

The Penrose Inequality and the Fluid/Gravity Correspondence

Yaron Oz and Michael Rabinovich

Raymond and Beverly Sackler School of Physics and Astronomy,

Tel-Aviv University, Tel-Aviv 69978, Israel

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Abstract

Motivated by the fluid/gravity correspondence, we consider the Penrose inequality in the framework of fluid dynamics. In general relativity, the Penrose inequality relates the mass and the entropy associated with a gravitational background. If the inequality is violated by some Cauchy data, it suggests a creation of a naked singularity, thus providing means to consider the cosmic censorship hypothesis. The analogous inequality in the context of fluid dynamics can provide a valuable tool in the study of finite-time blowups in hydrodynamics. We derive the inequality for relativistic and nonrelativistic fluid flows in general dimension. We show that the inequality is always satisfied at the ideal fluid order. At the leading viscous order, the inequality may be violated by relativistic fluid flows, while it is always satisfied by nonrelativistic incompressible flows. The inequality may be violated at the next to leading viscous order by both relativistic and nonrelativistic flows.

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I. INTRODUCTION

In this paper we will consider the issue of possible singularities in fluid dynamics. Compressible hydrodynamics allows singularities in the form of shock waves, that is a jump in the velocity vector field. Incompressible hydrodynamics does not exhibit shock waves, and by singularities one means blowups of the derivatives of the velocity vector field. It is often expected that the viscosity terms act as regulators that prevent the blowups.

The basic question concerning singularities in the hydrodynamic description, is whether starting with appropriate initial conditions, where the velocity vector field and its derivatives

are bounded, can the system evolve such that it will exhibit within a finite time a blowup of the derivatives of the vector field. Mathematically, this important question has been presented as one of the millennium problems posed by the Clay Mathematics Institute [1]. Physically, such singularities if present, indicate a breakdown of the effective hydrodynamic description at long distances and imply that some new degrees of freedom are required.

The issue of hydrodynamic singularities has an analogue in gravity. Given an appropriate Cauchy data, will the evolving space-time geometry exhibit a naked singularity, i.e. a blowup of curvature invariants and the energy density of matter fields at a point not covered by a horizon. The cosmic censorship conjecture in general relativity (for a review see [2]), states in its weak form that any space-time singularity is generically enclosed by an event horizon, that is a region that no light rays can escape from. Mathematically, naked singularities can exist, but physical considerations of causality seem to require that it should not be visible to distant observers. In its strong form, the conjecture states that even inside the black hole a falling observer will never "see" the singularity, i.e. time-like singularities never occur generically.

The Penrose inequality [3] (for a review see e.g. [4]) is a conjecture relating the mass and the horizon area of any Cauchy initial data that if violated leads to a space-time naked singularity. Let us briefly review Penrose's argument in asymptotically flat four-dimensional space-time. Consider a Cauchy data (\mathcal{M}^3, g, K) , where \mathcal{M}^3 is a 3-manifold with a complete Riemannian metric g and an extrinsic curvature K , satisfying the Gauss-Codazzi equations. In particular, let an asymptotically flat initial Cauchy data with a mass M_0 and event horizon area A_{H0} evolve with time. The solution to Einstein equations with this initial data is expected to settle down at long time to a Kerr black hole solution with mass M and horizon area A_H , where the inequality (we use $c = G_N = 1$)

$$M \geq \sqrt{A_H/16\pi} \tag{1}$$

holds. The inequality is saturated by the Schwarzschild black hole.

By the Hawking area theorem [5], the event horizon area does not decrease with time $A_H \geq A_{H0}$. The mass, on the other hand, cannot increase and may only decrease due to radiation loss $M \leq M_0$. Thus, we get the chain of inequalities

$$M_0 \geq M \geq \sqrt{A_H/16\pi} \geq \sqrt{A_{H0}/16\pi} , \tag{2}$$

and the initial Cauchy data should also satisfy the Penrose inequality $M_0 \geq \sqrt{A_{H0}/16\pi}$.

