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Warped compactification to de Sitter space

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ABSTRACT: We explore in detail the prospects of obtaining a four-dimensional de Sitter universe in classical supergravity models with warped and time-independent extra dimensions, presenting explicit cosmological solutions of the $(4+n)$-dimensional Einstein equations with and without a bulk cosmological constant term. For the first time in the literature we show that there may exist a large class of warped supergravity models with a noncompact extra dimension which lead to a finite 4D Newton constant as well as a massless 4D graviton localised on an inflating four-dimensional FLRW universe. This result helps establish that the 'no-go' theorem forbidding acceleration in 'standard' compactification of string/M-theory on physically compact spaces should not apply to a general class of warped supergravity models that allows at least one noncompact direction. We present solutions for which the size of the radial dimension takes a constant value in the large volume limit, providing an explicit example of spontaneous compactification.

KEYWORDS: de Sitter universe, warped extra dimensions, accelerating solutions.

1. Introduction

Since Kaluza's pioneering work in the early 1920s [1] there have been a wide range of speculations about a fifth dimension and also higher dimensions of space. In order to avoid gross violations with everyday experience and numerous tabletop and collider experiments, it is thought that extra dimensions would have to be compact and tiny, or curled-up very tightly so that they have an extremely small radius: 10^{-30} cm or less, and also that their effects on present day experiments become unobservably small. However, this may not be necessarily the case if the background (spacetime) geometry is non-factorizable or warped, and the theory allows at least one noncompact extra dimension [2–5]. Recent studies, see, e.g. [6–8], show that cosmological models with a noncompact extra space are consistent with known physics, and can help solve some vexing problems in particle physics and cosmology, including the mass hierarchy and cosmological constant problems.

The possibility of constructing four dimensional matter-gauge theories from higher dimensional theories of gravity [9] has led to a variety of viable models of the early universe cosmology, including the possibility of explaining cosmic inflation using branes (or brane-antibrane pairs) [10]. This idea has inspired many theoretical physicists to explore and investigate models of de Sitter cosmology in the context of string/M-theory as well as in simplest braneworld models [11–13]. The KKLT model [14] is an example 'flux compactification' in string theory - roughly, a model building in which certain background fluxes are turned on and non-perturvative effects are invoked for the purpose of stabilizing the extra dimensional volume, which is otherwise left unfixed within the KKLT approach.

It is worthwhile to realise a state of de Sitter expansion in four dimensions as an explicit solution to Einstein field equations in higher dimensions, first without introducing any stringy corrections to Einstein gravity. This seems to be difficult from the viewpoint of the 'no-go" theorem discussed by Gibbons [15], De Wit et al. [16], Maldacena and Nunez [17] and many others [18]. The theorem basically asserts that if we dimensionally compactify any string-derived supergravity models on a smooth compact manifold $\mathcal{M}$, then we find a flat Minkowski or anti de Sitter spacetime as a background solution of classical supergravities in $4+n$ dimensions unless that we also violate certain positivity conditions.

In recent years, many authors have constructed varieties of time-dependent solutions. One of the motivations for this has been that the original no-go theorem required time independence of the internal space, so one could look for time-dependent solutions in higher dimensions [19–25]. Through many examples discussed in the literature, we have learned that higher dimensional theories with time-dependent metric moduli generally give rise to a transient acceleration (of the universe). Moreover, cosmological solutions with time-dependent metric scalars possess some kind of metric singularities, especially, when the extra dimensional manifold is hyperbolic or contains a subspace that is negatively curved. Additionally, cosmological solutions obtained by allowing time-dependent metric moduli possess a particular drawback that some of the 'fundamental constants' in nature, such as, Yukawa couplings, can vary with time, when the model is coupled with matter fields. This result is not encouraging since experimental and astrophycical bounds applicable to such variations place strong constraints on (or even rule out) these models.

In this paper we find interest in models of warped compactifications for which the extra dimensions are time-independent. As a viable alternative to standard Kaluza-Klein type compactifications with physically compact extra dimensions, we choose to 'compactify' a class of string-inspired supergravity models in $4+n$ dimensions by allowing a non-compact dimension plus a $(n-1)$-dimensional compact manifold. Here we focus our discussions on a class of explicit cosmological solutions which give rise to a positive cosmological constant in the usual four dimensions. We explore de Sitter solutions within a general class of warped supergravity for which the 4D Newton constant is finite and the 4D massless graviton wavefunction is renormalizable in an inflating brane or FLRW universe. This result is new and quite remarkable. The original no-go theorems [15–17] are about (physically) compact internal spaces and their associated subtleties, whereas the model studied here has a noncompact direction. Our solutions nevertheless give a finite 4D Newton constant and a concrete realisation of four-dimensional Einstein gravity on a de Sitter brane.

We also show that a cosmological model with an inflating Friedmann-Robertson-Walker universe embedded in a five-dimensional de Sitter space gives rise to a finite 4D Planck mass similar to that in the Randall-Sundrum type braneworld models in a static AdS_5 spacetime. In spacetime dimensions $D \equiv 4+n \geq 7$, however, we find that a negative bulk cosmological term may be preferred over a positive cosmological term. In the former ($\Lambda_b < 0$) case, the warp factor can be regular everywhere for any Λ_b, while, in the latter case ($\Lambda_b > 0$), one is required to satisfy the bound $\Lambda_b < 2/[3(n+3)H^2]$, where n is the number of extra dimensions and H is the four-dimensional Hubble expansion parameter.

2. De Sitter solutions in various dimensions

We find interest in $(4+n)$-dimensional warped metrics that maintain the usual four-dimensional Poincaré symmetry, with general metric parametrization:

$$ds_D^2 = e^{2A(y)}\, \hat{g}_{\mu\nu} dX^\mu dX^\nu + e^{2B(y)} g_{mn}(y)\, dy^m dy^n, \tag{2.1}$$

where X^μ are the usual spacetime coordinates ($\mu, \nu = 0, 1, 2, 3$) and, $A(y)$ and $B(y)$ are some functions of one of the internal coordinates, y. We denote by $M, N, \cdots$ the D-dimensional indices, by $\mu, \nu, \cdots$ the 4-dimensional indices and by $m, n, p, q, \cdots$ the n-dimensional indices. The internal space metric i.e. $ds_n^2 = g_{mn}(y) dy^m dy^n$ may be taken to be a general Einstein space, not just a constant curvature manifold.

