On the Bogomol'nyi bound in Einstein-Maxwell-dilaton gravity

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It has been shown that the Einstein-Maxwell-dilaton theory allows a Bogomol'nyi-type inequality for an arbitrary dilaton coupling constant α, and that the bound is saturated if and only if the (asymptotically flat) spacetime admits a nontrivial spinor satisfying the gravitino and the dilatino Killing spinor equations. The present paper revisits this issue and argues that the dilatino equation fails to ensure the dilaton field equation unless the solution is purely electric/magnetic, or the dilaton coupling constant is given by $\alpha = 0, \sqrt{3}$, corresponding to the Brans-Dicke-Maxwell theory and the Kaluza-Klein reduction of 5-dimensional vacuum gravity, respectively. Bearing this remark in mind, we obtain all the supersymmetric solutions to the Einstein-Maxwell-dilaton gravity utilizing bilinears constructed from a Killing spinor. There appear two classes of solutions depending on whether the (Killing) vector field constructed from a Killing spinor is timelike or null. The timelike family of solution is very restrictive due to the dilatino equation. It turns out that the rotating and dyonic solution is allowed if and only if $\alpha = 0$ or $\sqrt{3}$. For other value of α, the timelike family of solutions is necessarily static and exhausted by the multiple generalization of Gibbons-Maeda solution. For the null family, the general solution is described by the plane-fronted wave with parallel rays (*pp*-wave). In each case, the spacetime preserves at least half of supersymmetries. Some characteristic properties of supersymmetric solutions are also explored.

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I. INTRODUCTION

Effective gravitational theories obtained via the Kaluza-Klein paradigm have attracted much attention and have continued to give a lot of physical insights into unified theories. In the low energy limit of string theory, one recovers Einstein's gravity with a dilatonic scalar field arising from dimensional reduction. A dilaton field naturally couples to several gauge fields with various rank, and its coupling constant depends on the underlying theory and the dimension of an internal space. A variety of physical phenomena may be influenced by a dilaton field. An illuminating example is the asymptotically flat, static and spherically symmetric black hole solutions to the Einstein-Maxwell-dilaton system [1–3]. They exhibit quite different aspects compared to the Reissner-Nordström solution in the Einstein-Maxwell theory: the inner "horizon" of a black hole is a spacelike singularity and the Hawking temperature in the "extreme" case can be non-vanishing. These properties alter significantly the spacetime structure [4, 5] and the evaporation process of the Hawking radiation [6]. Even with such unusual behaviors, the uniqueness theorem of static black holes continues to be valid in this theory, viz, the spherically symmetric solution found by Gibbons and Maeda [2] exhausts all the asymptotically flat, static black hole with a nondegenerate event horizon in the Einstein-Maxwell-dilaton theory [7, 8].

Despite the extensive work over the last two decades, a rotating black hole solution in this theory has been yet available with the exception of a slowly rotating approximate solution [9] and a Kaluza-Klein black hole [10]. A widely used formalism for obtaining a new solution is the solution-generating method for the stationary spacetime, in which certain gravitational theories are dimensionally reduced to 3-dimensional gravity coupled to scalar fields [11–13]. In the Einstein-Maxwell theory, the target space of the harmonic maps is described by Bergmann metric having the structure group isomorphic to coset $\mathrm{SU}(2,1)/\mathrm{S}[\mathrm{U}(1,1)\times\mathrm{U}(1)]$, which is large enough to contain the Ehlers-Harrison type transformations [14, 15]. If an additional axisymmetry is imposed the system becomes completely integrable, admitting a variety of generation techniques [16–18]. In the Einstein-Maxwell-dilaton theory, however, the target space is neither symmetric nor homogeneous (i.e., the coset representation is impossible and the isometry group does not act transitively) for a generic dilaton coupling [19]. Furthermore, the additional axisymmetry fails to render the system to be two-dimensionally integrable. This fact forbids us to get rotating black-hole solutions from simpler seed solutions following the conventional procedure. In this paper, we adopt an alternative strategy by focusing on supersymmetric solutions.

Supersymmetric solutions in supergravity have performed an invaluable rôle in the progression of non-perturbative regime of string theory and the anti-de Sitter/conformal field theory correspondence. The supersymmetric solu-

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tions saturate the Bogomol'nyi-Prasad-Sommerfield (BPS) bound and are characterized by the existence of a super-covariantly constant spinor referred to as a Killing spinor [20–22]. One can identify the Killing spinor equations as the "square root" of field equations, so that supersymmetric solutions can be obtained relatively easily just by solving linear equations. As a matter of fact, we can systematically classify and sometimes can obtain *all* supersymmetric solutions. An initiated work is due to Tod, who inventoried all the BPS solutions admitting a nontrivial Killing spinor in 4-dimensional $N = 2$ supergravity [23]. Although reference [23] shed some light on the whole picture of BPS solutions, his method lacks utility in higher dimensions since the Newman-Penrose formalism has been used therein. This difficulty can be overcome by the seminal work of Gauntlett *et al.* [24], where general supersymmetric solutions in 5-dimensional minimal supergravity were classified by making use of bilinears constructed from a Killing spinor. Thereafter the classification program has achieved a remarkable development in diverse supergravities in various dimensions [25–31]. This formulation has provided valuable tools for finding supersymmetric black holes [32] and black rings [33], and for proving uniqueness theorem of certain black holes [34]. It turns out that all the supersymmetric black-hole solutions have universal properties such as strict stationarity and mechanical equilibrium in the ungaged supergravities. This means that black holes fail to posses the trapped region (e.g., inside the Schwarzschild interior) and the ergoregion even if it has a nonvanishing angular momentum. The mechanical equilibrium condition allows a multiple collection of black holes, reflecting a "no force" situation between BPS objects [35]. The BPS configurations are thus very simple since supersymmetry prohibits any dynamical processes.

In this paper, we consider a simple model of Einstein-Maxwell-dilation gravity described by the action

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2(\nabla^\mu\phi)(\nabla_\mu\phi) - e^{-2\alpha\phi}F_{\mu\nu}F^{\mu\nu}\right] , \tag{1.1}$$

where ϕ is a dilaton field, $F = \mathrm{d}A$ is the Maxwell field and α controls the strength of the coupling of a dilaton to the Maxwell field. The critical coupling $\alpha = 1$ arises by the truncation of $N = 4$ supergravity [1–3]. Whereas, the $\alpha = \sqrt{3}$ case occurs via the Kaluza-Klein compactification of 5-dimensional vacuum gravity. Nevertheless, it has been shown that the theory admits a Bogomol'nyi-type inequality for an *arbitrary* coupling, and allows a nontrivial "Killing spinor" of gravitino and dilatino when the inequality is saturated [36]. This fact strongly encourages us to speculate that the theory (1.1) can be embedded into some supergravity theories for general coupling. In this paper we cast doubt on this promising outlook by examining the integrability condition of the dilatino Killing spinor equation. A basic belief for the fermionic supersymmetry transformations is that their integrability conditions guarantee the corresponding bosonic equations of motions. It is argued that this consistency condition is satisfied only for certain cases. The other subject of this article is to list all the supersymmetric vacua of this theory under the circumstances in which the consistency condition is satisfied. In the classification procedure we adopt a prescription of [24], which is adequate also in the proof for the variant of positive energy theorem described below.

The present paper is organized as as follows. In the next section, we give a brief overview on the Einstein-Maxwell-dilaton gravity and discuss the Bogomol'nyi inequality. Section III is devoted to the systematic construction of all BPS solutions, which fall into a timelike and a null family. Some properties of BPS solutions are analyzed in section IV. Section V concludes with several future prospects. Our conventions are summarized in appendix with some fruitful formulae used in the main text.

II. ENERGY BOUND IN EINSTEIN-MAXWELL-DILATON GRAVITY

A. Preliminaries

The gravitational field equations derived from the action (1.1) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{2.1}$$

$$= T^{(\phi)}_{\mu\nu} + T^{(\mathrm{em})}_{\mu\nu} , \tag{2.2}$$

where $T_{\mu\nu}$ is the total stress-energy tensor and

$$T^{(\phi)}_{\mu\nu} = 2\left[(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\nabla_\rho\phi)(\nabla^\rho\phi)\right] , \tag{2.3}$$

$$T^{(\mathrm{em})}_{\mu\nu} = 2e^{-2\alpha\phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) . \tag{2.4}$$

The conservation equations for each stress-energy tensor lead to the Maxwell equations

$$\nabla_\nu (e^{-2\alpha\phi} F^{\mu\nu}) = 0\,, \tag{2.5}$$

and the dilaton evolution equation

$$\nabla_\mu \nabla^\mu \phi + \frac{\alpha}{2} e^{-2\alpha\phi} F_{\mu\nu} F^{\mu\nu} = 0\,. \tag{2.6}$$

The action (1.1) is invariant under the discrete duality rotation,

$$\phi \to \tilde{\phi} = -\phi\,, \qquad F_{\mu\nu} \to \tilde{F}_{\mu\nu} = e^{-2\alpha\phi} \star F_{\mu\nu}\,. \tag{2.7}$$

The continuous electric-magnetic duality symmetry in Einstein-Maxwell theory is broken by the presence of a dilaton.

It should be emphasized that the constant dilaton reduces not to the Einstein-Maxwell system but to the Brans-Dicke-Maxwell theory with a Brans-Dicke constant $\omega = -1$. The ordinary Einstein-Maxwell system is recovered when $\phi =$ constant and $F_{\mu\nu}F^{\mu\nu} = 0$, or $\phi =$ constant and $\alpha = 0$; otherwise the dilaton field equation (2.6) is not satisfied. For $\alpha = 1$, the action (1.1) corresponds to the truncated action of $N = 4$ supergravity. The action for $\alpha = \sqrt{3}$ is the Kaluza-Klein reduction of five-dimensional vacuum gravity.

B. BPS inequality

At least for the aforementioned values of α ($= 1$, or $\sqrt{3}$) there is an underlying supergravity theory. Still, the Einstein-Maxwell-dilaton gravity (1.1) enjoys a Bogomol'nyi-type inequality for general values of α, as shown by Gibbons *et al.* [36]. We begin by a brief review about their argument and move to the detailed discussion about the BPS inequality.

Following the standard prescription of the positive energy theorem [20–22], define a Nester-like anti-symmetric tensor in terms of a super-covariant derivative $\hat{\nabla}_\mu$ acting on a (commuting) spinor ϵ as

$$\hat{E}^{\mu\nu} = -\mathrm{i}\left(\bar{\epsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon - \overline{\hat{\nabla}_\rho\epsilon}\gamma^{\mu\nu\rho}\epsilon\right)\,. \tag{2.8}$$

Here, the operator $\hat{\nabla}_\mu$ is defined by

$$\hat{\nabla}_\mu\epsilon = \left(\nabla_\mu + \frac{\mathrm{i}}{4\sqrt{1+\alpha^2}} e^{-\alpha\phi}\gamma^{ab}\gamma_\mu F_{ab}\right)\epsilon\,, \tag{2.9}$$

which specifies the "variation of gravitino." When acting on a spinor, the covariant derivative ∇_μ is given in terms of a torsion-free spin connection $\omega_{\mu ab}$ as

$$\nabla_\mu\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon\,, \tag{2.10}$$

which obeys the Leibniz rule

$$\begin{aligned}
\nabla_\mu(\bar{\epsilon}_1\gamma_{\mu_1}\cdots\gamma_{\mu_n}\epsilon_2) &= \overline{\nabla_\mu\epsilon_1}\gamma_{\mu_1}\cdots\gamma_{\mu_n}\epsilon_2 + \bar{\epsilon}_1\gamma_{\mu_1}\cdots\gamma_{\mu_n}\nabla_\mu\epsilon_2\,,\\
\nabla_\mu(\bar{\epsilon}_1\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_n}\epsilon_2) &= \overline{\nabla_\mu\epsilon_1}\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_n}\epsilon_2 + \bar{\epsilon}_1\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_n}\nabla_\mu\epsilon_2\,.
\end{aligned} \tag{2.11}$$

Observe that $\hat{E}^{\mu\nu}$ decompose as

$$\hat{E}^{\mu\nu} = E^{\mu\nu} + H^{\mu\nu}\,, \tag{2.12}$$

where $E^{\mu\nu} = -\mathrm{i}(\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon - \overline{\nabla_\rho\epsilon}\gamma^{\mu\nu\rho}\epsilon)$ is an ordinary Nester 2-tensor and $H^{\mu\nu}$ represents the electromagnetic contribution,

$$H^{\mu\nu} = -\frac{2e^{-\alpha\phi}}{\sqrt{1+\alpha^2}}\left(\bar{\epsilon}\epsilon F^{\mu\nu} - \mathrm{i}\bar{\epsilon}\gamma_5\epsilon \star F^{\mu\nu}\right)\,. \tag{2.13}$$

Reference [36] also introduced the "variation of dilatino" by[1]

$$\delta\lambda := \frac{1}{\sqrt{2}}\left(\gamma^\mu\nabla_\mu\phi - \frac{\mathrm{i}\alpha}{2\sqrt{1+\alpha^2}}e^{-\alpha\phi}\gamma^{ab}F_{ab}\right)\epsilon\,. \tag{2.14}$$

Here, the specific factors appearing in (2.20) and (2.21) have been chosen a posteriori in order to give an energy bound.

