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De Sitter brane-world, localization of gravity, and the cosmological constant

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Cosmological models with a de Sitter 3-brane embedded in a five-dimensional de Sitter spacetime (dS_5) give rise to a finite 4D Planck mass similar to that in Randall-Sundrum (RS) brane-world models in AdS_5 spacetime. Yet there arise a few important differences as compared to the results with a flat 3-brane or 4D Minkowski spacetime. For example, the mass reduction formula (MRF) $M_{\rm Pl}^2 = M_{(5)}^3 \ell_{\rm AdS}$ as well as the relationship $M_{\rm Pl}^2 = M_{{\rm Pl}(4+n)}^{n+2} L^n$ (with L being the average size or the radius of the n extra dimensions) expected in models of product-space (or Kaluza-Klein) compactifications get modified in cosmological backgrounds. In an expanding universe, a physically relevant MRF encodes information upon the four-dimensional Hubble expansion parameter, in addition to the length and mass parameters L, $M_{\rm Pl}$ and $M_{{\rm Pl}(4+n)}$. If a bulk cosmological constant is present in the solution, then the reduction formula is further modified. With these new insights, we show that the localization of a massless 4D graviton as well as the mass hierarchy between $M_{\rm Pl}$ and $M_{{\rm Pl}(4+n)}$ can be explained in cosmological brane-world models. A notable advantage of having a 5D de Sitter bulk is that in this case the zero-mass wavefunction is normalizable, which is not necessarily the case if the bulk spacetime is anti de Sitter. In spacetime dimensions $D \geq 7$, however, the bulk cosmological constant Λ_b can take either sign ($\Lambda_b < 0$, $= 0$, or > 0). The $D = 6$ case is rather inconclusive, in which case Λ_b may be introduced together with 2-form gauge field (or flux).

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I. INTRODUCTION

The universe endows with a number of cosmological mysteries, but the one that most vex physicists is the smallness of the observed vacuum energy density in the present universe and its effects on an accelerated expansion of the universe at a late epoch [1]. This cosmological enigma has so far defied an elegant and forthright explanation.

Brane-worlds are promising theories with extra spatial dimensions in which ordinary matter is localized on a (3+1) dimensional subspace [2]. To this end, the Randall-Sundrum (RS) models in five dimensions [3, 4] could be viewed as the simplest brane-world configurations with one extra dimension of space. The RS models and their generalizations in higher spacetime dimensions are known to have interesting consequences for gravitational physics [5–8] and cosmology, see, e.g. [9–12]; we refer to [13] for review and further references.

In the simplest Randall-Sundrum brane-world models, one has a flat 3-brane (or a 4D Minkowski spacetime) embedded in a five-dimensional anti-de Sitter spacetime, known as an AdS_5 bulk. In this simple setting there exist a massless graviton (or zero-mode) and massive gravitons (or Kaluza-Klein modes) of metric tensor fluctuations. The massless graviton mode reproduces the standard Newtonian gravity on the 3-brane, while the Kaluza-Klein modes, which arise as the effect of graviton fluctuations in extra dimension(s), give corrections to the Newton's force law [6]. The 5D bulk geometry is extremely warped in these models, as is reflected from a typical size of the 5D curvature radius, $\ell_{\rm AdS} < 0.1$ mm. Consequently the Newtonian gravity is recovered at distances larger than $\mathcal{O}(0.1$ mm$)$.

The requirement of an AdS_5 bulk spacetime in the original RS brane-world models may *not* be something that is totally unexpected since certain versions of 10D string theory, particularly, type IIB string theory, is known to contain AdS_5 space as a sub-background space of the full spacetime, which is $AdS_5 \times S^5$, and string theory itself is viewed as the most promising candidate for the unified theory of everything. However, the original RS models also predict a zero cosmological constant on the brane or the 4D spacetime. This result is not supported by recent cosmological observations, which favour a positive cosmological constant-like term in four dimensions.

To construct a natural theory of brane-world, we shall replace the flat 3-brane of the original RS setup by a dynamical brane or a physical $3 + 1$ dimensional hypersurface with a nonzero Hubble expansion parameter, for instance, by a Friedmann-Lamaître-Robertson-Walker metric. With such a simple modification of the original RS brane-world model, the zero mode graviton fluctuation is not guaranteed to be localised on the brane, if the 5D bulk spacetime is anti-de Sitter. However, if the 5D bulk spacetime is de Sitter or positively curved, then there always exists a normalizable zero mode graviton localised on a de Sitter brane. In such theories the smallness of the 4D cosmological constant term can be related to an infinitely large extension of the fifth dimension.

There is another motivation for considering a positively curved 5D background spacetime. When we consider compactifications of string/M theory or classical supergravity theories with more than one extra dimensions, then in a cosmological setting, and under the dimensional reduction from D dimensions to five, we generally find that the 5D spacetime is de Sitter, if we also insist on the existence of a four-dimensional de Sitter solution.

Even though AdS_5 is well motivated from some aspects of type IIB supergravity, for its role in the AdS/CFT correspondence, it is difficult to realise an AdS_5 background, while at the same time we also obtain a dS_4 solution (or an inflating FRW universe in four dimensions) by solving the full D-dimensional Einstein equations.

In this paper we show how brane-world models with a positively curved bulk spacetime (dS_5) can generate a four-dimensional cosmological constant in the gravity sector of the effective 4D theory with a finite 4D Newtons constant and also help explain the localization of a normalizable zero mode graviton in four dimensions. We also present some new insights on localization of gravity on a de Sitter brane embedded in a higher dimensional bulk spacetime with a nonzero bulk cosmological term, Λ_b. In dimensions $D \geq 7$, we find that Λ_b term can take either sign ($< 0, = 0$ or > 0), though a negative Λ_b may be preferred over a positive Λ_b for regularity of the metric. Our approach is motivated also from the fact that models with an expanding universe embedded in a higher-dimensional bulk spacetime naturally take into account the effects of self-gravity on the 3-brane.

The standard mass reduction formula (MRF) $M_{\rm Pl}^2 = M_{{\rm Pl}(4+n)}^{n+2} L^n$ relates the four-dimensional effective Planck mass $M_{\rm Pl}$ with the $(4+n)$-dimensional Planck mass $M_{{\rm Pl}(4+n)}$ (with L being the average size of the n extra dimensions). Here we show that this result, also known as Gauss formula, gets naturally modified in the presence of a bulk cosmological term and also due a nonzero Hubble expansion parameter in four dimensions.

