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The essential norm of a composition operator on Orlicz spaces

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Abstract

In this note we determine the lower and upper estimates for the essential norm of a composition operator on the Orlicz spaces under certain conditions.

Key Words: Orlicz spaces, composition operator, compact operators, essential norm.

1. Introduction and preliminaries

Let $\varphi : [0,\infty) \to [0,\infty)$ be a non-decreasing and convex function such that $\lim_{x\to\infty}\varphi(x) = \infty$ and $\varphi(x) = 0$ if and only if $x = 0$. Such a function is known as an Orlicz function. Let (X, Σ, μ) be a complete sigma finite measure space. We identify any two functions that are equal μ-almost everywhere on X. Let $L^\varphi(\mu)$ be the set of all measurable functions such that $\int_X \varphi(\alpha|f|)d\mu < \infty$ for some $\alpha > 0$. The space $L^\varphi(\mu)$ is called an Orlicz space and is a Banach space with the Luxemburg norm defined by

$$\|f\|_\varphi = \inf\{\delta > 0 : \int_X \varphi(\frac{|f|}{\delta})d\mu \le 1\}.$$

If $\varphi(x) = x^p$, $1 \le p < \infty$, then $L^\varphi(\mu) = L^p(\mu)$, the space of Lebesgue integrable functions. An Orlicz function φ is said to satisfy the Δ_2-condition if there exists $k > 0$ and $M \ge 0$ such that $\varphi(2t) \le k\varphi(t)$ for all $t \ge M$. It is known fact that if φ satisfies the Δ_2-condition, then simple functions are dense in $L^\varphi(\mu)$. Let $\tau : X \to X$ be a non-singular measurable transformation, that is, $\mu\circ\tau^{-1}(A) := \mu(\tau^{-1}(A)) = 0$ for each $A \in \Sigma$ whenever $\mu(A) = 0$. This condition means that the measure $\mu\circ\tau^{-1}$ is absolutely continuous with respect to μ. Let $h := d\mu\circ\tau^{-1}/d\mu$ be the Radon-Nikodym derivative. In addition, we assume that h is almost everywhere finite valued, or equivalently that $(X, \tau^{-1}(\Sigma), \mu)$ is σ-finite. An atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. Let A be an atom. Since μ is σ-finite, it follows that $\mu(A) < \infty$. Also, every Σ-measurable function f on X is constant almost everywhere on A. It is well known fact that every sigma finite measure space (X, Σ, μ) can be decomposed into two disjoint sets X_1 and X_2, such that μ is non-atomic over X_1 and X_2 is a countable collection of disjoint atoms (see [10]).

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With each Orlicz function φ we can associate another continuous convex function $\psi : [0,\infty) \to [0,\infty)$ defined as $\psi(y) = \sup\{x|y| - \varphi(x) : \ x \geq 0\}$. The function ψ is called the complementary function to φ. Let $\varphi,\ \psi$ be a pair of complementary Orlicz functions. Then each $g \in L^{\psi}(\mu)$ defines a bounded linear functional F_g on $L^{\varphi}(\mu)$ by $F_g(f) = \int fgd\mu,\ f \in L^{\varphi}(\mu)$. Moreover, the mapping $g \to F_g$ is an isometry from $L^{\psi}(\mu)$ onto $(L^{\varphi})^*(\mu)$, so the norm dual of $L^{\varphi}(\mu)$ can be identified with $L^{\psi}(\mu)$. It is well-known that, if $A \in \Sigma$ and $0 < \mu(A) < \infty$ then $\|\chi_A\| = 1/\varphi^{-1}(1/\mu(A))$, where $\varphi^{-1}(t) = \inf\{s > 0 : \varphi(s) > t\}$ is the right-continuous inverse of φ. Let $\varphi_1,\ \varphi_2$ be Orlicz functions. Then φ_1 is said to be essentially stronger than φ_2 (it is usually denoted $\varphi_1 \succ\succ \varphi_2$) if $\varphi_1^{-1}(t)/\varphi_2^{-1}(t) \to 0$ as $t \to \infty$. It is well-known that if $\|f\|_{\varphi} \leq 1$ then $I_{\varphi}(f) := \int \varphi(|f|)d\mu \leq \|f\|_{\varphi}$. So $\|f_n - f\|_{\varphi} \to 0$ implies that $I_{\varphi}(f_n - f) \to 0$ for a sequence $\{f_n\}$ in $L^{\varphi}(\mu)$. If φ satisfy the Δ_2-condition, then the converse of the above fact is also true (see Remark 3.10.3 in [4]).

