arXiv:1011.0100v2 [hep-th] 4 Dec 2010

# Two-dimensional topological field theories as taffy

Matt Ando[1], Eric Sharpe[2]

[1] Department of Mathematics
University of Illinois, Urbana-Champaign
1409 West Green Street
Urbana, IL 61801

[2] Physics Department
Robeson Hall (0435)
Virginia Tech
Blacksburg, VA 24061

mando@math.uiuc.edu, ersharpe@vt.edu

In this paper we use trivial defects to define global taffy-like operations on string world-sheets, which preserve the field theory. We fold open and closed strings on a space $X$ into open strings on products of multiple copies of $X$, and perform checks that the "taffy-folded" worldsheets have the same massless spectra and other properties as the original worldsheets. Such folding tricks are a standard method in the defects community; the novelty of this paper lies in deriving mathematical identities to check that $e.g.$ massless spectra are invariant in topological field theories. We discuss the case of the B model extensively, and also derive the same identities for string topology, where they become statements of homotopy invariance. We outline analogous results in the A model, B-twisted Landau-Ginzburg models, and physical strings. We also discuss the understanding of the closed string states as the Hochschild homology of the open string algebra, and outline possible applications to elliptic genera.

# Contents

# 1   Introduction

Defects in two-dimensional theories are forms of domain walls, boundaries which connect open strings on potentially different spaces, and so act as some type of two-dimensional domain wall, see for example [1, 2, 3, 4, 5, 6]. Defects have appeared in numerous papers in the physics literature recently, for a sample see for example [7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein. They also seem to be implicit in parts of the mathematics literature, in connection with "enriched" topological field theories described by higher categories (see *e.g.* [16]), as we shall review.

In this paper we use 'identity' ('trivial') defects to perform taffy-like reparametrizations of string worldsheets, folding worldsheets over themselves to create new, physically-equivalent but different-looking, worldsheet theories.

Using folding tricks locally around a defect is not new, indeed is a standard method for computing spectra of operators in the defects community. What is (we believe) novel to this paper is the development of explicit mathematical identities required to carefully demonstrate that global foldings of worldsheets leave the physics invariant. For example, given an open string on $X$, we develop mathematical identities required to show that one gets physically-equivalent open strings on products of any number of copies of $X$ by folding and flattening along multiple identity defects. One case of this is illustrated below:
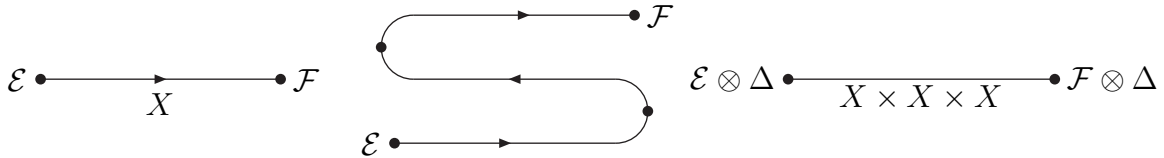


The diagram above illustrates an open string on $X^3$ corresponding to an open string on $X$, with boundaries determined in part by $\Delta$, a diagonal corresponding to the identity defect. From the general principles of folding, one expects that these diagrams should describe equivalent physics. We check that statement by comparing, for example, massless spectra in the B model, deriving the identity

$$\operatorname{Ext}^*_X (\mathcal{E}, \mathcal{F}) \ = \ \operatorname{Ext}^*_{X^3} \left( \mathbf{L}\pi_1^* \mathcal{E} \otimes^{\mathbf{L}} \Delta_{23}^\vee, \Delta_{12} \otimes^{\mathbf{L}} \mathbf{L}\pi_3^* \mathcal{F} \right)$$

confirming that the topological field theory is unchanged, at the level of *e.g.* D-branes described as objects in the derived category $D^b(X)$. We can also fold closed strings along identity defects and flatten; for example,
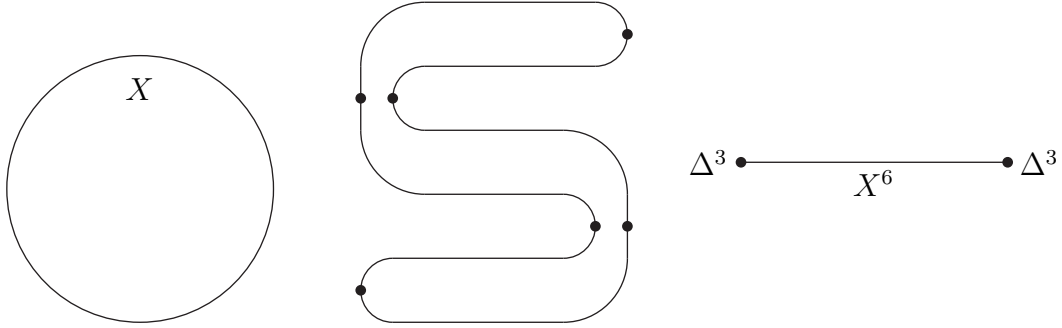
gives an open string on $X^6$ corresponding to a closed string on $X$. In the case of the B model, we check the identity above by comparing massless spectra:

$$H^*\left(X, \Lambda^* TX\right) \;=\; \mathrm{Ext}^*_{X^6}\left(\Delta^\vee_{12} \otimes^{\mathbf{L}} \Delta^\vee_{36} \otimes^{\mathbf{L}} \Delta^\vee_{45}, \Delta_{14} \otimes^{\mathbf{L}} \Delta_{23} \otimes^{\mathbf{L}} \Delta_{56}\right)$$

checking that topological field theories are, indeed, invariant under this operation. We can also include twists as we fold; for example,
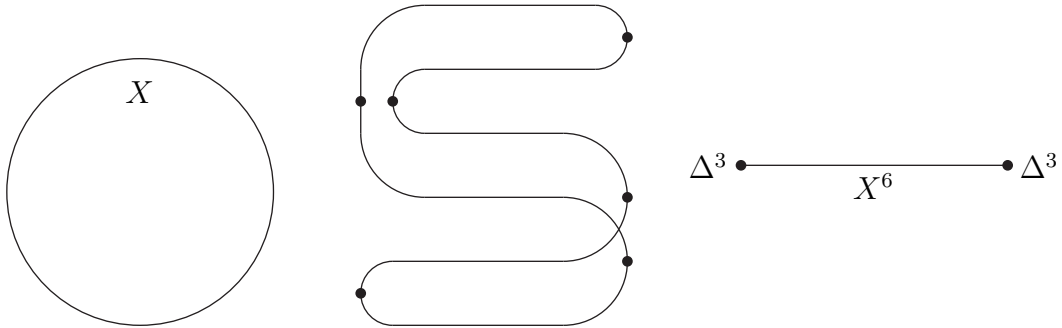


gives a different open string on $X^6$ corresponding to the same closed string on $X$. In this case, demanding that B model states be invariant under this operation implies the identity

$$H^*\left(X, \Lambda^* TX\right) \;=\; \mathrm{Ext}^*_{X^6}\left(\Delta^\vee_{12} \otimes^{\mathbf{L}} \Delta^\vee_{36} \otimes^{\mathbf{L}} \Delta^\vee_{45}, \Delta_{13} \otimes^{\mathbf{L}} \Delta_{24} \otimes^{\mathbf{L}} \Delta_{56}\right)$$

which we check rigorously.

As these operations involve folding worldsheets over onto themselves, we refer to them as taffy operations.

For readers not acquainted with such folding tricks, the philosophy is that inserting identity defects is, in principle, a trivial operation, so these global foldings, these taffy operations we describe, are no more than global reparametrizations, and so should encode equivalent physics. Much of this paper is devoted to carefully checking that hypothesis.

One important special case of these constructions involves taking a closed string on $X$ and constructing a physically-equivalent open string on $X \times X$ by folding and flattening along two identity defects:
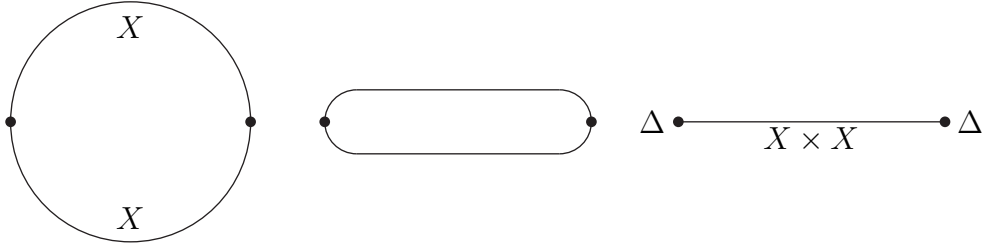
(a manipulation well-known in the defects community). The resulting identity relates Hochschild (co)homology (describing closed string states) to endomorphisms of the open string algebra, and for the B model is a well-known mathematical result, the "Hochschild-Kostant-Rosenberg isomorphism." Physically, this identity is a special case of the taffy operations; mathematically, this identity (and its analogues in other field theories) is one of the foundations on which these taffy identities are constructed. We discuss this in detail, and in particular argue that closed string states are more closely identified with Hochschild homology, instead of Hochschild cohomology, of the open string algebra.

The mathematical work in this paper for justifying taffy operations amounts to using homological algebra identities to reduce more complicated cases to the one above. An important point is homotopy invariance, as explained in section 5, which in some sense for this purpose is more fundamental than the Hochschild-Kostant-Rosenberg isomorphism or its analogues. The homological algebra identities we derive are the expressions of topological identities resulting from homotopy invariance.

Let us briefly illustrate what we mean by homotopy invariance. Consider the $B$-model on a Calabi-Yau manifold $X$. In that model, $D$-branes correspond to elements $\mathcal{E}$ of the derived category $D^b(X)$ of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules. Given two such complexes $\mathcal{E}$ and $\mathcal{F}$, the massless states of open strings emanating from $\mathcal{E}$ and terminating on $\mathcal{F}$ are measured by $\mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, while closed string states should be measured by the Hochschild cohomology $\mathrm{Hom}_{X \times X}(\mathcal{O}_X, \mathcal{O}_X)$. These two observations are simply related as follows. Consider a "closeable configuration" of open strings of the form
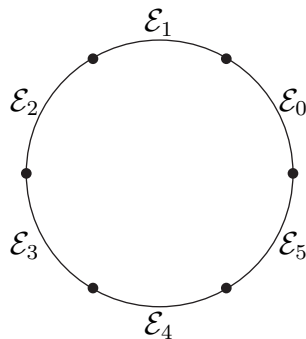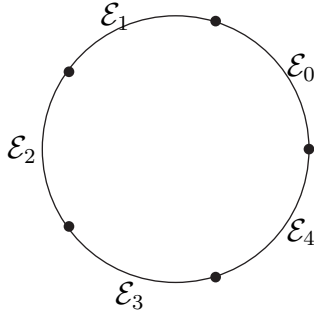


Given that the massless states for a string from $\mathcal{E}$ to $\mathcal{F}$ are given by $\mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, it is

natural to mathematically associate to this configuration the group

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_1) \times \mathbf{R}\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \times \cdots \times \mathbf{R}\mathrm{Hom}(\mathcal{E}_5, \mathcal{E}_0). \tag{1}$$

There is a certain redundancy in this description: products of open string states derived from



are related to those of the form (1) by the composition[1]

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}_4, \mathcal{E}_5) \times \mathbf{R}\mathrm{Hom}(\mathcal{E}_5, \mathcal{E}_0) \to \mathbf{R}\mathrm{Hom}(\mathcal{E}_4, \mathcal{E}_0)$$

in one direction, and by setting $\mathcal{E}_5 = \mathcal{E}_4$ and inserting the identity in the other. There is a standard mathematical way for encoding all this data, and, as we recall in §5.2, Keller and McCarthy [17, 18, 19] show that one can extract the Hochschild homology of $X$ from these data. This is part of what we mean by 'homotopy invariance,' above.

This observation raises several questions. For example, do diagrams of the form above have any physical sense? And can one use physical ideas to be more explicit about the relationship to Hochschild homology? We shall address these issues in this paper.

We begin in section 2 by briefly reviewing defects in two-dimensional quantum field theories. In section 3 we work through the taffy operations in detail for the B model, giving explicit descriptions and rigorously checking identities for relating folded strings to the original strings. Carefully studying these identities for the B model occupies the bulk of this paper. In section 4 we quickly rederive the same results in the context of 'string topology,' a mathematical abstraction of bosonic string field theory. In section 5 we discuss old lore relating closed string states to Hochschild homology (instead of cohomology) of the open string algebra, and briefly outline some generalizations. In sections 6, 7, 8, we outline analogous results and conjectures in the A model, B-twisted Landau-Ginzburg models, and critical strings, respectively. We conclude in section 9 by outlining some conjectural applications of this technology to elliptic genera.

---

[1]It is essential to use $\mathbf{R}$Hom instead of Ext here. As a related matter, composition of $\mathbf{R}$Hom is defined only up to homotopy, as we discuss in section 3.1.

# 2 Review of defects

## 2.1 Physics basics

A defect (see for example [1, 2, 3, 4, 5, 6]) is an open string boundary that connects open strings on two potentially distinct spaces. Let $X$, $Y$ be two spaces, then a defect is an open-string-type boundary defined by a submanifold of $X \times Y$, together with a bundle with connection over that submanifold.



Supersymmetry enforces the same conditions on that submanifold and bundle as one would have in an ordinary open string – for example, in the B model topological field theory, one has a complex submanifold of $X \times Y$ with a holomorphic vector bundle.

Let us work through some examples. Work locally on the complex plane, with a defect along the $x$-axis. Consider for example a string with target $\mathbf{C}$ on each side of the defect. We can describe this as a single complex boson $\phi$ defined over the complex plane, or a pair $(\phi_U, \phi_L)$ on the upper-half-plane. Boundary conditions such as, for example,

$$\lim_{y \to 0^+} \partial_n \phi \ = \ 0, \quad \lim_{y \to 0^-} \partial_t \phi \ = \ 0$$

or equivalently

$$\left[ \begin{array}{cc} \partial_n & 0 \\ 0 & \partial_t \end{array} \right] \left[ \begin{array}{c} \phi_U \\ \phi_L \end{array} \right] \ = \ 0$$

which would be described more formally in the B model by a sheaf on $\mathbf{C}^2$ with support along $(\mathbf{C}, 0) \subset \mathbf{C}^2$. Boundary conditions such as

$$\lim_{y \to 0^+} \partial_n \phi \ = \ \lambda \lim_{y \to 0^-} \partial_n \phi$$

would be described more formally by a sheaf on $\mathbf{C}^2$ with support along $\{(\lambda x, x) | x \in \mathbf{C}\} \subset \mathbf{C}^2$.

Special cases of these boundary conditions can be described by giving separate boundary conditions on $X$ and $Y$, and a map between the corresponding branes, but the general case is described by a sheaf on $X \times Y$.

One central idea from the study of defects, which we will use repeatedly throughout this paper, is that of folding tricks (see for example [2] and references therein). We have already used these implicitly in giving the data defining a defect. Given open strings on $X$, $Y$, meeting along a defect, as

$$\overline{\qquad\quad\bullet\quad\qquad}$$
$$X \qquad\qquad Y$$

we fold the combination of strings

$$Y \qquad\qquad\qquad Y$$
$$\text{or}$$
$$X \qquad\qquad\qquad X$$
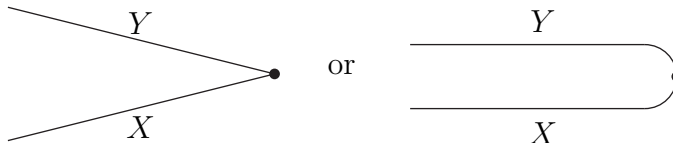
into a single open string on the product $X \times Y$,

$$\overline{\qquad\qquad\qquad}\bullet$$
$$X \times Y$$

so that the defect now appears as an ordinary boundary of an ordinary open string on $X \times Y$. This is one efficient way to understand why the data defining the defect should be ordinary Chan-Paton factors on the product of the spaces, and why the conditions for supersymmetry are the usual conditions, but applied to Chan-Paton factors over the product of spaces.

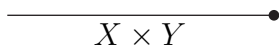There is one particular distinguished defect, the 'identity' or 'trivial' defect, which joins open strings on $X$ to open strings also on $X$. The trivial defect is defined by trivial rank one Chan-Paton factors along the diagonal embedding in $X \times X$. It is so named because one can insert it into a string worldsheet without changing the physics of that worldsheet theory.

In this paper, we insert identity defects and apply folding tricks globally on worldsheets, not just locally in the neighborhood of a defect, to give alternate (and physically equivalent) reparametrizations of worldsheets. We will check these methods extensively in the case of the B model, at the level of boundaries defined by branes, antibranes, and tachyons, as described mathematically by derived categories. (In fact, not only will this allow us to work in significant generality, but in addition, the formal manipulations required to check that massless spectra are preserved become significantly easier when working in derived categories than when working in, for example, an ordinary category of coherent sheaves.)

It is unclear at present whether defects can be consistently coupled to worldsheet gravity. One potential problem is that for general defects, one often gets infinite-dimensional moduli spaces of curves. For example, for a defect that starts in the middle of a closed string worldsheet, the shape of the defect is itself a modulus. If one could consistently couple to worldsheet gravity, and define critical string theories, the consequences appear to be somewhat radical. For example, defects link open strings on distinct spaces, and so, if (conformally invariant) defects can be consistently coupled to worldsheet gravity, and energy/momentum can flow across those defects, then those defects would define nonlocal interactions in quantum gravity. Just as a nonlocal interaction in ordinary quantum field theory violates local

energy/momentum conservation, this example of a nonlocal interaction in quantum gravity would violate local energy/momentum conservation across multiple spacetimes. In any event, we shall not attempt to couple defects to worldsheet gravity in this paper, and leave such considerations for other work.