II. DE SITTER BRANE-WORLDS

Five-dimensional de Sitter brane-worlds characterized by a single extra dimension where the bulk space-time is positively curved (instead of being flat or negative curved) are among some highly plausible approaches to explaining the smallness of the observed cosmological vacuum energy density and localization gravity.

The basic idea behind the existence of a four-dimensional de Sitter space solution (dS_4) supported by warping of extra spaces can be illustrated by considering a curved five-dimensional 'warped metric',

$$ds_5^2 = e^{2A(\phi)} \left(ds_4^2 + \rho^2 d\phi^2\right), \qquad (1)$$

where ρ is a free parameter with dimension of length and $e^{2A(\phi)}$ is the warp factor as a function of ϕ. Models with warped extra dimension(s) provide a new geometrical approach of dimensional reduction based on a strongly curved (rather than flat) extra dimension.

We look for solutions for which the four-dimensional line element takes the standard Friedmann-Lamaître-Robertson-Walker (FLRW) form

$$\begin{aligned} ds_4^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a^2(t) \left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega_2^2\right], \end{aligned} \qquad (2)$$

where κ is the 3D curvature constant with the dimension of inverse length squared, and the 5D background Riemann tensor satisfies

$${}^{(5)}R_{ABCD} = \frac{\Lambda_5}{6} \left({}^{(5)}g_{AC}\,{}^{(5)}g_{BD} - {}^{(5)}g_{AD}\,{}^{(5)}g_{BC}\right). \qquad (3)$$

The 5D Einstein-Hilbert action takes the form

$$S_{\rm grav} = M_{(5)}^3 \int d^5x \sqrt{-g}\,(R - 2\Lambda_5)\,, \qquad (4)$$

where $M_{(5)}$ is the 5D Planck mass and $\Lambda_5 \equiv 6/\ell^2$ and ℓ is the radius of curvature of the 5D bulk spacetime.

The gravitational action (4) may be supplemented with the following 3-brane action

$$S_{\rm brane} = \int_{\partial\mathcal{M}} \sqrt{-g_b}\,(-\tau), \qquad (5)$$

where τ denotes the brane tension. The 5D Einstein field equations are given by

$$G_{AB} = -\frac{\tau}{2} \frac{\sqrt{-g_b}}{\sqrt{-g}} g^b_{\mu\nu} \delta^\mu_A \delta^\nu_B \delta(\phi - \phi_0) - \Lambda_5 g_{AB}. \qquad (6)$$

The three independent equations of motion are

$$\frac{6A'^2}{\rho^2} = 6\left(\frac{\dot a^2}{a^2} + \frac{\kappa}{a^2}\right) - \Lambda_5\, e^{2A}, \qquad (7)$$

$$\frac{6A''}{\rho^2} = -\Lambda_5\, e^{2A} - \frac{\tau}{\rho M_{(5)}^3}\, \delta(\phi - \phi_0) e^A, \qquad (8)$$

$$\frac{\ddot a}{a} = \frac{\dot a^2}{a^2} + \frac{\kappa}{a^2}. \qquad (9)$$

Here we are interested in studying a theory without the orbifold boundary condition, that is a theory with infinite extent in both the positive and negative ϕ direction. We allow both even and odd functions of ϕ rather than the restriction to purely even functions demanded by the orbifold conditions in RS brane-world models.

A. A spatially flat universe

First, we take $\kappa = 0$ (spatially flat universe): The 5D Einstein equations are solved with the scale factor

$$a(t) = a_0\, e^{Ht} \qquad (10)$$

and the warp factor

$$A(\phi) = \ln(2\ell_0 H) - \ln\left(\exp(\rho H \phi) + \frac{\ell_0^2}{\ell^2}\exp(-\rho H \phi)\right), \qquad (11)$$

where ℓ_0 and H are two integration constants. The standard results in AdS_5 space, see, for example [14–16], are obtained by replacing ℓ^2 by $-\ell_{\rm AdS}^2$ or $6/\Lambda_5$.

From the explicit solution given above, we derive

$$S_{\rm eff}^{(D=4)} = M_{\rm Pl}^2 \int d^4x \sqrt{-g_4}\, R_4 - \int d^4x \sqrt{-g_4}\, K, \qquad (12)$$

where

$$M_{\rm Pl}^2 = 8\rho M_{(5)}^3 \ell_0^3 H^3 \int_{-\infty}^{\infty}\left(\mathrm{e}^{\rho H\phi}+\frac{\ell_0^2}{\ell^2}\,\mathrm{e}^{-\rho H\phi}\right)^{-3} d\phi$$
$$= 8M_5^3\ell_0^3 H^2\,\frac{\tan^{-1}(\ell_0/\ell)+\cot^{-1}(\ell_0/\ell)}{8\ell_0^3/\ell^3}$$
$$= \frac{\pi}{2} M_{(5)}^3 \ell^3 H^2, \qquad (13)$$

$$K \equiv \frac{M_{(5)}^3}{\rho}\int_{\infty}^{+\infty} e^{3A}\left(12A'^2+8A''+\frac{12\rho^2}{\ell^2}\,e^{2A}\right)d\phi$$
$$= 8M_{(5)}^3\ell_0^3 H^4\int_{-\infty}^{\infty}\frac{4\left(3\,e^{2\varphi}+3\lambda^2\,e^{-2\varphi}-2\lambda\right)}{\left(e^{\varphi}+\lambda\,e^{-\varphi}\right)^5}\,d\varphi$$
$$\equiv 8M_{(5)}^3\ell_0^3 H^4\int_{-\infty}^{\infty}\Lambda(\varphi), \qquad (14)$$

where we defined $\varphi \equiv \rho H\phi$ and $\lambda \equiv \ell_0^2/\ell^2$. This yields

$$K = 8M_{(5)}^3\ell_0^3 H^4\frac{3\pi}{8\lambda^{3/2}} = 6H^2 M_{\rm Pl}^2. \qquad (15)$$

A similar result was obtained in [17], taking $\lambda = 1$ and $\rho = 1$. We find interest only on smooth brane-world solutions, so we take $\lambda \equiv \ell_0^2/\ell^2 > 0$.