Consider an asymptotically flat spacetime to which an Arnowitt-Deser-Misner (ADM) 4-momentum can be assigned [20, 38]. Choose a spatial hypersurface Σ with a future-pointing unit normal n^μ and let $\partial\Sigma$ be its boundary at spatial infinity.[2] Assume that ϵ asymptotes to a constant spinor ϵ_∞ and that the dilaton falls off to zero at spatial infinity. Using Stokes' theorem, it is found that

$$\begin{aligned}
-\int_\Sigma \mathrm{d}\Sigma n_\mu \nabla_\nu \hat{E}^{\mu\nu} &= \frac{1}{2}\int_{\partial\Sigma}\mathrm{d}S_{\mu\nu}\hat{E}^{\mu\nu} \\
&= \frac{1}{2}\int_{\partial\Sigma}\mathrm{d}S_{\mu\nu}E^{\mu\nu} - \frac{1}{\sqrt{1+\alpha^2}}\int_{\partial\Sigma}\mathrm{d}S_{\mu\nu}\left(\bar\epsilon_\infty\epsilon_\infty F^{\mu\nu} - \mathrm{i}\bar\epsilon_\infty\gamma_5\epsilon_\infty \star F^{\mu\nu}\right) \\
&= -\mathrm{i}\bar\epsilon_\infty\gamma^\mu\epsilon_\infty P_\mu - \frac{1}{\sqrt{1+\alpha^2}}\bar\epsilon_\infty(Q_e - \mathrm{i}\gamma_5 Q_m)\epsilon_\infty\,,
\end{aligned} \tag{2.15}$$

where $\mathrm{d}S_{\mu\nu}$ is the element of 2-sphere at infinity. P_μ denotes the ADM 4-momentum [20, 38] and

$$Q_e = \int_{\partial\Sigma}\mathrm{d}S_{\mu\nu}F^{\mu\nu}\,, \qquad Q_m = \int_{\partial\Sigma}\mathrm{d}S_{\mu\nu}\star F^{\mu\nu}\,, \tag{2.16}$$

are the total electric and magnetic charges, respectively. A straightforward but rather tedious calculation shows that

$$\begin{aligned}
\nabla_\nu\hat{E}^{\mu\nu} =& 2\mathrm{i}\overline{\hat\nabla_\rho\epsilon}\gamma^{\mu\nu\rho}\hat\nabla_\nu\epsilon + 2\mathrm{i}\overline{\delta\lambda}\gamma^\mu\delta\lambda - \left(R^\mu{}_\nu - \frac{1}{2}R\delta^\mu{}_\nu - T^\mu{}_\nu\right)(\mathrm{i}\bar\epsilon\gamma^\nu\epsilon) \\
&- \frac{2}{\sqrt{1+\alpha^2}}\left[e^{\alpha\phi}\nabla_\nu(e^{-2\alpha\phi}F^{\mu\nu})\bar\epsilon\epsilon - e^{-\alpha\phi}(\nabla_\nu\star F^{\mu\nu})(\mathrm{i}\bar\epsilon\gamma_5\epsilon)\right]\,.
\end{aligned} \tag{2.17}$$

Relations (A3)–(A10) in appendix are of great help to derive this equation. The last three terms will vanish provided Einstein's equations, the Maxwell equations and the Bianchi identity are imposed. Then the volume integral of the left-hand side of (2.15) can be written as a sum of non-negative terms for ϵ satisfying the (modified) Dirac-Witten equation

$$\gamma^a\hat{D}_a\epsilon = 0\,, \tag{2.18}$$

where $\hat{D}$ is the projection of super-covariant derivative $\hat\nabla$ onto Σ. It follows that the right hand side of (2.15) has to have non-negative eigenvalues, giving rise to a suggestive inequality

$$M \geq \frac{1}{\sqrt{1+\alpha^2}}\sqrt{Q_e^2 + Q_m^2}\,, \tag{2.19}$$

where $M = \sqrt{-P_\mu P^\mu}$ is the ADM mass. In the context of supergravity, Q_e and Q_m enter the algebra of global supersymmetry transformations as central charges. The above lower bound is attained if and only if there exists a nontrivial spinor ϵ satisfying the gravitino Killing spinor equation

$$\left(\nabla_\mu + \frac{\mathrm{i}}{4\sqrt{1+\alpha^2}}e^{-\alpha\phi}\gamma^{ab}\gamma_\mu F_{ab}\right)\epsilon = 0\,, \tag{2.20}$$

and the dilatino Killing spinor equation

$$\left(\gamma^\mu\nabla_\mu\phi - \frac{\mathrm{i}\alpha}{2\sqrt{1+\alpha^2}}e^{-\alpha\phi}\gamma^{ab}F_{ab}\right)\epsilon = 0\,. \tag{2.21}$$

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[1] Note that the conventions of the present paper differs from [36], where the gamma matrix and the Dirac conjugate are defined by $\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}$ and $\bar\psi = \psi^\dagger\gamma^0$. Equation (2.14) also corrects the typo in [36].

[2] Although we have supposed that Σ has no interior boundary corresponding to the black hole horizon, this condition can be relaxed [36] (see also [37]).

These can be viewed as supersymmetry transformations which leave the bosonic background invariant.

The resulting energy bound (2.19) strongly implies that the theory (1.1) might be embedded into some supergravity theory [22]. We are now going to claim, however, that this might be too optimistic an estimate. To illustrate, let us consider the multiple black hole solution found in [1],

$$\mathrm{d}s^2 = -H_1^{-1}H_2^{-1}\mathrm{d}t^2 + H_1H_2\mathrm{d}\vec{x}^2\,, \tag{2.22}$$

where H_1 and H_2 are arbitrary harmonics on the Euclid 3-space $\mathrm{d}\vec{x}^2$, and

$$A = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}t}{H_1} + \vec{A}\cdot\mathrm{d}\vec{x}\right)\,, \qquad \vec{\nabla}\times\vec{A} = \vec{\nabla}H_2\,, \qquad \phi = -\frac{1}{2}\ln\left(\frac{H_1}{H_2}\right)\,. \tag{2.23}$$

Here and hereafter, the 3-dimensional vector notation will be used for quantities of 3-dimensional Euclid space $\mathrm{d}\vec{x}^2$. The metric (2.22) solves the field equations (2.2), (2.5) and (2.6) with $\alpha = 1$, which is the distinguished value predicted by string theory. Two functions H_1 and H_2 obey Laplace equations, so the feature of force balance is appropriately captured. At first sight, it therefore seems reasonable to expect that this solution would saturate the bound (2.19). Contrary to our intuition, this is not the case [39]. Consider multiple point sources

$$H_1 = 1 + \sum_k \frac{\sqrt{2}Q_e^{(k)}}{|\vec{x}-\vec{x}_{(k)}|}\,, \qquad H_2 = 1 + \sum_k \frac{\sqrt{2}Q_m^{(k)}}{|\vec{x}-\vec{x}'_{(k)}|}\,, \tag{2.24}$$

where $\vec{x}_{(k)}$ and $\vec{x}'_{(k)}$ represent the locus of point sources. One immediately finds that the metric is asymptotically flat, $Q_e = \sum_k Q_e^{(k)}$ and $Q_m = \sum_k Q_m^{(k)}$ correspond to the total electric and magnetic charges, in terms of which the ADM mass is given by $M = (Q_e + Q_m)/\sqrt{2}$. This is strictly above the lower bound (2.19).[3]

This puzzling issue is best understood as follows. Acting $\gamma^\nu\nabla_\nu$ to (2.21) and using (2.20) and (A2), we obtain

$$\begin{aligned}\Bigg[&\nabla_\mu\nabla^\mu\phi + \frac{\alpha}{2}e^{-2\alpha\phi}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}\gamma_5\alpha(\alpha^2-3)}{2(1+\alpha^2)}F_{\mu\nu}\star F^{\mu\nu} \\ &- \frac{\mathrm{i}\alpha e^{-\alpha\phi}}{2\sqrt{1+\alpha^2}}\left\{\gamma^{\mu\nu\rho}\nabla_{[\mu}F_{\nu\rho]} - 2e^{2\alpha\phi}\gamma_\mu\nabla_\nu(e^{-2\alpha\phi}F^{\mu\nu})\right\}\Bigg]\epsilon = 0\,. \end{aligned} \tag{2.25}$$

Accordingly, even if the Bianchi identity $\mathrm{d}F = 0$ and the Maxwell equations $\mathrm{d}\star(e^{-2\alpha\phi}F) = 0$ are satisfied, the integrability condition of the dilatino equation (2.25) does *not* guarantee the dilaton equations of motion (2.6) apart from $\alpha = 0, \sqrt{3}$ and $F_{\mu\nu}\star F^{\mu\nu} = 0$. In this sense, *the dilatino equation is not the proper "square root" of the dilaton field equation*. The dyonic solution (2.22) is not supersymmetric in spite of string motivated case $\alpha = 1$ since it does not satisfy $F_{\mu\nu}\star F^{\mu\nu} = 0$.

A major cause of this apparent variance may be attributed to the absence of the axion field in the theory. The effective theory of heterotic string indeed involves the axion field, which couples to $F_{\mu\nu}\star F^{\mu\nu}$ term in the Lagrangian. It therefore cannot be consistently truncated unless $F_{\mu\nu}\star F^{\mu\nu} = 0$ [39, 40] (see [41] for a proof of the Bogomol'nyi inequality in the Einstein-Maxwell-dilaton-axion system). This observation leads to speculate that the Gibbons solution (2.22) is the BPS solution to some truncation of different supergravity theory, rather than the truncation of Einstein-Maxwell-dilaton-axion gravity. It seems interesting to examine which supergravity has (2.22) as a BPS solution. But addressing this issue is beyond the scope of the present paper.

Nonetheless, the configurations which saturate the bound (2.19) can be identified as ground states that minimize the energy for fixed charges, irrespective of whether the Einstein-Maxwell-dilaton theory (1.1) has a supergravity origin or not. Besides, it is far from obvious whether there exist solutions saturating the BPS inequality other than the Gibbons-Maeda solution [2]. Bearing the above remark in mind, we will classify solutions admitting a Killing spinor which satisfies the 1st-order differential equations (2.20) and (2.21) with careful attention to the dilaton equation. We shall refer to such solutions as "supersymmetric" in what follows.

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[3] If the "scalar charge" Σ is introduced by the asymptotic value of the scalar field as $\phi \sim \pm\Sigma/|\vec{x}|$, the nonextremal metric in [1] admits an inequality $M^2 + \Sigma^2 \geq Q_e^2 + Q_m^2$, which is saturated by the BPS state (2.22). It is worthwhile to emphasize that this inequality differs from (2.19) in philosophy: the Bogomol'nyi inequality (2.19) is expressed only in terms of global charges, while the above force-balance condition involves the scalar charge which is inherently secondary since it is not defined covariantly by the two-sphere surface integral at infinity.

III. SUPERSYMMETRIC SOLUTIONS

A basic strategy for the classification of BPS solutions is to assume the existence of at least one Killing spinor and construct its bilinear tensor quantities. These satisfy a number of algebraic and differential conditions, which can be used to deduce the bosonic constituents. This program follows the work of Caldarelli and Klemm [27].

