

On some models of geometric noncommutative general relativity

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Abstract

Using Fedosov theory of deformation quantization of endomorphism bundle we construct several models of pure geometric, deformed vacuum gravity. Deformations of Einstein-Hilbert and Palatini actions are investigated. Field equations are derived, and in some cases corrections to the metric are pointed out. The relation to the theory of Seiberg-Witten map is also studied and the correspondence to the spacetime noncommutativity described by Fedosov $*$ -product of functions is explained.

1 Introduction

The present paper is dedicated to a study of some possible global and geometric models of relativity on noncommutative spacetimes within the framework of Fedosov deformation quantization of endomorphism bundle. The motivation for such investigation originates in the conviction that whatever “noncommutative gravity” would be, it should preserve the basic symmetry of the classical theory – the full diffeomorphism invariance. Presented analysis aims at showing that Fedosov quantization of endomorphism bundle can serve as a tool for building geometric field theories on noncommutative spacetimes.

The general strategy we are going to adopt can be summarized in the following steps.

- 1) Take some symplectic manifold and an action on it which leads to the general relativity.
- 2) Rewrite the action by representing Lagrangian as a product of endomorphisms of some bundle.

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- 3) Replace the product of endomorphisms by Fedosov $*$ -product of endomorphisms.
- 4) Replace the integral by Fedosov trace functional.
- 5) Do the variations to obtain field equations.
- 6) Observe that steps 3 and 4 together with results of [1] induce that the theory is locally equivalent to the theory with Seiberg-Witten map applied on endomorphisms.

There is vast literature concerning construction of noncommutative gravity by means of Moyal product and Seiberg-Witten map. Hence, one can point out series of works [2, 3, 4] based on combination of infinitesimal $so(3, 1)$ gauging with infinitesimal coordinate transformations, preserving (at first order of deformation) constant deformation parameter θ^{ij} . Another approach is given by [5, 6, 7, 8], where $SO(4, 1)$ (or $U(2, 2)$) symmetry is investigated. In such setting gauge potential carries information about both tetrad field and the usual $SO(3, 1)$ -connection. The standard gravity is recovered by the procedure of contraction of the gauge group. There are also investigations based on some variants of $SL(2, \mathbb{C})$ symmetry [9, 10]. The common feature of all of these approaches are vanishing first order corrections to the field equations. On the other hand, the common issue is the lack of diffeomorphism invariance. In [11] attempts were made to geometrize theories based on Moyal product and Seiberg-Witten map. The resulting structure is explicitly diffeomorphism invariant, but at the price of nonassociativity of the corresponding $*$ -product. One should also mention some other approaches to noncommutative gravity related somehow to $*$ -products and Seiberg-Witten map. These are [12, 13, 14], where the method of Lie algebra twisting has been used to represent deformation of diffeomorphism symmetry. Finally there are investigations which are strictly related to some particular models emerging in the context of the string theory, e.g. [15, 16].

The paper is organized as follows. In section 2 a brief overview of results of Fedosov construction is presented, and also some further conventions are fixed. In sections 3 and 4 deformations of Einstein-Hilbert and Palatini actions are investigated. Fifth section is devoted to analysis of interrelation between presented models and the theory of Seiberg-Witten map. We also clarify, how obtained results are related to the noncommutativity of the spacetime described by Fedosov $*$ -product of functions. Finally some concluding remarks (section 6) are given.

2 Preliminaries

2.1 Fedosov construction

The main tool used in this paper is Fedosov construction of deformation quantization of endomorphism bundle formulated in [17]. We are not going to concern technical or “internal”

details of this theory (which are interesting and beautiful on their own) but rather to make use of some of its particular results. Interested reader may find short exposition of Fedosov construction in its simplest, suitable for present purposes form in [1]. Further geometric and algebraic interpretations are provided by [18, 19]. Some other analysis and examples can be found in [20, 21, 22]. Thus, we limit ourselves to the very brief, notation-fixing description of Fedosov $*$ -product.