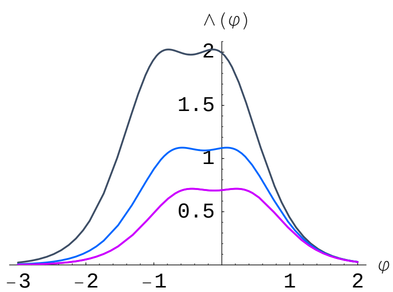

FIG. 1: The plot of the function $\Lambda(\varphi)$ with $\lambda = 0.4, 0.6$ and 0.8 (from top to bottom) (online: black, blue and pink). Like the warp factor e^A, $\Lambda(\varphi)$ is regular, has a peak at $\phi \equiv \phi_0$ and falls off rapidly away from the brane.

The four-dimensional effective action is given by

$$S_{\rm eff}^{(D=4)} = M_{\rm Pl}^2\int d^4x\sqrt{-g_4}\,(R_4-\Lambda_4)\,, \qquad (16)$$

where the 4D effective cosmological constant

$$\Lambda_4 = 6H^2. \qquad (17)$$

These results are different from that in the simplest RS brane worlds at least in two aspects. Firstly, the cosmological mass reduction formula (cMRF)

$$M_{(5)}^3 = \frac{2M_{\rm Pl}^2}{\pi\ell^3 H^2} \qquad (18)$$

is clearly different from that obtained in a static AdS_5 brane-world configuration [4], for which

$$M_{(5)}^3 = \frac{M_{\rm Pl}^2}{\ell_{\rm AdS}}. \qquad (19)$$

Secondly, and perhaps more importantly, no parameter of the action would have to be tuned to keep Λ_4 positive. The cosmological constant problem of the original RS brane world proposal, which is the question of why the background warps in the appropriate fashion without introducing an effective 4D cosmological constant, does not arise here. In fact, in a dynamical spacetime, the vacuum energy on the brane can naturally warp the bulk spacetime and introduce a nontrivial curvature in the bulk while maintaining a 4D de Sitter solution.

B. Inclusion of brane action

In the presence of a brane action, we have to consider the metric a step function in ϕ, while computing derivatives of $A(\phi)$ (with respect to ϕ). The solution valid for $-\infty \le \phi \le +\infty$ then implies that

$$A'' + \frac{4\lambda\rho^2H^2}{\Phi_+^2} + \frac{\rho H\Phi_-}{\Phi_+}\,(2\delta(\phi-\phi_0)) = 0, \qquad (20)$$

where $' \equiv \frac{d}{d\phi}$ and $\Phi_\pm \equiv e^{\rho H\phi} \pm \lambda\, e^{-\rho H\phi}$. One could think of a de Sitter brane as the location $\phi = \phi_0$ ($z = z_c$) where the zero-mode graviton wave-function is peaked.

The $\mu\nu$-components of the 5D Einstein equations yield

$$A'' + \frac{4\lambda\rho^2H^2}{\Phi_+^2} + \frac{2\rho H\ell_0}{3M_5^3\,\Phi_+}\,(\tau\delta(\phi-\phi_0)) = 0. \qquad (21)$$

By comparing eqs. (20) and (21), we get

$$\tau = \frac{3M_5^3}{\ell_0}\left(e^{\rho H\phi_0} - \lambda\, e^{-\rho H\phi_0}\right). \qquad (22)$$

Particularly, in the limit $\lambda \to 0$, the 5D spacetime becomes spatially flat and gravity is *not* localised in this case. Indeed, $\lambda > 0$ is required to keep the warp factor bounded from the below and above.

For $\lambda > 0$, by writing

$$\phi \equiv \frac{1}{\rho}\left(z + \ln\frac{\ell_0}{\ell}\right).$$

the solution for warp factor, Eq. (11), and the brane tension can be written in standard forms, i.e.

$$e^{A(z)} = \frac{\ell H}{\cosh Hz}, \quad \tau = \frac{6M_{(5)}^3}{\ell}\sinh Hz_c, \qquad (23)$$

where $z_c > 0$. The scale of warped compactification is

$$r_c \equiv \rho\, e^A = \frac{2\ell_0\rho H}{(e^{\rho H\phi}+\lambda e^{-\rho H\phi})} = \frac{\rho H\ell}{\cosh Hz}. \qquad (24)$$

The $\rho \to \infty$ limit gives rise to a theory with a semi-infinite extra dimension.

C. Non-flat universe

In a spatially non-flat universe ($\kappa \neq 0$), the 5D Einstein equations are explicitly solved when

$$a(t) = \frac{c_0^2 + \kappa\rho^2}{2c_0}\cosh\left(\frac{t}{\rho}\right) + \frac{c_0^2 - \kappa\rho^2}{2c_0}\sinh\left(\frac{t}{\rho}\right) \quad (25)$$

and

$$A(\phi) = \ln\left(\frac{2\ell_0}{\rho}\right) - \ln\left(\exp(\phi) + \frac{\ell_0^2}{\ell^2}\exp(-\phi)\right), \quad (26)$$

The Hubble-like parameter H (appeared in the $\kappa = 0$ case above) is no more arbitrary but it is fixed in terms of the length parameter ρ, i.e. $H \to 1/\rho$. The 4D effective action still takes the form of (12), but now

$$M_{\rm Pl}^2 = \frac{\pi}{2}\frac{M_{(5)}^3\ell^3}{\rho^2}, \qquad K = \frac{3\pi M_{(5)}^3\ell^3}{\rho^4}. \quad (27)$$

Note that, unlike in the simplest RS brane-world models, we do not require the Z_2 symmetry in order to get a finite 4D Planck mass, as long as $\lambda > 0$ or $\ell_0^2/\ell^2 > 0$.

In the $\kappa \neq 0$ case, equation (22) is modified as

$$\tau = \frac{3M_5^3}{\ell_0}\left(e^{\phi_0} - \lambda\, e^{-\phi_0}\right). \quad (28)$$

The scale of warped compactification is now

$$r_c \equiv \rho\, e^A = \frac{2\ell_0}{(e^\phi + \lambda e^{-\phi})} \equiv \frac{2\ell}{\cosh y}, \quad (29)$$

where $y \equiv \phi - \ln(\ell_0/\ell)$. This is exponentially suppressed as $y \to \pm\infty$. Clearly, there is no problem with taking the $\rho \to \infty$ limit of the background solution given above.

III. LINEARIZED GRAVITY

Brane-world models with one or more non-compact extra spaces are known to require the trapping of gravitational degrees of freedom on the brane [4, 8]. To determine whether the spectrum of linearized tensor fluctuations $\delta^{(5)}g_{AB}$ is consistent with four-dimensional experimental gravity, we shall consider the perturbations around the background solution given above.