A. Differential forms constructed from a Killing spinor

Given a commuting spinor ϵ, we can define the following bilinear bosonic differential forms [27]

$$\begin{aligned} &\textit{a scalar} && E := \bar\epsilon\epsilon\,, && (3.1)\\ &\textit{a pseudo scalar} && B := \mathrm{i}\bar\epsilon\gamma_5\epsilon\,, && (3.2)\\ &\textit{a vector} && V_\mu := \mathrm{i}\bar\epsilon\gamma_\mu\epsilon\,, && (3.3)\\ &\textit{a pseudo vector} && a_\mu := \mathrm{i}\bar\epsilon\gamma_5\gamma_\mu\epsilon\,, && (3.4)\\ &\textit{an anti-symmetric tensor} && \Phi_{\mu\nu} := \mathrm{i}\bar\epsilon\gamma_{\mu\nu}\epsilon\,, && (3.5)\end{aligned}$$

Here we have introduced the factor "i" to ensure these differential forms to be real in our convention. Since $\{\mathbf{1},\gamma_5,\gamma_\mu,\gamma_\mu\gamma_5,\gamma_{\mu\nu}\}$ span the basis of Clifford algebra, any other differential forms can be built from linear combination of above quantities.

A (Dirac) spinor ϵ has a real dimension 8, whereas $(E, B, V_\mu, a_\mu, \Phi_{\mu\nu})$ sum up to have 16 components. This means that these bilinears are not all independent. In fact, viewing $\epsilon\bar\epsilon$ as a 4×4 matrix, it can be expanded by gamma-matrix basis as

$$4\epsilon\bar\epsilon = E\mathbf{1} - \mathrm{i}V^\mu\gamma_\mu + \frac{\mathrm{i}}{2}\Phi^{\mu\nu}\gamma_{\mu\nu} + \mathrm{i}a^\mu\gamma_5\gamma_\mu - \mathrm{i}B\gamma_5\,, \tag{3.6}$$

which implies

$$\mathrm{i}V^\mu\gamma_\mu\epsilon = -\mathrm{i}a^\mu\gamma_5\gamma_\mu\epsilon = -(E+\mathrm{i}B\gamma_5)\epsilon\,, \qquad \mathrm{i}\Phi^{\mu\nu}\gamma_{\mu\nu}\epsilon = 2(E-\mathrm{i}B\gamma_5)\epsilon\,. \tag{3.7}$$

Contraction with $\bar\epsilon$ gives

$$f := -V^\mu V_\mu = a^\mu a_\mu = E^2 + B^2\,, \tag{3.8}$$

$$E^2 - B^2 = \tfrac{1}{2}\Phi_{\mu\nu}\Phi^{\mu\nu}\,. \tag{3.9}$$

We can find that V^μ is everywhere causal, while a^μ cannot be timelike. The possibility of $V^\mu\equiv 0$ can be eliminated by noticing $V^0 = \epsilon^\dagger\epsilon > 0$ for a nonvanishing Killing spinor. This also signifies that V^μ is future-directed. Contracting $\bar\epsilon\gamma_5$ to (3.7), it is shown that V and a are orthogonal $V^\mu a_\mu = 0$.

Using (3.6) and availing ourselves of the expressions summarized in appendix, it is straightforward to derive the following algebraic constraints

$$EV_\mu = \star\Phi_{\mu\nu}a^\nu\,, \qquad Ea_\mu = \star\Phi_{\mu\nu}V^\nu\,, \tag{3.10a}$$

$$BV_\mu = \Phi_{\mu\nu}a^\nu\,, \qquad Ba_\mu = \Phi_{\mu\nu}V^\nu\,, \tag{3.10b}$$

$$EB = -\tfrac{1}{4}\Phi_{\mu\nu}\star\Phi^{\mu\nu}\,, \tag{3.10c}$$

$$E\Phi_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}V^\rho a^\sigma + B\star\Phi_{\mu\nu}\,, \tag{3.10d}$$

$$\Phi_{(\mu}{}^\rho\star\Phi_{\nu)\rho} = \tfrac{1}{4}g_{\mu\nu}\Phi_{\rho\sigma}\star\Phi^{\rho\sigma}\,. \tag{3.10e}$$

Upon using (3.8) and (3.9), a little bit amount of calculation shows

$$\Phi_{\mu\rho}\Phi_\nu{}^\rho = V_\mu V_\nu - a_\mu a_\nu + g_{\mu\nu}E^2\,, \qquad \star\Phi_{\mu\rho}\star\Phi_\nu{}^\rho = V_\mu V_\nu - a_\mu a_\nu + g_{\mu\nu}B^2\,. \tag{3.11}$$

This relation will be of use for the classification of null class.

Let us turn to the analysis of differential relations. Now suppose that ϵ satisfies the Killing spinor equation (2.20).

Noticing (2.11), we can derive the following differential constraints

$$\nabla_\mu E = \frac{e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} F_{\mu\nu} V^\nu \,, \tag{3.12a}$$

$$\nabla_\mu B = -\frac{e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \star F_{\mu\nu} V^\nu \,, \tag{3.12b}$$

$$\nabla_\mu V_\nu = \frac{e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \left(-E F_{\mu\nu} + B \star F_{\mu\nu}\right) \,, \tag{3.12c}$$

$$\nabla_\mu a_\nu = \frac{e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \left(-2F_{(\mu}{}^{\rho} \star \Phi_{\nu)\rho} + \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} \star \Phi^{\rho\sigma}\right) \,, \tag{3.12d}$$

$$\nabla_\mu \Phi_{\nu\rho} = -\frac{e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \left(a_\mu \star F_{\nu\rho} + 2\epsilon_{\nu\rho\sigma[\mu} F_{\tau]}{}^{\sigma} a^\tau\right) \,. \tag{3.12e}$$

We can thus identify E and B as the electric and magnetic potentials, respectively. Equation (3.12c) indicates that V^μ is a Killing vector

$$\nabla_{(\mu} V_{\nu)} = 0 \,. \tag{3.13}$$

From (3.12d) we find $\nabla_{[\mu} a_{\nu]} = 0$, i.e., a_μ is a pure gradient vector.

Next, let us look into the dilatino equation (2.21). Contracting it with $\bar\epsilon$, $\bar\epsilon\gamma_5$, $\bar\epsilon\gamma_\mu$, $\bar\epsilon\gamma_5\gamma_\mu$ and $\bar\epsilon\gamma_{\mu\nu}$ we obtain the following relations

$$V^\mu \nabla_\mu \phi = 0 \,, \tag{3.14a}$$

$$\alpha \Phi_{\mu\nu} F^{\mu\nu} = 0 \,, \tag{3.14b}$$

$$a^\mu \nabla_\mu \phi + \frac{\alpha e^{-\alpha\phi}}{2\sqrt{1+\alpha^2}} F_{\mu\nu} \star \Phi^{\mu\nu} = 0 \,, \tag{3.14c}$$

$$E \nabla_\mu \phi - \frac{\alpha e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} F_{\mu\nu} V^\nu = 0 \,, \tag{3.14d}$$

$$\Phi_{\mu\nu} \nabla^\nu \phi - \frac{\alpha e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \star F_{\mu\nu} a^\nu = 0 \,, \tag{3.14e}$$

$$B \nabla_\mu \phi - \frac{\alpha e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \star F_{\mu\nu} V^\nu = 0 \,, \tag{3.14f}$$

$$\star\Phi_{\mu\nu} \nabla^\nu \phi - \frac{\alpha e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} F_{\mu\nu} a^\nu = 0 \,, \tag{3.14g}$$

$$2V_{[\mu} \nabla_{\nu]} \phi - \frac{\alpha e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \left(B \star F_{\mu\nu} + E F_{\mu\nu}\right) = 0 \,, \tag{3.14h}$$

$$\epsilon_{\mu\nu\rho\sigma} a^\rho \nabla^\sigma \phi + \frac{2\alpha e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} F_{[\mu}{}^{\rho} \Phi_{\nu]\rho} = 0 \,. \tag{3.14i}$$

Contraction with $\bar\epsilon\gamma_5\gamma_{\mu\nu}$ yields the duals of (3.14h) and (3.14i). When the Bianchi identity $\mathrm{d}F = 0$ and the Maxwell equations $\mathrm{d} \star (e^{-2\alpha\phi} F) = 0$ are satisfied, equations (3.12a), (3.12b), (3.14a), (3.14d) and (3.14f) give

$$\mathscr{L}_V F = 0 \,, \qquad \mathscr{L}_V \star F = 0 \,, \qquad \mathscr{L}_V \phi = 0 \,, \tag{3.15}$$

where $\mathscr{L}_V = \mathrm{d} i_V + i_V \mathrm{d}$ is the Lie derivative and i_V is the internal product. It turns out that a vector field V constructed from a Killing spinor generates the symmetry of all the bosonic constituents $(g_{\mu\nu}, F_{\mu\nu}, \phi)$. This is not an obvious result since the Killing symmetry just requires that $\mathscr{L}_V F$ is proportional to $\star F$ even in the Einstein-Maxwell system (see Theorem 11.1 in [15]).

To proceed, we will examine separately the cases where the Killing vector is timelike or null. The algebraic and differential constraints derived in this section are solved for each case.

B. Timelike family

Let us begin by the analysis for the case of timelike V, i.e., f is nowhere nonvanishing which we take $f > 0$. Equations (3.10a) and (3.10b) can be solvable for $\Phi_{\mu\nu}$, giving

$$\Phi_{\mu\nu} = \frac{1}{f}\left(2BV_{[\mu}a_{\nu]} - E\epsilon_{\mu\nu\rho\sigma}V^\rho a^\sigma\right), \qquad \star\Phi_{\mu\nu} = \frac{1}{f}\left(2EV_{[\mu}a_{\nu]} + B\epsilon_{\mu\nu\rho\sigma}V^\rho a^\sigma\right). \tag{3.16}$$

These expressions are consistent with other equations (3.9), (3.10c)–(3.10e) and (3.11). Analogously equations (3.12a) and (3.12b) combine to give

$$F_{\mu\nu} = \frac{e^{\alpha\phi}\sqrt{1+\alpha^2}}{f}\left(2V_{[\mu}\nabla_{\nu]}E + \epsilon_{\mu\nu\rho\sigma}V^\rho\nabla^\sigma B\right), \qquad \star F_{\mu\nu} = \frac{e^{\alpha\phi}\sqrt{1+\alpha^2}}{f}\left(-2V_{[\mu}\nabla_{\nu]}B + \epsilon_{\mu\nu\rho\sigma}V^\rho\nabla^\sigma E\right). \tag{3.17}$$

From these expressions, one can easily verify

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= \frac{2e^{2\alpha\phi}(1+\alpha^2)}{f}\left[(\nabla B)^2 - (\nabla E)^2\right], & F_{\mu\nu}\star F^{\mu\nu} &= \frac{4e^{2\alpha\phi}(1+\alpha^2)}{f}\nabla_\mu E\nabla^\mu B, \\ F_{\mu\nu}\star\Phi^{\mu\nu} &= \frac{\sqrt{1+\alpha^2}e^{\alpha\phi}}{f}a^\mu\nabla_\mu(B^2-E^2), & \Phi_{\mu\nu}F^{\mu\nu} &= -\frac{2\sqrt{1+\alpha^2}e^{\alpha\phi}}{f}a^\mu\nabla_\mu(EB). \end{aligned} \tag{3.18}$$

Substituting (3.16) and (3.17) into (3.12c) and (3.12d), we obtain

$$\nabla_\mu V_\nu = f^{-1}\left[-V_{[\mu}\nabla_{\nu]}f - \epsilon_{\mu\nu\rho\sigma}V^\rho(E\nabla^\sigma B - B\nabla^\sigma E)\right], \tag{3.19}$$

$$\nabla_\mu a_\nu = -\tfrac{1}{2}g_{\mu\nu}a^\rho\nabla_\rho(\ln f) + a_{(\mu}\nabla_{\nu)}\ln f - f^{-2}V_\mu V_\nu a^\rho\nabla_\rho f + 2f^{-2}V_{(\mu}\epsilon_{\nu)\rho\sigma\tau}(E\nabla^\rho B - B\nabla^\rho E)V^\sigma a^\tau. \tag{3.20}$$

Using (3.16), (3.17), (3.19) and (3.20), a lengthy calculation shows that (3.12e) is fulfilled automatically.