The argument relies on the Hawking area theorem and the relaxation at late times to a Kerr solution, both assume the weak censorship hypothesis. Therefore, finding Cauchy data that violates the Penrose inequality implies finding a solution to Einstein equations in which a naked singularity is created. Proving that the inequality is satisfied does not prove the cosmic censorship conjecture but provides an important support for it. Note, that there are other less likely ways to interpret a violation of the inequality, e.g. that the system never reaches equilibrium.

In fact, we can formulate an analog of the Penrose inequality for any dynamical system. Consider a system, where the total energy does not increase. We replace the area increase theorem by the second law of thermodynamics, that is the total entropy never decreases. The system is assumed to relax at late time to a thermodynamical equilibrium state. The inequality relation between the mass and the entropy is simply an equilibrium relation, while by repeating Penrose's argument, the inequality should hold away from equilibrium.

The fluid/gravity correspondence relates in a derivative expansion two non-linear systems, the fluid dynamics described by the Navier-Stokes equations and gravity described by general relativity. In one particular (conformal) setup of the correspondence [6], the Navier-Stokes equations in d space-time dimensions are identified with the constraint Einstein equations of a deformed black brane gravitational background in one higher dimension. Alternatively, they are identified with the Gauss-Codazzi equations governing the evolution of the black brane event horizon [7–9]. Motivated by the fluid/gravity correspondence, we consider the Penrose inequality in the framework of fluid dynamics, which can provide a valuable tool in the study of finite time blowups in hydrodynamics. We derive the inequality for relativistic and nonrelativistic fluid flows in general dimension and study its implications.

The paper is organized as follows. In section II we briefly review the fluid/gravity correspondence in the conformal case. We then derive the inequality for relativistic ideal hydrodynamics, where we consider both the conformal and non-conformal cases. We show that the inequality is always satisfied at the ideal fluid order, and present a class of solutions that develop a shock wave singularity, while still satisfying the inequality. This is curious, though of course there is no inconsistency, since a non-violation of the inequality is only a sufficient condition for not having a finite-time blowup and not a necessary one. In section III we derive the inequality at the first and second viscous order for relativistic fluids. The inequality is nontrivial and maybe be violated. In section IV we show that the inequality is

always satisfied by nonrelativistic incompressible fluids. However, this is not the case when considering subleading corrections to the incompressible Navier-Stokes equations. Section V is devoted to a discussion.

II. THE PENROSE INEQUALITY IN HYDRODYNAMICS

In this section we will formulate the Penrose inequality for fluid dynamics. We will use the fluid/gravity correspondence to motivate the inequality. However, the actual derivation does not require the use of the correspondence.

A. The fluid/gravity correspondence: conformal case

Consider the vacuum $(d+1)$ -dimensional Einstein equations with a negative cosmological constant

$$R_{MN} + dg_{MN} = 0, \quad R = -d(d+1) \quad (M, N = 0, \dots, d). \quad (3)$$

We will use the notation $X^M = (x^\mu, \mu = 0, \dots, d-1, x^d = r)$. One solution of these equations is the boosted black brane with AdS_{d+1} asymptotics

$$ds_0^2 = -2u_\mu dx^\mu dr + \frac{1}{b^d r^{d-2}} u_\mu u_\nu dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu, \quad (4)$$

with the d -dimensional velocity vector $u^\mu = (\gamma, \gamma v_i)$, $u_\mu u^\mu = -1$, b related to the Hawking temperature by

$$b = \frac{d}{4\pi T}, \quad (5)$$

and $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$ is the d -dimensional Minkowski metric.

The metric (4) is an exact solution of the equations (3) when b and u_μ are constant. This is the thermal equilibrium state. Next, we get out of equilibrium by deforming the solution and allowing $b(x^\mu)$, $u_\mu(x^\mu)$. With these, the metric (4) is no longer a solution of the equations (3).