2.1 The $D = 5$ case

The basic idea behind the existence of a four-dimensional de Sitter space solution (dS_4) supported by warping of extra spaces can be illustrated by considering a five-dimensional 'warped metric',

$$ds_D^2 = e^{2A(y)}\, \hat{g}_{\mu\nu} dX^\mu dX^\nu + e^{2B(y)} \rho^2\, dy^2 \tag{2.2}$$

and the 4D metric in the standard Friedmann-Lemaître-Robertson-Walker (FLRW) form

$$ds_4^2 \equiv \hat{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left[\frac{dX^2}{1-kX^2} + X^2(dY^2 + \sin^2 Y dZ^2)\right]. \tag{2.3}$$

Here the 3D curvature constant k is arbitrary *. It is not difficult to check that the metric Ansatz (2.2) becomes an exact solution of 5D Einstein equations, following from

$$S \propto \int d^5x\sqrt{-g}R, \tag{2.4}$$

when

$$B(y) = A(y) + \ln\frac{dA}{dy} - \ln c_1, \quad a(t) = \frac{1}{2}\left(c_0 \exp(c_1 t/\rho) + \frac{k\rho^2}{c_0}\exp(-c_1 t/\rho)\right). \tag{2.5}$$

The integration constant c_1 may be set to unity or absorbed in ρ. Further, c_0 may also be set to unity by using the freedom to rescale time, i.e. $t \to t - t_0$. This result shows that there can exist a large class of de Sitter solutions with different choices of $B(y)$. In the particular case that $B(y) = 0$, the solution is given by

$$e^{2A} = \frac{(y-y_0)^2}{y_1^2}. \tag{2.6}$$

Since the warp factor vanishes at $y = y_0$, this solution is not particularly interesting.

To obtain a completely regular solution, we have to make a suitable choice of the warp factor. For $A(y) = B(y)$, we can write the 5D metric as

$$ds_D^2 = e^{2A(y)}\left(\hat{g}_{\mu\nu}dX^\mu dX^\nu + \rho^2\beta^2\, dy^2\right), \tag{2.7}$$

where ρ is a compactification radius. The numerical constant β has been introduced just for convenience. The 5D Einstein equations are now explicitly solved when

$$e^{2A(y)} = e^{-2\beta(y+y_0)} \tag{2.8}$$

with the 4D scale factor as given above (cf eq.(2.5)). The slope of the warp factor is determined in terms of the constant β. In the above case, the warp factor diverges as $y \to -\infty$. As a result, the warped volume is not finite unless that one imposes a Z_2 symmetry along the $y = 0$ hypersurfaceor as in RS models.

To this end, we may introduce a bulk cosmological term into the 5D action

$$S = M_{(5)}^3 \int d^5x\sqrt{-g}\,(R - 2\Lambda_5), \tag{2.9}$$

where Λ_5 is the 5D bulk cosmological constant. With the metric Ansatz (2.7), the 5D Einstein equation are explicitly solved when

$$A(y) = \ln\left(\frac{b^2}{\rho^2}\right) - \frac{1}{2}\ln\left(\exp(\beta y) + \frac{\Lambda_5 b^4}{24\rho^2}\exp(-\beta y)\right)^2, \tag{2.10}$$

where b is a constant with length dimension of one, and the 4D scale factor is given by

$$a(t) = \frac{c_0^2 + k\rho^2}{2c_0}\cosh(t/\rho) + \frac{c_0^2 - k\rho^2}{2c_0}\sinh(t/\rho), \tag{2.11}$$

*Of course, one could choose to rescale k to 0 or ± 1, but in this case one would also have to rescale all other dimensionful quantities. There is no need to fix k in a particular way as long as one can solve the D-dimensional Einstein equations with an arbitrary k.

where c_0 is an integration constant. The constant k determines the asymptotic form of $a(t)$ in the usual way. In a spatially flat universe ($k = 0$), the scale factor grows as $a(t) \propto e^{(t/\rho)}$. In a non-flat universe ($k \neq 0$), the expansion is close to being de Sitter. In spacetime regions endowed with a strong gravitational potential, the curvature k is always non-vanishing, at least, in a local region. When one considers an infinitely large universe, with the Hubble radius $r \equiv cH_0^{-1} \sim 10^{27}$ cm, then $a(t) \simeq a_0\, e^{t/\rho}$. In fact, in the $k = 0$ case, with $a(t) \propto e^{Ht}$, the solution for the warp factor can be written in a slightly more general form

$$A(y) = \ln(bH) - \frac{1}{2}\left(e^{\rho H y} + \frac{\Lambda_5 b^2}{24}\, e^{-\rho H y}\right)^2, \tag{2.12}$$

where ρ and H are arbitrary constants.

Though the radial coordinate y is non-compact, because of a significant warping of the fifth dimension, we can get a finite 1d warped volume, especially, with $0 < \lambda < 1$. To be more precise, from the explicit solution given above above, we derive [26, 27])

$$\begin{aligned} M_{(5)}^3 \int d^5x \sqrt{g}\, R &= M_5^3\, \beta\rho \int e^{3A(y)} dy \int \sqrt{-g_4}\left(\hat{R}_4 - \frac{12A'^2}{\beta^2\rho^2} - \frac{8A''}{\beta^2\rho^2} - 2\Lambda_5 e^{2A}\right) \\ &= M_{\rm Pl}^2 \int d^4x \sqrt{-g_4}\, \hat{R}_4 - M_{(5)}^3\, \frac{\beta\, b^6}{\rho^5} \int d^4x \sqrt{-g_4} \int dy\, \Lambda(y), \end{aligned} \tag{2.13}$$

where

$$M_{\rm Pl}^2 = M_{(5)}^3\, \beta \frac{b^6}{\rho^5} \int_{-\infty}^{\infty} \frac{dy}{\left(\mathrm{e}^{\beta y} + \lambda\, \mathrm{e}^{-\beta y}\right)^3}, \quad \Lambda(y) \equiv \frac{4\left(3\, e^{2\beta y} + 3\lambda^2\, e^{-2\beta y} - 2\lambda\right)}{\rho^2\left(e^{\beta y} + \lambda\, e^{-\beta y}\right)^5}. \tag{2.14}$$

I. The $\lambda > 0$ case

In the $\lambda > 0$ (and hence $\Lambda_5 > 0$) case, we explicitly derive

$$M_{\rm Pl}^2 = M_{(5)}^3\, \frac{b^6}{\rho^5}\, \frac{\tan^{-1}\sqrt{\lambda} + \cot^{-1}\sqrt{\lambda}}{8\lambda^{3/2}}. \tag{2.15}$$

Since

$$\tanh^{-1}\sqrt{\lambda} + \cot^{-1}\sqrt{\lambda} = \frac{\pi}{2}, \quad \Lambda_5 \equiv \frac{6}{\ell_{\rm dS}^2}, \quad \lambda = \frac{\Lambda_5 b^4}{24\rho^2}, \tag{2.16}$$

the 4D effective Planck mass is simply given by

$$M_{\rm Pl}^2 = \frac{\pi}{2} M_{(5)}^3 \frac{1}{\rho^2}\left(\frac{6}{\Lambda_5}\right)^{3/2}. \tag{2.17}$$

The y coordinate can vary from $-\infty$ to $+\infty$ and the 5D spacetime is geodesically complete.

II. The $\lambda < 0$ case

In this case, the effective 4D Planck mass diverges when y is integrated from $-\infty$ to $+\infty$, independent of the sign of the coefficient β. To get a finite 4D Planck mass, $M_{\rm Pl}$, one may allow the y coordinate to range from 0 to $+\infty$ (for $\lambda > 0$) or from $-\infty$ to 0 (for $\beta < 0$). This is similar to that in simplest Randall-Sundrum braneworld models, where one considers only half of the AdS_5 space and replace the other half by its mirror image.