Inserting (3.17) into the Maxwell equation $\mathrm{d}\star(e^{-2\alpha\phi}F) = 0$ and the Bianchi identity $\mathrm{d}F = 0$, we find

$$f^2\nabla^\mu(f^{-1}\nabla_\mu E) + \Omega_\mu\nabla^\mu B - \alpha f\nabla_\mu\phi\nabla^\mu E = 0, \tag{3.21}$$

$$f^2\nabla^\mu(f^{-1}\nabla_\mu B) - \Omega_\mu\nabla^\mu E + \alpha f\nabla_\mu\phi\nabla^\mu B = 0, \tag{3.22}$$

where we have used an abbreviation

$$\Omega_\mu = 2(E\nabla_\mu B - B\nabla_\mu E), \tag{3.23}$$

which corresponds to the twist of V^μ, i.e., $\Omega_\mu = \epsilon_{\mu\nu\rho\sigma}V^\nu\nabla^\rho V^\sigma$. Equation (3.23) manifests that the supersymmetric solution can be rotating only in the dyonic case.

At this stage we introduce a local coordinate system. Since V is Killing $\nabla_{(\mu}V_{\nu)} = 0$, the most desirable choice is $V^\mu = (\partial/\partial t)^\mu$ for which the metric components are independent of the time coordinate t. Thus, the spacetime metric can be locally written as a twisted fibre bundle over the 3-space as

$$\mathrm{d}s^2 = -f(\mathrm{d}t + \omega)^2 + f^{-1}h_{mn}\mathrm{d}x^m\mathrm{d}x^n, \tag{3.24}$$

where $f^{-1}h_{mn}$ $(m, n, ... = 1, 2, 3)$ is the metric of the orbit space of the action of V. The 1-form ω corresponds to the rotation of V, which measures the gravito-electromagnetic Sagnac connection. Viewing $V = -f(\mathrm{d}t + \omega)$, equation (3.19) gives the governing equation for ω as

$$\nabla_{[\mu}\omega_{\nu]} = \frac{1}{2f^2}\epsilon_{\mu\nu\rho\sigma}V^\rho\Omega^\sigma, \tag{3.25}$$

which determines ω uniquely modulo a gradient of a scalar function.

Besides, there exists a local scalar z such that $a_\mu = \nabla_\mu z$ due to $\mathrm{d}a = 0$, which also can be used as one of the coordinate. Thus, the 3-metric h_{mn} may be decomposed as

$$h_{mn}\mathrm{d}x^m\mathrm{d}x^n = \tilde{h}_{MN}(\mathrm{d}x^M + k^M\mathrm{d}z)(\mathrm{d}x^N + k^N\mathrm{d}z) + \mathrm{d}z^2, \tag{3.26}$$

where the indices $M, N, ...$ range from 1 to 2 with $x^1 = x$ and $x^2 = y$.

Observe that the above metric form has a large degrees of gauge freedom. One may easily deduce that the metric is invariant under the change of coordinate

$$t \to t - \lambda(x^m)\,, \qquad \text{and} \qquad \omega \to \omega + \mathrm{d}\lambda(x^m)\,, \tag{3.27}$$

which is the gauge transformation of the Kaluza-Klein gauge field ω. This freedom will be used to eliminate the integration function arising from (3.25), so we remain it unspecified at present. In addition the coordinate transformation $x^M = x^M(x'^N, z)$ is permissible. Using this freedom we can always choose the coordinates x^M in such a way that

$$x^M = x^M(x'^N, z)\,, \qquad \text{with} \qquad \frac{\partial x^M}{\partial z} = -k^M\,, \tag{3.28}$$

which eliminates the vector k^M from the metric. In the following discussion we can, without loss of no generality, restrict the 3-metric h_{mn} to take the form,

$$h_{mn}\mathrm{d}x^m\mathrm{d}x^n = \tilde{h}_{MN}(x, y, z)\mathrm{d}x^M\mathrm{d}x^N + \mathrm{d}z^2\,. \tag{3.29}$$

We shall refer to this 3-dimensional Riemannian manifold as a "base space."

Let us turn to examine (3.20). Define a projection operator

$$\tilde{h}^\mu{}_\nu = f\delta^\mu{}_\nu + V^\mu V_\nu - a^\mu a_\nu\,, \tag{3.30}$$

which can be regarded as $\tilde{h}_{\mu\nu} = \tilde{h}_{MN}(\nabla_\mu x^M)(\nabla_\nu x^N)$. The nonvanishing components of (3.20) boil down to

$$\tilde{h}^\rho{}_\mu \tilde{h}^\sigma{}_\nu \nabla_\rho a_\sigma = 0\,. \tag{3.31}$$

We can view this equation as such that the level set $z =$ constant is a totally geodesic submanifold with respect to the base space $\tilde{h}_{MN}\mathrm{d}x^M\mathrm{d}x^N + \mathrm{d}z^2$, i.e., its extrinsic curvature is zero. This requires that $\tilde{h}_{MN}$ is independent of the coordinate z, $\partial_z\tilde{h}_{MN} = 0$.

We next investigate the dilatino equation (3.14), which are divided into two cases depending on $\alpha \neq 0$ or $\alpha = 0$. In the following subsections we shall examine these cases separately.

1. *The $\alpha \neq 0$ case*

Inspecting (3.12a) and (3.14d) one finds

$$\nabla_\mu(Ee^{-\phi/\alpha}) = 0\,. \tag{3.32}$$

Similarly, equations (3.12b) and (3.14f) give

$$\nabla_\mu(Be^{\phi/\alpha}) = 0\,. \tag{3.33}$$

These are easily solved as

$$E = c_{\mathrm{E}}e^{\phi/\alpha}\,, \qquad B = c_{\mathrm{B}}e^{-\phi/\alpha}\,, \tag{3.34}$$

where $(c_{\mathrm{E}}, c_{\mathrm{B}})$ are constants. Taking note of a useful relation

$$\nabla_\mu\phi - \frac{\alpha}{2f}\nabla_\mu(E^2 - B^2) = 0\,, \tag{3.35}$$

one can find that all other dilatino equations (3.14) are satisfied. Unlike the ordinary supergravities, we must check the dilaton field equation so as to keep the consistency with the dilatino equation. Substitution of (3.34) into (2.6) yields

$$D_m\left(f^{-(1+\alpha^2)/2}D^m\phi\right) = 0\,, \tag{3.36}$$

where D_m denotes the covariant derivative associated with the base space metric h_{mn}. Indices $m, n, ...$ are raised and lowered by h_{mn} and its inverse. The above equation (3.36) is to be compared with the equations for the gauge fields below.

Substituting (3.34), equations (3.21) and (3.22) simplify to

$$c_{\rm E}\left[D^2\phi+\left\{\frac{1-\alpha^2}{\alpha}-\frac{2(E^2-3B^2)}{\alpha(E^2+B^2)}\right\}(D\phi)^2\right]=0\,, \tag{3.37}$$

$$c_{\rm B}\left[D^2\phi-\left\{\frac{1-\alpha^2}{\alpha}+\frac{2(3E^2-B^2)}{\alpha(E^2+B^2)}\right\}(D\phi)^2\right]=0\,. \tag{3.38}$$

Consider first the case $(c_{\rm E}, c_{\rm B})\neq 0$ where the solution is dyonic ($F_{\mu\nu}\star F^{\mu\nu}\neq 0$). The above two equations yield

$$(3-\alpha^2)(D\phi)^2=0\,, \tag{3.39}$$

and

$$D^2\left[\left(c_{\rm E}^2e^{4\phi/\alpha}+c_{\rm B}^2\right)^{-1}\right]=0\,. \tag{3.40}$$

For the generic coupling ($\alpha\neq\sqrt{3}$), equation (3.39) implies that the supersymmetric dyonic solution has a constant dilaton field, whence $E=B=$ constant. Since the Maxwell field vanishes in the constant dilaton case [see (3.17)], this is nothing but a vacuum supersymmetric solution, i.e., the Minkowski spacetime.

In the dyonic case, equation (3.39) shows that a nontrivial dilaton arises only for $\alpha=\sqrt{3}$, which corresponds to the Kaluza-Klein compactification of 5-dimensional vacuum gravity. Indeed equations (3.36) and (3.40) are compatible if and only if $\alpha=\sqrt{3}$, as expected from (2.25). Furthermore, in the dyonic case, only the $\alpha=\sqrt{3}$ case is consistent with the integrability condition of (3.25):

$$\nabla_\mu\left(f^{-2}\Omega^\mu\right)=0\,. \tag{3.41}$$

From (3.17) the dilaton is given by

$$\phi=\frac{\sqrt{3}}{4}\ln\left(\frac{c_{\rm E}^2+c_{\rm B}^2-c_{\rm B}^2H}{c_{\rm E}^2H}\right)\,, \tag{3.42}$$

where H stands for a harmonic function on the base space $D^2H=0$. The lapse function f and the rotation form ω (3.25) are successively obtained as

$$f=\frac{c_{\rm E}(c_{\rm E}^2+c_{\rm B}^2)}{\sqrt{H(c_{\rm E}^2+c_{\rm B}^2-c_{\rm B}^2H)}}\,,\qquad \partial_{[m}\omega_{n]}=\frac{c_{\rm B}}{2c_{\rm E}(c_{\rm E}^2+c_{\rm B}^2)}{}^{(h)}\epsilon_{mnp}D^pH\,, \tag{3.43}$$

where ${}^{(h)}\epsilon_{mnp}$ is the volume-element compatible with the 3-metric h_{mn} of the base space (3.29) with $V\wedge{}^{(h)}\epsilon$ being positively oriented. From (3.40), we can find the gauge potential $F={\rm d}A$,

$$A=\frac{c_{\rm E}^2+c_{\rm B}^2}{2c_{\rm E}H}\left({\rm d}t+\omega\right)\,. \tag{3.44}$$

One can also obtain the corresponding dualized ones (2.7):

$$\tilde\phi=-\frac{\sqrt{3}}{4}\ln\left(\frac{c_{\rm E}^2+c_{\rm B}^2-c_{\rm B}^2H}{c_{\rm E}^2H}\right)\,,\qquad \tilde A=-\frac{c_{\rm B}(c_{\rm E}^2+c_{\rm B}^2)(1-H)}{2(c_{\rm E}^2+c_{\rm B}^2-c_{\rm B}^2H)}\left[{\rm d}t+\frac{c_{\rm E}^2\omega}{c_{\rm B}^2(1-H)}\right]\,, \tag{3.45}$$

where $\tilde F={\rm d}\tilde A$.

Although we have introduced two integration constants $(c_{\rm E}, c_{\rm B})$, only one of them is of physical relevance. Consider a scaling the Killing spinor

$$\epsilon\ \to\ C\epsilon\,, \tag{3.46}$$

where C is a complex constant. Then the metric (3.24) and the Maxwell field (3.17) transform as $f\to|C|^4f$, $\omega\to|C|^{-4}\omega$ and $F\to|C|^{-2}F$, that is to say, we can choose $c_{\rm E}$ or $c_{\rm B}$ to take any value we wish. The choice $c_{\rm E}={\rm sech}\sigma$ and $c_{\rm B}=\tanh\sigma$ ($\sigma\in\mathbb{R}$) is physically definitive provided H goes to unity at infinity, since $\phi\to 0$ and $f\to 1$ for the above value.

In the purely electric case, i.e., $c_{\rm E} \neq 0$ and $c_{\rm B} = 0$, one can set $c_{\rm E} = 1$ by the scaling freedom as described above. In this case, the Bianchi identity automatically holds and α can take any value since $F_{\mu\nu} \star F^{\mu\nu} = 0$ is satisfied. Then we find from (3.37) that the dilaton and the electromagnetic fields are given by

$$\phi = -\frac{\alpha}{1+\alpha^2} \ln H \,, \qquad A = \frac{{\rm d}t}{\sqrt{1+\alpha^2}H} \,. \tag{3.47}$$

Here H is a harmonic function on the base space, $D^2 H = 0$. For the purely magnetic case, setting $c_{\rm E} = 0$ and $c_{\rm B} \equiv 1$ amounts to the duality rotation (2.7) of the purely electric case:

$$\tilde{\phi} = \frac{\alpha}{1+\alpha^2} \ln H \,, \qquad \partial_{[m} \tilde{A}_{n]} = -\frac{1}{2\sqrt{1+\alpha^2}} {}^{(h)}\epsilon_{mnp} D^p H \,. \tag{3.48}$$

Since either electric field or magnetic field vanishes, $\Omega_\mu = 0$ holds, to wit ${\rm d}\omega = 0$. Hence ω is locally gradient of some scalar function, which can be made to vanish by incorporating into the definition of t by exploiting the gauge freedom (3.27). It follows that V is hypersurface orthogonal and the spacetime is static for the purely electric/magnetic case.

Remark that the 2-metric $\tilde{h}_{MN}(x, y)$ is still unrestricted at the current moment.