Throughout this note we assume that φ satisfies Δ_2-condition, τ is non-singular and h is a finite valued function. For more literature concerning Orlicz spaces, we refer to Kufner, John and Fuc̆ik [4] and Rao [7].

Any non-singular measurable transformation τ induces a linear operator C_{τ} from $L^{\varphi}(\mu)$ into the linear space of equivalent classes of Σ-measurable functions on X defined by $C_{\tau}f = f \circ \tau,\ f \in L^{\varphi}(\mu)$. Here, the non-singularity of τ guarantees that the operator C_{τ} is well-defined. If C_{τ} takes $L^{\varphi}(\mu)$ into itself, then we call C_{τ} a composition operator on $L^{\varphi}(\mu)$.

Let $\mathcal{B}$ be a Banach space and $\mathcal{K}$ be the set of all compact operators on $\mathcal{B}$. For $T \in \mathcal{L}(\mathcal{B})$, the Banach algebra of all bounded linear operators on $\mathcal{B}$ into itself, the essential norm of T means the distance from T to $\mathcal{K}$ in the operator norm, namely

$$\|T\|_e = \inf\{\|T - S\| : \ S \in \mathcal{K}\}.$$

Clearly, T is compact if and only if $\|T\|_e = 0$. As is seen in [8], the essential norm plays an interesting role in the compact problem of concrete operators. Many people have computed the essential norm of various concrete operators. For these studies about composition operators, refer to [6], [9] and [11].

The question of actually calculating the norm and essential norm of a composition operator on Orlicz spaces is not a trivial one. In spite of the difficulties associated with computing the essential norm exactly, it is often possible to find upper and lower bound for the essential norm $C_{\tau} : L^{\varphi}(\mu) \to L^{\varphi}(\mu)$ under certain conditions on τ and φ. In the next section we will determine the lower and upper estimates for the essential norm of a composition operator on $L^{\varphi}(\mu)$.

2. Main result

Compactness of composition operators on $L^{\varphi}(\mu)$ are being studied in [1], [2], [3] and [5]. In this note, first we give an equivalent condition for compactness of C_{τ} on $L^{\varphi}(\mu)$.

Proposition 2.1 *Let φ be an Orlicz function and τ be a non-singular measurable transformation from X into itself. Then the operator $C_{\tau} : L^{\varphi}(\mu) \to L^{\varphi}(\mu)$ is compact if and only if for any $\varepsilon > 0$, the set $N(h,\varepsilon) = \{x \in X : h(x) > \varepsilon\}$ consists of finitely many atoms.*

Proof. $(\Rightarrow)$ Assume the contrary. Then for some $\varepsilon > 0$, the set $N(h,\varepsilon)$ either contains a non-atomic subset or has infinitely many atoms. In both cases we can find a sequence of pairwise disjoint measurable subsets $\{A_n\}$

with $0 < \mu(A_n) < \infty$ for every n. Define $f_n = \varphi^{-1}(1/\mu(A_n))\chi_{A_n}$. Then $I_\varphi(f_n) = \int_X \varphi(|f_n|)d\mu = 1$, whence $f_n \in L^\varphi(\mu)$ and $\|f_n\|_\varphi = 1$. So we have

$$I_\varphi(f_n \circ \tau) = \int_X \varphi(|f_n \circ \tau|)d\mu = \int_X h\varphi(|f_n|)d\mu$$

$$= \int_{A_n} \varphi\left(\varphi^{-1}(\frac{1}{\mu(A_n)})\right) hd\mu \geq \varepsilon \int_{A_n} \frac{1}{\mu(A_n)} d\mu = \varepsilon \ ,$$

whence $C_\tau \in \mathcal{L}(L^\varphi(\mu))$ and $\|C_\tau f_n\| \geq \varepsilon$. Consequently we have for $m \neq n$:

$$I_\varphi(C_\tau f_m - C_\tau f_n) = \int_X \varphi(|f_m - f_n|)hd\mu$$

$$= \int_{A_m} \varphi(|f_m|)hd\mu + \int_{A_n} \varphi(|f_n|)hd\mu = I_\varphi(f_m \circ \tau) + I_\varphi(f_n \circ \tau) \geq 2\varepsilon \ .$$