III. The effect of brane action

If required, one may supplement the 5D gravity action with brane action

$$S_{\rm brane} = \int_{\partial \mathcal{M}_1} \sqrt{-g_{b1}}\,(-\tau_{b1}) + \int_{\partial \mathcal{M}_2} \sqrt{-g_{b2}}\,(-\tau_{b2}), \tag{2.18}$$

where τ_{b1}, τ_{b2} denote the brane tensions corresponding to the two 3-branes $b1$ and $b2$, and then solve the 5D Einstein equations by placing two 3-branes at $y = 0$ and $y = \pi$. In such a configuration, while computing derivatives of $A(y)$, we have to consider the metric a periodic function in y. The solution valid for $-\pi \le y \le \pi$ then implies that

$$A'' + \frac{\beta^2 \rho^2 b^4 \Lambda_5}{6\,K_+^2} + \frac{\beta K_-}{K_+}\,(2\delta(y) - 2\delta(y-\pi)) = 0, \tag{2.19}$$

where

$$K_\pm \equiv e^{\beta\, y} \pm \frac{\Lambda_5 b^4}{24\rho^2}\, e^{-\beta\, y}. \tag{2.20}$$

From the $\mu\nu$-components of the 5D Einstein equations we get

$$\begin{aligned} \frac{3}{\beta^2\rho^2}\left(A'' + A'^2 - \beta^2\right) + \Lambda_5\, e^{2A} + \frac{e^A}{2\beta\rho M_5^3}\,(\tau_{b1}\delta(y) + \tau_{b2}\delta(y-\pi)) &= 0 \\ \Rightarrow A'' + \frac{\beta^2\rho^2 b^4 \Lambda_5}{6\,K_+^2} + \frac{\beta b^2}{6\rho M_5^3\, K_+}\,(\tau_{b1}\delta(y) + \tau_{b2}\delta(y-\pi)) &= 0. \end{aligned} \tag{2.21}$$

By comparing eqs. (2.19) and (2.21), we get

$$\tau_{b1} = \frac{3M_5^3\,\rho}{b^2}\,(1-\lambda)\,, \quad \tau_{b2} = -\frac{3M_5^3\,\rho}{b^2}\left(e^{\beta\,\pi} - \lambda\, e^{-\beta\,\pi}\right), \tag{2.22}$$

where

$$\lambda \equiv \frac{\Lambda_5 b^4}{24\rho^2}. \tag{2.23}$$

The solution found in [28] corresponds to the choice $\lambda = -1$ (so that $\Lambda_5 < 0$), while that in [29] corresponds to $\lambda = 1$. In both these papers ρ was set to unity. In the discussion below, we will take $\lambda > 0$ and also send the second brane to infinity, or simply replace the 3-brane with a physical four-dimensional FLRW universe. A detailed discussion on localisation of gravity in five dimensions appears elsewhere [30].

In the following we will not compute the brane tension explicitly, but we note that the corresponding solutions give rise to a positive brane tension when a p-brane with $3 \le p \le (D-2)$ is introduced along with the gravitational action. The location of the brane may be viewed as the place where the zero mode graviton wavefunction is peaked.

2.2 The $D = 6$ case

With two extra dimensions, we may write the metric in the form

$$ds_6^2 = e^{2A(y)}\,\hat{g}_{\mu\nu} dx^\mu dx^\nu + e^{2B(y)}\rho^2\left(e^{2C(y)} dy^2 + e^{2D(y)} d\theta^2\right), \tag{2.24}$$

where θ is a periodic coordinate, $0 \le \theta \le 2\pi$. The internal space becomes physically compact when $C(y) = 0$ and $D(y) = \ln \sin y$. But in this case the warp factor becomes singular at $y = 0$. We shall make some other choices that yield a regular warp factor.

It is not difficult to check that the metric Ansatz (2.24) becomes an exact solution of 6D Einstein equations, following from

$$S \propto \int d^6x \sqrt{-g} R, \tag{2.25}$$

for instance, when

$$e^{2A(y)} = \frac{h^2}{X^{2p}}, \quad e^{2B+2C} = h^2 \, \frac{p^2 X'^2}{X^{2p+2}}, \quad e^{2B+2D} = h^2, \tag{2.26}$$

where $X \equiv X(y)$, $X' = dX(y)/dy$, h and p are dimensionless constants, and

$$a(t) = \frac{c_0^2 + k\rho^2}{2c_0} \cosh(t/\rho) + \frac{c_0^2 - k\rho^2}{2c_0} \sinh(t/\rho). \tag{2.27}$$

The 3D spatial curvature constant has a dimension of $(\text{length})^{-2}$. As is evident, the scale factor is regular everywhere with $k = 0$ and $k > 0$. Especially, in the case $k < 0$ (i.e., in an open universe), and with the choice $k\rho^2 = -c_0^2$, we get $a(t) = \sinh(\sqrt{|k|}t/c_0)$.

For instance, with $X \equiv \cosh(My)$, the six-dimensional solution takes the form

$$ds_6^2 = \frac{h^2}{X^{2p}} \left(\hat{g}_{\mu\nu} dx^\mu dx^\mu dx^\nu + \rho^2 \left(\frac{p^2 M^2 (X^2 - 1)}{X^2} \, dy^2 + X^{2p} \, d\theta^2 \right) \right) \tag{2.28}$$

In this case the internal two-dimensional space, given by the metric

$$ds_2^2 = \rho^2 \left(p^2 M^2 \frac{X^2 - 1}{X^2} \, dy^2 + X^{2p} \, dz^2 \right) \tag{2.29}$$

is negatively curved, $R_2 \equiv R_{mn} g^{mn} = -2/\rho^2 < 0$. However, when we take into account the effect of warping and write the internal 2d metric as

$$\widetilde{ds}_2^2 = \frac{h^2 \rho^2}{X^{2p}} \left(p^2 M^2 \frac{X^2 - 1}{X^2} \, dy^2 + X^{2p} \, dz^2 \right), \tag{2.30}$$

then we find that the 2d warped space is Ricci flat and its volume is effectively finite.