One can, however, correct the metric (4) by adding derivatives of $b(x^\mu)$, $u_\mu(x^\mu)$. We will have in general

$$ds^2 = ds_0^2 + ds_1^2 + ds_2^2 \dots \quad (6)$$

where the lower index of ds^2 denotes the order of the derivative terms that it contains and ds_0^2 is the metric (4). Requiring the deformed metric (6) to be a solution of (3) up to the neglected order in derivatives, is equivalent to the order by order derivative expansion of conformal hydrodynamics [6]. More specifically, the set of $d+1$ constraint Einstein equations can be written as the conservation law of a conformal hydrodynamic stress energy tensor in flat d -dimensional space with specific transport coefficients

$$\partial_\mu T^{\mu\nu} = 0 . \quad (7)$$

Thus, the gravity solution provides a dual description in terms of fluid dynamics. Equivalently, one arrives at the fluid/gravity correspondence starting from the AdS/CFT correspondence, and considering the hydrodynamics of the CFT and its gravitational dual description. Alternatively, the hydrodynamic equations are identified with the Gauss-Codazzi equations governing the evolution of the black brane null event horizon [7–9]. The various schemes are related by an RG flow in the radial direction [10].

The hydrodynamic stress energy tensor $T_{\mu\nu}$ can also be calculated at the d -dimensional boundary [11]. We write the $(d+1)$ -dimensional metric in an ADM-like decomposition

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr) , \quad (8)$$

where $\gamma_{\nu\mu}$ is the boundary ($r = \text{const.}$) metric, N and N^μ are the lapse and shift functions. The extrinsic curvature tensor of the boundary ∂M is defined by

$$K_{\mu\nu} = -\frac{1}{2N}(\partial_r \gamma_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu) , \quad (9)$$

and the boundary stress energy tensor reads

$$T_{\mu\nu} = \lim_{r \rightarrow \infty} \left[\frac{r^{d-2}}{64\pi^2} (K_{\mu\nu} - K \gamma_{\mu\nu} - (d-1)\gamma_{\mu\nu}) \right] . \quad (10)$$

B. The Penrose inequality in conformal hydrodynamics

The hydrodynamic stress energy tensor of an ideal relativistic neutral fluid is

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p\eta_{\mu\nu} , \quad (11)$$

where ϵ is the energy density and p is the pressure. Conformal invariance implies that the stress energy tensor is traceless, thus we have the equation of state $\epsilon = (d-1)p$. The

only dimensionfull parameter is the temperature T and we have that $\epsilon = (d - 1)p \sim T^d$. The boundary stress tensor (10) corresponding to the stationary metric (4) is that of ideal conformal neutral fluid and reads [12, 13]

$$T_{\mu\nu} = \frac{1}{16\pi b^d}(\eta_{\mu\nu} + du_\mu u_\nu) . \quad (12)$$

Let us derive the analog of the Penrose inequality in this case. To this end we need to define the energy (mass), the horizon area and the equilibrium relation. The energy ($E = M$) is given by

$$E = \int T_{00} d^{d-1}x = \frac{d\gamma^2 - 1}{16\pi b^d} \int d^{d-1}x , \quad (13)$$

while the horizon area is defined via the entropy S . Consider a conformal fluid at rest ($u_\mu = (1, 0, 0, 0)$) satisfying the thermodynamic relation

$$\epsilon + p = T s , \quad (14)$$

where s is entropy density, which in the conformal case reads

$$s = \frac{\epsilon}{T} \frac{d}{d-1} = \frac{1}{4b^{d-1}} , \quad (15)$$

and the horizon area is

$$A = 4S = \int \frac{1}{b^{d-1}} d^{d-1}x , \quad (16)$$

so we get that the equilibrium relation between energy and entropy

$$E = \frac{d-1}{16\pi} (4S)^{\frac{d}{d-1}} . \quad (17)$$

The fluid analog of the Penrose inequality is therefore

$$E \geq \frac{d-1}{16\pi} (4S)^{\frac{d}{d-1}} . \quad (18)$$

Consider as an example the case when the fluid is not at rest. The inequality is slightly generalized in this case by replacing the entropy density s by the time component of entropy current $s^\mu = s u^\mu$ [14], that is $s^0 = s\gamma$. For any $\gamma \geq 1$ we have