## 2.2 Typical mathematics application: higher categories

A typical mathematical application of defects is to give a physical realization of descriptions of topological field theories using higher categories, as in *e.g.* [16, 20]. In this section we will briefly review such constructions and applications, partly to contrast with the focus of this paper, which will be on taffy operations with defects.

In the B model, a defect defines a Fourier-Mukai transform. The data of the defect, an object $\mathcal{E}$ in $D^b(X \times Y)$, defines the kernel of a Fourier-Mukai transform, hence a functor

$$F_{\mathcal{E}}(-) \ \equiv \ \mathbf{R}p_{Y*} \left( \mathcal{E} \otimes^{\mathbf{L}} \mathbf{L}p_X^* - \right)$$

Such transforms can act on closed string states on either space to generate closed string states on the other: one pulls back a polyvector field to the product, wedges with the Chern character of the object along the defect, then pushes forward to the other space. (This has been previously proposed in *e.g.* [6][pp 43-44].)

Given a defect on $X \times Y$ and another on $Y \times Z$, if those two defect lines collide in parallel, the result is a defect along $X \times Z$. In the B model, which will be a focus of this paper, this composition is believed to be defined as follows [21][prop. 5.1]. Consider $X \times Y \times Z$ with its projections $p_{12}$ to $X \times Y$, $p_{13}$ to $X \times Z$, $p_{23}$ to $Y \times Z$. Given $\mathcal{E}$ an object in $D^b(X \times Y)$ and $\mathcal{F}$ an object in $D^b(Y \times Z)$, the composition is defined by the object

$$p_{13*} \left( p_{12}^* \mathcal{E} \otimes p_{23}^* \mathcal{F} \right)$$

in $D^b(X \times Z)$ (taking all operations to be derived).

In this language, there is an 'identity' defect, defined by the diagonal [21][section 6.5]. If $\Delta_Y$ is the diagonal on $Y \times Y$, then for any $\mathcal{E}$ on $X \times Y$,

$$p_{13*} \left( p_{12}^* \mathcal{E} \otimes p_{23}^* \Delta_Y \right) \ \cong \ \mathcal{E}$$

Suppose a defect line ends in the middle of a disk, as outlined below:
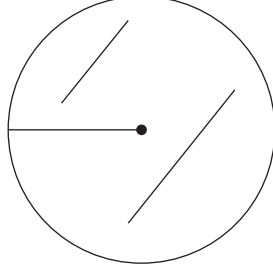
In the B model, if one is describing maps into a space $X$, then along the defect line one has an object $\mathcal{E} \in D^b(X \times X)$. At the endpoint one naturally has a boundary-condition-changing operator defined by an element[2] of

$$\mathrm{Ext}^*_{X \times X} (\Delta_* \mathcal{O}_X, \mathcal{E})$$

This has a natural categorical interpretation in terms of traces[3] in 2-categories. If we view $D^b(X)$ as a higher-categorical version of a vector space (a 2-vector space), then $D^b(X \times X)$ is some version of the space of endomorphisms of $D^b(X)$, and a trace map should be a map to the analogue of the ground field, which in this case is $D^b(\mathbf{C} - \mathrm{vect})$, the derived category of complex vector spaces. In this language, the trace functor sends any object $\mathcal{E}$ in $D^b(X \times X)$ to

$$\mathrm{Tr}\,\mathcal{E} \;\equiv\; \mathbf{R}\mathrm{Hom}_{X \times X} (\Delta_* \mathcal{O}_X, \mathcal{E})$$

and any morphism $f : \mathcal{E} \to \mathcal{F}$ in $D^b(X \times X)$ is mapped to a morphism $\mathrm{Tr}(f)$ defined by

$$\mathrm{Tr}(f)(\phi) \;\equiv\; f \circ \phi$$

This is a special case of the Ganter-Kapranov categorification of the notion of trace to any 2-category [22].

We should also note that open string defect diagrams seem to naturally correspond to "string diagram" pictures of 2-category structures (see for example [23] for a discussion of the mathematical notion of string diagrams). Briefly, in what mathematicians call a string diagram, objects are represented by 2-dimensional areas, 1-morphisms by boundaries

---

[2] As a check, boundary-condition-changing operators inserted at the intersection of the defect and the boundary would be counted, in the B model, by elements of $\mathrm{Ext}^*_{X^2} (\pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{F}^\vee, \mathcal{E})$, where $\mathcal{F} \in D^b(X)$ defines the disk boundary, and all operations assumed derived, conventions as explained later in this text. If we reel in the defect line, shrinking it to zero length, then the states at the triple intersection are composed with the states at the endpoint, namely elements of $\mathrm{Ext}^*_{X^2} (\mathcal{E}, \Delta_* \mathcal{O}_X)$ (which for $X$ Calabi-Yau matches the groups above), to form elements of

$$\mathrm{Ext}^*_{X^2} (\pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{F}^\vee, \Delta_* \mathcal{O}_X) \;=\; \mathrm{Ext}^*_X (\mathcal{F}, \mathcal{F})$$

as expected for insertions along the disk boundary, using a mathematical identity we shall derive later in this paper.

[3] E.S. would like to thank T. Pantev for explaining this material on 2-category traces.

11

between those areas, and 2-morphisms by boxes or marked points along the boundaries. Let us work through a simple example, both to illustrate string diagrams and to display their correspondence to physics. In the language of string diagrams, a commutative diagram in 2-categories

$$X \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightrightarrows}} Y \quad \Downarrow \alpha$$

would be represented by the (mathematical) string diagram



Physically, given open strings on $X$ and $Y$, defects joining the open strings are defined (in the B model, say) by objects $\mathcal{F}, \mathcal{G} \in D^b(X \times Y)$, each of which defines a Fourier-Mukai transform (hence a functor)

$$\mathcal{F}, \mathcal{G} : \ D^b(X) \ \longrightarrow \ D^b(Y)$$

Furthermore, along the defect, one can insert an operator $\alpha$, which will be an element of

$$\mathrm{Ext}^*_{X \times Y}(\mathcal{F}, \mathcal{G}) \ = \ H^* \left( \mathbf{R}\mathrm{Hom}_{X \times Y}(\mathcal{F}, \mathcal{G}) \right)$$

and so will define a natural transformation between the two functors defined by $\mathcal{F}$ and $\mathcal{G}$. Clearly the mathematical 'string diagram' above corresponds very closely to the physical picture of corresponding open strings joined along defects.

Although we will not work with mathematical string diagrams *per se* further in this paper, the picture above lends itself to higher categorical structures: a defect along a $k$-codimensional submanifold defines a $k$-morphism, and operators on the defect and smaller defects define higher-order morphisms. There is, in fact, a mathematical definition of ("enriched") topological field theories (see *e.g.* [16, 20, 24]), which involves representations not just of the category of $d$-dimensional bordisms of $d - 1$-manifolds, but of appropriate $d$-categories $(0, d)$–Bord (actually, "$(\infty, d)$-categories") of bordisms of manifolds of dimension 0 through $d$.

The Baez-Dolan cobordism hypothesis (now a family of theorems of Lurie [16]) says that an enriched topological field theory is determined by its value on a point. That is, to specify a representation of $(0, d)$–Bord in a $d$-category $\mathcal{C}$ is equivalent to specifying an object of $\mathcal{C}$,

which is required to satisfy some hypotheses depending on the precise flavor of enriched field theory one considers.

For example, ([16, Theorem 4.2.11]) to specify a 0—2-dimensional symmetric monoidal TFTs is equivalent to specifying a "Calabi-Yau" object in a symmetric monoidal 2-category (actually symmetric monoidal $(\infty,2)$-category). This is an object $X$ with a dual $X^\vee$, an evaluation map

$$\mathrm{ev}: \ X \otimes X^\vee \to \mathbf{1},$$

a coevaluation map

$$\mathrm{coev}: \ \mathbf{1} \to X \otimes X^\vee$$

(satisfying various properties), and so a map, sometimes labelled

$$\dim(X) \ = \ \mathrm{coev} \circ \mathrm{ev}: \ \mathbf{1} \to \mathbf{1}$$

This (strangely labelled) object $\dim(X)$ automatically carries an $S^1$-action (because, despite the name, it represents the massless states of a closed string). Finally, one should have an $S^1$-equivariant evaluation map

$$\eta: \ \dim(X) \to \mathbf{1}.$$

It is often equivalent ([16, 4.2.17]) to consider an $S^1$-invariant cotrace $\mathbf{1} \to \dim(X)$.

To obtain the B model on a Calabi-Yau $X$, consider the 2-category of differential graded categories, and for an object in it, take the $(\infty, 1)$–category of complexes of quasi-coherent $O_X$-modules on $X$ (whose homotopy category is the derived category). We interpret $\mathbf{1}$ as $\mathcal{O}_X$, the open string algebra, and the holomorphic top-form on the Calabi-Yau defines a map $\mathbf{1} \to \dim X$. For details, and an explanation of why $\infty$-categories are required, see §4.2 of [16].

In this paper, we will be concerned with naively distinct mathematical structures arising from defects, namely taffy operations.

# 3  String states and taffy operations in the B model

In this section we will describe string states in strings with defects, in the topological B model. We will compute the spectrum of string states using folding tricks for defects. Folding tricks have been widely discussed in the defects literature previously, see for example [2] and references therein. Moreover, some of the mathematical consequences of applying such folding tricks to topological field theories have been previously described elsewhere, see for example [24]. Our contribution is to work out and check explicit detailed expressions relating various (folded) open and closed strings.

We should emphasize at this point that this 'folding' is actually a trivial operation on the string worldsheet – in particular, despite the name, no folds in the sense of non-differentiable structures are being introduced into the maps into the target space. Rather, this is merely a reparametrization of the string worldsheet.

The result of this folding is to create physically-equivalent (but different-looking) string diagrams. As a consistency check, we will demonstrate that massless spectra of various physically-equivalent configurations agree.

## 3.1   Operations on derived categories

In our discussion of taffy-esque identities in the B model, we will be working extensively with objects in derived categories. When we perform operations such as duals, tensor products, and homomorphisms, we necessarily mean derived duals, derived tensor products, and so forth. As these concepts are not widely used in the physics community, in this section we briefly recall some of the basic facts about derived categories and the derived operations we use in this paper. We would also like to take this opportunity to thank A. Caldararu for numerous discussions of the identities in this and the next subsection.

Let $X$ be a ringed space, such as a Calabi-Yau manifold, with structure sheaf $\mathcal{O}_X$. By "$\mathcal{O}_X$-module" we will always mean a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We write $\mathrm{Mod}(X)$ for the category of $\mathcal{O}_X$-modules. If $E$, $F$, and $G$ are $\mathcal{O}_X$-modules, then we can form the $\mathcal{O}_X$-modules $E \otimes_X F$, $Hom_X(F,G)$, and $E^* = Hom_X(E, O_X)$. The sheaf $Hom_X(F,G)$ is sometimes called the "internal hom", because it is itself an $\mathcal{O}_X$-module: it is internal to the category of $O_X$-modules. Also, we can form the ring $R = \Gamma(X, \mathcal{O}_X)$ and the $R$-module $\Gamma E$. The essential properties of these operations are expressed by the identities

$$
\begin{aligned}
Hom_X(E \otimes_X F, G) &\cong Hom_X(E, Hom_X(F,G)) \\
E^* \otimes_X F &\cong Hom_X(E, F) \text{ if } E \text{ is finitely generated} \\
(E^*)^* &= E \text{ if } E \text{ is finitely generated} \\
E^* \otimes F^* &= (E \otimes F)^* \text{ if } E \text{ and } F \text{ are finitely generated} \qquad (2) \\
E &\cong Hom_X(\mathcal{O}_X, E) \\
\mathrm{Hom}_{\mathrm{Mod}(X)}(E, F) &\cong \Gamma(X, Hom(E, F)) \text{ and so} \\
\Gamma(X, E) &\cong \Gamma(X, Hom(\mathcal{O}_X, E)).
\end{aligned}
$$

Let $f : X \to Y$ be a map of ringed spaces. Associated to these we have operations

$$
\mathrm{Mod}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}(X)
$$

Some essential properties of these operations are[4]

$$f_*E(U) = E(f^{-1}(U)) \text{ if } U \text{ is an open set of } Y. \text{ In particular}$$
$$f_*E \cong \Gamma(X, E) \text{ if } Y \text{ is a one-point space.}$$
$$\mathrm{Hom}_{\mathrm{Mod}(X)}(f^*E, F) \cong \mathrm{Hom}_{\mathrm{Mod}(Y)}(E, f_*F)$$
$$f^*(E \otimes F) = f^*E \otimes f^*F$$
$$f^*(E^*) = (f^*E)^*$$

Let $D^b(X)$ be the the derived category of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules. An $\mathcal{O}_X$-module $E$ gives rise to an object of $D^b(X)$, namely the complex $\ldots 0 \to E \to 0 \cdots$ which is $E$ in degree 0 and 0 in other degrees. We simply write $E$ for this object of $D^b(X)$. The morphisms in the derived category from $\mathcal{E}$ to $\mathcal{F}$ form a cochain complex, denoted $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$. The derived category has a tensor product, denoted $\mathcal{E} \otimes_X^{\mathbf{L}} \mathcal{F}$, an internal Hom, denoted $\mathbf{R}Hom_X(\mathcal{E}, \mathcal{F})$, and a dual $\mathcal{E}^\vee = \mathbf{R}Hom(\mathcal{E}, \mathcal{O}_X)$. Moreover, given a complex of $\mathcal{O}_X$-modules $\mathcal{E}$ we can form the chain complex of $\Gamma(X, \mathcal{O}_X)$-modules $\mathbf{R}\Gamma(X, \mathcal{E})$. These constructions are related by identities much like those for $\mathrm{Mod}(X)$, namely

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E} \otimes_X \mathcal{F}, \mathcal{G}) \simeq \mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathbf{R}Hom_X(\mathcal{F}, \mathcal{G}))$$
$$\mathcal{E}^\vee \otimes_X^{\mathbf{L}} \mathcal{F} \simeq \mathbf{R}Hom_X(\mathcal{E}, \mathcal{F}) \text{ if } \mathcal{E} \text{ is finitely generated}$$
$$\mathcal{E} \simeq \mathbf{R}Hom_X(\mathcal{O}_X, \mathcal{E})$$
$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F}) \simeq \mathbf{R}\Gamma\mathbf{R}Hom(\mathcal{E}, \mathcal{F})$$
$$\mathbf{R}\Gamma\mathcal{E} \simeq \mathbf{R}\Gamma\mathbf{R}Hom(\mathcal{O}_X, \mathcal{E}).$$

In the above, the symbol $\simeq$ denotes quasi-isomorphism, meaning the cochain complexes have isomorphic cohomology. Familiar cohomological invariants may be accessed via the identities

$$\mathrm{Ext}_X^i(\mathcal{E}, \mathcal{F}) = H^i\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$$
$$H^i(X; \mathcal{E}) = H^i\mathbf{R}\Gamma(\mathcal{E}) \cong \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{E}).$$

**Remark 3.** The classical derived category $D^b(X)$ is obtained from the classical category $\mathrm{Ch}(X)$ of cochain complexes in $\mathrm{Mod}(X)$ by inverting the quasisomorphisms, and it is a triangulated category. The symbol $\simeq$ therefore indicates an isomorphism in the derived category.

These leads to some subtlely in viewing $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$ as the morphisms of $D^b(X)$. The problem is that, first of all, even to build $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$ involves a choice of injective

---

[4] The second line is a slight abuse of notation. The point is that if $*$ is a point, then $\mathcal{O}_*$ is the sheaf whose value global sections are just the complex numbers. Taking global sections is an equivalence of categories between $\mathcal{O}_*$-modules and $\mathbb{C}$-modules. On the left one has an $\mathcal{O}_C$-module, on the right one has the corresponding $\mathbb{C}$-module.

resolution of $\mathcal{F}$. This choice interacts with the definition of the composition

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F}) \times \mathbf{R}\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \to \mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{G}). \tag{4}$$

Homological algebra implies that $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$ is well-defined up to quasi-isomorphism, and so the groups $\mathrm{Ext}_X(\mathcal{E}, \mathcal{F})$ and the composition

$$\mathrm{Ext}_X(\mathcal{E}, \mathcal{F}) \times \mathrm{Ext}_X(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}_X(\mathcal{E}, \mathcal{G})$$

are well-defined on the nose.

For many purposes it is essential to deal with the category $\mathrm{Ch}(X)$ in a way that recognizes the special importance of quasi-isomorphisms without passing immediately to the derived category. The theory of $\infty$-categories provides a convenient framework for doing so. A key feature of $\infty$-categories is that they allow the cochain complex $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$ of morphisms to be defined only up to quasi-isomorphism, and they recognize that the composition of morphisms (4) is necessarily an $A_\infty$ operation: the choices involved in building an explicit model for the composition are analogous to the choice of how to divide up the unit interval to building a concatenation-of-paths operation

$$\mathrm{Map}([0,1], X) \times \mathrm{Map}([0,1], X) \to \mathrm{Map}([0,1], X).$$

The upshot is an $\infty$-category $D_\infty^b(X)$ of chain complexes of $\mathcal{O}_X$-modules, whose "homotopy category" is the classical triangulated derived category $D^b(X)$ of Verdier. J. Lurie develops the theory of the derived category from this point of view in [25].