Therefore, $\|C_\tau f_m - C_\tau f_n\|_\varphi \geq \varepsilon$ for $m, n \in N$ with $m \neq n$. This means that $\{C_\tau f_n\}$ contains no Cauchy subsequence, that is $C_\tau(U(L^\varphi(\mu))$ is not relatively compact, where $U(L^\varphi(\mu))$ is the unit ball of $L^\varphi(\mu)$. Consequently, the operator C_τ is not compact, and so this is a contradiction.

($\Leftarrow$) Let $\varepsilon > 0$ be given. Then under the hypothesis that $A := N(h, \varepsilon)$ consists finitely many atoms, $M_{\chi_{\tau^{-1}A}} C_\tau$ is a finite rank operator. Since $h < \varepsilon$ on $X \setminus A$, for each $f \in L^\varphi(\mu)$ with $\|f\|_\varphi \leq 1$ we have

$$I_\varphi(f \circ \tau - \chi_{\tau^{-1}A} f \circ \tau) = I_\varphi((1 - \chi_{\tau^{-1}A}) f \circ \tau)$$

$$= \int_X |(\chi_{X \setminus A} \circ \tau) f \circ \tau| d\mu = \int_{X \setminus A} h\varphi(|f|)d\mu$$

$$\leq \int_{X \setminus A} \varepsilon\varphi(|f|)d\mu \leq \varepsilon I_\varphi(|f|) \leq \varepsilon \|f\|_\varphi \leq \varepsilon.$$

It follows that

$$\|C_\tau - M_{\chi_{\tau^{-1}A}} C_\tau\| = \sup_{\|f\|_\varphi \leq 1} \|C_\tau f - \chi_{\tau^{-1}A} C_\tau f\|_\varphi \leq \varepsilon.$$

Thus C_τ is the limit of some finite rank operators and is therefore compact. □

Let $X = X_1 \cup X_2$ be the decomposition of X into non-atomic and atomic parts respectively. If $X_2 = \emptyset$ or $\mu(X) = +\infty$ and X_2 consists of finite many atoms, then by the Proposition 2.1, $L^\varphi(\mu)$ dose not admit a non-zero compact composition operator. Thus, in this case $\mathcal{K} = \{0\}$ and hence $\|C_\tau\|_e = \|C_\tau\|$.

Now, we present the main result of this paper.

Theorem 2.2 *Let X_2 consists of infinitely many atoms, φ be an Orlicz function and τ be a non-singular measurable transformation from X into itself. Put*

$$\alpha = \inf\{\varepsilon > 0 : N(h, \varepsilon) \text{ consists of finitely many atoms}\},$$

where $N(h, \varepsilon) = \{x \in X : h(x) > \varepsilon\}$. If $C_\tau : L^\varphi(\mu) \to L^\varphi(\mu)$ is a composition operator, then *(i)* $\|C_\tau\|_e = 0$ *if and only if* $\alpha = 0$.

(ii) $\|C_\tau\|_e \geq \alpha$ *if* $0 < \alpha \leq 1$ *and* $\varphi(x) \succ\succ x$.

(iii) $\|C_\tau\|_e \leq \alpha$ *if* $\alpha > 1$.

Proof. Proposition 2.1 implies that C_τ is compact if and only if $\alpha = 0$. So (i) is a direct consequence of Proposition 2.1.