The $\kappa \neq 0$ solutions discussed in the subsection are slightly more restrictive than the $\kappa = 0$ solution. So, henceforth, we focus our discussions to the $\kappa = 0$ case, for which ρ is arbitrary. The perturbations of the 5D metric $\delta^{(5)}g_{AB} \equiv h_{AB}$ may be written as

$$\delta\,^{(5)}g_{AB} = \left[\begin{array}{cc|c} -2e^{2A}\psi & e^{2A}a^2(\partial_i\mathcal{B} - S_i) & e^A\xi \\ e^{2A}a^2(\partial_j\mathcal{B} - S_j) & e^{2A}a^2\left\{2\mathcal{R}\delta_{ij} + 2\partial_i\partial_j\mathcal{C} + 2\partial_{(i}V_{j)} + h_{ij}\right\} & e^{2A}a^2(\partial_i\beta - \chi_i) \\ \hline e^A\xi & e^{2A}a^2(\partial_j\beta - \chi_j) & 2\rho^2 e^{2A}\zeta \end{array}\right], \quad (30)$$

where $\psi, \mathcal{R}, \mathcal{C}, \zeta, \beta, \xi$ are metric scalars, while S_i, V_i, χ_i are transverse 3D vector fields, and h_{ij} represent transverse traceless tensor modes. Here we focus on the analysis of tensor modes (we refer to [18] for the analysis of gauge-invariant scalar and vector perturbations of maximally symmetric spacetimes), see also [19].

The transverse-traceless tensor modes $h_{ij} \equiv \delta g_{ij} = \delta_i^\mu \delta_j^\nu h_{\mu\nu}(x,\phi)$ satisfy the following wave equation

$$e^{-2A}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\phi^2} + \frac{3A'}{\rho^2}\partial_\phi - \partial_t^2 - 3\frac{\dot a}{a}\partial_t + \frac{\vec\nabla^2}{a^2}\right)h_{ij} + \frac{\tau}{2M_{(5)}^3}\frac{e^{-A}}{\rho}\delta(\phi-\phi_0)h_{ij} = 0. \quad (31)$$

The last term above has arisen from the first term on the right hand side in Eq. (6). By separating the variables as

$$h_{ij}(x^\mu,\phi) \equiv \sum \alpha_m(t)u_m(\phi)\, e^{ik\cdot x}\,\hat e_{ij}, \quad (32)$$

where $e_{ij}(x^i)$ is a transverse, tracefree harmonics on the spatially flat 3-space, $\vec\nabla^2\hat e_{ij} = -k^2\hat e_{ij}$, we get

$$\ddot\alpha_m + 3\frac{\dot a}{a}\dot\alpha_m + \left(\frac{k^2}{a^2} + m^2\right)\alpha_m = 0, \quad (33a)$$

$$\left(\frac{1}{\rho^2}\frac{d^2}{d\phi^2} + \frac{3A'}{\rho^2}\frac{d}{d\phi} + \frac{3H\Phi_-}{\rho\Phi_+}\delta(\phi-\phi_0) + m^2\right)u_m = 0, \quad (33b)$$

where m is a 4D mass parameter and k is the co-moving wavenumber along the 4D hypersurface.

Let us first consider Eq. (33a). If we write $\alpha_m = \varphi_m/a(\eta)$ and use conformal time $\eta = -\int(dt/a)$, then the wave equation on the brane reads as

$$\frac{d^2\varphi_m}{d\eta^2} + \left[-\frac{1}{a}\frac{d^2a}{d\eta^2} + a^2m^2 + k^2\right]\varphi_m = 0. \quad (34)$$

With $a \propto e^{tH}$ (and hence $\eta = -1/(aH)$), we get

$$\frac{d^2\varphi_m}{d\eta^2} + \left[-\frac{2}{\eta^2} + \frac{m^2}{\eta^2 H^2} + k^2\right]\varphi_m = 0. \qquad (35)$$

The general solution is

$$\varphi_m(\eta;\vec{k}) = \sqrt{\eta k}\, Z(\lambda, \eta k), \qquad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \qquad (36)$$

where $Z(\nu, \eta k)$ is a linear combination of Bessel functions of order ν. One recovers the RS solution in the limit $a(\eta) \to \text{const} \equiv 1$, in which case $\varphi_m = \exp(\pm i\omega t)$, with $\omega^2 = k^2 + m^2$. The perturbations are over-damped for all light modes with $0 < m < 3H/2$, while all heavy modes with $m > 3H/2$ oscillate and decay more rapidly and the modes with $m^2 < 0$ do not exist. In a 4D de Sitter space, the eigen modes satisfying $m^2 > 0$ are not localised on an inflating (de Sitter) brane, see below.

Defining $u_m \equiv e^{-3A/2}\,\psi_m$, it is possible to rewrite Eq. (33b) in a Schrödinger-like form

$$\frac{d^2\psi_m}{d\phi^2} - V\,\psi_m + m^2\rho^2\psi_m = 0 \qquad (37)$$

with the scalar potential

$$\begin{aligned} V &\equiv \frac{9}{4}A'^2 + \frac{3}{2}A'' - \frac{3H\rho\Phi_-}{\Phi_+}\delta(\phi-\phi_0) \\ &= \frac{9\rho^2H^2}{4} - \frac{15\lambda\rho^2H^2}{\Phi_+^2} - \frac{6H\rho\Phi_-}{\Phi_+}\delta(\phi-\phi_0), \end{aligned} \qquad (38)$$

where we used the solution (11) and $\Phi_\pm = e^{\rho H\phi} \pm \lambda e^{-\rho H\phi}$. Introducing a new coordinate variable

$$z \equiv \rho\phi - \ln\sqrt{\lambda},$$

we get

$$\frac{d^2\psi_m}{dz^2} - V\psi_m = -m^2\psi_m, \qquad (39)$$

where

$$V = \frac{9H^2}{4} - \frac{15H^2}{4\cosh^2(Hz)} - 6H\tanh(Hz)\delta(z-z_c). \qquad (40)$$

The brane is now located at $z = z_c$. The zero-mode solution ($m^2 = 0$) is given by

$$\psi_0(z) = \frac{b_0}{(\cosh(Hz))^{3/2}}, \qquad (41)$$

which is clearly normalizable since

$$\int_{-\infty}^{\infty} |\psi_0(z)|^2 dz = \frac{\pi b_0^2}{2H}.$$

There is one more bound state solution, i.e.,

$$\psi_1(z) = b_1\,\frac{\sqrt{\cosh^2(Hz)-1}}{(\cosh(Hz))^{3/2}}, \qquad (42)$$

which is obtained by taking $m^2 = 2H^2$. This solution is also normalizable. However, only the zero-mode solution ($m^2 = 0$) is localised on the de Sitter brane. This can be seen by substituting (41) into Eq. (39) and comparing the delta-function terms.