In this paper, when we speak of the derived category we often implicitly mean the $\infty$-category $D_\infty^b(X)$ associated to $\mathrm{Mod}(X)$: For example, when we speak of $\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})$ as the morphisms in the derived category, we really mean the $\infty$-category. When we use the symbol $\simeq$, we are indicating an equivalence in the $\infty$-category which becomes an isomorphism in $D^b(X)$.

This $\infty$-category plays an prominent role in the current mathematical study of topological field theories. For example, Costello [20, Section 2.2] emphasizes that it is essential to use $D_\infty^b(X)$ to obtain the $B$-model in his framework.

Associated to a map $f: X \to Y$ of ringed spaces we have operations

$$D^b(Y) \underset{\mathbf{R}f_*}{\overset{f_!}{\underset{-\mathbf{L}f^*-}{\rightleftarrows}}} D^b(X)$$

By construction, these operations satisfy properties analogous to those for $f_*$ and $f^*$ above:

$$\mathbf{R}f_*\mathcal{E} \simeq \mathbf{R}\Gamma\mathcal{E} \text{ if } Y \text{ is a one-point space}$$
$$\mathbf{R}\mathrm{Hom}_X(\mathbf{L}f^*\mathcal{E}, \mathcal{F}) \simeq \mathbf{R}\mathrm{Hom}_Y(\mathcal{E}, \mathbf{R}f_*\mathcal{F})$$
$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathbf{L}f^*\mathcal{G}) \simeq \mathbf{R}\mathrm{Hom}_Y(f_!\mathcal{F}, \mathcal{G})$$

16

In particular, if $f : X \to *$, then

$$\mathbf{R}^i f_* \mathcal{E} \cong H^i(X; \mathcal{E}).$$

We recall a number of additional properties enjoyed by these operations.

## The projection formula

Via their tensor products $D^b(X)$ and $D^b(Y)$ can be thought of as categorical versions of commutative rings[5]. The pull-back $Lf^*$ preserves the tensor product, and so gives $D^b(X)$ in this sense the structure of a module over $D^b(Y)$. The *projection formula* says that $\mathbf{R}f_*$ is a homomorphism of modules over $D^b(Y)$: for $\mathcal{E} \in D^b(X)$ and $\mathcal{F} \in D^b(Y)$, we have

$$\mathbf{R}f_*(\mathcal{E} \otimes_X^{\mathbf{L}} Lf^*\mathcal{F}) \simeq (\mathbf{R}f_*\mathcal{E}) \otimes_Y^{\mathbf{L}} \mathcal{F}. \tag{5}$$

## Flat base change

Suppose that

$$\begin{array}{ccc} X' & \xrightarrow{\ g'\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\ g\ } & Y. \end{array}$$

is a pull-back diagram: that is, $f' : X' \to Y'$ is obtained from $f : X \to Y$ by base change along $g : Y' \to Y$.

**Proposition 6** ([26], prop. III.9.3; [21], section 2.7). *If $g$ is flat, then so is $g'$, and so $\mathbf{L}g^* = g^*$ and $(\mathbf{L}g')^* = (g')^*$. Moreover*

$$g^*\mathbf{R}f_* \simeq (\mathbf{R}f')_*(g')^* : D^b(X) \to D^b(Y).$$

## Grothendieck-Serre duality

The Grothendieck Duality theory describes the relationship between the derived dual and the derived pushforward. If $X$ is a smooth projective variety of dimension $d$, let $S_X = \Omega_X^d[d] = K_X[d]$ be its "dualizing complex." If $\mathcal{E}$ is a complex of coherent (=finitely generated quasi-coherent) $\mathcal{O}_X$-modules, let

$$\mathcal{E}^D = \mathcal{E}^\vee \otimes_X^{\mathbf{L}} S_X \simeq \mathcal{E}^\vee \otimes_X S_X \simeq \mathbf{R}Hom_X(\mathcal{E}, S_X).$$

---

[5] More generally, symmetric monoidal categories are categorical analogues of commutative rings. The category of $\mathcal{O}_X$ modules is a symmetric monoidal category, and, because of the tensor products mentioned above, the derived category $D^b(X)$ can also be given the structure of a symmetric monoidal category.

(if $X$ is smooth then $\Omega_X^d$ is projective, and so $\otimes^{\mathbf{L}}$ and $\otimes$ coincide). If $f : X \to Y$ be a proper map of smooth projective varieties, then the duality theory says that[6]

$$\mathbf{R}f_*(\mathcal{E}^D) \simeq (\mathbf{R}f_*\mathcal{E})^D. \tag{7}$$

**Remark 8.** More generally, the Grothendieck Duality theory concerns the existence and properties of a *right* adjoint $f^!$ of the functor $f_*$, so that, for $\mathcal{E} \in D^b(X)$ and $\mathcal{F} \in D^b(Y)$,

$$\mathbf{R}f_*\mathbf{R}Hom_X(\mathcal{E}, f^!\mathcal{F}) \simeq \mathbf{R}Hom_Y(\mathbf{R}f_*\mathcal{E}, \mathcal{F}). \tag{9}$$

For example, the theory asserts that if $f$ is proper and smooth of dimension $d$, then

$$f^!\mathcal{F} \simeq (f^*\mathcal{F}) \otimes^{\mathbf{L}} \Omega_{X/Y}^d[d] \simeq (f^*\mathcal{F}) \otimes \Omega_{X/Y}^d[d].$$

If $f : X \to Y$ is such that $f^!$ exists and is of the form

$$f^!\mathcal{F} \simeq (f^*\mathcal{F}) \otimes^{\mathbf{L}} S_{X/Y}$$

for some element $S_{X/Y}$ of $D^b(X)$, then $S_{X/Y}$ is called a *dualizing complex* for the map $f$. If $X$ and $Y$ are themselves smooth, then

$$S_{X/Y} \simeq S_X \otimes f^*S_Y^{-1}. \tag{10}$$

Assuming that $\mathcal{E}$ is finitely generated, and taking $\mathcal{F} = \mathcal{O}_Y$, (9) becomes

$$\mathbf{R}f_*(\mathcal{E}^\vee \otimes S_X \otimes f^*S_Y^{-1}) \simeq (\mathbf{R}f_*\mathcal{E})^\vee.$$

Using the projection formula (5), this becomes

$$\mathbf{R}f_*(\mathcal{E}^\vee \otimes S_X) \otimes S_Y^{-1}) \simeq (\mathbf{R}f_*\mathcal{E})^\vee,$$

so $\mathbf{R}f_*(\mathcal{E}^D) \simeq (\mathbf{R}f_*\mathcal{E})^D$, as asserted at (7).

**Example 11.** Consider for example the case that $X$ is compact and smooth and $Y = *$ is a the one-point space. Then $S_Y = \mathbb{C}$, considered as a sheaf over a point, and (7) becomes

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, K_X[d]) \simeq \mathrm{Hom}_Y(\mathbf{R}\Gamma\mathcal{E}, \mathbb{C}).$$

Taking cohomology, we find that

$$\mathrm{Ext}_X^{d-i}(\mathcal{E}, K_X) \cong \mathrm{Hom}(H^i(X; \mathcal{E}), \mathbb{C}),$$

which is Serre duality.

---

[6]We learned this formulation of Grothendieck duality from A. Neeman and A. Caldararu.

## 3.2   Comparison of derived and underived operations

In this paper, our arguments for taffy identities in the B model will technically revolve around manipulations in derived categories, where it becomes very simple to make strong statements. However, in many cases of physical interest, one really is using nonderived operations on sheaves, not derived operations. In this section, we will compare derived and underived operations, to uncover the physics hidden in the technology of derived categories.

In particular in this section we study derived operations in the special case that the derived category objects are pushforwards along embeddings of vector bundles of finite rank, which is the simplest description of a single set of D-branes. We shall discuss the relation between

1. Derived pushforwards $\mathbf{R}p_*$ and ordinary pushforwards $p_*$

2. Derived pullbacks $\mathbf{L}p^*$ and ordinary pullbacks $p^*$

3. Derived duals $\vee$ and ordinary duals $*$

4. Derived tensor products $\otimes^{\mathbf{L}}$ and ordinary tensor products $\otimes$

for such special objects in derived categories. We shall see, for example, that the Freed-Witten anomaly [27] is implicit in derived duals.

Before specializing to pushforwards of vector bundles, we mention in passing the simplest relationships of the derived operations to their underived analogues:

1. $\mathbf{R}f_*E \simeq f_*E$ if $E$ is injective or if $f$ is affine (*e.g.* a closed embedding);

2. $\mathbf{L}f^*E \simeq f^*E$ if $E$ is projective or if $f$ is flat;

3. $\mathbf{R}\mathrm{Hom}_X(E, F) \simeq \mathrm{Hom}_X(E, F)$ if either $E$ is projective or $F$ is injective;

4. $E \otimes_X F \simeq E \otimes_X^{\mathbf{L}} F$ if either $E$ or $F$ is projective (*e.g.* free).

**Derived pushforwards**

In the B model we can consider a $D$-brane on a closed submanifold $i : S \to X$ described by the sheaf $i_*E$, with $E$ a finite rank vector bundle over $S$. Since $i$ is a closed embedding, it follows that
$$\mathbf{R}i_*E \simeq i_*E$$

(and indeed for any sheaf on $S$). We shall make considerable use of the particular case

$$\Delta : X \to X \times X$$

of the diagonal embedding: $\mathbf{R}\Delta_* \simeq \Delta_*$.

## Derived pullbacks

In this paper we shall be considering primarily pullbacks along projections, *e.g.* $p : X \times Y \to X$. Projections are flat, and so derived pullbacks match ordinary pullbacks.

In addition, we sometimes use pullbacks along other maps in derivations. However, in other cases, derived and ordinary pullbacks typically do not match. For example, we occasionally discuss pullbacks along closed embeddings, such as the diagonal map $\Delta : X \to X \times X$. In the special case of pullbacks of vector bundles, the derived and ordinary pullbacks (along closed embeddings) do match. On the other hand, for more general sheaves, the derived and ordinary pullbacks do not match, essentially because closed embeddings are not flat. For example, $\Delta^* \mathcal{O}_\Delta = \mathcal{O}_X$; however, $\mathbf{L}\Delta^* \mathcal{O}_\Delta \neq \mathcal{O}_X$, as there are nonzero cohomology sheaves (roughly $\Omega^i$ in position $-i$).

## Derived duals

Given a vector bundle of finite rank $E$ over a submanifold $i : S \hookrightarrow X$ of codimension $d$, consider the derived dual

$$(\mathbf{R}i_* E)^\vee \equiv \mathbf{R}Hom(\mathbf{R}i_* E, \mathcal{O}_X) \simeq \mathbf{R}Hom(i_* E, \mathcal{O}_X).$$

(As we noted above, $\mathbf{R}i_* \simeq i_*$, as $i$ the inclusion of a submanifold.) We shall see that it is related to $i_*(E^*)$ by a combination of grading shifts and tensoring with canonical bundles. Specifically,

$$(i_* E)^\vee \simeq \mathbf{R}i_*(E^* \otimes S_{S/X}) \tag{12}$$

where $S_{S/X}$ is the dualizing sheaf of the embedding, and $*$ denotes the ordinary vector bundle dual. To see this, note that, using (7), we have

$$(i_* E)^D \simeq i_*(E^D).$$

That is,

$$\begin{aligned}
(i_* E)^\vee &\simeq (i_* E)^D \otimes S_X^{-1} \\
&\simeq i_*(E^D) \otimes S_X^{-1} \\
&\simeq i_*(E^\vee \otimes S_S \otimes i^* S_X^{-1}) \\
&\simeq \mathbf{R}i_*(E^* \otimes S_{S/X}).
\end{aligned}$$

20

In the preceding, we used the fact that $X$ and $S$ are smooth, so $S_S$ and $S_X$ are projective: thus derived tensor products and pull-backs reduce to ordinary ones. We used the fact that $E$ is a vector bundle, and so locally free, to conclude that

$$E^\vee = \mathbf{R}Hom_X(E, \mathcal{O}_X) \simeq Hom_X(E, \mathcal{O}_X) = E^*.$$

We used the projection formula to conclude that

$$i_*(\mathcal{F}) \otimes^{\mathbf{L}} S_X^{-1} \simeq i_*(\mathcal{F} \otimes^{\mathbf{L}} \mathbf{L}i^* S_X) \simeq i_*(\mathcal{F} \otimes i^* S_X).$$

It remains to compute $S_{S/X}$. From equation (10), we know that

$$S_{S/X} = S_S \otimes i^* S_X^{-1} = K_S[\dim S] \otimes i^* (K_X[\dim X])^{-1} = K_S \otimes (K_X|_S) [-d]$$

where recall $d$ is the codimension of $S$ in $X$. For completeness, note from the short exact sequence

$$0 \longrightarrow N_{S/X}^* \longrightarrow \Omega_X^1|_S \longrightarrow \Omega_S^1 \longrightarrow 0$$

it is straightforward to show that

$$\Lambda^{\text{top}} N_{S/X} \cong K_S \otimes (K_X|_S)^{-1}$$

hence

$$S_{S/X} = \Lambda^{\text{top}} N_{S/X}[-d].$$

Putting this together, we have that

$$(i_* E)^\vee \cong i_* \left( E^* \otimes K_S \otimes (K_X|_S)^{-1} \right) [-d] \cong i_* \left( E^* \otimes \Lambda^{\text{top}} N_{S/X} \right) [-d]. \qquad (13)$$

**Example 14.** Consider the diagonal embedding $\Delta : X \to X \times X$. Then $N_{X/X \times X} \cong T_X$, $K_{X \times X} = \pi_1^* K_X \otimes \pi_2^* K_X$, $K_{X \times X}|_X = K_X^{\otimes 2}$, and (13) becomes

$$(\Delta_* \mathcal{O}_X)^\vee \simeq \Delta_* \left( \mathcal{O}_X \otimes K_X \otimes \left( K_X^{\otimes 2} \right)^{-1} \right) [-\dim X] = \Delta_* K_X^{-1}[-\dim X]. \qquad (15)$$

(See also [23] for a discussion of the dual of the diagonal.)

Now, let us return to our discussion of the derived dual of $i_* E$. In this case, for $X$ a Calabi-Yau, equation (13) becomes

$$(i_* E)^\vee = i_* (E^* \otimes K_S) [-d].$$

Note that the factor of $K_S$ is ultimately due to the Freed-Witten anomaly [27]. Recall from [28] that because of the Freed-Witten anomaly, the sheaf $i_* E$ is associated with D-brane

Chan-Paton factors[7] $E \otimes K_S^{-1/2}$. If we dualize the Chan-Paton factors, they become

$$\left( E \otimes K_S^{-1/2} \right)^* \ = \ E^* \otimes K_S^{+1/2} = E^* \otimes K_S^{-1/2} \otimes K_S;$$

which means that the appropriate dual of the sheaf $i_*E$ (modulo grading shifts) should be $i_*(E^* \otimes K_S) \simeq i_*(E^D) \simeq (i_*E)^D$. In effect, this means that the Freed-Witten anomaly is baked into the formalism of derived categories, as it is automatically encoded in the natural dual in the sense of derived categories (the derived dual).

**Derived tensor products**

Next, let us compare the derived tensor product $\otimes^{\mathbf{L}}$ to the ordinary tensor product $\otimes$. Consider two D-branes wrapped on $i : S \hookrightarrow X$ and $j : T \hookrightarrow X$, $S$, $T$, and $X$ all assumed smooth. If the intersection of $S$ and $T$ is transversal (*e.g.* $\operatorname{codim} S/X + \operatorname{codim} T/X = \operatorname{codim}(S \cap T)/X$), then it is a result of Serre that

$$(i_*E) \otimes^{\mathbf{L}} (j_*F) \ \simeq \ (i_*E) \otimes (j_*F) \,.$$

(The point is to show that the sheaf $\underline{\operatorname{Tor}}^p(i_*E, j_*F)$ is zero for $p > 0$, which is precisely when the derived and underived tensor products will match. One reduces to the case that the ring of functions on $X$ is $A$, on $S$ is $A/I$, and $T$ is $A/J$ for prime ideals $I$ and $J$. With our hypotheses $\operatorname{Tor}^A(A/I, A/J)$ can be computed by a Koszul complex, and the transverse intersection assumption implies that this complex is acylic. The relevant results are p. 54 Prop 2, and p. 55 Cor 2 of [29].)

More generally, however, for non-transversal intersections, the derived tensor product will differ from the ordinary tensor product. Let $k : S \cap T \hookrightarrow X$ denote the natural embedding of the intersection, then the extra contributions arise from

$$k_* \left( E|_{S \cap T} \otimes F|_{S \cap T} \otimes \Lambda^* B \right)$$

where

$$B^* \ = \ TX|_{S \cap T} / \left( TS|_{S \cap T} + TT|_{S \cap T} \right)$$

(a bundle which also appeared in [28]). The bundle $B$ expresses the amount by which the intersection fails to be transversal.