$$\frac{d\gamma^2 - 1}{d-1} \geq \gamma^{\frac{d}{d-1}} . \quad (19)$$

Thus, by Hölder's inequality we get

$$\int \frac{d\gamma^2 - 1}{b^d(d-1)} \geq \int \frac{\gamma^{\frac{d}{d-1}}}{b^d} \geq \left(\int \frac{\gamma}{b^{d-1}} \right)^{\frac{d}{d-1}}. \quad (20)$$

The inequality (18) holds for any $\gamma \geq 1$, and is an equality only when $\gamma = 1$, that is for a fluid at rest.

Note, that although motivated by the fluid/gravity correspondence the argument that leads to the inequality for fluids is rather general and is independent of having a gravitational description. It assumes that at late times the system relaxes to an equilibrium state, whose thermodynamic relation gives the inequality relation. One then assumes that mass (energy) is a non-increasing, that is no energy is pumped into the system, and that the entropy is non-decreasing by the second law of thermodynamics. Thus, starting with initial data (E_0, S_0) one has

$$E_0 \geq E \geq \frac{d-1}{16\pi} (4S)^{\frac{d}{d-1}} \geq \frac{d-1}{16\pi} (4S_0)^{\frac{d}{d-1}}. \quad (21)$$

The Penrose inequality is a condition to be imposed on any initial spacetime configuration (Cauchy data). When this condition fails, the evolution in time is expected to exhibit a naked singularity. Likewise, The inequality (18) is a condition that we impose on any initial fluid data. If the condition fails, we expect the fluid to exhibit a finite-time blowup of the velocity field gradients. Note, however, that it is quiet important not to link directly singularities in the hydrodynamic description to naked singularities in gravity. The reason being that the fluid/gravity correspondence is established order by order in a derivatives expansion. An appearance of singularities signals the breakdown of a derivative expansion and for a direct link between the two descriptions one needs to go beyond the established relation.

C. Nonconformal fluids

We can easily generalize the inequality to non-conformal fluids with a general equation of state. As an example consider an equation of state of the form $E = \sigma p$ where $\sigma \neq d-1$. An example of a dual gravitational background for such an equation of state was constructed in [15]. A study of the universal statistics of such hydrodynamics in $d = 2$ has been carried

out in [16]. By dimensional analysis $p = aT^d$ and

$$\epsilon = \sigma aT^d, \quad s = \frac{\epsilon + p}{T} = (\sigma + 1)aT^{d-1}. \quad (22)$$

The relation between the energy and entropy reads

$$E = \frac{\sigma}{(\sigma + 1)^{\frac{d}{d-1}} a^{\frac{1}{d-1}}} S^{\frac{d}{d-1}}, \quad (23)$$

and the inequality in this case is

$$E \geq \frac{\sigma}{(\sigma + 1)^{\frac{d}{d-1}} a^{\frac{1}{d-1}}} S^{\frac{d}{d-1}}. \quad (24)$$

The inequality (18) is a special case of (24) with $\sigma = d - 1$ and $a = \frac{1}{16\pi} \left(\frac{4\pi T}{d}\right)^d$.

D. Simple relativistic waves

A violation of the Penrose inequality is sufficient condition for a creation of a naked singularity, but is not a necessary one. Likewise, the hydrodynamic inequality is only a sufficient condition for a singularity. In this subsection we give an example for an initial relativistic conformal ideal fluid flow configuration leading to a formation of a singularity, despite satisfying the Penrose inequality.