To quantify this, we shall consider the following dimensionally reduced action

$$\begin{aligned} \int d^6x \sqrt{-g} \, R &= pM\rho^2 \, h^4 \int \frac{\sqrt{X^2 - 1}}{X^{3p+1}} \, dy \int_0^{2\pi} d\theta \int d^4x \sqrt{\hat{g}} \left(\hat{R}_4 - \frac{12}{\rho^2} \right) \\ &= 2\pi p \rho^2 \, h^4 V_2^{\rm w} \int d^4x \sqrt{\hat{g}_4} \left(\hat{R}_4 - 2\Lambda_4 \right), \end{aligned} \tag{2.31}$$

where

$$V_2^{\rm w} = \frac{1}{3p} + \frac{\sqrt{\pi}\,\Gamma[3p/2]}{2\Gamma[(3p+1)/2]} + \frac{2^{3p+1}}{2+3p} \, {}_2F_1 \left[\frac{3p+2}{2}, 3p+1, \frac{3p+4}{2}, -1 \right] \tag{2.32}$$

and $\Lambda_4 \equiv 6/\rho^2$. Here we allow p to take a rational number such that $n \equiv 3p+1$ is a positive integer, excluding $n = 0$ and 1. We then get $V_2^{\rm w} = 2/3p$ and hence

$$M_{(6)}^4 \int d^6x\sqrt{-g}\,R = M_{\rm Pl}^2 \int d^4x\sqrt{\hat{g}_4}\left(\hat{R}_4 - 2\Lambda\right), \tag{2.33}$$

where

$$M_{\rm Pl}^2 = M_{(6)}^4 \mathcal{R}^2, \qquad \mathcal{R} \sim \rho\sqrt{\frac{4\pi}{3}}h^2, \tag{2.34}$$

with $\mathcal{R}$ being the effective size of the extra dimensions.

In fact, in dimensions $D \geq 6$, we can find an even more general class of solutions than the ones presented above. In the $D = 6$ case, the solution may be written as

$$ds_6^2 = \frac{h^2}{F(y)}\left[-dt^2 + a(t)^2 d\vec{x}_{3,k}^2 + \rho^2\left(G(y)\,dy^2 + H(y)\,d\theta^2\right)\right], \tag{2.35}$$

with the scale factor as given in (2.27) and

$$F(y) = \left(X + \sigma\sqrt{X^2-1}\right)^{2p} = H(y), \quad G(y) = p^2M^2\left(\frac{\sqrt{X^2-1}+\sigma X}{X+\sigma\sqrt{X^2-1}}\right)^2, \tag{2.36}$$

where $X = \cosh(My)$ and σ is a numerical constant.

One should note that the $D = 5$ case is special for which the Einstein equations enforce one to take $\sigma = 1$, which leads to a diverging warp factor at $y = -\infty$ unless that one imposes Z_2 symmetry along $y = 0$ or introduces a positive bulk cosmological term. In dimensions $D \geq 6$, however, there exist no such restrictions, although for generality of the model, one may introduce a bulk cosmological term and/or p-form gauge fields (or fluxes).

2.3 The $D = 7$ case

In spacetime dimensions $D = 7$ (or $n = 3$), the metric ansatz can be written as

$$ds_7^2 = \frac{h^2}{F(y)}\left(\hat{g}_{\mu\nu}dx^\mu dx^\nu + \rho^2\Big(G(y)\,dy^2 + H(y)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big)\right). \tag{2.37}$$

This metric becomes an exact solution of 7D Einstein equations, following from

$$S \propto \int d^7x\sqrt{-g}\,R, \tag{2.38}$$

for instance, when

$$F(y) = \left(X + \sigma\sqrt{X^2-1}\right)^{2p}, \quad G(y) = \frac{5p^2M^2}{3}\left(\frac{\sqrt{X^2-1}+\sigma X}{X+\sigma\sqrt{X^2-1}}\right)^2, \quad H(y) = \frac{1}{3}, \tag{2.39}$$

where $X \equiv \cosh(My)$. The 4D scale factor is still given by (2.27). It is not difficult to check that the internal 3-space, given by the metric

$$ds_3^2 = \frac{\rho^2}{3}\left(\frac{5}{4}\frac{F'^2}{F^2}\,dy^2 + d\Omega_2^2\right), \tag{2.40}$$

is positively curved, $R_3 = 6/\rho^2$. When we take into account the warp factor, i.e.,

$$\widetilde{ds}_3^{\,2} = \frac{h^2}{F(y)}\, ds_3^2, \tag{2.41}$$

then also we find that the 3d curvature is positive, $\widetilde{R}_3 = \tilde{R}_{mn}\tilde{g}^{mn} = 24F(y)/(5\rho^2 h^4) > 0$.

From the explicit 7D solution given above, we derive

$$\begin{aligned} \int d^7x\sqrt{-g}\, R &= h^5\, \frac{2\pi\sqrt{5}pM\rho^3}{3\sqrt{3}} \int \frac{u'\, dy}{u^{5p+1}} \int d^4x\sqrt{\hat{g}_4}\left(\hat{R}_4 - \frac{12}{\rho^2}\right) \\ &= h^5\, 2\pi \frac{\sqrt{5}\,\rho^3}{15\sqrt{3}}\left(\frac{-1}{u^{5p}}\right) \int d^4x\sqrt{\hat{g}_4}\,\left(\hat{R}_4 - \frac{12}{\rho^2}\right), \end{aligned} \tag{2.42}$$

where

$$u(y) \equiv \cosh(My) + \sigma|\sinh(My)|. \tag{2.43}$$

The function $u(y)$ is symmetric along $y = y_0$ † and hence $\left[-1/u^{5p}\right]_{-\infty}^{\infty} \simeq 2$, provided that $p > 0$ and $1 + \sigma > 0$. Finally, under the dimensional reduction from 7 to 4, we find

$$M_{\rm Pl(7)}^5 \int d^7x\sqrt{-g}\, R = M_{\rm Pl}^2 \int d^4x\sqrt{\hat{g}_4}\left(\hat{R}_4 - 2\Lambda\right), \tag{2.44}$$

where $\Lambda_4 = 6/\rho^2$ and

$$M_{\rm Pl}^2 = M_{\rm Pl(7)}^5 \mathcal{R}^3, \qquad \mathcal{R} \sim \rho\left(\frac{2V_2\sqrt{5}\,h^5}{15\sqrt{3}}\right)^{1/3}. \tag{2.45}$$

The numerical constant h in the metric does not have to be set to unity; its value is rather fixed by the D-dimensional dilaton coupling and it can be much smaller than unity.

2.4 Generalisation to D-dimensions

Next we shall begin with a general $(4+n)$-dimensional 'warped metric', maintaining the usual four-dimensional Poincaré symmetry,

$$ds_D^2 = \frac{h^2}{F(y)}\left(\hat{g}_{\mu\nu}dx^\mu dx^\nu + \rho^2\left(G(y)dy^2 + E(y)\, d\Omega_{n-1}^2\right)\right). \tag{2.46}$$

Here and henceforth we shall take $n \geq 3$ ‡. This metric Ansatz becomes an exact solution of D-dimensional Einstein equations, following from

$$S = M_{(D)}^{D-2} \int d^Dx\sqrt{-g}\, R, \tag{2.47}$$

†The actual value of y_0 depends on the choice of σ. One has $u = u_{\rm min}$ and $f = f_{\rm max}$ at $y = y_0$.