In this paper, we shall only need the simple transverse case. In every case in which we are initially reading off string states from a diagram, the sheaves will be supported on different

---

[7] It has been suggested by A. Caldararu that an alternative approach to the Freed-Witten anomaly would be, in part, to define a "good" dual to a sheaf $\mathcal{E}$ by, $\mathcal{E}^* \equiv \mathcal{E}^\vee \otimes \sqrt{S_X}$, when this makes sense. We shall not follow that path here, but thought it warranted mentioning.

factors in a product. Let $E$ be a vector bundle on $i : S \hookrightarrow X$, and $F$ be a vector bundle on $j : T \hookrightarrow Y$. Then the ordinary and derived tensor products match [31]:

$$p_1^* i_* E \otimes p_2^* j_* F \;=\; p_1^* i_* E \otimes^{\mathbf{L}} p_2^* j_* F$$

and if we let $k$ denote the inclusion $S \times T \hookrightarrow X \times Y$, then they both match

$$k_* \left( p_1^* E \otimes p_2^* F \right)$$

In the bulk of the rest of this paper, we will study examples of taffy-like foldings of string worldsheets, and derive mathematical identities to check that such taffy operations give alternative descriptions of physically-equivalent string worldsheets.

Our constructions in the B model necessarily take place in the derived category, and as such, all operations will necessarily be derived. For notational simplicity, in the rest of this paper after this subsection, we will use underived notation to implicitly mean derived constructions. In other words, we will omit the $\mathbf{R}$ and the $\mathbf{L}$ from the notation: thus Hom means $\mathbf{R}$Hom, $\otimes$ means $\otimes^{\mathbf{L}}$, $i_*$ means $\mathbf{R}i_*$, and so forth.

## 3.3   Open string states

Let $\mathcal{E}$, $\mathcal{F}$ be objects in $D^b(X)$, defining boundaries in the open string B model on $X$.

Here is our first example of a taffy operation. Consider folding an oriented open string



along a trivial defect inserted at the center into a U-shape:



We mean by the diagram above (and other such in this paper) to indicate that the original open string has been folded over and then collapsed onto a single new open string on $X \times X$, with one boundary determined by $\mathcal{E}$ and $\mathcal{F}$, and the other boundary determined by the identity defect, as represented by a diagonal embedding $\Delta : X \to X \times X$:

In order to convey more information, the previous diagram was 'expanded' vertically, to show the different layers of the original open string, though the reader should always interpret such diagrams to mean that a vertical contraction onto a single-layer open string on multiple copies of $X$ has taken place.

In effect, we are encoding worldsheet geometry in D-branes. See also [30] where something analogous was done in a different theory.

We are not quite done. We need to uniquely specify Chan-Paton factors on the folded string (*e.g.* $\mathcal{F}$ versus $\mathcal{F}^\vee$), and we also need to specify an orientation on the folded open string. It is straightforward to see that different reasonable choices differ only by grading shifts, so this is a matter of picking a convention, not something essential to the physics. We shall follow the following convention (which was chosen specifically to preserve gradings):

1. For diagonals, if the orientation on the folded string points *towards* the diagonal, we describe the Chan-Paton factors by the object $\Delta_* \mathcal{O}_X$ in the derived category, which (because it will appear commonly) we shall abbreviate $\Delta$. If the orientation on the folded string points *away* from the diagonal, we use instead the object $\Delta^\vee$. (Recall from equation (15) that on a Calabi-Yau, $\Delta$ differs from $\Delta^\vee$ merely by a grading shift, $\Delta^\vee = \Delta[-\dim X]$, so we are merely choosing conventions so as to preserve gradings.)

2. For other objects present before the fold (in the example above, $\mathcal{E}, \mathcal{F}$), if the orientation on the folded string is parallel to the orientation on the part of the original string going into that boundary, we use the original object; if antiparallel, we use its dual.

For example, if we pick the orientation on the folded string to be

$$\mathcal{E}, \mathcal{F} \bullet\!\!\xrightarrow{\hspace{4cm}}\!\!\bullet\, \Delta$$

then the left Chan-Paton factors are described by the derived category object $\pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee$, and the right Chan-Paton factors are described by the derived category object $\Delta$. In this case, the string states in the new (folded) open string are counted by

$$\mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta \right)$$

Applying the identities from section 3.1, we have

$$\mathbf{R}\mathrm{Hom}_{X \times X}(\pi_1^* \mathcal{E} \otimes_{X \times X} \pi_2^* \mathcal{F}^\vee, \Delta_* \mathcal{O}_X)$$
$$\simeq \quad \mathbf{R}\mathrm{Hom}_X(\Delta^*(\pi_1^* \mathcal{E} \otimes_{X \times X} \pi_2^* \mathcal{F}^\vee), \mathcal{O}_X)$$
$$\simeq \quad \mathbf{R}\mathrm{Hom}_X(\mathcal{E} \otimes \mathcal{F}^\vee, \mathcal{O}_X) \simeq \mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F}).$$

and taking cohomology yields the equality

$$\mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta \right) \; = \; \mathrm{Ext}^*_X \left( \mathcal{E}, \mathcal{F} \right) \tag{16}$$

so we see that in this convention, the open string states on the folded string precisely match the original open string states, without even a grading shift.

If we had picked the opposite orientation on the folded string:

$$\mathcal{E}, \mathcal{F} \, \bullet \!\!\! \longleftarrow \!\!\! \bullet \, \Delta$$

then the left Chan-Paton factors would be described by the derived category object $\pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{F}$, and the right Chan-Paton factors would be described by the derived category object $\Delta^\vee$. In this case, the open string states in the folded open string would be given by

$$\mathrm{Ext}^*_{X \times X} \left( \Delta^\vee, \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{F} \right)$$

but a trivial application of section 3.1 and identity (16) implies

$$\mathrm{Ext}^*_{X \times X} \left( \Delta^\vee, \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{F} \right) \; = \; \mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta \right) \; = \; \mathrm{Ext}^*_X \left( \mathcal{E}, \mathcal{F} \right)$$

so again we recover the states of the original open string, without even a grading shift.

One technical point is that in the description above, the diagonal defect $\Delta$ is implicitly assumed to have (trivial, rank 1) Chan-Paton factors. The fact that the boundaries have support along the diagonal corresponds to the fact that they correspond to folds, but, folds do not have Chan-Paton factors. However, the open string spectrum with trivial line bundles along the defect $\Delta : X \to X \times X$ is the same as the open string spectrum with no Chan-Paton factors added at all (a trivial consequence of *e.g.* [28]). Thus, we can equivalently interpret the Ext groups above as computing open string spectra between boundaries with no added Chan-Paton factors.

Now, let us perform a consistency check of our description of folded open strings. Consider the folded open string with orientation

$$\mathcal{E}, \mathcal{F} \, \bullet \!\!\! \longrightarrow \!\!\! \bullet \, \Delta$$
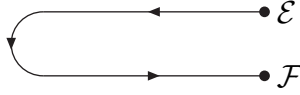
so that the string states are

$$\mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta \right) \; = \; \mathrm{Ext}^*_X \left( \mathcal{E}, \mathcal{F} \right)$$

Now that we have fixed an orientation on the folded open string corresponding to the orientation on the original string, we can ask about the effect of flipping the orientation. If we were to traverse the folded string in the opposite direction, the string states would be

$$
\begin{aligned}
\mathrm{Ext}^*_{X \times X} \left( \Delta, \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee \right) \; &= \; \mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{F}, \Delta^\vee \right) \\
&= \; \mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{F}, \Delta[-\dim X] \right) \quad \text{(for } X \text{ Calabi-Yau)} \\
&= \; \mathrm{Ext}^*_X \left( \mathcal{E}^\vee, \mathcal{F}^\vee[-\dim X] \right) \\
&= \; \mathrm{Ext}^*_X \left( \mathcal{F}, \mathcal{E}[-\dim X] \right)
\end{aligned}
$$

which, up to an irrelevant grading shift, are precisely the states one would get from traversing the original unfolded string in the opposite direction, as expected.

For completeness, let us also consider folding in the opposite direction:



If we collapse this to an open string on $X \times X$ with the following orientation:



then following our usual convention, string states are given by

$$\mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta \right)$$

and we have already seen that this matches $\mathrm{Ext}^*_X \left( \mathcal{E}, \mathcal{F} \right)$. Similarly for the opposite orientation convention on the folded string.

In order to show that similar results apply for more complicated folds, we will need a slight generalization of the identity (16), whose proof we learned from [31]. For any $\mathcal{E} \in D^b(X)$ and any finitely generated $\mathcal{S} \in D^b(X \times Y)$, we have

$$\mathrm{Ext}^*_{X \times X \times Y} \left( \pi_1^* \mathcal{E} \otimes \pi_{23}^* \mathcal{S}^\vee, \pi_{12}^* \Delta \right) \; = \; \mathrm{Ext}^*_{X \times Y} \left( \pi_1^* \mathcal{E}, \mathcal{S} \right) \tag{17}$$

This is our first "taffy identity." Intuitively, it allows us to equate the two diagrams
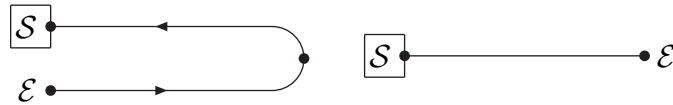


(where $\mathcal{S}$ is being used to encode any number of additional foldings) and hence can be used to unfold foldings.

To prove (17), note first of all that

$$\mathrm{Ext}^*_{X \times X \times Y} \left( \pi_1^* \mathcal{E} \otimes \pi_{23}^* \mathcal{S}^\vee, \pi_{12}^* \Delta \right) \; = \; \mathrm{Ext}^*_{X \times X \times Y} \left( \pi_1^* \mathcal{E}, \pi_{23}^* \mathcal{S} \otimes \pi_{12}^* \Delta \right)$$

Let

$$\Delta_{12} : \; (x, y) \; \mapsto \; (x, x, y), \quad \Delta : \; x \; \mapsto \; (x, x).$$

26

We have the pull-back diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\ \pi_1\ } & X \\
\downarrow{\scriptstyle \Delta_{12}} & & \downarrow{\scriptstyle \Delta} \\
X \times X \times Y & \xrightarrow{\ \pi_{12}\ } & X \times X
\end{array}
$$

with $\pi_{12}$ flat, and so by flat base change (prop. 6) we have

$$
\pi_{12}^* \Delta_* \mathcal{O}_X \simeq \Delta_{12*} \pi_1^* \mathcal{O}_X = \Delta_{12*} O_{X \times Y}.
$$

Now using the projection formula (equation (5)) and the fact that $\pi_{23} \circ \Delta_{12}(x, y) = (x, y)$, we have

$$
\begin{aligned}
\pi_{23}^* \mathcal{S} \otimes \pi_{12}^* \Delta &\simeq \pi_{23}^* \mathcal{S} \otimes \Delta_{12*} O_{X \times Y} \\
&\simeq \Delta_{12*}(\Delta_{12}^* \pi_{23}^* \mathcal{S}) \\
&\simeq \Delta_{12*} \mathcal{S}.
\end{aligned}
$$

Thus (noting that $\pi_1 \circ \Delta_{12} = \pi_1$)

$$
\begin{aligned}
\mathrm{Ext}^*_{X \times X \times Y} (\pi_1^* \mathcal{E} \otimes \pi_{23}^* \mathcal{S}^\vee, \pi_{12}^* \Delta) &= \mathrm{Ext}^*_{X \times X \times Y} (\pi_1^* \mathcal{E}, \Delta_{12*} \mathcal{S}) \\
&= \mathrm{Ext}^*_X (\mathcal{E}, \pi_{1*} \Delta_{12*} \mathcal{S}) \\
&= \mathrm{Ext}^*_X (\mathcal{E}, \pi_{1*} \mathcal{S}) \\
&= \mathrm{Ext}^*_{X \times Y} (\pi_1^* \mathcal{E}, \mathcal{S}),
\end{aligned}
$$

which is (17).

One trivial application of the result above is to recheck the U-shaped-strings discussed previously. There, we argued that

$$
\mathrm{Ext}^*_{X \times X} (\pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta) = \mathrm{Ext}^*_X (\mathcal{E}, \mathcal{F})
$$

directly from homological algebra. If instead we apply (17), taking $Y$ to be a point and $\mathcal{S} = \mathcal{F}$, then we immediately recover the same result.

A more interesting example is to fold an ordinary open string on $X$ with boundaries $\mathcal{E}$, $\mathcal{F}$ into an S-shape,



27
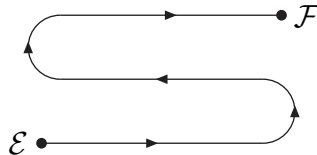
Pressing this down into a single open string on $X^3$, and picking an orientation on the folded string, we recover

$$\mathcal{E}, \Delta_{23} \bullet\!\!\longrightarrow\!\!\bullet \Delta_{12}, \mathcal{F}$$

with states

$$\mathrm{Ext}^*_{X \times X \times X}\left(\pi_1^* \mathcal{E} \otimes \pi_{23}^* \Delta^\vee, \pi_{12}^* \Delta \otimes \pi_3^* \mathcal{F}\right).$$

In order for the folded string to be equivalent to the original, we conjecture

$$\mathrm{Ext}^*_X\left(\mathcal{E}, \mathcal{F}\right) \;=\; \mathrm{Ext}^*_{X \times X \times X}\left(\pi_1^* \mathcal{E} \otimes \pi_{23}^* \Delta^\vee, \pi_{12}^* \Delta \otimes \pi_3^* \mathcal{F}\right). \qquad (18)$$

(See also $e.g.$ [2][fig. 6a] for a different discussion of this same diagram.)

The identity above can be derived by repeatedly applying identity (17). For notational convenience, define

$$\Delta_{ij} \;\equiv\; \pi_{ij}^* \Delta_* \mathcal{O}_X$$

where $\pi_{ij}$ is the projection onto the $i$th, $j$th factors of $X$ in a product of several copies. Then the massless states associated to the S-shape above are given by

$$\begin{aligned}
\mathrm{Ext}^*_{X^3}\left(\pi_1^* \mathcal{E} \otimes \Delta_{23}^\vee, \Delta_{12} \otimes \pi_3^* \mathcal{F}\right) &= \mathrm{Ext}^*_{X^3}\left(\pi_1^* \mathcal{E} \otimes \Delta_{23}^\vee \otimes \pi_3^* \mathcal{F}^\vee, \Delta_{12}\right) \\
&= \mathrm{Ext}^*_{X^3}\left(\pi_1^* \mathcal{E} \otimes \pi_{23}^*(\Delta \otimes \pi_2^* \mathcal{F})^\vee, \Delta_{12}\right)
\end{aligned}$$

Applying the identity (17), which means, unrolling the bottommost U-shape, this becomes

$$\mathrm{Ext}^*_{X^2}\left(\pi_1^* \mathcal{E}, \Delta \otimes \pi_2^* \mathcal{F}\right) \;=\; \mathrm{Ext}^*_{X^2}\left(\pi_1^* \mathcal{E} \otimes \mathcal{F}^\vee, \Delta\right),$$

or graphically
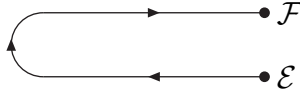
We have already seen that the massless states in this diagram are just $\mathrm{Ext}^*_X(\mathcal{E}, \mathcal{F})$, hence we recover (18).

The pattern can now be repeated *ad infinitum*, repeatedly folding open strings over themselves to produce open strings on higher-dimensional spaces. To drive home that point, let us work through one last example of these manipulations. Consider refolding an open string over itself into a five-layer pattern, as shown:
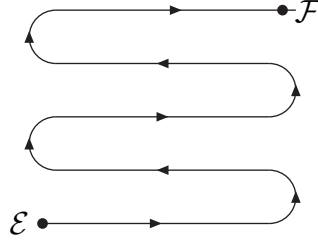
The result of this folding is an open string on the product of five copies of $X$, with states counted by

$$\text{Ext}^*_{X^5}\left(\pi_1^*\mathcal{E} \otimes \Delta_{23}^\vee \otimes \Delta_{45}^\vee, \Delta_{12} \otimes \Delta_{34} \otimes \pi_5^*\mathcal{F}\right)$$

Since this is just a folded version of a string between $\mathcal{E}$ and $\mathcal{F}$, the states should be counted by $\text{Ext}^*_X\left(\mathcal{E}, \mathcal{F}\right)$, which we shall now check by successive unfolding operations. We can begin by unfolding the bottommost U-shape, using identity (17), revealing that the states above are the same as

$$\text{Ext}^*_{X^4}\left(\pi_1^*\mathcal{E}, \Delta_{12} \otimes \Delta_{34} \otimes \Delta_{23} \otimes \pi_4^*\mathcal{F}\right) \;=\; \text{Ext}^*_{X^4}\left(\pi_1^*\mathcal{E} \otimes \Delta_{12}^\vee \otimes \Delta_{34}^\vee, \Delta_{23} \otimes \pi_4^*\mathcal{F}\right)$$

or graphically
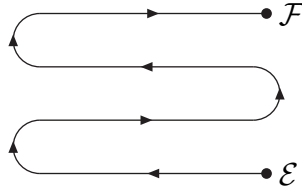


Unrolling the next bottom U-shape, applying identity (17), we see that the states above are the same as

$$\text{Ext}^*_{X^3}\left(\pi_1^*\mathcal{E} \otimes \Delta_{23}^\vee, \Delta_{12} \otimes \pi_3^*\mathcal{F}\right)$$

or graphically



29
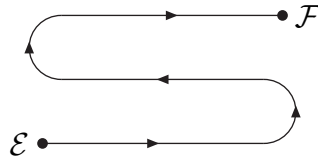
This is identical to the S-shape discussed earlier, so we can now conclude that the states in this folded open string are indeed counted by $\text{Ext}_X^*(\mathcal{E}, \mathcal{F})$, as expected.