The hydrodynamic equations for a relativistic conformal ideal fluid can be written in the form

$$\partial_\mu u^\mu + (d - 1)D \ln T = 0, \quad a_\sigma + P_\sigma^\mu \partial_\mu \ln T = 0, \quad (25)$$

where $D = u^\mu \partial_\mu$, $a_\sigma = u^\mu \partial_\mu u_\sigma$. One obtains this form of the equations by the projection of the stress energy conservation equations as $u_\nu \partial_\mu T^{\mu\nu}$ and $P_\nu^\sigma \partial_\mu T^{\mu\nu}$, with

$$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu. \quad (26)$$

Consider the simple relativistic waves presented in [17] (here we generalize to d space-time dimensions)

$$u^\mu = (\gamma(t, z), 0, \dots, 0, \gamma v(t, z)), \quad v(t, z) = f\left(z - \frac{v + c_s}{1 + v c_s} t\right), \quad b = \left(\frac{1 + v}{1 - v}\right)^{\frac{c_s}{2}}, \quad (27)$$

where b is related to the temperature T as in (5), f is an arbitrary function specifying the initial wave profile, and c_s is the speed of sound

$$c_s = \left(\frac{\partial p}{\partial \epsilon} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{d-1}} . \quad (28)$$

This configuration is a solution to equations (25), and develops a singularity for any regular initial conditions function $f(z)$. The inequality (18) is satisfied by the same argument as (20), and gives no indication of the singularity that will develop in the future.

III. THE VISCOUS ORDER

A. first order in derivatives

The correction to the relativistic hydrodynamic ideal stress-energy tensor (11) at the first viscous order reads

$$T_{\mu\nu}^{(1)} = -2\eta\sigma_{\mu\nu} - \zeta\partial_\alpha u^\alpha(\eta_{\mu\nu} + u_\mu u_\nu) , \quad (29)$$

where we work in the Landau frame $u^\mu T_{\mu\nu}^{(1)} = 0$. $\sigma_{\mu\nu}$ is the shear tensor

$$\sigma_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu + \partial_\nu u_\mu) + \frac{1}{2}(u_\mu u^\alpha \partial_\alpha u_\nu + u_\nu u^\alpha \partial_\alpha u_\mu) - \frac{1}{d-1}\partial_\alpha u^\alpha(\eta_{\mu\nu} + u_\mu u_\nu) . \quad (30)$$

$\eta \geq 0$ and $\zeta \geq 0$ are the shear and bulk viscosities, respectively. In the conformal case, $T_\mu^\mu = 0$ implies that $\zeta = 0$. For the black brane gravitational background, the stress-energy tensor is

$$T_{\mu\nu} = \frac{1}{16\pi b^d}(\eta_{\mu\nu} + du_\mu u_\nu) - \frac{1}{8\pi b^{d-1}}\sigma_{\mu\nu} , \quad (31)$$

and the ratio of the shear viscosity to the entropy density is $\frac{\eta}{s} = \frac{1}{4\pi}$. In the following we consider the conformal case.

The energy decreases due to shear effects and reads

$$E = \int \left(\frac{d\gamma^2 - 1}{16\pi b^d} - 2\eta\sigma_{00} \right) d^{d-1}x , \quad (32)$$

where the first and second terms are the ideal and viscous contributions, respectively. The decrease in the energy is written explicitly

$$- \int 2\eta\sigma_{00} d^{d-1}x = -2 \int \eta \left(\partial_t \gamma + \gamma(-\gamma \partial_t + v^i \partial_i) \gamma - \frac{\gamma^2 - 1}{d-1}(-\partial_t \gamma + \partial_i v^i) \right) d^{d-1}x . \quad (33)$$

The entropy, on the other hand, does not change at this order, but is no longer conserved $\partial_\mu s^\mu \sim \frac{\eta}{T} \sigma_{\mu\nu} \sigma^{\mu\nu}$. On the gravitational side, the horizon area of the deformed black brane is not corrected at the first viscous order [6].