‡To perform a consistent truncation in dimensions $D \equiv 4 + n \geq 7$, we can write the internal $(D-4)$-dimensional space as a general warped product of a non-compact direction and a compact Einstein space X of dimensions $(D-5)$. In this set up, one just has to find a physical way to limit the growth in the radial direction, leading to a finite warped volume.

when

$$G(y)=\frac{n+2}{12}\left(\frac{F'}{F}\right)^2, \qquad E(y)=\frac{(n-2)}{3}=\text{const}, \tag{2.48}$$

and

$$a(t)=\frac{c_0^2+k\rho^2}{2c_0}\cosh(t/\rho)+\frac{c_0^2-k\rho^2}{2c_0}\sinh(t/\rho). \tag{2.49}$$

As is evident, there can exist a large class of four-dimensional de Sitter solutions with different choices of $F(y)$. In view of this result, we may comfortably conclude that the usual 'no-go' theorem forbidding acceleration in 'standard' compactification of string/M-theory on physically compact spaces should not apply to a general class of warped supergravity models that allows a noncompact direction.

In the following discussion, we take [§]

$$F(y)=\left(X+\sigma\sqrt{X^2-1}\right)^{2p}, \qquad X\equiv\cosh(My), \tag{2.50}$$

where M is a constant with mass dimension and p is a numerical constant. Hence

$$G(y)=\frac{(n+2)p^2M^2}{3}\,\frac{\sqrt{X^2-1}+\sigma X}{X+\sigma\sqrt{X^2-1}}. \tag{2.51}$$

The result reported in [33] is obtained by taking $n=6$, and setting $M=1$. We can also also obtain the result found in [34] by taking $n=6$, $\sigma=0$ and choosing the coefficient p appropriately, but $p<0$. In this last case, however, the six-dimensional warped volume is not finite unless one also introduces some elements of the Randall-Sundrum type braneworld models and/or introduces an ultraviolet (UV) cutoff in the y-direction.

In the particular case that $\sigma=-1$, the function $F(y)\to 0$ as $y\to\infty$. We will actually discard this choice since in this case the warp volume can be infinitely large. To get a physical model, we shall take $1+\sigma>0$ and also take $p>0$. This choice helps us to get a finite 4D Planck mass as well as a normalisable zero mode graviton wavefunction.

From the explicit solution given above, we derive

$$\begin{aligned}\int d^{4+n}x\sqrt{-g}\,R &= h^{2+n}\rho^n\,pM\sqrt{\frac{(n+2)(n-2)^{(n-1)}}{3^n}}\\ &\quad\times V_{n-1}\times I(y)\times\int d^4x\sqrt{\hat{g}_4}\left(\hat{R}_4-\frac{12}{\rho^2}\right),\end{aligned} \tag{2.52}$$

where V_{n-1} is the volume of S^{n-1} sphere, $\Lambda_4\equiv 6/\rho^2$ and

$$I(y)\equiv\int_0^\infty\frac{\sqrt{\cosh^2(My)-1}+\sigma\cosh(My)}{\left(\cosh(My)+\sigma\sqrt{\cosh^2(My)-1}\right)^{p(n+2)+1}}\,dy\simeq\frac{1}{p(n+2)M}. \tag{2.53}$$

[§]This form of warp factor can be supported, for instance, by a D-dimensional bulk scalar field, with a nonzero scalar potential, see, for example, refs. [31, 32].

Upon the dimensional reduction, we get

$$\int d^{4+n}x\sqrt{-g}\,R = h^{2+n}\rho^n\sqrt{\frac{(n-2)^{(n-1)}}{(n+2)\,3^n}}\,V_{n-1}\int d^4x\sqrt{\hat{g}_4}\,\left(\hat{R}_4 - 2\Lambda_4\right). \tag{2.54}$$

Equivalently,

$$M_{\rm Pl\,(4+n)}^{2+n}\int d^{4+n}x\sqrt{-g}\,R = M_{\rm Pl}^2\int d^4x\sqrt{\hat{g}_4}\left(\hat{R}_4 - 2\Lambda_4\right), \tag{2.55}$$

where

$$M_{\rm Pl}^2 = M_{{\rm Pl},(4+n)}^{n+2}\rho^n\,V_{(n-1)}\,h^{n+2}\sqrt{\frac{(n-2)^{(n-1)}}{(n+2)3^n}}. \tag{2.56}$$

For instance, with $n = 3$ (i.e. $D = 7$), this yields

$$M_{\rm Pl}^2 = \frac{2\pi}{3\sqrt{15}}\,M_{\rm Pl(7)}^5\,\rho^3 h^5. \tag{2.57}$$

This may be further analysed by taking $\rho \sim 10^{27}$ cm, especially, if one wants to tune Λ_4 to the present value of 4D cosmological constant. One then has to allow h to take an extremely small value, $h \lesssim 10^{-21} \ll 1$, so that the 7D Planck mass $M_{\rm Pl(7)} \gtrsim$ TeV. A constraint like this becomes weaker in the presence of a bulk cosmological term and/or (supergravity) background fluxes, and also when one applies the model to explain the early universe inflation. For instance, with $\rho^{-1} \sim 10^{-5}\, M_{\rm Pl}$, we get $M_{\rm Pl} \sim 10^{-3} M_{\rm Pl(7)} h$.

In [35], Gibbons and Hull explored the possibility of constructing a model of de Sitter universe within a context of 10 and 11 dimensional supergravity, by allowing a noncompact extra dimension. The authors, however, did not find a physical model of 4D de Sitter cosmology that leads to a finite 4D Planck mass. Here we have achieved this goal.

2.5 Effects of a bulk cosmological term

The explicit solutions given above can easily be generalised so as to include the effect of a bulk cosmological term. For generality, here we shall keep the number of extra dimensions n unspecified (except that $n \geq 3$) and write the action as

$$S = M_{\rm Pl(n+4)}^{2+n}\int d^{4+n}x\left(R - 2\Lambda_b\right). \tag{2.58}$$

The $(4+n)$-dimensional metric ansatz may be written in the form

$$ds_{4+n}^2 = \frac{h^2}{F(y)}\left(-dt^2 + a(t)^2 d\vec{x}_{3,k}^2 + G(y)dy^2 + E(y)\,d\Omega_{n-1}^2\right). \tag{2.59}$$

The Einstein field equations following from the action (2.58) are explicitly solved when

$$a(t) = \frac{1}{2}e^{H(t-t_0)} + \frac{k}{2H^2}e^{-H(t-t_0)}, \tag{2.60}$$

where t_0 is a constant, and

$$G(y)=\frac{(n+2)(n+3)F'(y)^2}{12(n+3)H^2F(y)^2-8\Lambda_b F(y)},\quad E(y)=\frac{(n-2)}{3H^2}=\text{const}, \tag{2.61}$$

where $F'=dF(y)/dy$. In dimensions $D\geq 7$ (or $n\geq 3$), it looks suggestive to take a negative bulk cosmological term, $\Lambda_b<0$, in which case $G(y)$ is regular everywhere.