2. The $\alpha = 0$ case

Next, we discuss the $\alpha = 0$ case. Contraction of V^μ to (3.14h) gives $\phi =$ constant. It follows that the Brans-Dicke-Maxwell system reduces to a usual Einstein-Maxwell theory due to supersymmetry. Thus its timelike family of supersymmetric solution is given by the Israel-Wilson-Perjés (IWP) solution [42]. For completeness we shall also discuss this case within the present framework, which should recover the result in [23]. Let $\Psi = E - {\rm i}B$ denote a complex Ernst-Maxwell potential [11]. Then the Maxwell equations ${\rm d} \star F = 0$ (3.21) and the Bianchi identity ${\rm d}F = 0$ (3.22) are combined to give the 3-dimensional (complex) Laplace equation

$$D^2 \Psi^{-1} = 0 \,. \tag{3.49}$$

Looking at (3.23), the solution can be rotating only in the dyonic case. The undetermined 2-metric $\tilde{h}_{MN}$ will be found by the integrability condition of the Killing spinor equation, as demonstrated below.

3. Integrability condition

So far we have been discussed constraints on the geometry and matter fields which are necessary for the existence of Killing spinor. We have exhausted the equations satisfied by bosonic quantities, leaving the 2-metric $\tilde{h}_{MN}$ undetermined. We shall next look at the Killing spinor equations and examine whether further restriction is imposed. Adopting the tetrad frame

$$e^0 = f^{1/2}({\rm d}t + \omega) \,, \qquad e^I = f^{-1/2} \hat{e}^I \quad (I = 1, 2) \,, \qquad e^3 = f^{-1/2} {\rm d}z \,, \tag{3.50}$$

where $\tilde{h}_{MN} = \delta_{IJ} \hat{e}^I{}_M \hat{e}^J{}_N$, equation (3.7) reads

$${\rm i}\gamma^0 \epsilon = f^{-1/2}(E + {\rm i}B\gamma_5)\epsilon \,. \tag{3.51}$$

Under this condition, the time and spatial components of Killing spinor equation are written as

$$\partial_t \epsilon = 0 \,, \qquad \left[D_m - \omega_m \partial_t + \frac{1}{4f} \left(-\partial_m f + 2{\rm i}\Omega_m \gamma_5 \right) \right] \epsilon = 0 \,, \tag{3.52}$$

where we have treated the spatial components at once instead of discriminating components x, y from z. The first equation shows that the Killing spinor is time-independent. Defining chiral spinors

$$\epsilon^\pm := \frac{1 \pm \gamma_5}{2\sqrt{E \mp {\rm i}B}} \epsilon \,, \tag{3.53}$$

with $\gamma_5\epsilon^\pm = \pm\epsilon^\pm$, the second equation of (3.52) reduces to

$$D_m\epsilon^\pm = 0\,, \tag{3.54}$$

viz, $\epsilon^\pm$ are covariantly constant spinors for the base space. It follows that the the solution of the Killing spinor equation is given by

$$\epsilon = \sqrt{E - \mathrm{i}B}\epsilon^+ + \sqrt{E + \mathrm{i}B}\epsilon^- \tag{3.55}$$

where $\epsilon^\pm$ are the spatially parallel and chiral spinors satisfying $\gamma_5\epsilon^\pm = \pm\epsilon^\pm$. In the purely electric or magnetic case where α is arbitrary, it is further simplified to

$$\epsilon = H^{-1/[2(1+\alpha^2)]}\epsilon_\infty\,, \tag{3.56}$$

where H is harmonic (3.47) and ϵ_∞ is the spatially parallel spinor independent of t, corresponding to the asymptotic value of ϵ and satisfying $\mathrm{i}\gamma^0\epsilon_\infty = \epsilon_\infty$. It is worth commenting that the condition $\mathrm{i}\gamma_5\gamma^3\epsilon = f^{-1/2}(E + \mathrm{i}B\gamma_5)\epsilon$ is not used to derive (3.55) and (3.56).

The integrability condition of (3.54) is

$$0 = [D_m, D_n]\epsilon^\pm = \frac{1}{2}\left(h_{m[p}{}^{(h)}S_{q]n} - h_{n[p}{}^{(h)}S_{q]m}\right)\gamma^{pq}\epsilon^\pm\,, \tag{3.57}$$

where we have replaced the Riemann tensor by the Schouten tensor ${}^{(h)}S_{mn} := {}^{(h)}R_{mn} - (1/4){}^{(h)}Rh_{mn}$ for the 3-metric h_{mn}. Contracting with $\bar\epsilon$ and $\bar\epsilon\gamma_5$, we obtain

$${}^{(h)}S^p{}_{[m}\Phi_{n]p} = 0\,, \qquad {}^{(h)}S^p{}_{[m}\star\Phi_{n]p} = 0\,. \tag{3.58}$$

Combined with (3.16), (3.29) and (3.31), we come to the conclusion that the base space (3.29) is Ricci flat (${}^{(h)}R_{mn} = 0$), thence flat since it is 3-dimensional. This means that the spacetime is conforma-stationary [15] and $(\epsilon^\pm, \epsilon_\infty)$ are constant spinors. We can also find that the dilatino equation (2.21) is satisfied automatically under the projection (3.51). Since equation (3.51) is the only restriction, the solution preserves at least half of supersymmetries.

We have only solved the gravitino and dilatino Killing equations, the dilaton equation of motion, the Maxwell equations and the Bianchi identity. We have nowhere used Einstein's equations, but they automatically hold as an integrability condition for the Killing spinor equation. From (2.20), we get

$$\nabla_{[\mu}\nabla_{\nu]}\epsilon = \frac{1}{8}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\epsilon = -\frac{\mathrm{i}}{4\sqrt{1+\alpha^2}}\gamma^{\rho\sigma}\gamma_{[\nu}\nabla_{\mu]}(e^{-\alpha\phi}F_{\rho\sigma})\cdot\epsilon - \frac{e^{-2\alpha\phi}}{16(1+\alpha^2)}\gamma^{\rho\sigma}\gamma_{[\nu}\gamma^{\lambda\tau}\gamma_{\mu]}F_{\rho\sigma}F_{\lambda\tau}\epsilon\,. \tag{3.59}$$

Contracting γ^ν to this equation and *using the dilatino equation* (2.21) and the first Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$, we find

$$\left[\mathcal{E}_{\mu\nu}\gamma^\nu + \frac{\mathrm{i}}{8\sqrt{1+\alpha^2}}e^{-\alpha\phi}\left\{\gamma^{\nu\rho\sigma}\gamma_\mu\nabla_{[\nu}F_{\rho\sigma]} - 2e^{2\alpha\phi}\gamma_\nu\gamma_\mu\nabla_\rho(e^{-2\alpha\phi}F^{\nu\rho})\right\}\right]\epsilon = 0\,, \tag{3.60}$$

where we have used the identity (A2) and we have defined

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - 2(\nabla_\mu\phi)(\nabla_\nu\phi) - T^{(\mathrm{em})}_{\mu\nu}\,. \tag{3.61}$$

Here, $\mathcal{E}_{\mu\nu} = 0$ is equivalent to Einstein's equations (2.2). From (3.60), when the Bianchi identity and the Maxwell equations for F are satisfied, we deduce that

$$\mathcal{E}_{\mu\nu}\gamma^\nu\epsilon = 0\,. \tag{3.62}$$

Contracting with $\bar\epsilon$, one finds

$$\mathcal{E}_{\mu\nu}V^\nu = 0\,. \tag{3.63}$$

If we dot it with $\mathcal{E}_{\mu\rho}\gamma^\rho$, we get

$$\mathcal{E}_{\mu\nu}\mathcal{E}_\mu{}^\nu = 0\,, \qquad (\text{no sum on } \mu)\,. \tag{3.64}$$

In the orthonormal frame, equation (3.63) implies $\mathcal{E}_{00} = \mathcal{E}_{0i} = 0$ where $i, j, .. = 1, 2, 3$ and (3.64) implies $\mathcal{E}_{ij} = 0$, as we desired to show. This has been already demonstrated in (3.58).

4. Summary

Let us encapsulate our results. The timelike family of supersymmetric solutions in Einstein-Maxwell-dilaton system where the dilatino equation implies the dilaton equation are either of the followings:

(i) *A dyonic and rotating solution for* $\alpha=\sqrt{3}$: the metric is written as a conforma-stationary form

$$\mathrm{d}s^2=-f\,(\mathrm{d}t+\vec{\omega}\cdot\mathrm{d}\vec{x})^2+f^{-1}\mathrm{d}\vec{x}^2\,. \tag{3.65}$$

where H is harmonic on the base space $\vec{\nabla}^2H=0$, f and $\vec{\omega}$ are given by (3.43) and the dilaton and the gauge fields are (3.42), (3.44) or (3.45). The solution of the Killing spinor equation is given by (3.55) where E and B are given by (3.34).

(ii) *A purely electric or magnetic static solution for arbitrary* α: the spacetime is the Gibbons-Maeda solution [2],

$$\mathrm{d}s^2=-H^{-2/(1+\alpha^2)}\mathrm{d}t^2+H^{2/(1+\alpha^2)}\mathrm{d}\vec{x}^2\,, \tag{3.66}$$

where the dilaton and the gauge fields are given by (3.47) for the electric case and by (3.48) for the magnetic case. The solution of the Killing spinor equation is given by (3.56).

(iii) *A dyonic and rotating solution for* $\alpha=0$: this reduces to the BPS solution in the Einstein-Maxwell system and described by the IWP metric [42],

$$\mathrm{d}s^2=-|\Psi|^2(\mathrm{d}t+\vec{\omega}\cdot\mathrm{d}\vec{x})^2+|\Psi|^{-2}\mathrm{d}\vec{x}^2\,, \tag{3.67}$$

where Ψ is a complex harmonic function $\vec{\nabla}^2\Psi=0$ and $\vec{\omega}$ is given by quadrature (3.25) as

$$\vec{\nabla}\times\vec{\omega}=\mathrm{i}\left(\bar{\Psi}^{-1}\vec{\nabla}\Psi^{-1}-\Psi^{-1}\vec{\nabla}\bar{\Psi}^{-1}\right)\,. \tag{3.68}$$

The solution of the Killing spinor is given by (3.55) with E and B obeying (3.49).

C. Null family

In this section we study the case in which V^μ is null, i.e., $E=B=0$. The Maxwell field $F_{\mu\nu}$ and $\Phi_{\mu\nu}$ satisfy

$$\begin{aligned}&i_VF=0\,,\qquad i_V\star F=0\,,\qquad i_V\Phi=0\,,\qquad i_V\star\Phi=0\,,\\&\Phi_{\mu\nu}\Phi^{\mu\nu}=0\,,\qquad \Phi_{\mu\nu}\star\Phi^{\mu\nu}=0\,,\qquad \Phi_{(\mu}{}^{\rho}\star\Phi_{\nu)\rho}=0\,.\end{aligned} \tag{3.69}$$

These relations are sufficient to establish

$$F_{\mu\nu}F^{\mu\nu}=0\,,\qquad F^{\mu\mu}\star F^{\mu\nu}=0\,,\qquad F_{\mu\nu}\Phi^{\mu\nu}=0\,,\qquad F_{\mu\nu}\star\Phi^{\mu\nu}=0\,. \tag{3.70}$$

As opposed to the timelike case, $F_{\mu\nu}\star F^{\mu\nu}=0$ always holds for the null case. We are thus not concerned with the dilaton field equation since it is ensured by dilatino equation. The dilatino equation (3.14) imposes a single restriction

$$V\wedge\mathrm{d}\phi=0\,. \tag{3.71}$$

Equation (3.12c) means that the vector field V^μ is covariantly-conserved $\nabla_\mu V_\nu=0$, i.e., the spacetime of the null family describes a *pp*-wave [15]. The *pp*-wave spacetime always belongs to the Petrov type N. Since V is closed $\mathrm{d}V=0$ and tangent to the affine parametrized geodesic $V^\mu\nabla_\mu V^\nu=0$, V can be written as

$$V_\mu=-\nabla_\mu u\,,\qquad V^\mu=\left(\frac{\partial}{\partial v}\right)^\mu\,, \tag{3.72}$$

where u is some scalar function and v is an affine parameter of the geodesics. Then the metric is independent of v and can be cast into the form [15]

$$\mathrm{d}s^2=-2\mathrm{d}u\left(\mathrm{d}v+\mathcal{H}\mathrm{d}u+\beta_i\mathrm{d}x^i\right)+\tilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{3.73}$$

Here $\mathcal{H}$, β_i and $\tilde{g}_{ij}$ $(i, j = 1, 2)$ are functions of u and x^i. Using the coordinate transformation of x^i, the 2-dimensional metric $\tilde{g}_{ij}$ can be written in a conformally flat form,

$$\tilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j = \Omega^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2\right) . \tag{3.74}$$

Equation (3.71) now implies that the dilaton is a function only of u, $\phi = \phi(u)$. One may thence regard the scalar field as a "null dust" since the stress-tensor takes the form $T^{(\phi)}_{\mu\nu} = 2(\mathrm{d}\phi/\mathrm{d}u)^2 V_\mu V_\nu$.