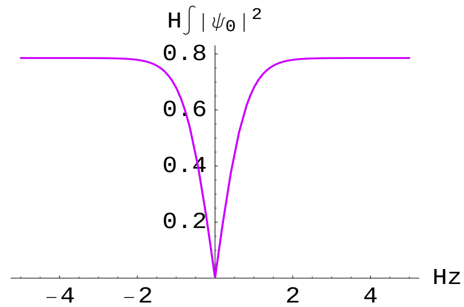

FIG. 2: The plot of the function $H\int|\psi_0|^2$ with $b_0 = 1$.

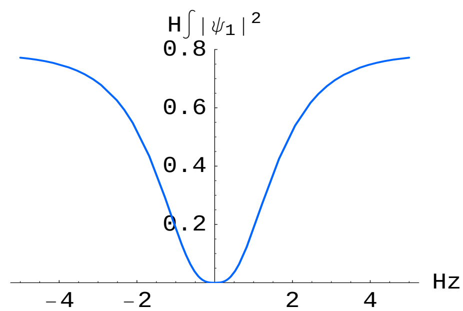

FIG. 3: The plot of the function $H\int|\psi_1|^2$ with $b_1 = 1$.

We also find that

$$\int \frac{|\psi_0|^2}{b_0^2}dz = \frac{\tan^{-1}\tanh\frac{|Hz|}{2}}{H} + \frac{\text{sech}(Hz)\tanh\frac{|Hz|}{2}}{2H}, \qquad (43a)$$

$$\int \frac{|\psi_1|^2}{b_1^2}dz = \frac{\tan^{-1}\tanh\frac{|Hz|}{2}}{H} - \frac{\text{sech}(Hz)\tanh\frac{|Hz|}{2}}{2H}, \qquad (43b)$$

The zero-mode graviton is localised on the brane. The first excited state with mass $m^2 = 2H^2$ is normalizable but this mode is not localised on the brane.

The general solution to Eq. (39) is given by

$$\begin{aligned} \psi(z) &= c_1\, X^{5/2}\, {}_2F_1\left(\frac{5+2i\mu}{4}, \frac{5-2i\mu}{4}; \frac{1}{2}; 1-X^2\right) \\ &\quad + c_2\sqrt{X^2-1}\,X^{5/2} \\ &\qquad {}_2F_1\left(\frac{7+2i\mu}{4}, \frac{7-2i\mu}{4}; \frac{3}{2}; 1-X^2\right), \end{aligned} \qquad (44)$$

where $X = \cosh(Hz)$ and $\mu \equiv \sqrt{\frac{m^2}{H^2} - \frac{9}{4}} = \pm i\nu$. The allowed values of m are quantised in units of H or the index $\gamma \equiv m^2/H^2$. Around the brane's position at $z \equiv z_c$, satisfying $Hz \ll 1$, we have

$$\psi(z) = c_1 P_\mu + c_2 Q_\mu, \tag{45}$$

where

$$P_\mu = 1 - \frac{3+2\gamma}{4}(Hz)^2 + \frac{39+12\gamma+4\gamma^2}{96}(Hz)^4 + \cdots, \tag{46a}$$

$$Q_\mu = Hz - \frac{3+2\gamma}{12}(Hz)^3 + \frac{99+12\gamma+4\gamma^2}{480}(Hz)^5 + \cdots. \tag{46b}$$

The inequality $Hz \ll 1$ signifies a cosmological scale for which $H^{-1} \gg z$, i.e. the Hubble radius is much larger than the radial extension of the fifth dimension.

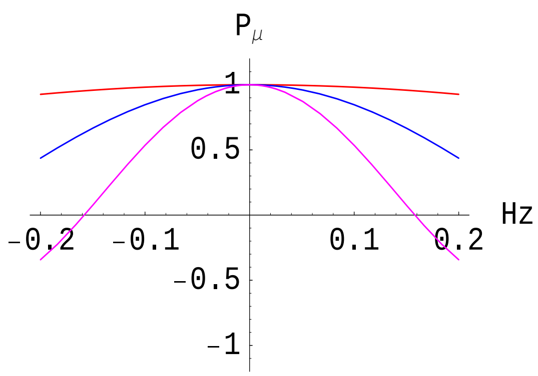

FIG. 4: The plot of the function $P_\mu(z)$ with $\gamma = 9/4, 30, 100$ (top to bottom).

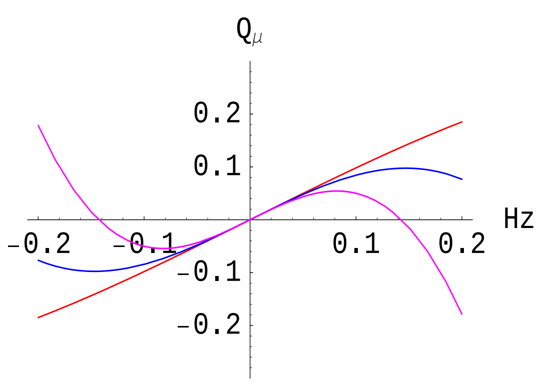

FIG. 5: The plot of the function $Q_\mu(z)$ with $\gamma = 9/4, 30, 100$ (top to bottom, in the range $Hz > 0$).

Using the following property

$${}_2F_1(a,b;c;z) \;=\; (1-z)^{-b}\,{}_2F_1(c-a,b;c;\frac{z}{z-1}),$$

the eigenfunctions for the massive continuous modes with $m^2 \geq 2H^2$ may be given by

$$\begin{aligned}\psi_m(z) \;=\; & c_1\, X^{i\mu} \\ & \times {}_2F_1\left(-\frac{3+2i\mu}{4}, \frac{5-2i\mu}{4}; \frac{1}{2}, \frac{X^2-1}{X^2}\right) \\ & + c_2 \frac{\sqrt{X^2-1}}{X} X^{i\mu} \\ & \times {}_2F_1\left(-\frac{1+2i\mu}{4}, \frac{7-2i\mu}{4}; \frac{3}{2}, \frac{X^2-1}{X^2}\right),\end{aligned} \tag{47}$$

where $X = \cosh(Hz)$. In the large zH limit, we get

$$\psi_{Hz\to\infty} = c_1\, e^{i\mu zH} + c_2\, e^{-i\mu zH}, \tag{48}$$

where $\mu \equiv \sqrt{\frac{m^2}{H^2} - \frac{9}{4}}$. With $c_1 = 0$, all heavy modes with $\mu > 0$ become oscillating plane waves, which represent the de-localised KK massive gravitons. The time-evolution of the mode functions of these heavy modes (cf Eq. (36)) shows that they remain underdamped at late times $|\eta| \to 0$.