In the discussion below, we simplify the model by taking $F\propto(\cosh My)^{2p}\equiv X^{2p}$ ($p>0$). In dimensions $D=7$, the metric solution then takes the form

$$ds_7^2=\frac{h^2}{X^{2p}}\left(-dt^2+a(t)^2d\vec{x}_{3,k}^2+\frac{15p^2M^2(X^2-1)}{9H^2X^2-\Lambda_b X^{2(1-p)}}\,dy^2+\frac{1}{3H^2}\,d\Omega_2^2\right), \tag{2.62}$$

which is regular everywhere with $\Lambda_b<9H^2$. In the $\Lambda_b<0$ case, the ratio $(-\Lambda_b)/9H^2$ can take any value between 0_+ and ∞. In either case, the size of the fifth dimension stabilizes in the large y limit, implying that the model has a feature of spontaneous stabilization.

The difference we now have as compared to the result (2.52) is that $\rho=1/H$ and

$$I(y)=\int_{y_0}^{y_\infty}\frac{\sqrt{X^2-1}}{X^{(n+1)p+1}\sqrt{X^{2p}+\mathcal{B}}}\,dy,\qquad \mathcal{B}\equiv\frac{|-2\Lambda_b|}{3(n+3)H^2}, \tag{2.63}$$

where $X=\cosh(My)$. For brevity, let us make the simplest choice that $p=1$. Hence

$$I(y)=\frac{1}{16M\mathcal{B}^{5/2}}\left[2\mathcal{B}Y\,\mathrm{sech}^2(My)\left(3-2\mathcal{B}\,\mathrm{sech}^2(My)\right)-6\tanh^{-1}Y\right]_{y_0}^{y_\infty}, \tag{2.64}$$

where $Y\equiv\sqrt{(1+2\mathcal{B}+\cosh(My))/2\mathcal{B}}$. Without loss of generality, one may take $y_0=0$ and $y_\infty=\infty$, and consider two specific cases: $\mathcal{B}\ll 1$ and $\mathcal{B}\gg 1$. In fact, the condition $|\mathcal{B}|\ll 1$ is a reasonable approximation in the early universe, while, in the present universe, or in a large cosmological scale, e.g. $cH^{-1}\sim 10^{27}$ cm, one can have $\mathcal{B}\gg 1$ ¶ and hence

$$I\simeq\frac{H}{(\Lambda_b)^{1/2}\,M},\qquad M_{\rm Pl}^2\simeq\frac{h^5M_{\rm Pl,7}^5}{H^2\sqrt{|\Lambda_b|}}\times\frac{2\pi\sqrt{5}}{3\sqrt{3}}. \tag{2.65}$$

That is, in a cosmological background, unlike in the simplest braneworld models in a static AdS_5 background, the 4D Plank mass may well depend on the observed value of the Hubble parameter H or the 4D cosmological constant, other than on the fundamental $(4+n)$-dimensional Planck mass $M_{\rm Pl,4+n}$ and the bulk cosmological term Λ_b.

A couple of remarks are in order. From the above discussion, it should be clear that $F(y)$ can be a completely general function of y, especially, for the background D-dimensional Einstein equations. However, when one supplements a higher dimensional supergravity action with one or more external sources, such as, D-dimensional scalar field, p-form gauge fields or fluxes, then the warp factor might take a particular form. We will not elaborate here on which form of the warp factor corresponds to (or supported by) what sort of bulk source(s), but only consider a few more specific examples, such as,

$$F(y)\propto\frac{\sinh^2(My)}{M^2y^2},\quad F(y)\propto(y^2+b^2),\quad F(y)=\left(c_0+c_1e^{M|y|}\right)^{c_2},$$

where M, b and c_i are integration constants, for which the warp factor is regular everywhere.

¶In terms of the bulk curvature radius $\ell\equiv\sqrt{42/(-\Lambda_b)}$, the condition $\mathcal{B}\gg 1$ implies $cH^{-1}\gg\ell$.

3. Linearized gravity

Higher dimensional theories with one or more non-compact extra dimensions require the trapping of gravitational degrees of freedom on the brane, or a physical 4D universe. To determine whether the spectrum of linearized tensor fluctuations δg_{AB} is consistent with four-dimensional experimental gravity, we shall consider the perturbations around the background solutions given above [‖]. For brevity, we take $k = 0$ so that the usual 3D space is spatially flat, and hence $a(t) \propto e^{Ht}$.

In a general D dimensional spacetime (with $D = n + 4 \geq 7$), the transverse-traceless tensor modes $h_{ij} \equiv \delta g_{ij} = \delta_i^A \delta_j^B h_{AB}(x^\lambda, y)$ satisfy the following wave equation

$$\left(\partial_y^2 - \frac{(n+2)F'}{2F}\partial_y - \frac{G'}{2G}\partial_y - G(y)\left(\partial_t^2 + 3\frac{\dot{a}}{a}\partial_t - a^{-2}\vec{\nabla}^2\right)\right) h_{ij} = 0. \tag{3.1}$$

In the presence of a brane source or a localised object at $y = y_0$, there can appear delta function term(s) on the right hand side of this equation. These are easily computed once the brane action is known, but their explicit forms are not important for our analysis.

We begin with a canonical model with $n = 3$ (or $D = 7$), for which

$$\left(\partial_y^2 - \frac{5F'}{2F}\partial_y - \frac{G'}{2G}\partial_y - G(y)\left(\partial_t^2 + 3H\partial_t - e^{2Ht}\vec{\nabla}^2\right)\right) h_{ij} = 0. \tag{3.2}$$

Indeed, the $n = 3$ is the case with a minimal number of extra dimensions that allows the two-dimensional compact manifold to have positive, negative or zero curvature. For simplicity, we have taken the 2-dimensional compact space to be a usual two-sphere, S^2. By separating the variables as

$$h_{ij}(x^\mu, y) \equiv \sum \alpha_m(t)\psi_m(y)\, e^{ik\cdot x}\,\hat{e}_{ij}, \tag{3.3}$$

where $e_{ij}(x^i)$ is a transverse, tracefree harmonics on the spatially flat 3-space, $\vec{\nabla}^2\hat{e}_{ij} = -k^2\hat{e}_{ij}$, we get

$$\frac{d^2\psi_m}{dy^2} - \left(\frac{5F'}{2F} + \frac{G'}{2G}\right)\frac{d\psi_m}{dy} + m^2\psi_m = 0, \tag{3.4}$$

$$\ddot{\alpha}_m + 3\frac{\dot{a}}{a}\dot{\alpha}_m + \left(\frac{k^2}{a^2} + m^2\right)\alpha_m = 0, \tag{3.5}$$

where m is a 4D mass parameter and k is the co-moving wavenumber along the 4D hypersurface.

3.1 Tensor perturbation equation with $\Lambda_b = 0$

In the $\Lambda_b = 0$ case, the background equations are explicitly solved for

$$G(y) = \frac{15F'^2}{36H^2F^2} \qquad (D = 7). \tag{3.6}$$

[‖] We refer to [36] for the analysis of metric perturbations in higher dimensions with a nontrivial warp factor.

