ij}$ from (2.21). Then we have the two arbitrary spatial functions: $\ell^{(0)}$ and $C^{(2)}$, which are related to the number of physical degrees of freedom for the initial conditions. Therefore $\ell^{(0)}$ and $C^{(2)}$ correspond to the initial conditions determined by matching a solution of n-th order perturbation solved inside the horizon to this superhorizon solution at $\eta = \eta_*$. Notice that $\ell^{(0)}$ represents δN term as seen in (2.22) and $C^{(2)}$ originally comes from the decaying mode of the fluctuation of the scalar field [20].

III. GROWING AND DECAYING MODES

In this section, firstly, let us consider the growing and decaying modes in the linear theory. In the linear theory, the curvature perturbation on comoving hypersurfaces follows (2.2). As usual, we consider it in Fourier space,

$$\mathcal{R}_c^{\mathrm{Lin}\,\prime\prime} + 2\frac{z'}{z}\mathcal{R}_c^{\mathrm{Lin}\,\prime} + c_s^2 k^2\, \mathcal{R}_c^{\mathrm{Lin}} = 0 \,. \quad (3.1)$$

Real space expressions (2.2) will be recovered by the replacement $k^2 \to -\Delta$. This equation has two independent solutions, conventionally called a growing mode and a decaying mode.

The growing mode is a constant at the leading-order in the long-wavelength approximation or equivalently the spatial gradient expansion. Then in terms of the growing mode solution u, the decaying mode solution v can be given as [25]

$$v(\eta) = u(\eta)\frac{\tilde{D}(\eta)}{\tilde{D}(\eta_*)}, \tilde{D}(\eta) = 3\mathcal{H}_* \int_\eta^0 d\eta' \frac{z^2(\eta_*)u^2(\eta_*)}{z^2(\eta')u^2(\eta')} \quad (3.2)$$

Note that this expression is correct for any order in the gradient expansion in the linear theory.

The general solution of a curvature perturbation is written in terms of their linear combinations as

$$\mathcal{R}_c^{\mathrm{Lin}}(\eta) = \alpha^{\mathrm{Lin}} u(\eta) + \beta^{\mathrm{Lin}} v(\eta) \,, \quad (3.3)$$

where the coefficients α^{Lin} and β^{Lin} may be assumed to satisfy $\alpha^{\mathrm{Lin}} + \beta^{\mathrm{Lin}} = 1$ without loss of generality. Note that the assumption of the gradient expansion (2.7) corresponds to the condition,

$$\beta^{\mathrm{Lin}} = 1 - \alpha^{\mathrm{Lin}} = O(\epsilon^2) \,. \quad (3.4)$$

This means, as mentioned before, that the decaying mode at leading-order in the gradient expansion has already decayed after horizon crossing.

Therefore the decaying mode solutions can be automatically obtained as following (3.2), if we obtain the growing mode solutions. Let us solve for the growing mode solution. In accordance with the gradient expansion, we set

$$u(\eta) = \sum_{n=0}^{\infty} u_n(\eta) k^{2n} \,. \quad (3.5)$$

At the leading-order in the gradient expansion, the growing mode solution $u^{(0)}$ is just a constant. Then inserting the above expansion with $u^{(0)}$ =const. to the equation of motion (3.1) gives iteratively

$$u''_{n+1} + 2\frac{z'}{z}u'_{n+1} = -c_s^2 u_n \,. \tag{3.6}$$

As shown in [25], $O(k^2)$ corrections to $u^{(0)}$ can be written as

$$u^{(2)} = u^{(0)}\left[C_1^{(2)} + C_2^{(2)} D(\eta) + k^2 F(\eta)\right], \tag{3.7}$$

where the integrals $D(\eta)$ and $F(\eta)$ have been given in (2.27), satisfying (2.28) and (2.29), respectively, as similar to the nonlinear theory, and $C_1^{(2)}$ and $C_2^{(2)}$ are constants of integration. We fix the two constants as $C_1^{(2)} = C_2^{(2)} = 0$ so that $u^{(2)}$ is proportional to the integral $F(\eta)$ at $O(k^2)$ accuracy [1]. Hence we find

$$u^{(2)}(\eta) = k^2 u^{(0)} F(\eta)\,. \tag{3.8}$$

As for the decaying mode, because of (3.4) we only need the leading-order solution. Since we may replace $\tilde{D}$ with D in (3.2), we immediately find

$$v^{(0)} = u^{(0)}\frac{D(\eta)}{D_*}\,. \tag{3.9}$$

Thus from (3.8) and (3.9), the general linear solution valid up through $O(\epsilon^2)$ is obtained as linear combination of constant $u^{(0)}$, growing mode $u^{(2)}$ and decaying mode $v^{(0)}$, which are proportional to $F(\eta)$ and $D(\eta)$, respectively.