It should be clear that this process can be continued for arbitrarily many folds; regardless of the number of foldings introduced, if we start with an open string from $\mathcal{E}$ to $\mathcal{F}$, then after foldings the open string states will still be given by $\text{Ext}_X^*(\mathcal{E}, \mathcal{F})$.

## 3.4   Closed string states

Now, let us apply the same ideas to closed strings. Begin with a closed string on $X$,
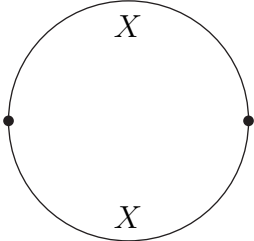


with trivial defects inserted as shown. By flattening the diagram above, as



and picking an orientation, say,



we predict that the closed string states are the same as elements of

$$\text{Ext}_{X \times X}^*\left(\Delta^\vee, \Delta\right)$$

As we shall discuss in section 5, this is the "Hochschild homology" of $X$, and for smooth $X$, the Hochschild-Kostant-Rosenberg (HKR) isomorphism [21][theorem 6.3] says that

$$\text{Ext}_{X \times X}^*\left(\Delta^\vee, \Delta\right) = \bigoplus_{p-q=*} H^p\left(X, \Omega_X^q\right)$$

30

and so we see that the Ext groups on $X \times X$ are differential forms, as expected for closed string states. Using the fact that

$$\Omega_X^q \;=\; K_X \otimes \Lambda^{n-q} TX$$

where $n$ is the dimension of $X$, we have that

$$H^p\left(X, \Omega_X^q\right) \;=\; H^p\left(X, K_X \otimes \Lambda^{n-q} TX\right)$$

and so

$$\mathrm{Ext}_{X \times X}^*\left(\Delta^{\vee}, \Delta\right) \;=\; \bigoplus_{p-q=*} H^p\left(X, \Omega_X^q\right) \;=\; \bigoplus_{p+q=n-*} H^p\left(X, K_X \otimes \Lambda^q TX\right)$$

When $X$ is Calabi-Yau, the states above are well-known to match closed string states on $X$ [32].

The Hochschild-Kostant-Rosenberg (HKR) theorem above will form the intellectual basis of the taffy identities for closed strings. In effect, we will use homological algebra to reduce all diagrams obtained by folding and twisting closed strings to the diagram above, then apply HKR to argue that the results match closed string states.

For our first example of a folded closed string, consider folding a closed string into a U-bar shape, as
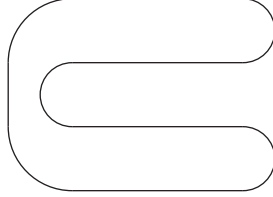


then we get a prediction that the closed string states are given by

$$\mathrm{Ext}_{X \times X \times X \times X}^*\left(\Delta_{14}^{\vee} \otimes \Delta_{23}^{\vee}, \Delta_{12} \otimes \Delta_{34}\right). \tag{19}$$

More generally, for $\mathcal{S} \in D^b(X^2 \times Y)$, it can be shown that

$$\mathrm{Ext}_{X^4 \times Y}^*\left(\Delta_{12}^{\vee}, \Delta_{14} \otimes \Delta_{23} \otimes \pi_{34Y}^* \mathcal{S}\right) \;=\; \mathrm{Ext}_{X^2 \times Y}^*\left(\Delta_{12}^{\vee}, \mathcal{S}\right) \tag{20}$$

which is our second "taffy identity" (and, for $Y$ a point and $\mathcal{S} = \Delta$, together with the HKR isomorphism, implies that the states (19) match closed string states).

The second taffy identity (20) can be checked as follows [31]. First, the left-hand-side can be written as the cohomology of

$$\mathbf{R}\Gamma\left(X^4 \times Y, \Delta_{12} \otimes \Delta_{23} \otimes \Delta_{14} \otimes \pi_{34Y}^* \mathcal{S}\right)$$

However,
$$\Delta_{12} \otimes \Delta_{23} \otimes \Delta_{14} \;=\; \Delta_{\text{small}\,*}\mathcal{O}_{X\times Y}$$

where
$$\Delta_{\text{small}} : \; (x, y) \;\mapsto\; (x, x, x, x, y)$$

(Intuitively, this $\Delta_{\text{small}}$ is at least analogous to a small loop between the various copies of $X$.) Furthermore,
$$\Delta_{\text{small}\,*}\mathcal{O}_{X\times Y} \otimes \pi_{34Y}^{*}\mathcal{S} \;=\; \Delta_{\text{small}\,*}\left(\Delta_{\text{small}}^{*}\pi_{34Y}^{*}\mathcal{S}\right)$$

Thus, the left-hand-side of (20) is the cohomology of
$$\mathbf{R}\Gamma\left(X \times Y, \Delta_{\text{small}}^{*}\pi_{34Y}^{*}\mathcal{S}\right)$$

Similarly, the right-hand-side of (20) can be written
$$\mathbf{R}\Gamma\left(X^{2} \times Y, \Delta_{12} \otimes \mathcal{S}\right)$$

where $\Delta_{12} \equiv \Delta_{12*}\mathcal{O}_{X\times Y}$ and
$$\Delta_{12} : \; (x, y) \;\mapsto\; (x, x, y)$$

Now,
$$\mathcal{S} \otimes \Delta_{12*}\mathcal{O}_{X\times Y} \;=\; \Delta_{12*}\left(\Delta_{12}^{*}\mathcal{S}\right)$$

so the right-hand-side of (20) is
$$\mathbf{R}\Gamma\left(X \times Y, \Delta_{12}^{*}\mathcal{S}\right)$$

Finally, the left- and right-hand-sides match because $\pi_{34Y} \circ \Delta_{\text{small}} = \Delta_{12}$:



Thus, we have established our second taffy identity.

Note that as a special case, if we take $Y$ to be the empty set, and $\mathcal{S} = \Delta$, then identity (20) reduces to
$$\text{Ext}_{X^{4}}^{*}\left(\Delta_{14}^{\vee} \otimes \Delta_{23}^{\vee}, \Delta_{12} \otimes \Delta_{34}\right) \;=\; \text{Ext}_{X^{2}}^{*}\left(\Delta^{\vee}, \Delta\right) \qquad (21)$$

as expected, which together with HKR verifies that (19) describe closed string states.

For another application of the second taffy identity, consider folding twice, to get
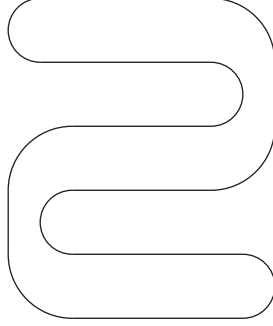
and press down to form an open string on $X^6$, then we get a prediction that the closed string states are given by

$$\text{Ext}^*_{X^6}\left(\Delta^\vee_{14} \otimes \Delta^\vee_{23} \otimes \Delta^\vee_{56}, \Delta_{12} \otimes \Delta_{36} \otimes \Delta_{45}\right)$$

We can show this by repeatedly applying identity (20) to successively unfold the diagram above. Applying it once yields

$$\text{Ext}^*_{X^6}\left(\Delta^\vee_{14} \otimes \Delta^\vee_{23} \otimes \Delta^\vee_{56}, \Delta_{12} \otimes \Delta_{36} \otimes \Delta_{45}\right) = \text{Ext}^*_{X^4}\left(\Delta^\vee_{12}, \Delta_{34} \otimes \Delta_{23} \otimes \Delta_{14}\right)$$

which we have already demonstrated to match closed string states.

It is straightforward to check that additional folds can be straightened; we leave the details as an exercise for the reader.

In addition to folds, we can also stretch out sections. For example, consider the diagram
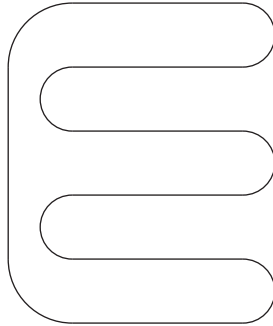


The string states in this case are

$$\text{Ext}^*_{X^6}\left(\Delta^\vee_{16} \otimes \Delta^\vee_{23} \otimes \Delta^\vee_{45}, \Delta_{12} \otimes \Delta_{34} \otimes \Delta_{56}\right)$$

We can apply identity (20) to show that the states above match closed string states. First, relabel the states above by exchanging 4 and 6, to get

$$\text{Ext}^*_{X^6}\left(\Delta^\vee_{14} \otimes \Delta^\vee_{23} \otimes \Delta^\vee_{65}, \Delta_{12} \otimes \Delta_{36} \otimes \Delta_{54}\right)$$
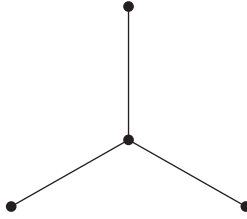
then, identity (20) implies that this is the same as

$$\mathrm{Ext}^*_{X^4}\left(\Delta^\vee_{12}, \Delta_{34} \otimes \Delta_{14} \otimes \Delta_{23}\right)$$

which we have already shown matches closed string states.

Next, consider the three-pointed star:
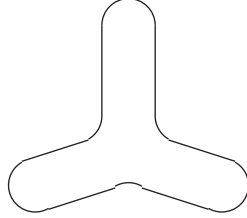


On the one hand, we can contract this to three open strings joined at the center, with diagonal defects at each outer corner and a more complicated diagonal defect at the center:



On the other hand, we can 'rolodex' this star into the fat E-shape figure described previously, which gives us the states and also tells us that they do match closed string states, as expected.

More generally, given an $n$-pointed star of the form above, the states are given as

$$\mathrm{Ext}^*_{X^{2n}}\left(\otimes^n_{i=1}\Delta^\vee_{2i,2i-1}, \otimes^n_{j=1}\Delta_{2j-1,2j-2}\right)$$

and by repeatedly unravelling with identity (20), we can show that these match closed string states, as expected.

In addition to folds, we can also twist strings. Consider first flattening a circle into a long flat oval, then twisting the oval into a figure eight. To understand the spectrum, fold at the twist to get a diagram of the form[8]

---

[8] There is no information pertinent for these considerations contained in whether lines cross over or under one another, hence we have not tried to distinguish crossings in the picture shown.
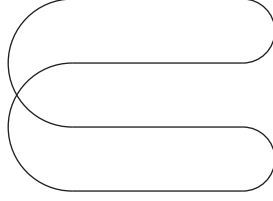
then we get a prediction that the closed string states are given by

$$\mathrm{Ext}^*_{X^4}\left(\Delta^\vee_{13}\otimes\Delta^\vee_{24},\Delta_{12}\otimes\Delta_{34}\right) \tag{22}$$

which we will show momentarily to be correct. More generally, it can be shown that [31]

$$\mathrm{Ext}^*_{X^4\times Y}\left(\Delta^\vee_{12},\Delta_{13}\otimes\Delta_{24}\otimes\pi^*_{34Y}\mathcal{S}\right) = \mathrm{Ext}^*_{X^2\times Y}\left(\Delta^\vee_{12},\mathcal{S}\right) \tag{23}$$

for any $\mathcal{S}\in D^b(X^2\times Y)$. This is our third taffy identity. This can be derived in almost exactly the same fashion as the second taffy identity (20); the only difference is that we utilize the result that

$$\Delta_{12}\otimes\Delta_{13}\otimes\Delta_{24} = \Delta_{\mathrm{small}}$$

When $Y$ is the empty set and $\mathcal{S}=\Delta_{34}$, this implies that

$$\mathrm{Ext}^*_{X^4}\left(\Delta^\vee_{12},\Delta_{13}\otimes\Delta_{24}\otimes\Delta_{34}\right) = \mathrm{Ext}^*_{X^2}\left(\Delta^\vee,\Delta\right)$$

which is precisely the conjecture (22).

For another example, consider taking a closed string, squeezing to a flattened oval, then twisting twice and folding at each twist. The resulting diagram is
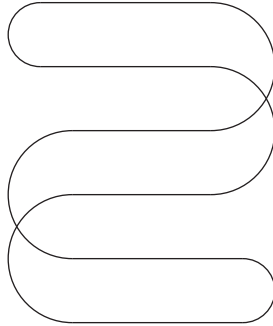


and the corresponding open string states are

$$\mathrm{Ext}^*_{X^6}\left(\Delta^\vee_{13}\otimes\Delta^\vee_{24}\otimes\Delta^\vee_{56},\Delta_{12}\otimes\Delta_{35}\otimes\Delta_{46}\right)$$

We can repeatedly apply the identity (23) to successively unroll and untwist layers to show that these are the same as closed string states on $X$. After one application, one has that the states above are the same as

$$\mathrm{Ext}^*_{X^4}\left(\Delta^\vee_{12},\Delta_{34}\otimes\Delta_{13}\otimes\Delta_{24}\right)$$

which is the case previously discussed, and so matches closed string states.

Because the taffy identities are insensitive to crossings, no knot invariants can be built from these constructions.

## 3.5  More general open-closed strings from defects

For completeness, and later applications, let us make some easy general observations on other more general examples of defects in the B model.

Consider, for our first example, the case of a closed string on $X$ with a single defect inserted:
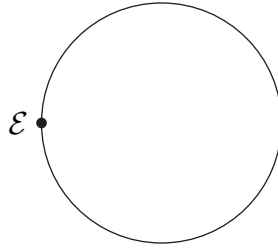


defined by $\mathcal{E} \in D^b(X)$. We can compute the open string states in this case by inserting a trivial fold at the opposite side of the circle:
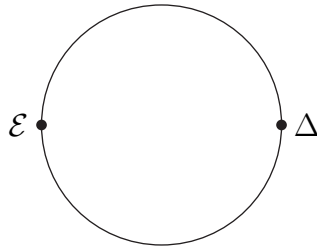


then from the previous analysis, we see that the open string specturm is given by

$$\mathrm{Ext}^*_{X \times X}\left(\Delta_* \mathcal{E}, \Delta\right)$$

Next, let us generalize to "closed strings" formed from gluing together open strings on distinct spaces along defects. To begin, consider an infinite cylinder split lengthwise into a pair of semi-infinite strips joined along two edges, one string on $X$ and the other on $Y$, so that a spacelike cross-section is
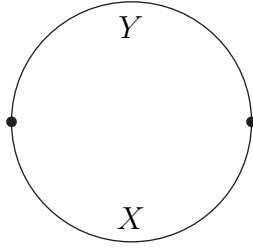
with defects defined by $\mathcal{S}_1, \mathcal{S}_2 \in D^b(X \times Y)$. This is physically equivalent to an ordinary open string on $X \times Y$:
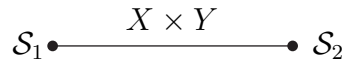


In particular, in both cases the open string states (inserted in the infinite future or past) are

$$\text{Ext}^*_{X \times Y}(\mathcal{S}_1, \mathcal{S}_2) \tag{24}$$

It is somewhat tempting to incorrectly speculate that the open string states in the first picture arise from both

$$\text{Ext}^*_X\left(\pi_{1*}\mathcal{S}_1, \pi_{1*}\mathcal{S}_2\right), \quad \text{Ext}^*_Y\left(\pi_{2*}\mathcal{S}_1, \pi_{2*}\mathcal{S}_2\right) \tag{25}$$

however, this is incorrect, as the corresponding operators are inserted at points where $X$ and $Y$ come together, so one cannot meaningfully distinguish in the form implied in (25). (Moreover, there is no mathematical equivalence between (24) and (25).)

With larger numbers of segments, additional folding tricks are possible. Suppose there are four segments, so that a cross-section of the partitioned cylinder is
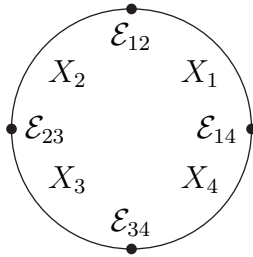


where $\mathcal{E}_{ij} \in D^b(X_i \times X_j)$, which folds into the equivalent diagram



37
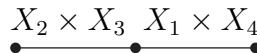
with middle defect on

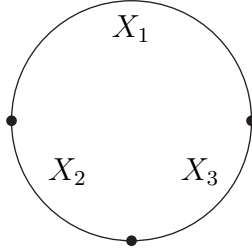$$\pi_{12}^* \mathcal{E}_{12} \otimes \pi_{34}^* \mathcal{E}_{34}$$

which folds again into an open string on

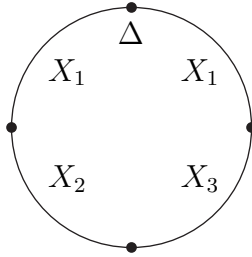$$Y \equiv X_1 \times X_2 \times X_3 \times X_4$$

with open string states

$$\mathrm{Ext}_Y^* \left( \pi_{12}^* \mathcal{E}_{12} \otimes \pi_{34}^* \mathcal{E}_{34}, \pi_{23}^* \mathcal{E}_{23}^\vee \otimes \pi_{14}^* \mathcal{E}_{14} \right)$$
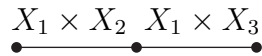
Another folding trick involves a cylinder formed from three open strings:



We first split the top open string by inserting $\Delta_* \mathcal{O}_{X_1}$, as



and then fold it into



which, after another fold, has states counted by

$$\mathrm{Ext}_{Y'}^* \left( \pi_{12}^* \mathcal{E}_{12} \otimes \pi_{31}^* \mathcal{E}_{31}^\vee, \pi_{23}^* \mathcal{E}_{23} \otimes \pi_{11}^* \Delta_* \mathcal{O}_{X_1} \right)$$

where
$$Y' \equiv X_1 \times X_1 \times X_2 \times X_3$$

Note that we can efficiently specify locations of parallel defects along a cylinder through a simplex: specify the location of each defect by what fraction of the circumference it sits at, relative to the previous defect. If there are $k$ defects, then that gives us $k$ real numbers between 0 and 1 whose sum is necessarily 1, which is precisely a simplex.