Thus, at the first viscous order the inequality reads

$$\begin{aligned} & \int \frac{d\gamma^2 - 1}{16\pi b^d} d^{d-1}x - 2 \int \eta \left(\partial_t \gamma + \gamma(-\gamma \partial_t + v^i \partial_i) \gamma - \frac{\gamma^2 - 1}{d-1} (-\partial_t \gamma + \partial_i v^i) \right) d^{d-1}x \\ & \geq \frac{d-1}{16\pi} \left(\int \frac{\gamma}{b^{d-1}} d^{d-1}x \right)^{\frac{d}{d-1}}, \end{aligned} \quad (34)$$

and may be violated. Shock waves regularized by the viscosity term of the type discussed in [18], may be possible candidates for violating the inequality (34).

Note, that a fluid flow configuration that violates (34) will also imply by the fluid/gravity correspondence that the deformed black brane background at first order in the derivatives will exhibit a naked singularity. This implies on the gravity side a need for the inclusion of the higher order derivative terms, possibly an infinite number of them.

B. Bjorken flow

As a simple example one can check that the Bjorken flow (for its description see e.g. [19]) satisfies the inequality at the viscous order. The Bjorken flow is a one-dimensional expansion of plasma with boost invariance symmetry along the expansion axis. It is convenient to parametrize the flow by the proper time τ and the rapidity y related to the lab frame coordinates by

$$x^0 = \tau \text{Cosh}[y], \quad x^1 = \tau \text{Sinh}[y]. \quad (35)$$

In equilibrium the energy density and the entropy density are $\epsilon = \epsilon_0 T^d$ and $s = \frac{d}{d-1} \epsilon_0 T^{d-1}$. The inequality can be read from (24) with $\sigma = d-1$ and $a = \frac{\epsilon_0}{d-1}$. The solution of the hydrodynamic equations is obtained as an expansion in $\tau^{-\alpha}$, $\alpha = \frac{d-2}{d-1}$. It is straightforward to check that the inequality is satisfied at the subleading order in this expansion, which captures the first order viscous corrections.

C. Second order in derivatives

Consider next the inequality at the second viscous order. It is straightforward to consider the most general stress-energy tensor at this order. However for simplicity we consider the

particular form of the stress-energy tensor corresponding to the deformed black brane gravitational background, that is a conformal hydrodynamics with a particular set of transport coefficients. The energy reads

$$E = \int \frac{1}{16\pi b^d} (d\gamma^2 - 1 - 2b\sigma_{00} - 2b\tau_\omega(u^\lambda D_\lambda \sigma_{00} + 2\omega_0^\lambda \sigma_{\lambda 0}) + 2b^2(u^\lambda D_\lambda \sigma_{00} + \sigma_0^\lambda \sigma_{\lambda 0} - (\gamma^2 - 1)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1})) d^{d-1}x . \quad (36)$$

τ_ω is a new transport coefficient that appears at this order. $\omega_{\mu\nu}$ in the vorticity tensor

$$\omega_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu) + \frac{1}{2}(u_\mu u^\alpha \partial_\alpha u_\nu - u_\nu u^\alpha \partial_\alpha u_\mu) , \quad (37)$$

and D_μ is the Weyl-covariant derivative that acts on the velocity vector as

$$D_\mu u^\nu = \sigma_\mu^\nu + \omega_\mu^\nu . \quad (38)$$

Then the entropy current at this order reads [6]

$$S = \int \frac{1}{4b^{d-1}} [\gamma + b^2\gamma(A_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2\omega_{\alpha\beta}\omega^{\alpha\beta} + A_3R) + b^2(B_1D_\lambda\sigma^{0\lambda} + B_2D_\lambda\omega^{0\lambda})] d^{d-1}x , \quad (39)$$

where

$$R = 2(d-1)\partial_\mu A^\mu - (d-2)(d-1)A_\mu A^\mu , \quad (40)$$

with

$$A_\mu = u^\alpha \partial_\alpha u_\mu - \frac{\partial_\alpha u^\alpha}{d-1} u_\mu . \quad (41)$$

The coefficients A 's and B 's are defined in [6] (Eq. (2.8) and Eq. (5.16)).