Due to $a^\mu a_\mu = f = 0$, the pseudo-vector a^μ is null or identically zero. Since $V \wedge a = 0$, there exists a function $\kappa = \kappa(u, x^i)$ such that

$$a_\mu = \kappa V_\mu \,. \tag{3.75}$$

From $\mathrm{d}a = 0$, one finds $\kappa = \kappa(u)$, hence $\nabla_\mu \kappa = -[\mathrm{d}\kappa(u)/\mathrm{d}u]V_\mu$. It follows that equations (3.12d) and (3.12e) simplify to

$$\frac{\mathrm{d}\kappa}{\mathrm{d}u} V_\mu V_\nu = \frac{2e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} F_{(\mu}{}^{\rho} \star \Phi_{\nu)\rho} \,, \tag{3.76}$$

$$\nabla_\mu \Phi_{\nu\rho} = \frac{\kappa e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \left(-V_\mu \star F_{\nu\rho} + \epsilon_{\nu\rho\sigma\tau} V^\tau F_\mu{}^\sigma\right) . \tag{3.77}$$

Let us introduce a tetrad frame

$$e^+ = \mathrm{d}u \,, \qquad e^- = \mathrm{d}v + \mathcal{H}\mathrm{d}u + \beta_i \mathrm{d}x^i \,, \qquad e^i = \Omega \mathrm{d}x^i \,, \tag{3.78}$$

which obey the orthogonality relations

$$\eta_{ab} e^a{}_\mu e^b{}_\nu = g_{\mu\nu} \,, \tag{3.79}$$

with $\epsilon_{-+12} = 1$, where $\eta_{+-} = \eta_{-+} = -1$, $\eta_{ij} = \delta_{ij}$ and other components vanish. Then the condition $i_V F = i_V \star F = 0$ determines the form of Maxwell fields as

$$F = F_{+i} e^+ \wedge e^i \,, \qquad \star F = -\epsilon_{ij} F_{+i} e^+ \wedge e^j \,, \tag{3.80}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$ and the summation over $i, j, ...$ is understood. Noting $\phi = \phi(u)$, the Bianchi identity $\mathrm{d}F = 0$ and the Maxwell equation $\mathrm{d}(e^{-2\alpha\phi} \star F) = 0$ reduce to

$$\partial_{[i} \left(\Omega F_{j]+}\right) = 0 \,, \qquad \partial_i \left(\Omega F_{i+}\right) = 0 \,. \tag{3.81}$$

It follows that there exists a function $\mathcal{F} = \mathcal{F}(u, x^i)$ such that

$$F_{+i} = -\Omega^{-1} \partial_i \mathcal{F} \,, \qquad \Delta \mathcal{F} = 0 \,, \tag{3.82}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. Thus, $\mathcal{F}$ is a harmonic function on a flat 2-space $\mathrm{d}x^2 + \mathrm{d}y^2$ with a u-dependence.

Equation (3.77) now implies

$$\mathrm{d}\Phi = 0 \,, \qquad \mathrm{d} \star \Phi = 0 \,. \tag{3.83}$$

Noticing $i_V \Phi = i_V \star \Phi = 0$, we can conclude by the parallel argument as above that there exists a harmonic function $\mathcal{P} = \mathcal{P}(u, x^i)$ such that

$$\Phi = -\Omega^{-1} \partial_i \mathcal{P} e^+ \wedge e^i = -\mathrm{d}u \wedge \mathrm{d}\mathcal{P} \,, \qquad \Delta \mathcal{P} = 0 \,. \tag{3.84}$$

Substituting (3.84) back into (3.77), we obtain

$$\Omega \partial_i \partial_j \mathcal{P} - 2\partial_{(i} \mathcal{P} \partial_{j)} \Omega = -\delta_{ij} \partial_k \mathcal{P} \partial_k \Omega \,, \tag{3.85}$$

$$\Omega \partial_u \left(\Omega^{-1} \partial_i \mathcal{P}\right) + \frac{1}{2\Omega^2} W_{ij} \partial_j \mathcal{P} = \frac{1}{\sqrt{1+\alpha^2}} e^{-\alpha\phi} \kappa \epsilon_{ij} \partial_j \mathcal{F} \,, \tag{3.86}$$

where $W_{ij} := \partial_i \beta_j - \partial_j \beta_i = (\partial_x \beta_y - \partial_y \beta_x)\epsilon_{ij}$. Inserting (3.82) and (3.84) into (3.76) gives

$$\sqrt{1+\alpha^2} \frac{\mathrm{d}\kappa}{\mathrm{d}u} e^{\alpha\phi} = 2\Omega^{-2} \epsilon_{ij} \partial_i \mathcal{F} \partial_j \mathcal{P} \,, \tag{3.87}$$

where the left hand side is dependent only on u. Multiplying $\partial_i\mathcal{P}$ to (3.86) and using (3.87), we find

$$\partial_u\left[\Omega^{-2}\partial_i\mathcal{P}\partial_i\mathcal{P}+\frac{1}{2}\kappa^2\right]=0\,. \tag{3.88}$$

Equation (3.11) now reduces to

$$\Omega^{-2}\partial_i\mathcal{P}\partial_i\mathcal{P}=1-\kappa^2\,. \tag{3.89}$$

Comparing (3.88) and (3.89), we arrive at $\kappa=$ constant.

Thus far, we have proceeded in a quite general metric form (3.73). The metric (3.73) is invariant under the three kind of coordinate transformations [15]. Letting $\zeta=x+{\rm i}y$ and $W=(\beta_x-{\rm i}\beta_y)/\sqrt{2}$, the metric-form remains intact under $\zeta\to\zeta'=h(\zeta,u)$ with

$$\Omega'^2=\frac{\Omega^2}{\partial_\zeta h\partial_{\bar\zeta}\bar h}\,,\qquad W'=\frac{W}{\partial_\zeta h}+\frac{\Omega^2\partial_u\bar h}{\partial_\zeta h\partial_{\bar\zeta}\bar h}\,,\qquad \mathcal{H}'=\mathcal{H}-\frac{\Omega^2\partial_u h\partial_u\bar h+W\partial_u h\partial_{\bar\zeta}\bar h+\bar W\partial_u\bar h\partial_\zeta h}{\partial_\zeta h\partial_{\bar\zeta}\bar h}\,, \tag{3.90}$$

where h is analytic in ζ. Using the above freedom, we can always adopt $\mathcal{P}$ as a one of the coordinate of the wave surface as $\mathcal{P}=x$. Then, equations (3.85) and (3.88) imply that Ω is constant, which can be taken as $\Omega\equiv1$ without losing any generality by means of a simple scaling $u\to u'=\Omega u$, $v\to v'=\Omega^{-1}v$ and $\zeta\to\zeta'=\Omega^{-1}\zeta$ with $\mathcal{H}'=\Omega^{-2}\mathcal{H}$, which also leaves Φ invariant. With these choices ($\mathcal{P}=x$ and $\Omega=1$), $\kappa=0$ is obtained from (3.89), i.e., $a_\mu=0$.

Equation (3.87) then leads to $\mathcal{F}=\mathcal{F}(x,u)$. Since $\mathcal{F}$ is harmonic, it is restricted to the form

$$\mathcal{F}=\mathcal{F}_0(u)x+\mathcal{F}_1(u)\,, \tag{3.91}$$

where $(\mathcal{F}_0,\mathcal{F}_1)$ are arbitrary functions of u. The function $\mathcal{F}_1$ can be gauged away since the Maxwell field strength F is not affected by this term. From (3.86) we can obtain $W_{ij}=0$, implying that β_i is a local gradient. This function can be set to zero by the transformation $v\to v'=v+g(\zeta,\bar\zeta,u)$ with

$$W'=W-\partial_\zeta g\,,\qquad \mathcal{H}'=\mathcal{H}-\partial_u g\,, \tag{3.92}$$

which corresponds to the choice of the $v=0$ surface.

Finally, the remaining function $\mathcal{H}$ can be obtained by use of the $(+,+)$-component of Einstein's equation. Other components of Einstein's equations are ensured to hold automatically as an integrability of the Killing spinor equation. Working in the basis (3.78), (3.63) implies $\mathcal{E}_{-i}=0$ and (3.64) implies $\mathcal{E}_{+i}=\mathcal{E}_{ij}=0$, as desired. The $(+,+)$-component of the Ricci tensor for the metric (3.73) reads

$$R_{++}=\frac{1}{2\Omega^4}\left[2\Omega^2\left(\Delta\mathcal{H}-\partial_u\partial_i\beta_i\right)+\frac{1}{2}W_{ij}W_{ij}-4\Omega^3\partial_u^2\Omega\right]\,. \tag{3.93}$$

Setting $\beta_i=0$ and $\Omega=1$, we arrive at the governing equation for $\mathcal{H}$:

$$\Delta\mathcal{H}(u,x,y)=2e^{-2\alpha\phi(u)}\mathcal{F}_0(u)^2+2\left(\frac{{\rm d}\phi}{{\rm d}u}\right)^2\,. \tag{3.94}$$

To sum up, the necessary condition for the supersymmetry in the null class requires that the spacetime is *pp*-wave [15] described by the metric

$${\rm d}s^2=-2{\rm d}u\left[{\rm d}v+\mathcal{H}(u,x,y){\rm d}u\right]+{\rm d}x^2+{\rm d}y^2\,, \tag{3.95}$$

with a Maxwell field of the form

$$F=-\mathcal{F}_0(u){\rm d}u\wedge{\rm d}x\,, \tag{3.96}$$

where $\mathcal{F}_0(u)$ is an arbitrary function characterizing the strength of the radiative Maxwell field. $\mathcal{H}$ is determined by the Poisson equation (3.94) for a given dilaton profile $\phi(u)$. Remark that (3.94) determines $\mathcal{H}$ up to another arbitrary harmonic function $\mathcal{H}_0$ with an arbitrary u-dependence.

Equation (3.7) implies

$$\gamma^+\epsilon=0\,. \tag{3.97}$$

Writing out the the Killing spinor equation for the metric (3.95) and using (3.97), we have

$$\left(\partial_u + \frac{\mathrm{i}}{\sqrt{1+\alpha^2}} e^{-\alpha\phi(u)} \mathcal{F}_0(u)\gamma^1\right)\epsilon = 0\,, \qquad \partial_v \epsilon = 0\,, \qquad \partial_i \epsilon = 0\,, \tag{3.98}$$

which can be solved as

$$\epsilon = \exp\left[-\frac{\mathrm{i}}{\sqrt{1+\alpha^2}} \int^u \mathrm{d}u e^{-\alpha\phi(u)} \mathcal{F}_0(u)\gamma^1\right]\epsilon_0\,, \tag{3.99}$$

where ϵ_0 is a constant spinor obeying $\gamma^+\epsilon_0 = 0$. The dilatino equation imposes no further condition. Since the projection (3.97) is a unique restriction, the solution preserves at least half of supersymmetries.

IV. NOVEL PROPERTIES OF BPS SOLUTIONS

We explore some characteristic properties of supersymmetric solutions obtained in the previous section. The following subsection enumerates all the maximally supersymmetric solutions. In the next two subsections, we study several aspects of BPS solutions from the viewpoints of conserved charges, sigma models and the Kaluza-Klein embedding. The dyonic solution in the timelike family is not entirely new, since it can be generated by the 5-dimensional transformations.