(ii) Suppose that $0 < \alpha \leq 1$ and $\varphi(x) \succ\succ x$. Take $0 < \varepsilon < 2\alpha$ arbitrarily. The definition of α implies that $F = N(h, \alpha - \frac{\varepsilon}{2})$ either contains a non-atomic subset or has infinitely many atoms. If F contains a non-atomic subset, then there are measurable sets E_n, $n \in N$, such that $E_{n+1} \subseteq E_n \subseteq F$, $0 < \mu(E_n) < \frac{1}{n}$. Define $f_n = \varphi^{-1}(1/\mu(E_n))\chi_{E_n}$. Then $\|f_n\|_\varphi = 1$ for all $n \in N$. We claim that $f_n \to 0$ weakly. For this we show that $\int_X f_n g \to 0$ for all $g \in L^\psi(\mu)$, where ψ is the complementary function to φ. Let $A \subseteq F$ with $0 < \mu(A) < \infty$ and $g = \chi_A$. Since $\varphi(x) \succ\succ x$, then we have

$$\left|\int_X f_n \chi_A d\mu\right| = \varphi^{-1}\left(\frac{1}{\mu(E_n)}\right)\mu(A \cap E_n)$$

$$\leq \varphi^{-1}\left(\frac{1}{\mu(E_n)}\right)\mu(E_n) = \frac{\varphi^{-1}(1/\mu(E_n))}{1/\mu(E_n)} \longrightarrow 0, \qquad \text{as } n \to \infty.$$

Since simple functions are dense in $L^\psi(\mu)$, thus f_n is proved to converge to 0 weakly. Now assume that F consists of infinitely many atoms. Let $\{E_n\}_{n=0}^\infty$ be disjoint atoms in F. Again put f_n as above. If $\mu(E_n) \to 0$, then by using the similar argument we had above, $\int_X f_n \chi_A d\mu \to 0$. Otherwise, $\mu(F) \geq \mu(\cup E_n) = \Sigma\mu(E_n) = +\infty$ and it follows that for each $A \subseteq F$ with $0 < \mu(A) < \infty$ we have $\mu(A \cap E_n) \to 0$, as $n \to \infty$. Hence in both cases $\int_X f_n g \to 0$. Now, we claim that $\|C_\tau f_n\|_\varphi \geq \alpha - \frac{\varepsilon}{2}$. Since $0 < \alpha - \varepsilon/2 < 1$ we see that

$$\|C_\tau f_n\|_\varphi = \inf\left\{\delta > 0 : \int_X \varphi\left(\frac{|f_n \circ \tau|}{\delta}\right) d\mu \leq 1\right\}$$

$$= \inf\left\{\delta > 0 : \int_X h\varphi\left(\frac{|f_n|}{\delta}\right) d\mu \leq 1\right\}$$

$$\geq \inf\left\{\delta > 0 : \int_X (\alpha - \frac{\varepsilon}{2})\varphi\left(\frac{|f_n|}{\delta}\right) d\mu \leq 1\right\}$$

$$\geq \inf\left\{\delta > 0 : \int_X \varphi\left(\frac{(\alpha - \frac{\varepsilon}{2})|f_n|}{\delta}\right) d\mu \leq 1\right\}$$

$$= (\alpha - \frac{\varepsilon}{2}) \inf\left\{\delta > 0 : \int_X \varphi\left(\frac{|f_n|}{\delta}\right) d\mu \leq 1\right\} = \alpha - \frac{\varepsilon}{2}.$$

Finally, take a compact operator T on $L^\varphi(\mu)$ such that $\|C_\tau - T\| < \|C_\tau\|_e + \frac{\varepsilon}{2}$. Then we have

$$\|C_\tau\|_e > \|C_\tau - T\| - \frac{\varepsilon}{2} \geq \|C_\tau f_n - Tf_n\|_\varphi - \frac{\varepsilon}{2}$$

$$\geq \|C_\tau f_n\|_\varphi - \|Tf_n\|_\varphi - \frac{\varepsilon}{2} \geq (\alpha - \frac{\varepsilon}{2}) - \|Tf_n\|_\varphi - \frac{\varepsilon}{2}$$

for all $n \in N$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|Tf_n\|_\varphi \to 0$. Hence $\|C_\tau\|_e \geq \alpha - \varepsilon$. Since ε was arbitrary, we obtain $\|C_\tau\|_e \geq \alpha$.