The starting point is given by Fedosov manifold $(\mathcal{M}, \omega, \partial^S)$ i.e. $2n$ -dimensional symplectic manifold (\mathcal{M}, ω) with some fixed symplectic (torsionless and preserving ω) connection ∂^S [23, 24]. The corresponding Poisson tensor (given by the inverse of ω_{ij}) is going to be denoted as Λ^{ij} . These data generate¹ global, geometric and associative deformation of product of functions on \mathcal{M} . Its explicit form can be computed up to arbitrary power of deformation parameter \hbar (which has nothing to do with Planck constant in our context) by means of Fedosov's recursive techniques.

For the vector bundle \mathcal{E} over \mathcal{M} , equipped with a connection $\partial^\mathcal{E}$ one can construct global, geometric and associative deformation of product of sections of $\text{End}(\mathcal{E})$. Locally it can be understood as a deformation of product of matrices. Denoting by $\partial = \partial^S \otimes 1 + 1 \otimes \partial^\mathcal{E}$ the connection in $T\mathcal{M} \otimes \mathcal{E}$ (and by the same symbol its natural extension to any other tensor product of $T\mathcal{M}$, $T^*\mathcal{M}$, \mathcal{E} and \mathcal{E}^*) one may calculate that for arbitrary two sections $F, G \in C^\infty(\text{End}(\mathcal{E}))[[\hbar]]$ the Fedosov $*$ -product is given up to \hbar^2 by the expression

$$F * G = FG - \frac{i\hbar}{2} \Lambda^{ab} \partial_a F \partial_b G + \frac{\hbar^2}{8} \Lambda^{ab} \Lambda^{cd} \left(\{\partial_b F, R_{ac}^\mathcal{E}\} \partial_d G + \partial_b F \{R_{ac}^\mathcal{E}, \partial_d G\} + \partial_{(a} \partial_c) F \partial_{(b} \partial_d) G \right) + O(\hbar^3) \quad (1)$$

where $R_{ab}^\mathcal{E} = \frac{\partial}{\partial x^a} \Gamma_b^\mathcal{E} - \frac{\partial}{\partial x^b} \Gamma_a^\mathcal{E} + [\Gamma_a^\mathcal{E}, \Gamma_b^\mathcal{E}]$ (for $\partial_i^\mathcal{E} = \frac{\partial}{\partial x^i} + \Gamma_i^\mathcal{E}$) is the curvature of $\partial^\mathcal{E}$, and $\{\cdot, \cdot\}$ stands for the anticommutator. It is clear that in above formula usual product of endomorphisms (noncommutative from the beginning) has been used. For the special case of flat $\partial^\mathcal{E}$ and the local frame with $\Gamma^\mathcal{E} \equiv 0$, the Fedosov $*$ -product of endomorphisms becomes product of matrices with commutative multiplication of entries replaced by noncommutative Fedosov product of functions. Such product of matrices is going to be denoted as $*_{\mathcal{S}}$. (The same symbol will be used for the Fedosov product of functions). If additionally ∂^S is flat and we work in local Darboux coordinates for which coefficients of ∂^S vanish, then Fedosov product of functions becomes Moyal product $*_{\mathcal{T}}$. Thus, in such special case, we are dealing with multiplication used in [25] for the description of deformed gauge transformations.

The object which needs some more attention is Fedosov trace functional ([17] section 5.6). Given some Fedosov product $*$ one is able to construct trace functional tr_* taking

¹Precisely, one has also to fix curvature and normalizing condition for Abelian connection generating Fedosov $*$ -product [17]. Within this paper, standard normalization $\mu \equiv 0$ and curvature $\Omega = -\omega$ are used.

values in $\mathbb{C}[[\hbar]]$ and acting on compactly supported sections belonging to $C^\infty(\text{End}(\mathcal{E}))[[\hbar]]$, with the property

$$\text{tr}_*(F * G) = \text{tr}_*(G * F). \quad (2)$$

If one requires additionally, that for arbitrary (global or local) isomorphism M between $*$ -algebras with products $*_1$ and $*_2$ (i.e. for M fulfilling $M(F *_1 G) = M(F) *_2 M(G)$) the relation

$$\text{tr}_{*_1}(F) = \text{tr}_{*_2}(M(F)) \quad (3)$$

holds, then it follows that the trace functional is unique up to a constant normalizing factor. The proof of this fact relies on the observation that for the Moyal product $*_T$ the trace is given by