The off-brane wave equation for tensor fluctuations, i.e. (3.4), reduces to

$$\frac{d^2\psi_m}{dy^2} - \left(\frac{3F'}{2F} + \frac{F''}{F'}\right)\frac{d\psi_m}{dy} + m^2\psi_m = 0. \tag{3.7}$$

By defining

$$\psi_m(y) \propto F(y)^{3/4}\sqrt{\frac{dF(y)}{dy}}\,\Psi_m(y), \tag{3.8}$$

we can bring eq. (3.7) into the standard Schrödinger form

$$\frac{d^2\Psi_m}{dy^2} - V(y)\,\Psi_m = -m^2\Psi_m, \tag{3.9}$$

where

$$V(y) = \frac{21}{16}\left(\frac{F'}{F}\right)^2 + \frac{3}{4}\left(\frac{F''}{F'}\right)^2 - \frac{1}{2}\frac{F'''}{F'}. \tag{3.10}$$

Below we consider a few explicit examples for which $F(y)$ is regular everywhere.

Take

$$F(y) = (\cosh(My))^{2p} \equiv X^{2p} \tag{3.11}$$

with $p > 0$. The off-brane graviton wave equation takes the form

$$\frac{d^2\psi_m}{dy^2} - M\left((5p-1)\frac{\sqrt{X^2-1}}{X} + \frac{X}{\sqrt{X^2-1}}\right)\frac{d\psi_m}{dy} + m^2\psi_m = 0. \tag{3.12}$$

By defining

$$\psi_m(y) = \left(\frac{X^2-1}{X^2}\right)^{1/4} X^{5p/2}\,\Psi_m(y), \tag{3.13}$$

we can bring eq. (3.12) into the standard Schrödinger form (3.9) with

$$V(y) = \frac{25p^2M^2}{4}\left(\frac{\cosh^2(My)}{\sinh^2(My)} + \frac{4}{\sinh^2(2My)} - \frac{2}{\sinh^2(My)}\right) + \frac{M^2}{\sinh^2(My)}\left(1 - \frac{1}{4\cosh^2(My)}\right). \tag{3.14}$$

Note that $V(y) \to 25p^2M^2/4$ as $My \to \infty$ (or simply when $My \gg 1$). There is a mass gap, which depends on the size of fifth dimension via p and also on the mass parameter M. The massless zero mode wave function is given by

$$\Psi_0(y) = c_1\,\frac{(\cosh(My))^{(1-5p)/2}}{\left(\cosh^2(My)-1\right)^{1/4}}., \tag{3.15}$$

which is normalizable with $p > 0$ and $0 \lesssim \epsilon < |My| < \infty$. Of course, in the presence of a δ-function type brane source at $y = y_c$, there would arise an addition term in (3.14) proportional to $\delta(y - y_c)$. In such a case we need satisfy the following boundary condition

$$\partial_y\Psi_0(y_c) = -\left(\frac{3}{4}\frac{F'}{F} + \frac{1}{2}\frac{F''}{F'}\right)\Psi_0(y_c). \tag{3.16}$$

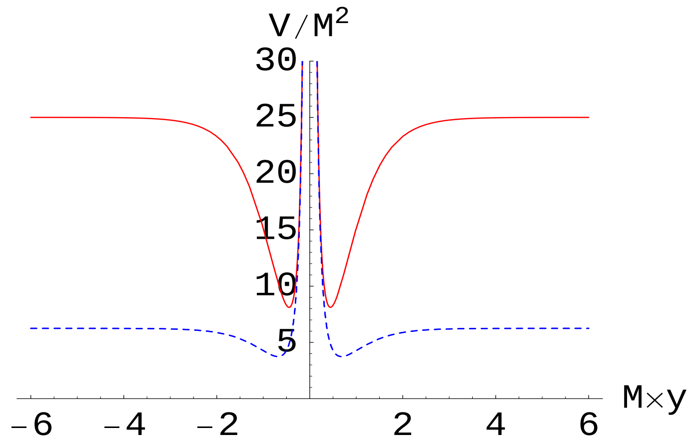

Figure 1: The plot of $V(y)/M^2$ with $p = 2$ (red, solid line) and $p = 1$ (blue, dotted line).

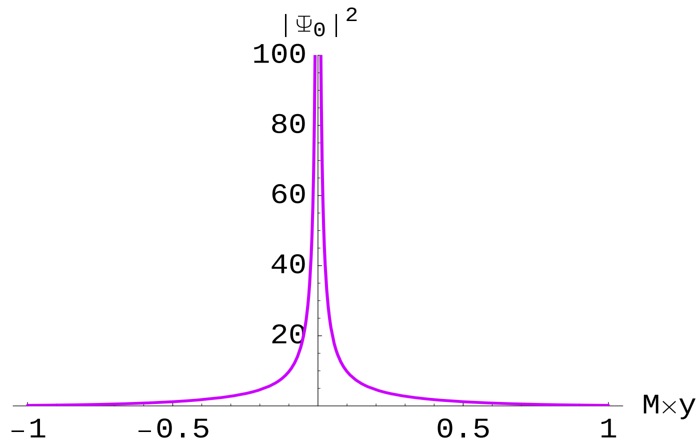

Figure 2: The plot of the function $|\Psi_0|^2$ with $N_0 = 1$ and $p = 1$.

The precise discrete modes are given in terms of the hypergeometric functions, and are a linear combination of

$$X^{c_-}\left(X^2-1\right)^{3/4}\ {}_2F_1\left(1+b_+,1+b_-,c_+,X^2\right)$$

and

$$X^{c_+}\left(X^2-1\right)^{3/4}\ {}_2F_1\left(1-b_-,1-b_+,c_-,X^2\right),$$

where

$$b_\pm = \frac{5p}{4} \pm \sqrt{\frac{25p^2}{16} - \frac{m^2}{4M^2}}, \quad c_\pm = \frac{2 \pm 5p}{2}, \quad X = \cosh(My).$$

Note that, unlike in five dimensions, the mass of KK modes is determined in terms of an arbitrary parameter M (rather than the Hubble parameter H). All massive modes with mass $m \geq 5pM/2$ behave as oscillating plane waves at infinity ($My \to \infty$), which are delocalised KK modes. There is only one light mode with $m = 2pM$, but this mode is non-renormalisable and hence it cannot be localised on a de Sitter brane.

Further, if we take

$$F(y) = \frac{y^2 + by + c^2}{L^2}, \tag{3.17}$$

where c is a resolution parameter, then the zero-mode wavefunction is given by

$$\Psi_0(y) \propto \frac{F^{-3/4}}{\sqrt{dF/dy}} = \frac{L^5\,(y^2+by+c^2)^{-3/4}}{(b+2y)^{1/2}}, \tag{3.18}$$

which is normalizable, provided that $-b/2 < y < \infty$. For brevity, we may set $b = 0$ and hence restrict the y coordinate in the range $0 \lesssim \epsilon < y < \infty$. Similarly, with

$$F(y) \equiv \frac{\sinh^2(My)}{M^2y^2}, \tag{3.19}$$

the zero mode wavefunction is given by

$$\Psi_0(y) = N_0 \frac{M^{5/2}y^3}{|\sinh(My)|^{3/2}\sqrt{\sinh(My)\,(My\cosh(My) - \sinh(My))}}. \tag{3.20}$$

which is normalisable, provided that $0 \lesssim \epsilon \le y < \infty$.