Another folding trick involves a diagram that cannot be understood as just a cylinder, with cross section
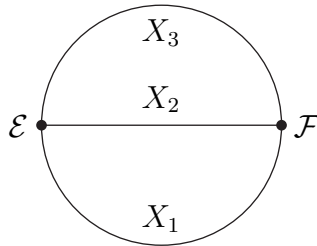


where $\mathcal{E}, \mathcal{F} \in D^b(X_1 \times X_2 \times X_3)$. This folds into the equivalent open string on $X_1 \times X_2 \times X_3$:
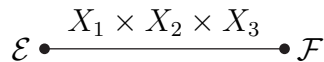


from which we see that the string states arising from this diagram are

$$\mathrm{Ext}^*_{X_1 \times X_2 \times X_3}(\mathcal{E}, \mathcal{F})$$

## 3.6  Correlation functions

So far we have only checked that massless states match after performing taffy operations. In this section, we shall outline how correlation function matching should occur. Unlike the case of massless states, for correlation functions we do not have rigorous proofs, so we will only outline conjectures.

Let us begin by considering correlation functions on a disk, describing an open string on $X$, with the same boundary conditions everywhere along the edge of the disk, corresponding to a object $\mathcal{E}$ in $D^b(X)$.
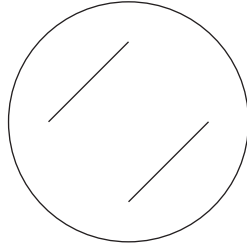
Now, split the disk in half along the identity defect, as
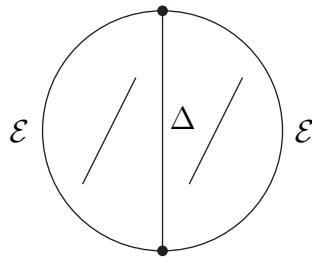


To perform such a splitting, we need to insert boundary-condition-changing operators at top and bottom intersection points, which in the present case will be elements of

$$\mathrm{Ext}^*_{X \times X} \left( \pi_1^* \mathcal{E}^\vee \otimes \pi_2^* \mathcal{E}, \Delta \right) \tag{26}$$

This group is the same as

$$\mathrm{Ext}^*_X \left( \mathcal{E}, \mathcal{E} \right) \tag{27}$$

which matches the intuition that inserting the diagonal defect is equivalent to doing nothing, since if we did not insert the defect, the boundary operators would have been counted by (27).

We can then fold the diagram above, to obtain the disk diagram
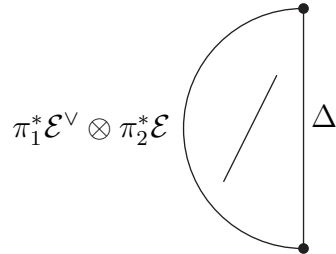


on $X \times X$.

In principle, we expect correlation functions on the folded disk to match correlation functions on the original disk.

*Conjecture:* The open string theory formed by taking a closed string worldsheet and triangulating into open strings with defect boundaries, is equivalent to the original closed string theory.

This would be in analogy with the behavior of two-dimensional QCD (see for example [33]). In the two-dimensional QCD story, one triangulates a Riemann surface, roughly, by inserting traces over group representations along edges. Here, by contrast, one is not inserting a complete set of states, but rather is merely inserting a trivial defect, creating a trivial reparametrization of the worldsheet, no more.

The conjecture above implies that one could fold all closed string diagrams into open string disk diagrams. For example, a sphere on $X$ could be flattened to a disk on $X \times X$ with the diagonal $\Delta$ along the edge:
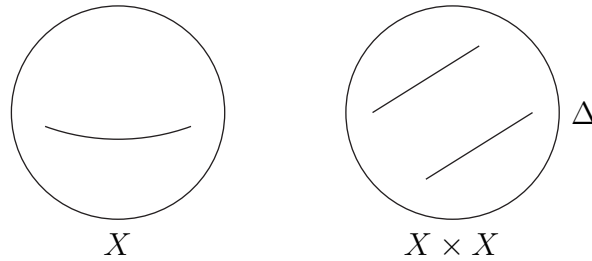


Similarly, a two-torus on $X$ can be flattened into an annulus on $X \times X$:
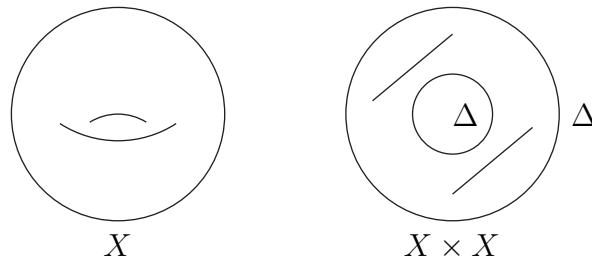


which can then be folded into a disk on $X^4$:



41

Similar manipulations can be performed at higher genera, reducing all such diagrams to disk diagrams on products of copies of $X$.

# 4 String topology

The taffy identities in §3 arose from studying the homological algebra of the category of chain complexes of $\mathcal{O}_X$-modules, where $X$ was a Calabi-Yau manifold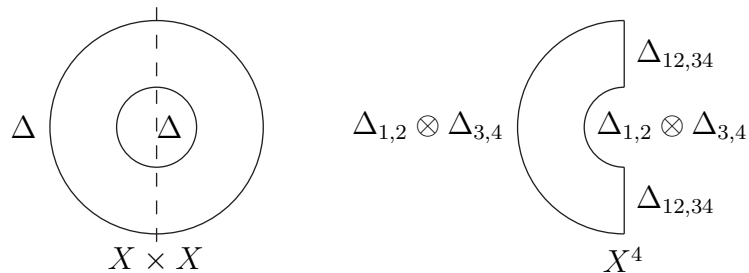. The ingredients which lead to these identities are available in other contexts. In this section we shall outline the details for the case of 'string topology.' This can be described as a mathematical abstraction of bosonic string field theory (see *e.g.* [35, 36]), that is well-known in the homotopy community. In this section we shall derive precise analogues of the taffy identities for string topology.

Let $X$ be a manifold, and let $LX$ be its free loop space. For simplicity, we will assume in this section that $X$ is simply-connected [34]. (See [16][remark 4.2.17] for an outline of the non-simply-connected case.) Costello ([20]; see also [16, section 4.2]) explains how the string topology operations of Chas and Sullivan on $C^*LX$, the rational cochains on the free loop space, can be understood as arising from an open-closed TFT on $C^*X$. In this language, the closed string states are[9] $H^*(LX)$.

Let $C^*X$ be its algebra of rational singular cochains, which plays the role of the open string algebra. There is an $\infty$-category $M_X$ of $C^*X$-modules, which plays the role for $C^*X$ analogous to the $\infty$-category $D_b^\infty(X)$ of quasi-coherent sheaves, whose homotopy category is the derived category (see Remark 3). In other words, (complexes of) D-branes are elements of $M_X$, $C^*X$ modules. In this language, open string states between two (complexes of) D-branes $\mathcal{E}, \mathcal{F} \in M_X$ are given by

$$\mathbf{R}\mathrm{Hom}\,(\mathcal{E}, \mathcal{F})$$

(Unlike algebraic geometry, here there is no distinction between local and global Hom; working over rational cochains is more closely analogous to working on an affine scheme, where the module defining the sheaf is the same as the global sections. Therefore, we use the same Hom to describe both $M_X$ modules and also the derived functor of global sections.)

Now, we shall start working out taffy identities. If $f : X \to Y$ is a map of spaces, then associated to the pull-back of singular cochains

$$f^* : C^*Y \longrightarrow C^*X$$

we have the derived pushforward

$$f_* : M_X \longrightarrow M_Y,$$

---

[9] We have not constructed vertex operators to physically realize such in a CFT; rather, we are saying that formally, the object playing the role of closed string states is $H^*(LX)$. A similar statement is true for open string states here.

which has a left adjoint
$$f^* : \; M_Y \; \longrightarrow \; M_X.$$

If $\mathcal{E}$ and $\mathcal{F}$ are $C^*X$-modules, then we write $\mathbf{R}\operatorname{Hom}_X(\mathcal{E}, \mathcal{F})$ for $\mathbf{R}\operatorname{Hom}_{M_X}(\mathcal{E}, \mathcal{F})$; it is again a $C^*X$-module. Moreover, we can define $E^\vee = \mathbf{R}\operatorname{Hom}_X(\mathcal{E}, C^*X)$. Provided that $\mathcal{E}$ is in a suitable sense finitely generated over $C^*X$ (we will use the term "dualizable"), we have equivalences

$$(\mathcal{E}^\vee)^\vee \; \simeq \; \mathcal{E}$$
$$\mathcal{E}^\vee \otimes \mathcal{F} \; \simeq \; \mathbf{R}\operatorname{Hom}_X(\mathcal{E}, \mathcal{F}).$$

Let $\mathcal{E}$ and $\mathcal{F}$ be dualizable $C^*X$-modules, and let $\Delta : X \to X \times X$ be the diagonal. Then

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_{X \times X}(\pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee, \Delta_* C^*X) \; &\simeq \; \mathbf{R}\operatorname{Hom}_X(\Delta^*(\pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{F}^\vee), C^*X) \\
&\simeq \; \mathbf{R}\operatorname{Hom}_X(\mathcal{E} \otimes \mathcal{F}^\vee, C^*X) \\
&\simeq \; \mathbf{R}\operatorname{Hom}_X(C^*X, \mathcal{E}^\vee \otimes \mathcal{F}) \\
&\simeq \; \mathbf{R}\operatorname{Hom}_X(\mathcal{E}, \mathcal{F}),
\end{aligned}$$

which is the analogue of the taffy identity (16).

The analogue the projection formula (5) holds in this context. If $f : X \to Y$, $\mathcal{E}$ is a $C^*X$-module, and $\mathcal{G}$ is $C^*Y$-module, then

$$f_*(\mathcal{E} \otimes_X f^*\mathcal{G}) \; \simeq \; (f_*\mathcal{E}) \otimes_Y \mathcal{G},$$

and, at least for products, the Eilenberg-Zilber Theorem provides the analog of the flat base change theorem (Prop 6). These considerations immediately yield the taffy identity analogous to (17)

$$\mathbf{R}\operatorname{Hom}_{X \times X \times Y}(\pi_1^* E \otimes \pi_{23}^* S^\vee, \pi_{12}^* \Delta) \; \simeq \; \mathbf{R}\operatorname{Hom}_{X \times Y}(\pi_1^* E, S)$$

for any dualizable $C^*(X \times Y)$-module $S$.

Similarly, if $S$ is a $C^*(X^2 \times Y)$-module, and if $\Delta$ refers to the $C^*(X^2 \times Y)$-module

$$\Delta = \Delta_*(C^*(X \times Y))$$

obtained by pushing forward along $(x, y) \mapsto (x, x, y)$, and if

$$\Delta_{ij} = \pi_{ij}^* \Delta$$

with $\pi_{ij}$ projection to the indicated factors

$$\pi_{ij} : X^4 \times Y \to X^2 \times Y,$$

then the taffy identities (20) and (23) become

$$\mathbf{R}\mathrm{Hom}_{X^4 \times Y}(\Delta_{12}^\vee, \Delta_{14} \otimes \Delta_{23} \otimes \pi_{34}^* S) \simeq \mathbf{R}\mathrm{Hom}_{X^2 \times Y}(\Delta^\vee, S)$$
$$\mathbf{R}\mathrm{Hom}_{X^4 \times Y}(\Delta_{12}^\vee, \Delta_{13} \otimes \Delta_{24} \otimes \pi_{34}^* S) \simeq \mathbf{R}\mathrm{Hom}_{X^2 \times Y}(\Delta^\vee, S).$$

The proofs follow the same pattern as the proof we gave of (20), using the projection formula and the flat base change theorem already mentioned.

A more compelling observation is that the the closed string taffy identity associated to the flattening



predicts that the closed string algebra associated to $C^*X$ should be

$$\mathbf{R}\,\mathrm{Hom}_{X \times X}(\Delta^\vee, \Delta) \simeq \Delta \otimes_{C^*(X \times X)}^{\mathbf{L}} \Delta.$$

As we shall outline below, it is well-known that this chain complex is equivalent both to the Hochschild chains on $C^*X$, $HC_*(C^*X; C^*X)$, and to the cochains on the free loop space $LX = \mathrm{Map}(S^1, X)$, $C^*(LX)$. This will give us the string-topology-analogue of the Hochschild-Kostant-Rosenberg isomorphism, that played a crucial role in the B model.

Let us consider this situation mathematically. It turns out that the mathematics of the situation is quite close to the defects we discuss here. By introducing two points (defects) on the closed string word sheet, we can identify the space of closed strings ("loops") in $X$ with the space of pairs of paths in $X$, joined at their endpoints. That is, the loop space $LX = \mathrm{Map}(S^1, X)$ is the pull-back of the diagram

$$
\begin{array}{ccc}
 & & PX \hspace{3cm} (28) \\
 & & \downarrow {\scriptstyle ev_{01}} \\
PX & \xrightarrow{\ ev_{01}\ } & X \times X.
\end{array}
$$

Here $PX = \mathrm{Map}([0,1], X)$ is the space of maps of the unit interval to $X$, and $ev_{ij} : PX \to X^2$ is the map obtained by evaluating the 0-endpoint into the $i$ factor, and the 1-endpoint into the $j$ factor in the product.

Homotopically, it is equivalent to shrink the paths to constant length, yielding the diagram

$$
\begin{array}{ccc}
 & & X \hspace{3cm} (29) \\
 & & \downarrow {\scriptstyle \Delta} \\
X & \xrightarrow{\ \Delta\ } & X \times X.
\end{array}
$$

The point-set pull-back in the diagram (29) is not $LX$, but merely $X$. Since homotopy functors, such as cohomology, $K$-theory, and the derived category, view these diagrams as equivalent, they cannot possibly preserve pull-backs. It is necessary to consider in this situation a homotopy-invariant "derived pull-back".

It turns out that one can construct an object which encodes all possible homotopy deformations of the diagram (29): it is the "cosimplicial space" $B^\bullet(X, X \times X, X)$,

$$
\begin{array}{c}
\cdots \\
\uparrow \uparrow \uparrow \uparrow \\
X \times Y^2 \times X \\
\uparrow \uparrow \uparrow \\
X \times Y \times X \\
\uparrow \uparrow \\
X \times X
\end{array}
$$

Here we have written $Y$ for $X \times X$. The maps are

$$
\begin{aligned}
d^0(x_0, y_1, \ldots, y_{n-1}, x_n) &= (x_0, \Delta(x_0), y_1, \ldots) \\
d^n(x_0, y_1, \ldots, y_{n-1}, x_n) &= (x_0, y_1, \ldots, y_{n-1}, \Delta(x_n), x_n) \\
d^i(x_0, y_1, \ldots, y_{n-1}, x_n) &= (x_0, y_1, \ldots, y_i, y_i, \ldots, y_{n-1}, x_n) \qquad 1 \le i \le n-1.
\end{aligned}
$$

A basic theorem in homotopy theory ([37],[38, esp. p. 268],[39]) implies that the situation gets no more complicated than the pull-back diagram (28):

**Proposition 30.** *If $X$ is simply connected, then*

$$
B^\bullet(X, X \times X, X) \simeq LX.
$$

By applying a homotopy functor which preserves derived pull-backs, we get more familiar results. For example, applying singular cochains $C^*(-)$, we find that

$$
B_\bullet(C^*X, C^*(X \times X), C^*X) \simeq C^*(LX). \tag{31}
$$

The left-hand side in (31) refers to the bar complex which calculates Tor, and so taking cohomology on both sides yields

$$
\operatorname{Tor}^{C^*(X \times X)}(C^*\Delta, C^*\Delta) = H^*(LX).
$$

Using the relation between Tor and Ext groups, we can rewrite this as

$$
\mathbf{R}\operatorname{Hom}_{X \times X}(C^*\Delta^\vee, C^*\Delta) = H^*(LX)
$$

Moreover, it is a theorem of Cartan and Eilenberg [40][section IX.6] that the chain complex associated to $B_\bullet(C^*X, C^*(X \times X), C^*X)$ is equivalent to the cyclic complex which calculates Hochschild homology:

$$HC_*(C^*X) \simeq B_\bullet(C^*\Delta, C^*(X \times X), C^*\Delta),$$

and so we have the following (well-known, see for example [41][theorem 1.5.1], [42, 43]) result.

**Proposition 32.** *If $X$ is simply connected, then*

$$HH_*(C^*X) \;=\; H^*(LX) \;=\; \mathrm{Tor}^{C^*(X \times X)}(C^*\Delta, C^*\Delta) \;=\; \mathbf{R}\mathrm{Hom}_{X \times X}\left(C^*\Delta^\vee, C^*\Delta\right).$$

This is precisely the analogue of the Hochschild-Kostant-Rosenberg isomorphism, expressing closed string states on $X$ in terms of open string states on $X \times X$, and in terms of the Hochschild homology of the open string algebra $C^*X$.

Hopkins and Lurie ([16, 24]; see also Blumberg, Cohen, and Teleman [44]) have shown that $C^*X$ is the open string algebra of an open-closed TFT, whose closed string algebra is $C^*(LX)$; and they have shown that the resulting structure on $C^*(LX)$ includes the string topology operations of Chas and Sullivan.