$$\text{tr}_{*_T}(F) = \text{const} \int_{\mathbb{R}^{2n}} \text{Tr}(F) \frac{\omega^n}{n!} \quad (4)$$

where Tr stands for the trace of a matrix, and on possibility of representing tr_* in terms of traces on Moyal algebras by a partition of unity $\{\rho_i\}$ and a compatible set of local isomorphisms $\{M_i\}$ between $*$ and the Moyal product. It turns out that tr_* is independent of particular choice of $\{\rho_i\}$ and $\{M_i\}$. Unlike convention of [17], we fix the normalizing constant to be equal 1. Construction presented in [17] enables calculation of explicit form of tr_* . Up to \hbar^2 it reads²

$$\text{tr}_*(F) = \int_{\mathcal{M}} \text{Tr} \left(F + \frac{i\hbar}{2} \Lambda^{ab} R_{ab}^\mathcal{E} F + \hbar^2 \left(-\frac{3}{8} \Lambda^{[ab} \Lambda^{cd]} R_{ab}^\mathcal{E} R_{cd}^\mathcal{E} + s_2 1 \right) F + O(\hbar^3) \right) \frac{\omega^n}{n!} \quad (5)$$

where 1 is the identity endomorphism and the scalar³

$$s_2 = \frac{1}{64} \Lambda^{[ab} \Lambda^{cd]} \overset{S}{R}{}^k{}_{lab} \overset{S}{R}{}^l{}_{kcd} + \frac{1}{48} \Lambda^{ab} \Lambda^{cd} \partial_e^S \partial_a^S \overset{S}{R}{}^e{}_{bcd}$$

has been introduced for sake of simplicity of further notations. In above formula $\overset{S}{R}{}^i{}_{jab}$ stands for the curvature tensor of ∂^S . It is useful to write down explicit form of $\text{tr}_*(F * G)$.

²The computation leading to (5) is quite laborious as one has to deal with connection coefficients which in final step massively cancel and the remaining terms can be grouped to yield tensorial expressions. Large parts of this work has been performed with the significant use of xAct tensor manipulation package [26].

³Index ₂ corresponds to the presence s_2 at \hbar^2 . Such defined scalar is a symplectic part of what is called *trace density* in [17]. With similar conventions $s_1 = 0$.

Substitution of (1) into (5) after some manipulations yields⁴

$$\begin{aligned} \text{tr}_*(F * G) = \int_{\mathcal{M}} \text{Tr} \left(FG + \frac{i\hbar}{4} \Lambda^{ab} R_{ab}^{\mathcal{E}} \{F, G\} + h^2 \left[s_2 FG + \frac{1}{8} \Lambda^{ab} \Lambda^{cd} \left(R_{ab}^{\mathcal{E}} [\partial_c F, \partial_d G] + \right. \right. \right. \\ \left. \left. \left. - \partial_{(a} \partial_{c)} F \partial_{(b} \partial_{d)} G - \frac{3}{2} R_{[ab}^{\mathcal{E}} R_{cd]}^{\mathcal{E}} \{F, G\} \right) \right] + O(\hbar^3) \right) \frac{\omega^n}{n!} \quad (6) \end{aligned}$$

2.2 Some further conventions

The important problem related to the programme presented in the introduction is the incompatibility of the volume forms. In (5) the symplectic volume form $\text{vol}_S = \frac{\omega^n}{n!}$ must be used, and in general relativity the metric one $\text{vol}_M = \sqrt{-g} dx^1 \wedge \dots \wedge dx^{2n}$ more or less explicitly appears. Since the two must be proportional one can write $\text{vol}_M = v \text{vol}_S$ defining the function $v : \mathcal{M} \rightarrow \mathbb{R}$ this way. The above mentioned incompatibility should be handled somehow, and in what follows two possible approaches are investigated. First, one can simply rescale one of the endomorphisms by multiplying them by v . Thus, let us fix the convention that $\check{F} = vF$. The other option is given by introducing endomorphism $V = v1$ which multiplies endomorphism under the action. Both methods are completely equivalent at the undeformed level, but become different after deformation.