A. Maximal supersymmetry

The maximally supersymmetric solutions in this theory can be obtained as follows. To restore the complete supersymmetries, the dilatino equation must impose no algebraic constraints. This means that terms in the basis $\{\mathbf{1}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_{\mu\nu}\}$ of the gamma matrix must vanish separately. We are then led to

$$\phi = \phi_0\,, \qquad F_{\mu\nu} = 0\,, \tag{4.1}$$

for $\alpha \neq 0$ and $\phi = \phi_0$ for $\alpha = 0$. The $\alpha \neq 0$ case is then tantamount to the vacuum case, so that the maximally supersymmetric solution is only the Minkowski spacetime. For $\alpha = 0$, the maximally supersymmetric solutions in Einstein-Maxwell theory are obtained, which are the Minkowski spacetime, the Nariai-Bertotti-Robinson spacetime $\mathrm{AdS}_2 \times S^2$ [43],

$$\mathrm{d}s^2 = -\frac{r^2}{Q^2}\mathrm{d}t^2 + \frac{Q^2}{r^2}\mathrm{d}r^2 + Q^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\,, \qquad F = Q^{-1}\mathrm{d}t\wedge\mathrm{d}r\,, \tag{4.2}$$

where Q is a constant corresponding to the Maxwell charge, and the Kowalski-Glikman *pp*-wave [44],

$$\mathrm{d}s_2^2 = -2\mathrm{d}u\left[\mathrm{d}u + \tfrac{1}{2}\lambda^2(x^2+y^2)\mathrm{d}u\right] + \mathrm{d}x^2 + \mathrm{d}y^2\,, \qquad F = \lambda \mathrm{d}u\wedge\mathrm{d}x\,, \tag{4.3}$$

where λ is a constant. All of these backgrounds are conformally flat $C_{\mu\nu\rho\sigma} = 0$ and the Maxwell field is covariantly constant $\nabla_\mu F_{\nu\rho} = 0$.

B. Force balance

Each BPS solution is specified by a single harmonic function H on a flat base space, which is taken to be the multi-center point sources

$$H = 1 + \sum_k \frac{q^{(k)}}{|\vec{x} - \vec{x}_{(k)}|}\,. \tag{4.4}$$

The Gibbons-Maeda metric (3.66) with $\alpha \neq 0$ is asymptotically flat, and saturates the BPS inequality (2.19). It describes the collection of naked singularities, instead of the multiple configuration of black holes. A single charge cannot anchor the black hole to have a nonvanishing horizon. The IWP family ($\alpha = 0$) also describes naked singularities with the exception of the Reissner-Nordström solution [35].

The dyonic solution (3.65) is not singular at the point sources $\vec{x} = \vec{x}_{(k)}$, but they do not correspond to the locus of horizons since the circumferential radius vanishes there. Furthermore the dyonic solution is not asymptotically flat in the strict sense due to the NUT charge, thereby the solution fails to satisfy the BPS bound (2.19). Instead, the spacetime is asymptotically locally flat, wherein a NUT charge plays an interesting role as a gravitational dyon. Letting (r, θ, φ) be the polar coordinates at infinity, we shall define (in an appropriate gauge) the scalar charge Σ and the NUT charge N by

$$\phi \sim \pm \frac{\Sigma}{r}\,, \qquad g_{t\varphi} \sim \pm 2N g_{tt} \cos\theta\,, \tag{4.5}$$

as $r \to \infty$. It can be easily verified that the dyonic solution (3.65) with $c_{\rm E} = {\rm sech}\sigma$ and $c_{\rm B} = \tanh\sigma$ satisfies the "anti-gravity condition" of Scherk [49],

$$M^2 + \Sigma^2 + N^2 = Q_e^2 + Q_m^2\,, \tag{4.6}$$

where M, Q_e and Q_m have been read off from the "monopole terms" for the metric and the gauge potential. This equation just encodes the superposition principle, which is distinguished from the BPS condition (2.19) expressed only in terms of global charges.

Let us consider an additional implication of the relations between (2.19) and (4.6). From (2.13) and (3.19), the electromagnetic parts of the Nester 2-form can be rewritten as $H_{\mu\nu} = 2\nabla_\mu V_\nu$, thence its integral gives

$$-\frac{1}{2}\int_{\partial\Sigma} {\rm d}S_{\mu\nu} H^{\mu\nu} = M_{\rm Komar}\,. \tag{4.7}$$

This accords precisely with the expression of Komar integral for the timelike Killing vector V^μ [45]. It follows that the failure of the saturation of the BPS inequality (2.19) stems from the disagreement of the ADM mass and the Komar charge. This is of course outside the reach of supersymmetry, which is essentially local whilst the conserved charge in the gravitating system is a global notion. If the spacetime is asymptotically flat in the usual sense, the ADM mass and the Komar energy coincide $M = M_{\rm Komar}$, as expected.

Since the timelike family of solutions is necessarily stationary, it is also enlightening to discuss the relation to the non-BPS, stationary Einstein-Maxwell-dilaton system, which dimensionally reduces to the gravity-coupled sigma model. The sigma model analysis will reveal that BPS solutions occupy a distinguished position compared to non-BPS solutions.

A spacetime in Einstein-Maxwell-dilaton gravity admitting a timelike group of motions generated by a Killing vector V^μ with norm $V^\mu V_\mu = -f(<0)$ is described by the action,

$$S_3 = \int {\rm d}^3x\sqrt{h}\left[{}^{(h)}R - \mathscr{G}_{AB}(\Phi^C)h^{mn}(D_m\Phi^A)(D_n\Phi^B)\right]\,. \tag{4.8}$$

where $f^{-1}h_{mn}$ is the metric orthogonal to the orbits of V as (3.24) and ${}^{(h)}R$ is the Ricci curvature of h_{mn}. Here, Φ^A $(A = 1, ..., 5)$ constitutes the five real scalars [19]

$$\Phi^A = (f, \psi, v, a, \phi)\,, \tag{4.9}$$

where ϕ is a dilaton and

$$\partial_m v = \sqrt{2}F_{m\mu}V^\mu\,, \qquad \partial_m a = -\sqrt{2}e^{-2\alpha\phi}\star F_{m\mu}V^\mu\,, \qquad \partial_m\psi = \Omega_m - (v\partial_m a - a\partial_m v)\,, \tag{4.10}$$

with $\Omega = -\star(V\wedge {\rm d}V)$ being the twist of V. The target space metric $\mathscr{G}_{AB}$ reads [19]

$$\mathscr{G}_{AB}{\rm d}\Phi^A{\rm d}\Phi^B = \frac{1}{2f^2}\left[{\rm d}f^2 + ({\rm d}\psi + v{\rm d}a - a{\rm d}v)^2\right] - \frac{e^{-2\alpha\phi}{\rm d}v^2 + e^{2\alpha\phi}{\rm d}a^2}{f} + 2{\rm d}\phi^2\,, \tag{4.11}$$

which is symmetric iff $\alpha = 0, \sqrt{3}$ and Einstein iff $\alpha = \sqrt{3}$. The Euler-Lagrange equations derived from the action (4.8) define a harmonic map from the base space to the target space.

Comparing with the timelike class of supersymmetric solution, the above scalars take the form,

$$f = E^2 + B^2\,, \qquad \psi = 0\,, \tag{4.12}$$

and

$$E = c_{\rm E}e^{\phi/\sqrt{3}}\,, \qquad B = c_{\rm B}e^{-\phi/\sqrt{3}}\,, \qquad v = \frac{c_{\rm E}}{\sqrt{2}}e^{4\phi/\sqrt{3}}\,, \qquad a = \frac{c_{\rm B}}{\sqrt{2}}e^{-4\phi/\sqrt{3}}\,, \tag{4.13}$$

for the dyonic case, and

$$E = e^{\phi/\alpha}\,, \qquad B = 0\,, \qquad v = \sqrt{\frac{2}{1+\alpha^2}}e^{[(1+\alpha^2)/\alpha]\phi}\,, \qquad a = 0\,, \tag{4.14}$$

for the purely electric case. The purely magnetic case is obtained by $E \leftrightarrow B$, $a \leftrightarrow v$ and $\phi \leftrightarrow -\phi$. In every case, the supersymmetric solutions correspond to the null geodesics of the target space $\mathscr{G}_{AB}\mathrm{d}\Phi^A\mathrm{d}\Phi^B = 0$ with a harmonic being its affine parameter. Since the target space metric acts as a source of 3-dimensional Euclidean gravity (4.8), this implies that the 3-metric h_{mn} is flat, which appears to be responsible for producing a state of equipoise [39, 46].

Incidentally, the multiple solution (2.22) is not described by null geodesics of the target space aside from the Majumdar-Papapetrou solution ($H_1 = H_2$, i.e, $\phi = 0$ and $F_{\mu\nu}F^{\mu\nu} = 0$). We can deduce that this may also due to (2.25). It is therefore reasonable to infer that there exist other multi-soliton solutions which are not described by null geodesics on the target space (4.11). Moreover, there indeed exist multi-center solutions that are described by null geodesics on the target space (4.11) but not the BPS solutions to the Einstein-Maxwell-dilaton gravity. In order to gain further insight into equilibrium solutions, the analysis of all geodesics is required. Unfortunately the approach given in [39, 46] seems inapplicable since the sigma-model representation on coset spaces has been fully exploited wherein. The direct evaluation of geodesics for the space (4.11) seems more promising for this purpose. The interrelation among between BPS solutions and equilibrium states is somewhat obscure and deserves further detailed investigation. We hope to visit this issue elsewhere.

C. Liftup to 5-dimensional BPS solutions

Since the dyonic solution has $\alpha = \sqrt{3}$, the solution can be uplifted into five-dimensional vacuum gravity via the Kaluza-Klein ansatz,

$$\mathrm{d}s_5^2 = e^{-4\phi/\sqrt{3}}\left(\mathrm{d}x^5 + 2A_\mu \mathrm{d}x^\mu\right)^2 + e^{2\phi/\sqrt{3}}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{4.15}$$

We discuss the supersymmetric solutions with $\alpha = \sqrt{3}$ obtained in the previous section from the 5-dimensional perspective. The BPS solutions in Einstein-Maxwell-dilaton gravity with $\alpha = \sqrt{3}$ should constitute a subset of 5-dimensional vacuum BPS solutions with a spatial isometry.

The timelike family of BPS solutions for 5-dimensional vacuum gravity is static[4] and given by the direct product of a flat time-direction and a hyper-Kähler manifold [24],

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}s^2_{\rm HK}\,. \tag{4.16}$$

As a hyper-Kähler manifold, we choose the Gibbons-Hawking space [47], which naturally leads to dimensional reduction. Then, the 5-dimensional metric reads

$$\mathrm{d}s_5^2 = -\mathrm{d}t^2 + h^{-1}\left(\mathrm{d}x^5 + \vec{\chi}\cdot\mathrm{d}\vec{x}\right)^2 + h\mathrm{d}\vec{x}^2\,, \qquad \vec{\nabla}\times\vec{\chi} = \vec{\nabla}h\,, \tag{4.17}$$

where the metric is independent of t and x^5. h is harmonic on the flat 3-dimensional space $\mathrm{d}\vec{x}^2$ and $\partial/\partial x^5$ preserves the 3 complex structures. This metric describes a (multiple generalization of) Gross-Perry-Sorkin monopole [48]. Compactifying along x^5 via the ansatz (4.15), we find that the 4-dimensional Einstein metric $g_{\mu\nu}$ is the magnetic Gibbons-Maeda solution with $\alpha = \sqrt{3}$.

Applying the Lorentz boost along (t, x^5)-plane,

$$\mathrm{d}t \;\to\; \mathrm{d}t\cosh\sigma + \mathrm{d}x^5\sinh\sigma\,, \qquad \mathrm{d}x^5 \;\to\; \mathrm{d}x^5\cosh\sigma + \mathrm{d}t\sinh\sigma\,, \tag{4.18}$$

where $\tanh\sigma$ controls the boost velocity, we obtain a rotating metric from (4.17),

$$\begin{aligned}\mathrm{d}s_5^2 = &\left(h^{-1}\cosh^2\sigma - \sinh^2\sigma\right)\left[\mathrm{d}x^5 + \frac{\cosh\sigma\sinh\sigma(1-h)}{\cosh^2\sigma - h\sinh^2\sigma}\left(\mathrm{d}t + \frac{\vec{\chi}\cdot\mathrm{d}\vec{x}}{(1-h)\sinh\sigma}\right)\right]^2 \\ &- \frac{1}{\cosh^2\sigma - h\sinh^2\sigma}\left(\mathrm{d}t + \sinh\sigma\vec{\chi}\cdot\mathrm{d}\vec{x}\right)^2 + h\mathrm{d}\vec{x}^2\,. \end{aligned} \tag{4.19}$$

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[4] Setting $F = 0$ in (3.3) of reference [24] leads to $f =$ constant, $G^+ = G^- = 0$. Hence $\omega = 0$ is concluded.