We emphasize that the identification of $LX$ as the derived-pull back of

$$X \xrightarrow{\Delta} X \times X \xleftarrow{\Delta} X$$

encodes the closed string taffy identities we consider in §3, at the level of topological spaces: they then have instances in many algebraic settings by applying homotopy functors. To give an example, consider a loop in $X$ viewed as assembled from four paths, as in
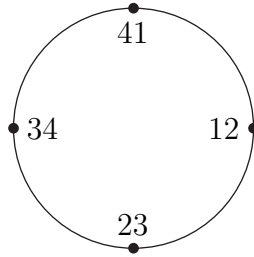


That is, the loop space is the pull-back in the diagram

$$
\begin{array}{ccc}
 & & PX \times PX \\
 & & \downarrow{\scriptstyle ev_{12} \times ev_{34}} \\
PX \times PX & \xrightarrow{\;ev_{14} \times ev_{23}\;} & X^4.
\end{array}
$$

But this diagram is homotopy equivalent to

$$
\begin{array}{c}
X \times X \\
\downarrow {\scriptstyle \Delta_{12} \times \Delta_{34}} \\
X \times X \xrightarrow{\ \Delta_{14} \times \Delta_{23}\ } X^4.
\end{array}
$$

The same analysis as Proposition 30 shows that, if $X$ is simply connected, then

$$
B^{\bullet}(\Delta_{12} \times \Delta_{34}, X^4, \Delta_{14} \times \Delta_{23}) \simeq LX,
$$

and so by applying singular cochains we find that

$$
\mathrm{Tor}^{C^*(X^4)}(C^*(\Delta_{12}) \otimes C^*(\Delta_{34}), C^*(\Delta_{14}) \otimes C^*(\Delta_{23})) = \mathrm{Tor}^{C^*(X^2)}(C^*(\Delta), C^*(\Delta)) = H^*(LX),
$$

with all of these agreeing with the Hochschild homology of the signular cochains, $HH_*(C^*X)$. This is the analogue, for string topology, of the taffy identity (21) for closed strings in the $B$-model.

# 5   Hochschild (co)homology and closed string states

It is sometimes said that closed string states are the Hochschild (co)homology of the open string algebra. In the physics literature, closed string states are often related to Hochschild cohomology; see for example [45][section 2], [46][section 3.3] and references therein. In the mathematics community, closed string states are often related to Hochschild homology[10], see work of *e.g.* Kontsevich, Costello, Hopkins–Lurie[11], Blumberg–Cohen–Teleman [16, 20, 24, 44]. In this section we shall examine this correspondence. Among other things, we shall argue that, based on the taffy identities, the natural relation is between the closed string states and Hochschild homology instead of cohomology.

It is also worth pointing out specifically that Costello, Hopkins, and Lurie [16, 20, 24] have developed an approach to topological field theory in which the rotational symmetry of the closed string states is reflected in Connes' cyclic structure on the Hochschild complex, and so cyclic homology also naturally appears in their framework (see *e.g.* [16][section 4.2]). In physics, the relationship between cyclic homology and closed (bosonic) strings was recently discussed in [47]. (As string topology is a mathematical extraction of bosonic string field theory, there are close parallels between their work and parts of our discussion of string topology; our proposition 32, for example, is essentially [47][equ'n (95)].)

---

[10] M.A. first learned that closed string states are related to open string states in this way from D. Berenstein in 2001.

[11] The work of Hopkins and Lurie places this relationship in the context of enriched topological field theory, which we outlined in section 2.2.

## 5.1 B model

In the language of the B model topological field theory, closed string states can be related to Hochschild (co)homology via the "Hochschild-Kostant-Rosenberg (HKR) isomorphism,"

$$H^* (X, \Lambda^* TX) = \operatorname{Ext}^*_{X \times X} (\Delta, \Delta)$$

for a Calabi-Yau $X$. One, somewhat vague, reason sometimes stated for this relationship between closed and open strings is that if one accepts that closed strings should be derived from open strings, then closed string states should arise as some sort of cohomology operation on the open string algebra, and Hochschild cohomology arises very naturally in this role. A more refined intuition is sometimes stated, in terms of folding closed strings into open strings.

In section 3.4 we learned how to make such intuitions physically precise for the B model. In the process, we learned that the closed string states are naturally given by Hochschild homology, not cohomology. In this section we will review Hochschild homology and cohomology, and their relationship.

In general, for an algebra $A$, the Hochschild homology is [48][prop. 1.1.13]

$$HH_*(A) = \operatorname{Tor}_*^{A \otimes A^{op}} (A, A)$$

and the Hochschild cohomology is [48][prop. 1.5.8]

$$HH^*(A) = \operatorname{Ext}^*_{A \otimes A^{op}} (A, A)$$

In the case of the B model, the open string algebra is $\mathcal{O}_X$, as the B-branes are coherent $\mathcal{O}_X$-sheaves. Since the local Ext groups reduce to Hochschild cohomology for algebras, one defines the Hochschild cohomology of $X$ to be [21][def'n 6.2]

$$\operatorname{Ext}^*_{X \times X} (\Delta, \Delta)$$

where, as usual, $\Delta$ denotes $\Delta_* \mathcal{O}_X$. A natural guess would be that the Hochschild homology should be defined similarly as a global Tor group; however, global Tor groups for sheaves do not seem to be consistently defined. It is true, however, true that

$$\operatorname{Ext}^*_X (A^\vee, B) = \mathbf{R}\Gamma (X, A \otimes B)$$

for any two elements $A, B \in D^b(X)$, and the right-hand-side of that expression is morally, if not literally, a global Tor group. Thus, the reader should not be surprised to learn that Hochschild homology of $X$ is defined to be [21][def'n 6.2]

$$HH_*(X) = \operatorname{Ext}^{-*}_{X \times X} (\Delta^\vee, \Delta)$$

(The change in sign of the grading is the convention in this context. Although Hochschild cohomology of a space is always in positive degrees, Hochschild homology is in both positive and negative degrees.)

In section 3.4 we saw from folding operations that the closed string states in the B model are given by Hochschild homology,

$$HH_*(X) = \mathrm{Ext}^{-*}_{X \times X} (\Delta^\vee, \Delta)$$

ultimately because a closed string can be flattened into an open string on $X \times X$ with diagonal boundary conditions. Let us compare this to more standard expressions for states in the closed string B model.

The Hochschild homology and cohomology can be related to differential forms [21][section 6.4], thanks to the "Hochschild-Kostant-Rosenberg isomorphism" [49, 50, 51, 52], which says that for $X$ a smooth quasi-projective variety over $\mathbf{C}$,

$$HH^n(X) \equiv \mathrm{Ext}^n_{X \times X} (\Delta, \Delta) \cong \bigoplus_{p+q=n} H^p (X, \Lambda^q TX) \tag{33}$$

$$HH_n(X) \equiv \mathrm{Ext}^{-n}_{X \times X} (\Delta^\vee, \Delta) \cong \bigoplus_{q-p=n} H^p (X, \Omega^q_X) \tag{34}$$

Using the fact that
$$\Omega^q_X = K_X \otimes \Lambda^{n-q} TX$$

where $n$ is the dimension of $X$, we see that

$$H^p (X, \Omega^q_X) = H^p \left( X, K_X \otimes \Lambda^{n-q} TX \right)$$

and so

$$HH_*(X) = \bigoplus_{q-p=*} H^p (X, \Omega^q_X) = \bigoplus_{q-p=*} H^p \left( X, K_X \otimes \Lambda^{n-q} TX \right)$$
$$= \bigoplus_{p+q=n-*} H^p \left( X, K_X \otimes \Lambda^q TX \right)$$

When $X$ is Calabi-Yau, we see from the above that

$$HH_*(X) = HH^{n-*}(X)$$

If $K_X$ is nontrivial but 2-torsion, then the Hochschild homology and cohomology no longer have a simple relationship.

## 5.2 String topology

In §3 we used taffy identities built from defects to explain why the closed string in a topological field theory should be the Hochschild homology of the open string states.

We discussed an analogue of the Hochschild-Kostant-Rosenberg isomorphism for string topology in section 4. Specifically, we argued there that

$$HH_*(C^*X) \;=\; H^*(LX) \;=\; \mathbf{R}\mathrm{Hom}_X\left(C^*\Delta^\vee, C^*\Delta\right)$$

This tells us, in part, that open string states on $X \times X$ match closed string states on $X$, but more to the point, it identifies closed string states with the Hochschild homology (instead of Hochschild cohomology) of $C^*X$, the open string algebra. Just as in the B model, the closed string states are naturally associated to Hochschild homology instead of cohomology, in this framework.

## 5.3 Generalized cohomology

Our analysis of the relationship between Hochschild homology and closed strings has the virtue that one can insert other homotopy functors which preserve derived pull-backs, yielding potentially other topological field theories. For example, replacing singular cochains with maps to $\mathbb{Z} \times BU$, one can hope to build a topological field theory based on $K$-theory, whose closed string algebra is the Hochschild homology $HH_*(K^X)$. In fact Hopkins announced a result like this at the Fields Institute [24]. Note that some care must be taken in interpreting this assertion: for example the usual analysis of Hochschild homology and loop spaces requires the space $X$ to be simply connected; in this context, it also suggests that one might have to use "connective" $K$-theory.

One situation to which this analysis of Hochschild homology does *not* apply directly is the $B$-model considered in §3. We could analyze that case in this language by using the ideas outlined in the introduction. Specifically, consider a "closeable" configuration of $n$ open strings of the form



50
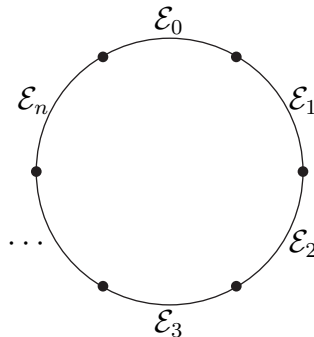
to which we associate

$$\text{Hom}(\mathcal{E}_0, \mathcal{E}_1) \times \text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \times \cdots \times \text{Hom}(\mathcal{E}_n, \mathcal{E}_0)$$

(Note this is not the same thing as the massless spectrum of the diagram, but rather is a more abstract quantity. Also note that in this discussion, the Homs should be taken in the $\infty$-category $D_b^\infty(X)$: they are really $\mathbf{R}$Homs.) A similar configuration containing only $n - 1$ open strings should be associated a product of Hom's related to the one above via the composition

$$\text{Hom}(\mathcal{E}_{n-1}, \mathcal{E}_n) \times \text{Hom}(\mathcal{E}_n, \mathcal{E}_0) \longrightarrow \text{Hom}(\mathcal{E}_{n-1}, \mathcal{E}_0)$$

in one direction, and by setting $\mathcal{E}_n = \mathcal{E}_{n-1}$ with the identity operator in the other direction. This gives rise to a complex

$$\bigoplus\nolimits_{\mathcal{E}_0, \ldots, \mathcal{E}_n} \text{Hom}(\mathcal{E}_0, \mathcal{E}_1) \times \text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \times \cdots \times \text{Hom}(\mathcal{E}_n, \mathcal{E}_0)$$

$$\cdots$$

$$\bigoplus\nolimits_{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2} \text{Hom}(\mathcal{E}_0, \mathcal{E}_1) \times \text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \times \text{Hom}(\mathcal{E}_2, \mathcal{E}_0)$$

$$\downarrow \downarrow \downarrow$$

$$\bigoplus\nolimits_{\mathcal{E}_0, \mathcal{E}_1} \text{Hom}(\mathcal{E}_0, \mathcal{E}_1) \times \text{Hom}(\mathcal{E}_1, \mathcal{E}_0)$$

$$\downarrow \downarrow$$

$$\bigoplus\nolimits_{\mathcal{E}_0} \text{Hom}(\mathcal{E}_0, \mathcal{E}_0)$$

In this complex the structure maps are those of the Hochschild complex, and (modulo important technical subtleties) McCarthy and Keller [17, 18, 19] use this complex to define the Hochschild homology of the category of $\mathcal{O}_X$-modules, in such a way that $HH_*(\text{Mod}(\mathcal{O}_X))$ is the Hochschild homology of $X$.

# 6   A model

So far we have discussed the B model and string topology in this paper. Let us now turn to the analogous constructions in the A model, and briefly outline the highlights. A complete analysis would involve working through the analogous constructions in derived Fukaya categories, which we have not done; instead, we will work solely in a large-radius limit (hence turning off quantum corrections), and only consider a few special cases.

Let us first consider folds in the A model. Open strings in the A model have boundaries on either Lagrangian or co-isotropic submanifolds, depending upon the Chan-Paton factors. If we let $\omega$ denote the symplectic form on $X$, then because the folding operation reverses the orientation on the second sheet[12], the symplectic form on $X \times X$ is

$$\pi_1^* \omega \; - \; \pi_2^* \omega$$

With respect to this[13] symplectic form on $X \times X$, the diagonal embedding $\Delta : X \to X \times X$ defines a Lagrangian[14] submanifold of $X \times X$ for any $X$, and so partly as a result that submanifold together with a trivial line bundle define a supersymmetric boundary for the A model. That boundary will play the same role here that it did for the B model, *i.e.* it will play the role of an identity in folding operations.

For the moment, let us assume all boundaries are on Lagrangian submanifolds, for simplicity. Omitting worldsheet instanton corrections, the massless spectrum of an A model open string between boundaries $(L_1, \mathcal{E}_1)$ and $(L_2, \mathcal{E}_2)$, where $L_1$, $L_2$ are Lagrangian submanifolds of $X$, and $\mathcal{E}_1$, $\mathcal{E}_2$ are flat vector bundles with connection over those Lagrangian submanifolds, is given by

$$H_d^i \left( L_1 \cap L_2, \mathcal{E}_1^* \otimes \mathcal{E}_2 \right)$$

Let us first consider open string states and folding tricks, and try to re-derive a few of our results from the B model and string topology.

Consider first a closed string on $X$ in the A model. If we crush it to an open string on $X \times X$, as



then (ignoring orientation issues on the boundaries for the moment), we would expect that closed string states should match

$$H_d^* (\Delta, \mathcal{O}) \; = \; H_{DR}^*(X)$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal in $X \times X$. Such a case has been considered previously in [53]. In fact, that reference also considered the effect of quantum corrections (meaning, for

---

[12] We can see this by following the orientation on a string; across a fold, the direction flips. In the B model, this is a complex conjugation, and this is one way of understanding why the Chan-Paton factors on one sheet are defined by the dual bundle.

[13] It is straightforward to check that without the relative sign, *i.e.* for $\pi_1^* \omega + \pi_2^* \omega$, the diagonal is not Lagrangian.

[14] More generally, if $W$, $Y$ are Lagrangian in $X$, then $W \times Y$ is Lagrangian in $X \times X$.

example, that open string states are Hom's in a derived Fukaya category), and conjectured that the product structure on open string Hom's should match the quantum cohomology ring of $X$.
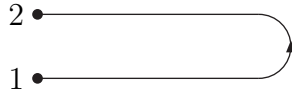
Let us also outline some of the basic manipulations of open strings, also for simplicity in a large-radius limit where worldsheet instanton corrections have been turned off. Start with an oriented open string

$$1 \bullet\!\!\longrightarrow\!\!\bullet 2$$

between $(L_1, \mathcal{E}_1)$ and $(L_2, \mathcal{E}_2)$, with states given by

$$H_d^* \left( L_1 \cap L_2, \mathcal{E}_1^* \otimes \mathcal{E}_2 \right)$$

Now, fold this into a U-shape:

$$
\begin{array}{l}
2 \bullet \\
1 \bullet
\end{array}
$$

Following the usual procedure, open string states on the U-shaped diagram should be given by

$$H_d^* \left( L \cap \Delta, \pi_1^* \mathcal{E}_1^* \otimes \pi_2^* \mathcal{E}_2 \right)$$

where $L$ is the Lagrangian submanifold of $X \times X$ given by

$$L \;=\; \{\, (x_1, x_2) \in X \times X \,|\, x_1 \in L_1,\, x_2 \in L_2 \,\}$$

i.e., $L = L_1 \times L_2 \subset X \times X$.

As the physics matches, it should be true that

$$H_d^* \left( L_1 \cap L_2, \mathcal{E}_1^* \otimes \mathcal{E}_2 \right) \;=\; H_d^* \left( L \cap \Delta, \pi_1^* \mathcal{E}_1^* \otimes \pi_2^* \mathcal{E}_2 \right)$$

and indeed, this is trivial to check.

Similarly, if we fold the same oriented string in the opposite direction to get

$$
\begin{array}{l}
\bullet 1 \\
\bullet 2
\end{array}
$$

then the corresponding open string states should again be

$$H_d^* \left( L \cap \Delta, \pi_1^* \mathcal{E}_1^* \otimes \pi_2^* \mathcal{E}_2 \right)$$

So far we have merely outlined how the simplest taffy manipulations would work in the A model, in the large-radius limit. More generally, we conjecture that there exist analogues of the B model taffy identities (17), (20), (23) for derived Fukaya categories.

One quick check that we shall mention is to compare to open string Gromov-Witten invariants. We are implicitly predicting that open string Gromov-Witten invariants of open strings on $X \times X$, with the boundary conditions determined by the diagonal, should match closed string Gromov-Witten invariants. (For example, a rational curve on $X$ would be a calzone-shaped object on $X \times X$.) Furthermore, this would also test whether the same ideas are applicable after coupling to worldsheet gravity. We have been informed [54] that, indeed, open string Gromov-Witten invariants on $X \times X$ as above do in fact match closed string Gromov-Witten invariants on $X$, and that there also it is merely an unravelling of definitions.