A. Linearized bulk equation

The perturbations of the 5D metric can be analysed also by considering a wave equation for a master variable $\Omega \equiv \Omega(x^\mu;\phi)$ introduced in [18], which in a 5D de Sitter background defined by (1) reads as

$$\frac{1}{\rho^2}\left(\frac{e^{-3A}}{a^3}\Omega'\right)' - \left(\frac{e^{-3A}}{a^3}\,\dot{\Omega}\right)^{\dot{}} + \left(\frac{\vec{\nabla}^2}{a^2} - \frac{e^{2A}}{\ell^2}\right)\frac{e^{-3A}}{a^3}\Omega = 0. \tag{49}$$

By using the following change of variable

$$\Omega \equiv a(t)^3 e^{3A(\phi)}\,\tilde{\Omega}, \tag{50}$$

Eq. (49) can be written as

$$\frac{1}{\rho^2}\left(\tilde{\Omega}'' + 3A'\tilde{\Omega}' + 3A''\tilde{\Omega}\right) - \left(\ddot{\tilde{\Omega}} + 3\frac{\dot{a}}{a}\dot{\tilde{\Omega}}\right) + \left(\frac{\vec{\nabla}^2}{a^2} - \frac{e^{2A}}{\ell^2}\right)\tilde{\Omega} = 0. \tag{51}$$

By separating the newly defined master variable as $\tilde{\Omega}(x^\mu,\phi) \equiv \sum \alpha_m(t) u_m(\phi)\, e^{ik\cdot x}$, we get

$$\ddot{\alpha}_m + 3\frac{\dot{a}}{a}\dot{\alpha}_m + \left(\frac{k^2}{a^2} + m^2\right)\alpha_m = 0, \tag{52a}$$

$$\frac{d^2u_m}{d\phi^2} + 3A'u_m' + \left(3A'' + m^2\rho^2 - \frac{\rho^2}{\ell^2}\,e^{2A}\right)u_m = 0. \tag{52b}$$

The first equation above is the same as (33a), so we only have to consider the second equation.

Defining $u_m \equiv e^{-3A/2}\,\Phi_m$, we get

$$\frac{d^2\Phi_m}{d\phi^2} - V\,\Phi_m + m^2\rho^2\Phi_m = 0, \tag{53}$$

with the off-brane potential V of the form

$$\begin{aligned} U &\equiv \frac{9}{4}A'^2 - \frac{3}{2}A'' + \frac{\rho^2}{\ell^2}\,e^{2A} \\ &= \frac{9\rho^2H^2}{4} + \frac{\lambda\rho^2H^2}{\Phi_+^2}, \end{aligned} \tag{54}$$

where we used the solution (11) and $\Phi_\pm = e^{\rho H\phi} \pm \lambda e^{-\rho H\phi}$. Defining $z \equiv \rho\phi - \ln\sqrt{\lambda}$, as before, we get

$$-\frac{d^2\Phi_m}{dz^2} + V\,\Phi_m = m^2\Phi_m, \tag{55}$$

with the scalar potential V of the form

$$V = \frac{9H^2}{4} + \frac{H^2}{4\cosh^2(Hz)}. \tag{56}$$

This equation can also be obtained directly from Eq. (49) but using $\Omega \equiv \sum \alpha_m(t)u_m(\phi)$ and $u_m \equiv e^{-3A/2}\Phi_m$, see, e.g. [18]. The main difference, as compared to the result in AdS_5 spacetimes, is the sign of the second term above. The general solution to (55) is given by

$$\begin{aligned} \Phi_m(z) &= c_1\,\left(\cosh(Hz)\right)^{1/2} \\ &\times {}_2F_1\left(\frac{1+2\nu}{4}, \frac{1-2\nu}{4}; \frac{1}{2}; -\sinh^2(Hz)\right) \\ &+ c_2\,|\sinh(Hz)|\left(\cosh(Hz)\right)^{1/2} \\ &\times {}_2F_1\left(\frac{3+2\nu}{4}, \frac{3-2\nu}{4}; \frac{3}{2}; -\sinh^2(Hz)\right), \end{aligned} \tag{57}$$

where $\nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$. Again, there are two bound state solutions: (i) $\nu = 3/2$ ($m^2 = 0$) and

$$\begin{aligned} \Phi_0(z) &= c_1\sqrt{X}\sqrt{X^2-1} \\ &+ c_2\sqrt{X}\left(1 - \sqrt{X^2-1}\tan^{-1}\frac{1}{\sqrt{X^2-1}}\right), \end{aligned} \tag{58}$$

(ii) $\nu = 1/2$ ($m^2 = 2H^2$) and

$$\Phi_1(z) = \sqrt{X}\left(c_1 + c_2\tan^{-1}\frac{1}{\sqrt{X^2-1}}\right), \tag{59}$$

where $X \equiv \cosh(Hz)$. Both these solutions are non-normalisable. This result is desirable as it implies that a massless bulk scalar mode may not be localised on a de Sitter brane. Note that there are no any tachyonic or growing modes localised on the brane either.

B. Projected Weyl tensor

The non-existence of arbitrarily light KK excitations can be seen also by considering the wave equation for a projected 5D Weyl tensor, which is given by [14]

$$\left[e^A\frac{\partial}{\partial z}e^{-A}\frac{\partial}{\partial z} + \Box_4 - 4H^2\right]\hat{E}_{\mu\nu} = 0, \tag{60}$$

where $\hat{E}_{\mu\nu} \equiv e^{2A}\,E_{\mu\nu}$, $E_{\mu\nu} \equiv C^A{}_{\mu B\nu}n^An^B$ is the projected 5D Weyl tensor, n^A is the vector unit normal to the brane, $\Box_4 \equiv g^{\mu\nu}D_\mu D_\nu$ is the 4-dimensional d'Alembertian with respect to the metric $g_{\mu\nu}$ and D_μ is the covariant derivative. With a separation of variable $\hat{E}_{\mu\nu} = \Psi(\phi)Y^{(m)}_{\mu\nu}(x^\mu)$, Eq. (60) yields

$$\left[e^A\frac{\partial}{\partial z}e^{-A}\frac{\partial}{\partial z} - 2H^2\right]\Psi_m = -m^2\Psi_m, \tag{61a}$$

$$\left[\Box_4 - m^2 - 2H^2\right]Y^{(m)}_{\mu\nu} = 0, \tag{61b}$$

where m is a 4D mass parameter, which has been introduced here as a separation constant. In this formalism, we clearly see that there is a mode $\Psi =$ constant with $m^2 = 2H^2$ that trivially satisfies the equation.