The dimensional reduction gives the dyonic supersymmetric solution (3.43) and (3.45) with $H = h$, $c_{\rm E} = {\rm sech}\sigma$ and $c_{\rm B} = \tanh\sigma$.

The Kaluza-Klein embedding can be applied for the null case as well. The general null class of 5-dimensional vacuum BPS solution is the *pp*-wave [24],

$$ds_5^2 = -2{\rm d}u\left[{\rm d}v + \mathcal{H}(u,\vec{x}){\rm d}u\right] + \left[{\rm d}\vec{x} + \vec{x}\times\vec{\omega}(u){\rm d}u\right]^2 , \qquad \vec{\nabla}^2\mathcal{H} = 0 , \tag{4.20}$$

where we have included for convenience the cross-term ${\rm d}u{\rm d}\vec{x}$, which can be made to vanish by the isometry of 3-dimensional Euclid space ${\rm d}\vec{x}^2$. Compactification along v with $\vec{\omega} = 0$ gives rise to the electrically charged Gibbons-Maeda solution (3.66) [1]. Applying the Lorentz boost in the (v, z)-plane simply generates gauge transformation and does not alter the 4-dimensional solution.

In order to obtain the 4-dimensional *pp*-wave geometry (3.95) form (4.20), consider a coordinate transformation

$$\begin{aligned} &{\rm d}u = \Omega_1(u')^2{\rm d}u' , \qquad x = \Omega_1(u')x' , \qquad y = \Omega_1(u')y' , \qquad z = \Omega_3(u')z' , \\ &v = v' + \frac{1}{2}\left[\Omega_1^{-1}\dot{\Omega}_1(x'^2 + y'^2) + \Omega_1^{-2}\Omega_3\dot{\Omega}_3 z'^2\right] - \Omega_1\Omega_3\omega_2' x'z' , \end{aligned} \tag{4.21}$$

with $\omega_1' = \omega_3' = 0$ and $\omega_2' = \Omega_1(u')^{-5}\mathcal{F}_0(u')$, where $\vec{\omega}'(u') = \vec{\omega}(u)$. The dot denotes the derivative with respect to u'. Then the 5-dimensional metric (4.20) translates into

$$ds_5^2 = \Omega_1(u')^2\left[-2{\rm d}u'\left({\rm d}v' + \mathcal{H}'{\rm d}u'\right) + {\rm d}x'^2 + {\rm d}y'^2\right] + \Omega_3(u')^2\left[{\rm d}z' + 2\mathcal{F}_0(u')x'{\rm d}u'\right]^2 \tag{4.22}$$

where

$$\begin{aligned} \mathcal{H}' = \Omega_1^2\mathcal{H} - \frac{1}{2\Omega_1^3}\Big[&(x'^2 + y'^2)(-\Omega_1\ddot{\Omega}_1 + 2\Omega_1\dot{\Omega}_1^2) + \omega_2^2\Omega_1^5(-3x'^2\Omega_1^2 + z'^2\Omega_3^2) \\ &+ z'^2\Omega_3(2\dot{\Omega}_1\dot{\Omega}_3 - \Omega_1\ddot{\Omega}_3) + 2x'z'\Omega_1^4(\Omega_3\dot{\omega}_2 + 2\omega_2\dot{\Omega}_3)\Big] . \end{aligned} \tag{4.23}$$

Since it is always possible to choose $\mathcal{H}'$ to be independent of coordinate z', the dimensional reduction along z' gives the desired metric (3.95) by taking $\Omega_3 = \Omega_1^{-2} = e^{-2\phi(u')/\sqrt{3}}$.

V. CONCLUDING REMARKS

We investigated the supersymmetric solutions in 4-dimensional Einstein-Maxwell-dilaton theory with an arbitrary coupling constant α. The primary motivation to examine this theory comes from the fact that properties of static (nonextremal) black hole solutions are very sensitive to the coupling constant, and that the rotating black hole solution has not been found yet. In the light of sigma model, the target space metric becomes homogeneous only for $\alpha = 0, \sqrt{3}$, in which a coset representative is possible. For other values of α the nontrivial transformation is unavailable. Still, in the case of $\alpha \le \sqrt{3}$ the sectional curvature of the potential space (4.11) is negative semi-definite, which can be used to prove the uniqueness theorem of (yet to be discovered) rotating and nonextremal black holes [50]. This encourages us to inquire the extremal limits of these solutions. In this paper, we considered the supersymmetric solutions satisfying the gravitino and the dilatino Killing spinor equations.

Before passing to the main business of classifying the supersymmetric solutions, we reverted back to these 1st-order Killing spinor equations. We found that the dilatino equation does not imply the dilation field equation except for $\alpha = 0, \sqrt{3}$, which correspond respectively to the Brans-Dicke-Maxwell theory and the Kaluza-Klein reduction of 5-dimensional vacuum gravity, and $F_{\mu\nu}\star F^{\mu\nu} = 0$ for which the solution is purely electric or magnetic. Otherwise, the Einstein-Maxwell-dilaton gravity would not be embedded into supergravity theory. We may attribute this to the fact that the axion field resulting from the 10 dimensional heterotic string theory cannot be truncated consistently unless $F_{\mu\nu}\star F^{\mu\nu} = 0$. Hence the static dyonic multiple solution (2.22) is not the BPS solution to the Einstein-Maxwell-dilaton gravity, although it enjoys the superposition principle. This is also related to the fact that the solution (2.22) does not have the null geodesic description on the target space. The same is true for the dyonic Reissner-Nordström solution, which is given by $H_1 = H_2$ in equation (2.22) with a trivial dilaton. Since this metric fulfills $F_{\mu\nu}F^{\mu\nu} = 0$, it is an exact solution in the Einstein-Maxwell-dilaton system. Though, it is not the supersymmetric solution to this theory since it does not satisfy the dilatino equation despite being mechanical equilibrium. We should regard it as a BPS solution of the Einstein-Maxwell gravity, rather than the Einstein-Maxwell-dilaton theory.

Keeping these issues in mind, we attempted to classify the supersymmetric solutions and enumerated all explicit forms of bosonic quantities. As in the case of preceding studies on various supergravities, the supersymmetric solutions fall into two classes depending on whether $V^\mu = {\rm i}\bar{\epsilon}\gamma^\mu\epsilon$ is timelike or null.

The solutions in the timelike class can be rotating if and only if the solution is dyonic, which occurs only for $\alpha = 0, \sqrt{3}$. Since the $\alpha = 0$ case is nothing but the Einstein-Maxwell theory, our primary interest is the $\alpha = \sqrt{3}$ case, which can be oxidized into 5-dimensions. Looking from 5-dimensions, the dyonic solutions are generated via boosting the purely magnetic Gross-Perry-Sorkin monopole solution. It has been argued that the nonexistence of multi-spinning configurations may be related to the discrepancy of gyromagnetic ratio between the probe particle and the background spacetime [51]. The results in the present paper are not inconsistent with the claim of [51] since the dyonic metric (3.42) is not asymptotically flat due to the NUT charge. For the purely electric or magnetic case, the solution is inevitably static and exhausted by the multiple Gibbons-Maeda solution. Unfortunately, all the solutions in the timelike family do not describe regular black holes. It should be noted that this does not mean the nonexistence of nonextremal rotating dilatonic black holes. Curiously, it appears that the extremal limit of rotating black hole has a regular horizon in 5-dimensional Einstein-Maxwell-dilaton gravity. The detailed analysis will be reported elsewhere.

We demonstrated that the BPS solutions belonging to the null family are given by the *pp*-wave, in which the Killing vector V is covariantly constant. For the null family, the dilatino equation automatically implies the dilaton equation of motion provided the Maxwell equations and the Bianchi identity are satisfied. Both families of solutions preserve at least half of supersymmetries. The full restoration of supersymmetries occurs only for the Minkowski spacetime for $\alpha \neq 0$.

The present work can be extended into several directions. The result [52] strongly implies that the the theory (1.1) can be "gauged" to include the exponential Liouville-type potential. It is interesting to see whether the gauged dilaton gravity admits a Bogomol'nyi-type inequality. The classification of pseudo supersymmetric solutions in dilatonic "fake supergravity" also seems to be a plausible generalization to the present work.

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Appendix A: Conventions and useful formulae

In this paper, we used the mostly plus metric convention with the orientation $\epsilon_{0123} = 1$. Greek indices $\mu, \nu, ...$ denote the spacetime indices, whereas Roman indices $a, b, ...$ refer to those in tangent space. The Hodge dual is denoted by star $\star F_{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

Gamma matrix γ_μ satisfies the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. The antisymmetrized product is understood to be unit weight, e.g., $\gamma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2$ and so on. The chiral matrix is given by $\gamma_5 = -(\mathrm{i}/4!)\epsilon_{abcd}\gamma^{abcd}$, so

$$\gamma_{\mu\nu\rho} = \mathrm{i}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5\,, \qquad \gamma_{\mu\nu} = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\gamma_5\,. \tag{A1}$$

We define the Dirac conjugate by $\bar{\psi} := \mathrm{i}\gamma^0\psi^\dagger$.

For any anti-symmetric 2-form $\mathcal{F}_{\mu\nu} = \mathcal{F}_{[\mu\nu]}$, the following identity holds

$$\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\alpha\nu}\mathcal{F}^{\rho\sigma} = \frac{1}{4}\epsilon_{\tau\nu\rho\sigma}\mathcal{F}^{\tau\nu}\mathcal{F}^{\rho\sigma}\delta^\alpha{}_\mu\,. \tag{A2}$$

which has been used to derive (2.25) and (3.60).

The differential forms (3.1)–(3.5) constructed from a commuting spinor ϵ satisfy

$$\bar{\epsilon}\gamma_\mu\gamma_\nu\epsilon = Eg_{\mu\nu} - \mathrm{i}\Phi_{\mu\nu}\,, \tag{A3}$$
$$\bar{\epsilon}\gamma_5\gamma_\mu\gamma_\nu\epsilon = -\mathrm{i}Bg_{\mu\nu} + \star\Phi_{\mu\nu}\,, \tag{A4}$$
$$\bar{\epsilon}\gamma_{\mu\nu}\gamma_\rho\epsilon = -\epsilon_{\mu\nu\rho\sigma}a^\sigma - 2\mathrm{i}V_{[\mu}g_{\nu]\rho}\,, \tag{A5}$$
$$\bar{\epsilon}\gamma_5\gamma_{\mu\nu}\gamma_\rho\epsilon = -\epsilon_{\mu\nu\rho\sigma}V^\sigma - 2\mathrm{i}a_{[\mu}g_{\nu]\rho}\,, \tag{A6}$$
$$\bar{\epsilon}\gamma_{\mu\nu}\gamma_{\rho\sigma}\epsilon = -B\epsilon_{\mu\nu\rho\sigma} + 2\mathrm{i}(\Phi_{\mu[\rho}g_{\sigma]\nu} - g_{\mu[\rho}\Phi_{\sigma]\nu}) - 2Eg_{\mu[\rho}g_{\sigma]\nu}\,, \tag{A7}$$
$$\bar{\epsilon}\gamma_5\gamma_{\mu\nu}\gamma_{\rho\sigma}\epsilon = -\mathrm{i}E\epsilon_{\mu\nu\rho\sigma} + 2\epsilon_{\mu\nu\lambda[\rho}\Phi^\lambda{}_{\sigma]} + 2\mathrm{i}Bg_{\mu[\rho}g_{\sigma]\nu}\,, \tag{A8}$$
$$\bar{\epsilon}\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\epsilon = -B\epsilon_{\mu\nu\rho\sigma} - 2\mathrm{i}\Phi_{\mu[\rho}g_{\sigma]\nu} - 2\mathrm{i}g_{\mu[\rho}\Phi_{\sigma]\nu} - \mathrm{i}g_{\mu\nu}\Phi_{\rho\sigma} + 2Eg_{\mu[\rho}g_{\sigma]\nu}\,, \tag{A9}$$
$$\bar{\epsilon}\gamma_5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\epsilon = -\mathrm{i}E\epsilon_{\mu\nu\rho\sigma} - 2\mathrm{i}Bg_{\mu[\rho}g_{\sigma]\nu} + 2\epsilon_{\mu\nu\lambda[\rho}\Phi^\lambda{}_{\sigma]} + g_{\mu\nu}\star\Phi_{\rho\sigma}\,, \tag{A10}$$

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