# 7 Matrix factorizations

In a Landau-Ginzburg theory, we can understand a defect joining two open strings as follows [4, 5]. If one space is $X$ with superpotential $W_X$, and the other space is $Y$ with superpotential $Y$, then denoting the projections from $X \times Y$ to $X$, $Y$ by $p_X$, $p_Y$, respectively, the defect is defined by a matrix factorization in the superpotential

$$p_X^* W_X \; - \; p_Y^* W_Y$$

over $X \times Y$.

We can understand the identity defect as follows. Over the image of the diagonal embedding $\Delta : X \hookrightarrow X \times X$, the superpotential

$$W \; \equiv \; p_1^* W_X \; - \; p_2^* W_X$$

vanishes. A matrix factorization is defined over a submanifold $S$ by a pair of vector bundles $\mathcal{E}$, $\mathcal{F}$ over $S$ with maps $f : \mathcal{E} \to \mathcal{F}$, $g : \mathcal{F} \to \mathcal{E}$ such that $f \circ g = (W|_S) \, \mathrm{Id}_{\mathcal{E}}$, $g \circ f = (W|_S) \, \mathrm{Id}_{\mathcal{F}}$. In the present case, for $S$ the diagonal submanifold, $W|_S = 0$, so we can take $\mathcal{E} = \mathcal{O}_X$, $\mathcal{F} = 0$, and $f = g = 0$.

Given the structure above, we make the following conjectures for matrix factorizations:

1. The hypercohomology groups

$$\mathbf{H}^* \left( X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} \mathcal{O}_X \right)$$

which count closed string states in Landau-Ginzburg models [55] match **R**Hom's on $X \times X$ in the category of matrix factorizations

$$\mathbf{R}\mathrm{Hom}_{X \times X, p_1^* W - p_2^* W} \left( (\Delta_* \mathcal{O}_X, 0), (\Delta_* \mathcal{O}_X, 0) \right)$$

as would be suggested from the general considerations of section 5. This has been confirmed for the local case (meaning, affine $X$, isolated critical points) in [56] in the mathematics literature and [45] in the physics literature. The more general form above has also been conjectured by others.

2. Analogues of the taffy identities (17), (20), (23) hold.

# 8   Critical strings, supercritical dimensions

So far we have spoken exclusively about two-dimensional topological field theories, but identical ideas apply, with some caveats, to full string theories. After all, the folding operation is trivial, it is an artifact of the worldsheet description, and does not itself convey any physics. (The catch, the caveat, is the coupling to worldsheet gravity. We do not understand how to couple theories with defects to worldsheet gravity. In this section, we will consider physical untwisted strings, but not coupled to worldsheet gravity.)

This does lead to some counterintuitive results, however. For example, consider a closed string on a 10-manifold $X$. If we fold the closed string to an open string on $X \times X$, then we have discovered that, for certain special boundary conditions, open strings on 20-dimensional spaces behave like critical strings.

If we track through the physics in detail, we find that this is largely correct (albeit with subtleties involving coupling to gravity). Consider an ordinary closed string bosonic field $\phi^\mu$, canonically quantized as

$$\hat{\phi}(\tau, \sigma) \; = \; x \; + \; \frac{p}{4\pi} \; + \; \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n e^{-in(\tau-\sigma)} \; + \; \tilde{\alpha}_n e^{-in(\tau+\sigma)} \right)$$

Suppose these describe a local description of some manifold $X$ of dimension $k$. An open string on $X \times X$ with boundary conditions in the diagonal would be described by operators

$$\hat{\phi}^\mu(\tau, \sigma) \; = \; x^\mu \; + \; \frac{p^\mu}{4\pi} \; + \; \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n^\mu e^{-in(\tau-\sigma)} \; + \; \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)} \right)$$

with $\mu \in \{1, \cdots, 2k\}$, and the boundary conditions

$$\hat{\phi}^{\mu > k}(\sigma = 0, \pi) \; = \; \hat{\phi}^{\mu - k}(\sigma = 0, \pi)$$

force

$$x^{\mu>k} \;=\; x^{\mu-k}, \; p^{\mu>k} \;=\; p^{\mu-k}, \; \alpha_n^{\mu>k} \;=\; \alpha_n^{\mu-k}, \; \tilde{\alpha}_n^{\mu>k} \;=\; \tilde{\alpha}_n^{\mu-k}$$

With these constraints, however, $\hat{\phi}$ is equivalent to a closed string field on $X$.

So far we have demonstrated that in canonical quantization on the worldsheet, operators on an open string on $X \times X$ with diagonal boundary conditions are equivalent to operators on closed strings on $X$. It is tempting to therefore conclude that physical open strings on $X \times X$ as above are therefore equivalent to physical closed strings on $X$; however, the result should be interpreted with a certain amount of care. For example, the computation of the bulk central charge in the open string theory is unaffected by the boundary conditions, and so will be twice its value for the corresponding closed string theory. In the present case, that might possibly signal[15] that the number of ghost fields on the open string worldsheet should also be doubled, also with diagonal boundary conditions.

To properly understand the issues above would involve understanding how defects are coupled to worldsheet gravity, which we shall not attempt here.

In passing, we feel we should mention the tangentially-related fact that there exist tachyon-free closed string theories in dimensions $8k+2$ [57][section 5].

# 9  Conjectures on elliptic genera

Let us now apply some of these ideas to try to gain some insight into elliptic genera. Physically, an elliptic genus is the torus partition function of a (half-twisted) closed string on a space $X$. Now, insert two identity defects along the sides, along which the torus can be cut open like a bagel, and glue together the two sides to get an annulus diagram on $X \times X$.

Now, from the Cardy condition, an annulus diagram can be thought of as either a closed string propagating between two boundary states, or an open string propagating in a loop, and furthermore that either method of computation should give the same result.

For example, for an ordinary annulus diagram in the B model, the Cardy condition reduces to Hirzebruch-Riemann-Roch [23]. In the present case, we have a half-twisted closed string one-loop diagram on $X$, represented as an annulus diagram on $X \times X$. Following the same philosophy of annulus diagrams, it is natural to try to interpret the partition function of the annulus as describing an index computation on $X \times X$.

For the right notion of open string states, index theory should recover elliptic genera.

---

[15] We would like to thank S. Hellerman for suggesting this possibility.

For example, following [58], a prototype for the elliptic genus is the expression

$$\sum q^{k/2} R_k = \bigotimes_{k=1/2,3/2,5/2,\cdots} \Lambda_{q^k} TX \bigotimes_{\ell=1,2,3,\cdots} S_{q^\ell} TX$$

where

$$
\begin{aligned}
R_0 &= 1 \\
R_1 &= TX \\
R_2 &= \Lambda^2 TX \oplus TX \\
R_3 &= \Lambda^3 TX \oplus (TX \otimes TX) \oplus TX
\end{aligned}
$$

and so forth. The $R_i$ are defined by the string states; the expression above defines an index of the form

$$\sum_{i,k} q^{k/2}(-)^i \dim H^i(X, R_i) = \int_X \mathrm{Td}(TX) \wedge \mathrm{ch}\left( \bigotimes_{k=1/2,3/2,5/2,\cdots} \Lambda_{q^k} TX \bigotimes_{\ell=1,2,3,\cdots} S_{q^\ell} TX \right)$$

The right-hand side we could interpret as closed strings on $X \times X$ propagating between two copies of the diagonal; the left-hand side we could interpret as the partition function of open string states.

So far, however, this picture of elliptic genera is not particularly noteworthy. We can try to significantly generalize this picture as follows. Instead of starting with an ordinary closed string, begin with $n$ open strings on $X$ joined together along defects, defined by pushforwards along the diagonal of $n$ elements of K theory of $X$ (and so sitting on the diagonal embedding in $X \times X$).

We can understand those $n$ elements of K theory as giving a finite approximation to K theory on the loop space, in the spirit of [43]. (Given that elliptic genera are understood [58] in terms of index theory on loop spaces, this is already pertinent.)

Now, imagine propagating those defects around in a circle, to form a torus with slanted sides. We can use the folding tricks discussed earlier in this paper to collapse such a diagram down to an annulus, with collections of K theory elements on the boundaries. If one could compute those annulus amplitudes explicitly, and furthermore 'trace over' the boundary K theory elements, then one would be able to construct an explicit map to ordinary elliptic genera, giving an new way to understand the relationship between elliptic genera and K theory on loop spaces.

Although we do not have anything definitive to say here, we feel we should observe that the work [59] may be relevant to such questions.

# 10    Conclusions

In this paper we have worked out mathematical identities to verify that B model and string topology are invariant under taffy-like operations, which involve folding and twisting worldsheets into different-appearing yet physically-equivalent forms. Those identities are essentially consequences of homotopy invariance of the underlying theory. We have also outlined analogous results and conjectures in other contexts, including the A model, B-twisted Landau-Ginzburg models, and physical strings, and presented some conjectured applications to the study of elliptic genera.

One natural speculation for future work concerns other applications of trivial defects. For example, although one typically only thinks of using differential forms to compute (real-valued) cohomology on manifolds, differential forms can also be applied to more general topological spaces, see for example [60]. It is natural to speculate whether ideas similar to those of [60] might be applied physically to understand string states in higher-order defect junctions.

Another direction for future work is to apply the same ideas to topological field theories in higher dimensions, where defects have also appeared. Are there analogous taffy-like constructions there?

Yet another directions is to understand whether the taffy identities presented here are a consequence of general mathematical axiomatic frameworks for (enriched) topological field theories.

In a different direction, U-shaped branes have appeared in discussions of holographic duals to chiral symmetry breaking, see *e.g.* [61] and references therein. We do not see a direct connection – our work is concerned with worldsheet reparametrizations and alternate descriptions, whereas there a brane is physically being bent – but it would certainly be interesting if a direct link could be found.

# 11    Acknowledgements

# References

[1] M. Oshikawa, I. Affleck, "Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line," `arXiv: cond-mat/9612187`.

[2] C. Bachas, J. de Boer, R. Dijkgraaf, H. Ooguri, "Permeable conformal walls and holography," `arXiv: hep-th/0111210`.

[3] L. Rozansky, "Topological A models on seamed Riemann surfaces," `arXiv: hep-th/0305205`.

[4] L. Rozansky, "Topological Landau-Ginzburg models on a worldsheet foam," Adv. Theor. Math. Phys. **11** (2007) 233-260, `arXiv: hep-th/0404189`.

[5] M. Khovanov, L. Rozansky, "Matrix factorizations and link homology," `arXiv: math/0401268`.

[6] I. Brunner, D. Roggenkamp, "B-type defects in Landau-Ginzburg models," `arXiv: 0707.0922`.

[7] B. Fiol, "Defect CFT's and holographic multiverse," `arXiv: 1004.0618`.

[8] A. Kapustin, K. Setter, K. Vyas, "Surface operators in four-dimensional topological gauge theory and Langlands duality," `arXiv: 1002.0385`.

[9] A. Bartels, C. Douglas, A. Henriques, "Conformal nets and local field theory," `arXiv: 0912.5307`.

[10] J. Fuchs, C. Schweigert, C. Stigner, "The classifying algebra for defects," `arXiv: 1007.0401`.

[11] A. Davydov, L. Kong, I. Runkel, "Invertible defects and isomorphisms of rational CFTs," `arXiv: 1004.4725`.

[12] G. Sarkissian, "Defects in $G/H$ coset, $G/G$ topological field theory and discrete Fourier-Mukai transform," `arXiv: 1006.5317`.

[13] N. Carqueville, I. Runkel, "Rigidity and defect actions in Landau-Ginzburg models," `arXiv: 1006.5609`.

[14] A. Kapustin, N. Saulina, "Topological boundary conditions in abelian Chern-Simons theory," `arXiv: 1008.0654`.

[15] A. Kapustin, K. Setter, "Geometry of topological defects of two-dimensional sigma models," `arXiv: 1009.5999`.

[16] J. Lurie, "On the classification of topological field theories," `arXiv: 0905.0465`.

[17] R. McCarthy, "The cyclic homology of an exact category," J. Pure. Appl. Alg. **93** (1994) 251-296.

[18] B. Keller, "On the cyclic homology of ringed spaces and schemes," Documenta Mathematica **3** (1998) 231-259.

[19] B. Keller, "On the cyclic homology of exact categories," J. Pure. Appl. Alg. **136** (1999) 1-56.

[20] K. Costello, "Topological conformal field theories and Calabi-Yau categories," `arXiv: math/0412149`.

[21] A. Caldararu, "Derived categories of sheaves: a skimming," `arXiv: math.AG/0501094`.

[22] N. Ganter, M. Kapranov, "Representation and character theory in 2-categories," `arXiv: math/0602510`.

[23] A. Caldararu, S. Willerton, "The Mukai pairing I: a categorical approach," `arXiv: 0707.2052`.

[24] M. Hopkins, J. Lurie, unpublished work, described in a 2005 talk of M. Hopkins at the Fields Institute

[25] J. Lurie, "Stable infininty categories", `arXiv: math/0608228`

[26] R. Hartshorne, *Algebraic geometry*, Graduate texts in math. 52, Springer, New York, 1977.

[27] D. Freed, E. Witten, "Anomalies in string theory with D-branes," `arXiv: hep-th/9907189`.

[28] S. Katz, E. Sharpe, "D-branes, open string vertex operators, and Ext groups," Adv. Theor. Math. Phys. **6** (2003) 979-1030, `arXiv: hep-th/0208104`.

[29] J. P. Serre, *Local algebra*, Springer monographs in mathematics, Springer, Berlin, 2000.

[30] E. Witten, talk at Strings 2010.

[31] A. Caldararu, private communication.

[32] E. Witten, "Mirror manifolds and topological field theory," `arXiv: hep-th/9112056`.

[33] S. Cordes, G. Moore, S. Ramgoolam, "Lectures on 2d Yang-Mills theory, equivariant cohomology, and topological field theories," `arXiv: hep-th/9411210`.

[34] T. Goodwillie, "Cyclic homology, derivations and the free loop space," Topology **24** (1985) 187-215.

[35] D. Sullivan, "Open and closed string field theory interpreted in classical algebraic topology," `arXiv: math/0302332`.

[36] D. Sullivan, "String topology background and present state," `arXiv: 0710.4141`.

[37] J. F. Adams, "On the cobar construction," Proc. Nat. Acad. Sci. USA **42** (1956), 409-412.

[38] A. Bousfield and D. Kan, *Homotopy limits, completions, and localiations.* Springer Lecture Notes in Mathematics 304, 1972.

[39] D. Rector, "Steenrod operations in the Eilenberg–Moore spectral sequence", Comm. Math. Helv. **45** (1970), 540-552.

[40] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.

[41] R. Cohen, K. Hess, A. Voronov, *String topology and cyclic homology*, Advanced courses in mathematics CRM Barcelona, Birkhäuser, Basel, 2006.

[42] J. D. S. Jones, "Cyclic homology and equivariant homology," Invent. Math. **87** (1987) 403-423.

[43] E. Getzler, J. Jones, S. Petrack, "Differential forms on loop spaces and the cyclic bar complex," Topology **30** (1991) 339-371.

[44] A. J. Blumberg, R. L. Cohen, and C. Teleman, "Open-closed field theories, string topology, and Hochschild homology," `arXiv: 0906.5198`.

[45] A. Kapustin, L. Rozansky, "On the relation between open and closed topological strings," `arXiv: hep-th/0405232`.

[46] G. Moore, G. Segal, "D-branes and K-theory in 2d topological field theory," `arXiv: hep-th/0609042`.

[47] N. Moeller, I. Sachs, "Closed string cohomology in open string field theory," `arXiv: 1010.4125`.

[48] J. L. Loday, *Cyclic homology*, second edition, Springer, Berlin, 1992, 1998.

[49] G. Hochschild, B. Kostant, A. Rosenberg, "Differential forms on regular affine algebras," Trans. AMS **102** (1962) 383-408.

[50] R. G. Swan, "Hochschild cohomology of quasiprojective schemes," J. Pure Appl. Algebra **110** (1996) 57-80.

[51] M. Kontsevich, "Deformation quantization of Poisson manifolds, I," `arXiv: q-alg/9709040`.

[52] A. Yekutieli, "The continuous Hochschild cochain complex of a scheme," Canad. J. Math. **54** (2002) 1319-1337.

[53] M. Kontsevich, "Homological algebra of mirror symmetry," `arXiv: alg-geom/9411018`.

[54] J. Solomon, private communication.

[55] J. Guffin, E. Sharpe, "A-twisted Landau-Ginzburg models," J. Geom. Phys. **59** (2009) 1547-1580, `arXiv: 0801.3836`.

[56] T. Dychkerhoff, "Compact generators in categories of matrix factorizations," `arXiv: 0904.4713`.

[57] S. Hellerman, I. Swanson, "Charting the landscape of supercritical string theory," Phys. Rev. Lett. **99** (2007) 171601, `arXiv: 0705.0980`.

[58] E. Witten, "Elliptic genera and quantum field theory," Comm. Math. Phys. **109** (1987) 525-536.

[59] R. Schimmrigk, "Emergent spacetime from modular motives," `arXiv: 0812.4450`.

[60] P. Griffiths, J. Harris, *Rational homotopy theory and differential forms*, Birkhäuser, Boston, 1981.

[61] V. Niarchos, "Hairpin-branes and tachyon-paperclips in holographic backgrounds," `arXiv: 1005.1650`.