Defining $\Psi_m \equiv e^{A/2}\Phi_m$, we can rewrite the off-brane wave equation (61a) as

$$-\frac{d^2\Phi_m}{dz^2} + V\,\Phi_m = m^2\Phi_m, \tag{62}$$

where

$$V \equiv \frac{A'^2}{4} - \frac{A''}{2} + 2H^2 = \frac{9H^2}{4} + \frac{H^2}{4\cosh^2(Hz)}. \tag{63}$$

This is the same potential as in (55). The mode $m^2 = 2H^2$ translates to $\Phi_m \propto e^{-A/2} \propto \cosh(Hz)$, and it is the first eigenmode and is obtained from (59) with $c_2 = 0$.

C. Correction to Newton's law

In order to estimate the correction to Newton's force law, generated by a discrete tower of Kaluza-Klein modes, one may go to the thin brane limit, i.e. $H^{-1} \to 0$, but keeping the ratio z_c/H^{-1} finite. One also assumes that the matter fields in the four-dimensional theory is smeared over the width of the brane and the brane thickness is smaller compared with the bulk curvature, $H^{-1} < \ell$, so $H\ell > 1$. Under this approximation, the gravitational potential between two point-like sources of masses M_1 and M_2 located on the brane is modified via exchange of gravitons living in five-dimensions as

$$\begin{aligned} U(r) &= G_4\frac{M_1M_2}{r} + M_{(5)}^{-3}\int_m^\infty dm\,\frac{M_1M_2\,e^{-mr}}{r}\,|\Phi_m(z_c)|^2 \\ &\simeq \frac{G_4M_1M_2}{r}\left[1 + \frac{H^2\ell^3}{\alpha r}\sum_i e^{-m_ir}\left(1 + \mathcal{O}\left(\frac{1}{Hr}\right)\right)\right], \end{aligned} \tag{64}$$

where $m_i \geq \sqrt{2}H$, r is the distance between the two point-like sources and α is a constant of order unity. This result qualitatively agrees with that given in [17]. Note that, as compared to the results one has with a static 3-brane, the corrections are suppressed by a factor of e^{-mr}. There is no restriction in taking $\ell \gg r$, provided that the KK modes are sufficiently heavy. For instance, with $m_i \gtrsim$ TeV $\sim 10^{-15}$ cm, $\alpha \sim \mathcal{O}(1)$, the correction term may not show up unless we probe a sufficiently small distance scale, like $r \sim 10^{-12}$ cm.

In the presence of matter or gauge fields, one may require a more complete analysis, involving the effects of the overlap of the gravitational modes with the matter modes, but we will not consider this case here.

IV. GENERALIZATION TO HIGHER DIMENSIONS

For a consistent description of gravity plus gauge field theories, one may require models with more than one extra extra dimensions, and the world around us could have up to $n = 7$ extra spatial dimensions, if our universe is described by string/M theory. In the following, as some canonical examples, we will consider the $n = 2$ and $n = 3$ cases, but it is straightforward to generalise the discussion below to $D = 10$ or $D = 11$ dimensions.

First, we allow two extra dimensions ($n = 2$) and write the 6D action as

$$S_{\rm grav} = M_{(6)}^4 \int d^6x\sqrt{-g}\, R, \tag{65}$$

where $M_{(6)}$ is the 6D Planck mass. It is not difficult to check that the metric ansatz

$$ds_6^2 = \frac{1}{K^{2p}}\left(-dt^2 + a(t)^2 d\vec{x}_{3,k}^2 + \frac{p^2 K'^2}{H^2 K^2} dz^2 + L^2 K^{2p} d\theta^2\right) \tag{66}$$

where $K' = dK/dz$ and $0 \leq \theta \leq 2\pi$, solves the 6D Einstein equations with the 4D scale factor

$$a(t) = \frac{c_0}{2}\, e^{Ht} + \frac{k}{2c_0 H^2}\, e^{-Ht}. \tag{67}$$

In the above, $K(z)$ is an arbitrary function of z.

As a simple example, we take $K(z) \equiv \cosh(Mz)$, where M has a mass dimension of one. Then, under the dimensional reduction, from $D = 6$ to $D = 4$, we get

$$M_{(6)}^4 \int d^6x\sqrt{-g}R = M_{\rm Pl}^2 \int d^4x\sqrt{-\hat{g}_4}\left(R_{(4)} - \Lambda_4\right), \tag{68}$$

where $\Lambda_4 = 6H^2$ and

$$M_{\rm Pl}^2 = M_{(6)}^4 \frac{pL}{H}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} \frac{K'\, dz}{K^{3p+1}} = \frac{4\pi L M_{(6)}^4}{3H}. \tag{69}$$

Especially, in $D = 6$, if we introduce a bulk cosmological term, then we will also require another source term, e.g., 2-form gauge field, in order have an explicit solution. In dimensions $D \geq 7$, however, a bulk cosmological constant can be introduced into the Einstein action without considering other source terms.

The following seven-dimensional metric

$$ds_7^2 = \frac{1}{F(z)}\left(-dt^2 + a(t)^2 d\vec{x}_{3,k}^2 + G(z)dz^2 + E d\Omega_2^2\right), \tag{70}$$

solves the Einstein field equations following from

$$S = M_{(7)}^5 \int d^7x\sqrt{-g}\left(R - 2\Lambda_b\right), \tag{71}$$

when

$$G(z) = \frac{15F'(z)^2}{36H^2F(z)^2 - 4\Lambda_b F(z)}, \quad E = \frac{1}{3H^2}, \tag{72}$$

where $F' = dF(z)/dz$. There can exist a large class of 4D de Sitter solutions with different choices of $F(z)$.

Below, we will consider two physically interesting examples.