The inequality at second order in derivatives is

$$\begin{aligned} & \int \frac{1}{16\pi b^d} (d\gamma^2 - 1 - 2b\sigma_{00} - 2b\tau_\omega(u^\lambda D_\lambda \sigma_{00} + 2\omega_0^\lambda \sigma_{\lambda 0}) \\ & + 2b^2(u^\lambda D_\lambda \sigma_{00} + \sigma_0^\lambda \sigma_{\lambda 0} - (\gamma^2 - 1)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1})) d^{d-1}x \geq \\ & \frac{d-1}{16\pi} \left(\int \frac{1}{b^{d-1}} [\gamma + b^2\gamma(A_1\sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2\omega_{\alpha\beta}\omega^{\alpha\beta} + A_3R) \right. \\ & \left. + b^2(B_1D_\lambda\sigma^{0\lambda} + B_2D_\lambda\omega^{0\lambda})] d^{d-1}x \right)^{\frac{d}{d-1}} . \end{aligned} \quad (42)$$

IV. NONRELATIVISTIC FLOWS

A. Navier-Stokes equations

The incompressible Navier-Stokes equations can be obtained in the nonrelativistic limit of relativistic hydrodynamics, where the fluid flow velocity $v \ll c$. Specifically we can consider relativistic CFT hydrodynamics with the stress-energy tensor (31). The speed of sound in relativistic CFT hydrodynamics $v_s = \frac{c}{\sqrt{d-1}}$ and thus, we are considering in the nonrelativistic limit also the low mach limit of velocities much smaller than the speed of sound. This is the reason for obtaining incompressible flows.

The conservation law equations of the stress-energy tensor (31), projected on u_ν and P_ν^σ read

$$\begin{aligned} \partial_\mu u^\mu + (d-1)D \ln T &= \frac{1}{2\pi T} \sigma_{\mu\nu} \sigma^{\mu\nu} , \\ a_\sigma + P_\sigma^\mu \partial_\mu \ln T &= \frac{1}{2\pi T} P_\sigma^\mu (\partial_\alpha \sigma_\mu^\alpha - (d-1)\sigma_\mu^\alpha a_\alpha) . \end{aligned} \quad (43)$$

In the nonrelativistic we expand $u^\mu = (1 - v^2/2 + \dots, v^i)$, $T = T_0(1 + P + \dots)$ where we scale $v^i \sim \varepsilon$, $\partial_i \sim \varepsilon$, $\partial_t \sim \varepsilon^2$, $P \sim \varepsilon^2$, where $\varepsilon \sim 1/c \sim 1/v_s$. The first equation in (43) gives the incompressibility condition $\partial_i v^i = 0$, while the second equation gives [20, 21]

$$\partial_t v^i + v^j \partial_j v^i = -\partial^i P + \nu \Delta v^i , \quad (44)$$

where $\nu = \frac{1}{4\pi T_0}$.

B. First order inequality

The NS equations (44) are of order ε^3 . The inequality at the first viscous order is of order ε^2 . Let us derive it. We use $b = b_0(1 - P)$, $b_0 = \frac{d}{4\pi T_0} = d\nu$. The energy up to order ε^2 reads

$$E = \frac{d-1}{16\pi b_0^d} \int \left(1 + \frac{d}{d-1} v^2 + dP \right) d^{d-1}x , \quad (45)$$

where we put $c = 1$. The entropy up to order ε^2 reads

$$4S = \frac{1}{b_0^{d-1}} \int \left(1 + \frac{v^2}{2} + (d-1)P \right) d^{d-1}x . \quad (46)$$

It is easy to see that to this order the inequality is simply

$$\int v^2 d^{d-1}x \geq 0 , \quad (47)$$

and is always satisfied. This result can also be seen by noting that the viscous order correction to the energy (33) is of order ε^3 . Alternatively, we might have anticipated the above result on general ground. At order ε^2 the only scalar quantities that we can build are v^2 , P and $\partial_i v_i$. The pressure P appears in the NS equations via a derivative, thus, we can expect that it will not appear by itself, while $\partial_i v_i = 0$.