As for the nonlinear theory of cosmological perturbations, the solution of (2.26) is also shown as growing and decaying modes, respectively. We can find that the function $D(\eta)$ and $F(\eta)$ satisfy (2.28) and (2.29), respectively and they take the same forms, respectively as in the linear theory. Therefore we can interpret that they correspond to the decaying mode in the long-wavelength limit, and the growing mode taken the form $1 + k^2F(\eta) + O(k^4)$, where $F(\eta)$ is the k^2 correction to the growing (i.e., constant) mode, respectively. In our nonlinear theory, note that time derivative takes the same form as shown in (2.1) and (2.2), hence the decaying mode can not couple with the growing mode as similar to the linear theory because of the method of gradient expansion (i.e. time derivative takes as a linear operator). The difference from the linear theory is the source term , i.e. the Ricci scalar of the leading order metric $K^{(2)}$, which can be reduced to $k^2\mathcal{R}_c^{\rm Lin}$ in Fourier space as the source term in the linear theory (3.1).

[1] If we fix the two arbitrary constants as $C_1^{(2)} = 0$ and $C_2^{(2)} = -k^2F_*/D_*$ so that $u(\eta_*) = u^{(0)}$ holds at $O(k^2)$ accuracy, it is the case of [25] in which they discussed an enhancement of curvature perturbation on superhorizon scales due to suddenly change of the inflaton potential's slope, and its nonlinear effect also can be studied by matching the linear solution of [25] to our nonlinear solution in [20]

IV. CROSSING OF $\dot{\phi} = 0$

We consider the case when $\dot{\phi}$ (or z) crosses zero in our nonlinear theory. The gradient expansion allows us to discuss in a similar way as the linear theory [12, 25]. For simplicity, we assume that z changes the sign only once at $\eta = \eta_0$. Hereafter, we consider a canonical single scalar field, however, the same discussion can be done in the case of a non-canonical single scalar field, when $P_X X \approx 0$.

In the vicinity of $\eta = \eta_0$, z can be expressed as

$$z = z'_0(\eta - \eta_0), \tag{4.1}$$

where $z'_0 = z'(\eta_0)$. Hence the equation for $\mathcal{R}_c^{\rm NL}$ becomes

$$\left[\frac{d^2}{d\eta^2} + \frac{2}{\eta - \eta_0}\frac{d}{d\eta}\right]\mathcal{R}_c^{\rm NL} = -\frac{1}{4}K^{(2)}[\,\mathcal{R}_c^{\rm NL}\,] + O(\epsilon^4). \tag{4.2}$$

The two independent solutions in the linear theory (not guaranteed in the nonlinear theory) can be found as

$$u \approx \ell^{(0)}\left(1 - \frac{1}{6}K^{(2)}[\,\mathcal{R}_c^{\rm NL}\,](\eta - \eta_0)^2 + \cdots\right), \tag{4.3}$$

$$v \approx C^{(2)}\left(\frac{1}{\eta - \eta_0} - \frac{1}{2}K^{(2)}[\,\mathcal{R}_c^{\rm NL}\,](\eta - \eta_0) + \cdots\right). \tag{4.4}$$

We consider u and v should be chosen as the growing mode and decaying mode, respectively, and u remains constant across the epoch $\eta = \eta_0$. The second term in (4.3) can be obtained by the integral $F(\eta)$, which in this case is given by

$$F(\eta) \propto \lim_{\eta \to \eta_0}(\eta - \eta_0)^2, \tag{4.5}$$

and shown to be still well defined in the crossing of $\dot{\phi} = 0$. The final value of the growing (or non-decaying) mode at late times will take a constant $\ell^{(0)}$ (i.e. δN term).

The singularity will appear in the first term in (4.4), arising from the integral $D(\eta)$. It can be expressed as in the case of linear theory and for $\eta > \eta_0$, we obtain as

$$D(\eta) \propto \int_\eta^0 \frac{d\eta'}{z^2} \approx \frac{1}{{z'_0}^2(\eta - \eta_0)}. \tag{4.6}$$

We conclude that this term diverges in the limit $\eta \to \eta_0 + 0$, however, this is just a decaying mode, then it will vanish definitely at late times as

$$D(\eta) \propto a^{-3} \to 0, \quad \text{with} \;\; \eta \to 0. \tag{4.7}$$

Hence we can see that no problem will occur for both the linear and nonlinear theory.

V. CONCLUDING REMARKS

We clarify what happens when the inflaton stops during inflation for nonlinear cosmological perturbation theory. We focus on the evolution on the superhorizon scales and review our nonlinear theory, called the *beyond* δN-formalism for a general single scalar field as the next-leading order in the gradient expansion. In our nonlinear theory, we can deal with the time evolution in contrast to δN-formalism where curvature perturbations remain just constant.

As a summary of our formula, note that time derivative takes the same form as shown in (2.1) and (2.2), hence the decaying mode can not couple with the growing mode as similar to the linear theory because of the method of gradient expansion, i.e. time derivative takes as a linear operator. The difference from the linear theory is the source term , i.e. the Ricci scalar of the leading-order metric $K^{(2)}$, which can be reduced to $k^2\mathcal{R}_c^{\rm Lin}$ in Fourier space as the source term in the linear theory (3.1).

We can conclude that although the decaying mode diverges in the limit of time when $\dot{\phi}$ vanishes, there appears no trouble for both the linear and nonlinear theory since this mode will vanish definitely at late times.

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