(iii) Let $\alpha > 1$ and take $\varepsilon > 0$ arbitrarily. Put $K = N(h, \alpha + \varepsilon)$. The definition of α implies that K consists of finitely many atoms. So we can write $K = \{E_1, E_2, \ldots, E_m\}$, where $E_1, E_2, \ldots, E_m$ are distinct. Since $(M_{\chi_K} C_\tau f)(X) = \sum_{i=1}^m \chi_K(E_i) f(\tau(E_i))$, for all $f \in L^\varphi(\mu)$, hence $M_{\chi_K} C_\tau$ has finite rank. Now, let $F \subseteq X \setminus K$ such that $0 < \mu(F) < +\infty$. Then we have

$$\mu \circ \tau^{-1}(F) = \int_F h d\mu \leq (\alpha + \varepsilon)\mu(F).$$

Since $\alpha + \varepsilon > 1$ and φ^{-1} is a concave function, we obtain that

$$\varphi^{-1}\left(\frac{1}{\mu \circ \tau^{-1}(F)}\right) \geq \frac{1}{\alpha + \varepsilon}\ \varphi^{-1}\left(\frac{1}{\mu(F)}\right).$$

That is

$$\left\{\varphi^{-1}\left(\frac{1}{\mu \circ \tau^{-1}(F)}\right)\right\}^{-1} \leq (\alpha + \varepsilon)\ \left\{\varphi^{-1}\left(\frac{1}{\mu(F)}\right)\right\}^{-1}.$$

It follows that $\|\chi_F \circ \tau\|_\varphi \leq (\alpha + \varepsilon)\|\chi_F\|_\varphi$. Since simple functions are dense in $L^\varphi(\mu)$, we obtain

$$\sup_{\|f\|_\varphi \leq 1} \|\chi_{X\setminus K} f \circ \tau\|_\varphi \leq (\alpha + \varepsilon) \sup_{\|f\|_\varphi \leq 1} \|\chi_{X\setminus K} f\|_\varphi \leq \alpha + \varepsilon\ .$$

Finally, since $M_{\chi_K} C_\tau$ is a compact operator, we get

$$\|C_\tau - M_{\chi_K} C_\tau\| = \sup_{\|f\|_\varphi \leq 1} \|(1 - \chi_K) C_\tau f\|_\varphi = \sup_{\|f\|_\varphi \leq 1} \|\chi_{X\setminus K} C_\tau f\|_\varphi \leq (\alpha + \varepsilon)\ .$$

It follows that $\|C_\tau\|_e \leq \alpha + \varepsilon$ and, consequently, $\|C_\tau\|_e \leq \alpha$. □

Example 2.3 (i) Let φ be an Orlicz function and let $X = (-\infty, 0] \cup N$, where N is the set of natural numbers. Let μ be the Lebesque measure on $(-\infty, 0]$ and $\mu(\{n\}) = 1/2^n$, if $n \in N$. Define $\tau : N \to N$ as: $\tau(1) = 2$, $\tau(2) = \tau(3) = 3$, $\tau(4) = \tau(5) = \tau(6) = 4$, $\tau(n) = n$ for $n \geq 7$, and $\tau(x) = 2/3x$, for all $x \in (-\infty, 0]$. Direct computation shows that $h = 2\chi_{\{2\}} + 3\chi_{\{3\}} + 7/4\chi_{\{4\}} + 1\chi_{\{n:\ n\geq 7\}} + 3/2\chi_{(-\infty,0]}$, and $\alpha = 3/2$. So $\|C_\tau\|_e \leq 3/2$ on $L^\varphi(\mu)$.

(ii) Let φ be an Orlicz function such that $\varphi^{-1}(2^n)/2^n \to 0$ as $n \to \infty$. Put X and μ as above. Define $\tau(1) = \tau(2) = \tau(3) = 1$, $\tau(4) = 2$, $\tau(5) = \tau(6) = 3$, $\tau(2n+1) = 5$, for $n \geq 3$, $\tau(2n) = 2n - 2$, for $n \geq 4$, and $\tau(x) = 5x$ for all $x \in (-\infty, 0]$. Then a simple computation gives $h = 7/4\chi_{\{1\}} + 1/4\chi_{\{2\}} + 3/8\chi_{\{3\}} + 1/3\chi_{\{2n+1:\ n\geq 3\}} + 1/4\chi_{\{2n:\ n\geq 4\}} + 1/5\chi_{(-\infty,0]}$, and $\alpha = 1/3$. Thus $\|C_\tau\|_e \geq 1/3$ on $L^\varphi(\mu)$.

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