Example 1

Take [20]

$$F(z) \equiv F_0^2 \cosh^2(Mz). \tag{73}$$

The dimensionally reduced action reads as

$$M_{(7)}^5 \int d^7x\sqrt{-g}(R - 2\Lambda_b) = M_{\rm Pl}^2 \int d^4x\sqrt{-\hat{g}_4}\left(R_{(4)} - \Lambda_4\right), \tag{74}$$

where $\Lambda_4 = 6H^2$,

$$M_{\rm Pl}^2 = M_{(7)}^5 \frac{2\pi M\sqrt{15}}{9F_0^5 H^3} I(z) \tag{75}$$

and

$$I(z) \equiv \int_{-\infty}^{\infty} \frac{\sqrt{\cosh^2(Mz) - 1}\, dz}{\cosh^5(Mz)\sqrt{\cosh^2(Mz) - \beta}}, \tag{76}$$

with

$$\beta \equiv \frac{\Lambda_b}{9H^2F_0^2}.$$

With $\Lambda_b = 0$, so $\beta = 0$, one can easily evaluate the above integral and find that $I(z) = 2/(5M)$. In the $\Lambda_b > 0$ case, we require $0 < \beta \leq 1$. With $\beta = 1$, we get $I(z) = 3\pi/(8M)$. That is, provided that $0 < \beta < 1$, the integral converges and its value ranges between $2/(5M)$ and $3\pi/(8M)$.

In the case the Hubble expansion parameter H is large, or equivalently, when $|\beta| < 1$, the mass reduction formula is well approximated by

$$M_{\rm Pl}^2 \sim \frac{M_{(7)}^5}{H^3 F_0^5}. \qquad (77)$$

However, when H becomes small, or one considers a large cosmological distance scale, then one should allow a negative bulk cosmological term. The integral converges for any negative value of β. In the limit $|\beta| \gg 1$, the mass reduction formula is approximated by [20]

$$M_{\rm Pl}^2 \sim \frac{M_{(7)}^5}{H^2 F_0^4} \frac{1}{(-\Lambda_b)^{1/2}}. \qquad (78)$$

This result may be analysed further by taking $H \sim 10^{-60}\ M_{\rm Pl}$, especially, if one wants to tune Λ_4 to the present value of 4D cosmological constant. To satisfy phenomenological constraints, such as, $M_{(7)} \gtrsim$ TeV and $\left(-\Lambda_b\right)^{1/2} \lesssim M_{(7)}$, one then has to allow F_0 to take a reasonably large value, $F_0 \gtrsim 10^{14} \gg 1$. A constraint like this becomes much weaker when one applies the model to explain the early universe inflation. For instance, with $H \sim 10^{-5}\ M_{\rm Pl}$, we get $M_{\rm Pl} \sim 10^{-3} M_{\rm Pl(7)}/F_0$, in which case F_0 may be taken to be small, say $F_0 \sim \mathcal{O}(1)$.

To say anything further in a concrete way, we need to have some physical information about the constant F_0, which might actually be related to the D-dimensional dilaton coupling constant, which is assumed (rather implicitly) to be a constant in the present study.

Example 2

Take

$$F(z) \equiv F_0^2 \exp\left(2\arctan\left(e^{Mz}\right)\right). \qquad (79)$$

The warp factor $F(z)$ and the function $G(z)$ are regular everywhere. The integral (76) is now modified as

$$I(z) \equiv \int \frac{(1-\beta(z))^{-1/2}\ dz}{4\cosh(Mz)\ \exp\left(5\arctan\left(e^{Mz}\right)\right)}, \qquad (80)$$

where

$$\beta(z) \equiv \frac{\Lambda_b}{9H^2F_0^2} \exp\left(-2\arctan\left(e^{Mz}\right)\right). \qquad (81)$$

Since $\arctan(x) \to \pm\frac{\pi}{2}$ as $x \to \pm\infty$, the above integral gives a finite result provided that $\Lambda_b < 9H^2F_0^2\, e^{\pi}$, leading to a dynamical mechanism of compactfication. The choice $\Lambda_b < 0$ is pretty safe from the viewpoint of metric regularity, or the smoothness of $G(z)$.

All the results above can easily be generalised to higher dimensions, including the 10 and 11 dimensional models inspired by strong/M-theory [21, 22]. We refer to [20] for further discussions on localization of gravity. The method can also be extended to a class of thick domain walls (or de Sitter brane solutions) in gravity coupled to a bulk scalar field [23–25].

V. CONCLUSION

Finally, we summarise the main results in the paper.

Simple five-dimensional brane-world models defined in an AdS_5 spacetime have been known to provide a rich phenomenology for exploring some of the intriguing ideas that are emerging from string/M-theory, such as, AdS/CFT correspondence, AdS holography and mass hierarchies.

The replacement of AdS_5 spacetime by dS_5 spacetime, along with replacement of a flat 3-brane by a physical 4D universe, gives us new problems and new possibilities. The problem is that the embedding of dS_4 in dS_5 may be viewed as a $O(4)$ symmetric bubble described by a Coleman-De Luccia instanton [26], in which case the size of the bubble may not exceed the radius of dS_5. As a result, a constraint like $cH^{-1} \le \ell_{\rm dS}$ could bring the 5D Planck mass down to TeV scale or even much lower. This problem can easily be overcome by introducing two or more extra dimensions, along with a higher dimensional bulk cosmological term and background fluxes.

Another issue could be that dS_5 allows a foliation by a flat space but that is a spacelike hypersurface. There is no way to cut dS_5 by Minkowski spacetime. That is, in a cosmological setting, a flat 4D Minkowski spacetime is not a solution to 5D Einstein equations, if the background (bulk) spacetime is de Sitter. This is contrast to the results in Randall-Sundrum brane-world models in AdS_5 spacetimes. But this is anyway not a real problem since the universe has probably never gone through a phase of being close to a static universe or a flat 3-brane.

There is perhaps no necessity of having a Minkowski spacetime embedded in a dS_5 spacetime as long as the massless graviton wavefunction becomes normalizable on a 4D de Sitter spacetime, which is indeed the case within the model considered in this paper. We have shown that the effective four-dimensional Planck mass derived from the fundamental D-dimensional Planck mass can be finite because of the large but finite warped volume and also the large world-volume of a de Sitter brane (i.e. the physical universe), implying that a Z_2 symmetry available to 5D brane-world models in an AdS background can be simply relaxed.

We have followed the most general approach for obtaining a de Sitter solution from the higher dimensional Einstein equations, which could yield characteristic linear 4-dimensional spacetime sizes of many orders of magnitude bigger than linear sizes in extra coordinates.

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