A non-violation of the inequality is only a necessary condition for not having finite-time blowups, still is it curious that it holds so simply for all incompressible flows. On the gravity side, we may expect that the gravitational backgrounds dual to the incompressible Navier-Stokes equations will not develop a naked singularity. Note also in comparison, that for the relativistic flows which are compressible the inequality is not obviously satisfied. This presumably suggests that singularities are more likely to be expected in compressible fluid motions.

Finite-time blowups of three-dimensional nonrelativistic incompressible flows have been demonstrated with pure imaginary vector fields in [22]. It is amusing that in these cases the Penrose inequality is violated, unless we consider the absolute value of the flow field.

C. Second order viscous flows

Let us consider the first subleading correction in the $1/c$ expansion to the inequality of the incompressible case, which is now at order ε^4 . The fluid flows are now compressible. For simplicity we will analyze the black brane hydrodynamics and use (42). The energy reads

$$E = \frac{d-1}{16\pi b_0^d} \int \left(1 + \frac{dv^2}{d-1} + dP + \frac{dv^4}{d-1} + \frac{d^2 P v^2}{d-1} + \frac{(d+1)b_0}{(d-1)^2} v^2 \partial_i v^i \right) d^{d-1}x . \quad (48)$$

Note that at this order we cannot use the incompressibility condition and the last term does not vanish. The new transport coefficient τ_ω does not appear at this order.

The entropy in the zeroth order of derivatives is

$$S = \frac{1}{4b_0^{d-1}} \int \left(1 + \frac{v^2}{2} + (d-1)P + \frac{3v^4}{8} + \frac{(d-1)Pv^2}{2} \right) d^{d-1}x . \quad (49)$$

The entropy is corrected at second order in derivatives (39)

$$\int \frac{1}{4b_0^{d-3}} (A_1 \sigma_{ij} \sigma^{ij} + A_2 \omega_{ij} \omega^{ij} + A_3 R + B_1 D_i \sigma^{0i} + B_2 D_i \omega^{0i}) d^{d-1}x . \quad (50)$$

The inequality reads

$$\int \frac{5v^4}{8} + \frac{dPv^2}{2} + \frac{b_0(d+1)}{d} v^2 \partial_i v^i d^{d-1}x \geq b_0^2 \int \left(\frac{A_1 + A_2}{2} \right) \partial_i v_j \partial^i v^j + \left(\frac{A_1 - A_2}{2} \right) \partial_i v_j \partial^j v^i - \frac{A_1}{2(d-1)} (\partial_i v^i)^2 d^{d-1}x, \quad (51)$$

and is not obviously satisfied.

V. DISCUSSION

In this paper we derived an inequality, analogous to the Penrose inequality, for relativistic and nonrelativistic fluid flows in a general dimension. The inequality provides means to study singular flows. As in gravity, a non-violation of the inequality is only a necessary condition for the absence of singularities. Indeed, we saw that the inequality is always satisfied at the ideal fluid order, and in particular is not violated by relativistic shock waves.

At the leading viscous order, the inequality may be violated by relativistic fluid flows, A construction of an example that violates the inequality or a proof that it is always satisfied is still an open problem. We saw that the inequality at this order is always satisfied by nonrelativistic incompressible flows. Hence, on the gravity side, we may expect that the gravitational backgrounds dual to the incompressible Navier-Stokes equations will not develop a naked singularity. The inequality may be violated at the next to leading viscous order by both relativistic and nonrelativistic compressible flows. This presumably suggests that singularities are more likely to be expected in compressible fluid motions.

The derivation of the inequality is rather general and can be applied to any closed dynamical system that relaxes to equilibrium at late times. In the context of fluid dynamics it can be easily applied also to charged hydrodynamics. An interesting arena for a future study of the inequality is time dependent nonequilibrium relativistic dynamics. In this case the inequality may depend crucially on the definition of the entropy, which is not uniquely defined away from equilibrium (see e.g. [23]). Finally, there are other interesting cases where one can consider the Penrose inequality, e.g. black hole horizons with more complicated topology.

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