Finally, we consider a particular example with

$$F(y) = \left(f_0\, e^{2My} + f_1\right)^{2/5}, \tag{3.21}$$

where $0 \le y < \infty$, for which the tensor perturbation equation greatly simplifies, i.e.

$$\frac{d^2\psi_m}{dy^2} - 2M\frac{d\psi_m}{dy} + m^2\psi_m = 0. \tag{3.22}$$

By defining $\psi_m \equiv e^{My}\,u_m(y)$, it can be brought into a standard Schrödinger form

$$\frac{d^2u_m(y)}{dy^2} - (M^2 - m^2)u_m(y) = 0. \tag{3.23}$$

The massless zero-mode graviton wavefunction is given by

$$u_0(y) = c_1\, e^{-My}, \tag{3.24}$$

which is clearly normalisable when $0 < y < \infty$. All massive KK modes with mass $m > M$ are delocalised and behave as oscillatory plane waves.

In summary, we have found a class of physically viable models of accelerating universe for which the 4D Newton constant is finite and the 4D massless graviton wavefunction is normalisable. This is quite remarkable result and it is the first of such kind in the literature that gives an explicit realisation of 4D Einstein gravity. The results above can easily be generalised in higher dimensions, including 10 and 11 dimensional supergravity, with or without a bulk cosmological term.

3.2 Tensor perturbation equation with $\Lambda_b < 0$

With $|\Lambda_b| \neq 0$, the perturbation equation can in principle be analysed by making a generic choice of the warp factor. Here we shall simplify the model by taking

$$F(y) \equiv e^{My}, \quad G(y) = \frac{15M^2 F(y)}{36H^2 F(y) - 4\Lambda_b} \quad (n = 3)$$

and restricting the y coordinate in the range $0 \leq y < \infty$. By defining

$$\psi_m(y) \equiv \frac{e^{3My/2}}{(9H^2 F - \Lambda_b)^{1/4}} \Psi_m(y),$$

we can bring eq. (3.4) in the standard Schrödinger equation (3.9), with

$$V(y) = \frac{M^2}{16} \frac{\left(2025H^4F^2 - 576F\Lambda_b + 36\Lambda_b^2\right)}{(9H^2F - \Lambda_b)^2}. \tag{3.25}$$

In the limit $My \gg 1$, $V(y) \to \frac{25M^2}{16}$. The mass gap now depends on the value of M alone.

The massless zero-mode wavefunction, which is given by

$$\Psi_0(y) = N_0\, e^{-3My/2} \left(9H^2 e^{My} - \Lambda_b\right)^{1/4}, \tag{3.26}$$

where N_0 is the normalisation constant, is renormalisable. In the $\Lambda_b = 0$ case, we have

$$\int |\Psi_0|^2 = N_0^2 \frac{6H}{5M} \left[1 - e^{-\frac{5}{2}My_\infty}\right]. \tag{3.27}$$

In the the $\Lambda_b < 0$ case, we define $\mathcal{B} \equiv \left(-\Lambda_b\right)/9H^2$. We then find

$$\int_0^\infty |\Psi_0|^2 \simeq \frac{N_0^2 H}{8Mc^2}\Big[(2\mathcal{B} - 1)\sqrt{\mathcal{B}+1}(4\mathcal{B}+3) + \frac{3}{\sqrt{\mathcal{B}}} \ln\left(\sqrt{\mathcal{B}+1} + \sqrt{\mathcal{B}}\right)\Big]. \tag{3.28}$$

In the limit $\mathcal{B} \to 0$, this yields

$$\int |\Psi_0|^2 = \frac{N_0^2 H}{8M}\left(\frac{48}{5} + \frac{24\mathcal{B}}{7} - \frac{2\mathcal{B}^2}{3} + \mathcal{O}(\mathcal{B}^3)\right) \simeq \frac{6N_0^2 H}{5M}.$$

With $\mathcal{B} \gg 1$, we have $\int |\Psi_0|^2 \simeq N_0^2 H\sqrt{\mathcal{B}}/M = N_0^2(-\Lambda_b)^{1/2}/(3M)$. The amplitude of zero-mode wavefunction is suppressed when the mass-like parameter M is large.

The discrete KK modes are given in terms of the Legendre functions, i.e.,

$$\Psi_m = N_m\sqrt{3H}\left(e^{My} + c\right)^{1/4}\left[P_\nu^n(z) + c_1 Q_\nu^n(z)\right]. \tag{3.29}$$

where N_m is the normalisation constant and

$$n \equiv \sqrt{\frac{25}{4} - \frac{4m^2}{M^2}} - \frac{1}{2}, \quad \nu \equiv \sqrt{9 - \frac{4m^2}{M^2}}, \quad z \equiv \sqrt{1 + \frac{e^{My}}{\mathcal{B}}}.$$

For n, ν integers and z real, the Legendre function of the first kind simplifies to a polynomial, called the Legendre polynomial. This is the case only with $m = 0$.

All heavy modes with $m > 3M/2$ become oscillating plane waves, which represent the de-localised KK massive gravitons. The time-evolution of the mode functions of these heavy modes shows that they remain underdamped at late times ($t \to \infty$).

4. Conclusion

In the standard Kaluza-Klein approach to string theory compactification, one aims to connect the observed four-dimensional world with the 10-dimensional physics that arises naturally in string theory, by assuming that six of the ten-dimensions are very tiny and compact. In this context, there are no-go theorems, as discussed in [17, 18], setting the constraints that a given background has to satisfy in order to allow for solutions where the 4D spacetime is Minkowski or de Sitter. If fluxes are present in the solution, then some negative sources, like orientifold planes, must be present, or one should allow for metric singularities. That means, the problem of finding four-dimensional de Sitter solutions is well posed, if all extra dimensions are physically compact.

In string theory, the extra dimensions of spacetime are conjectured to take the form of a 6-dimensional CalabiYau manifold. Calabi-Yau manifolds are sometimes defined as compact Kähler manifolds whose canonical bundle is trivial. However, in general, Calabi-Yau spaces are noncompact and they are also known to allow at least one noncompact direction, admitting a warped throat geometry or a conifold type singularity.

One can study higher-dimensional supergravity theories by allowing a noncompact dimension. This is a perfectly viable option. In this paper we have addressed the problem of finding de Sitter solutions in the usual four dimensions, by studying a general class of warped compactifications that allows one of the extra spaces to be large in size. We have found explicit cosmological solutions of the $(4+n)$-dimensional Einstein equations with and without a bulk cosmological term, which could lead to a finite 4D Newton constant as well as a massless 4D graviton localised in an inflating four-dimensional FLRW universe. A remarkable feature of the solutions presented in this paper is also that the size of the radial dimension takes a constant value in the large volume limit, providing an explicit example of spontaneous compactification. Our examples look similar to that in Randall-Sundrum braneworld models, but they are presented in a cosmological context.

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