Let us also mention the following issue concerning the tangent bundle $T\mathcal{M}$. In presented models it appears in two distinct roles. First as a "component" of bundle \mathcal{E} , and then as an object which carries information about the symplectic structure and the covariant derivations producing quantization formalism. This distinction becomes important when applying covariant derivation to tensors involving indices from both copies of $T\mathcal{M}$. The ∂ connection acts in such case by means of $\partial^{\mathcal{E}}$ (which is going to be chosen as a metric connection) and the symplectic connection ∂^S respectively. Thus, one needs some way of "marking" indices which should be differentiated by $\partial^{\mathcal{E}}$ and we are going to put prime at them (e.g. $R^{\mathcal{E} a'}_{b'lm}$). The ambiguity may be also postponed by using index-free notation for endomorphisms, and this approach is also used. Finally, the primes are omitted in the field equations, as they are no longer needed and may tend to obscure the result.

Finally, let us recall that all indices in subsequent sections are manipulated by means of

⁴The formula has been rearranged to explicitly exhibit symmetry $\text{tr}_*(F * G) = \text{tr}_*(G * F)$. For the term at h this can be done quickly using integration by parts and definition of $R^{\mathcal{E}}$. For terms at h^2 one can proceed in a following manner. 1) Take what appears at h^2 after simple substitution of (1) into (5). Let it call $h^2 Q(F, G)$. 2) Rewrite it as $h^2/2(Q(F, G) + Q(G, F)) + h^2/2(Q(F, G) - Q(G, F))$. Drop the antisymmetric part. 3) Check that discarded terms are indeed equal to zero (as it should be, by the construction). When integrated by parts, the terms with single covariant derivative of $R^{\mathcal{E}}$ vanish in virtue of Bianchi identity, while the ones with double ∂ can be replaced by $R^{\mathcal{E}}$ and in turn, sum up with remaining terms to give 0. Such calculation can be treated as an additional verification of the formula (5).

corresponding metrics.

3 Einstein-Hilbert action

Now, let us analyze some possible applications of Fedosov theory in the general relativity on noncommutative spacetime. We are going to proceed using programme sketched in the introduction and to assume, that symplectic form ω and compatible symplectic connection ∂^S are fixed. First, let us focus on the Einstein-Hilbert action. Thus, there is a metric g_{ab} with determinant g , its torsionless Levi-Civita connection ∇ , Riemann curvature tensor $R^{a'}_{b'cd}$ (also used with all indices primed⁵ $R^{a'}_{b'c'd'}$), Ricci tensor $R_{a'b'} = R^{c'}_{a'c'b'}$ and Ricci scalar R . Field equations are going to be derived by the variation of the metric.

Let us introduce the notation $\underline{R}^{a'}_{b'} = R^{a'}_{b'}$ and $\underline{\underline{R}}^{a'b'}_{c'd'} = R^{a'b'}_{c'd'}$. (This becomes convenient when distinguishing between endomorphisms $\underline{\underline{R}}$, \underline{R} and the scalar R). Also, let

$$\begin{aligned} Y^{ijk'l'} &= \Lambda^{[ij} \Lambda^{ab]} R^{k'l'}_{ab} \\ X^{i'j'k'l'} &= \Lambda^{[ab} \Lambda^{cd]} R^{i'j'}_{ab} R^{k'l'}_{cd} = R^{i'j'}_{ab} Y^{abk'l'} \\ Z &= \frac{1}{\sqrt{-g}} \Lambda^{ij} \Lambda^{kl} \partial_i^S \partial_k^S \partial_j^S \partial_l^S \sqrt{-g} \end{aligned}$$

3.1 \check{R} as an endomorphism of $T\mathcal{M}$

The Einstein-Hilbert action can be quickly rewritten as

$$\mathcal{S}_{EH_{1A}} = \int_{\mathcal{M}} \text{Tr} \frac{\check{R} \omega^n}{n!}$$

Thus, we are going to treat redefined Ricci tensor $\check{R}^{a'}_{b'}$ as an endomorphism of $\mathcal{E} = T\mathcal{M}$. In order to define $*$ -product of endomorphisms one needs some connection in \mathcal{E} . Let us fix $\partial^{\mathcal{E}} = \nabla$ and consequently $R^{\mathcal{E}}$ is given by the Riemann tensor. The corresponding $*$ -product is going to be denoted by $*_{EH_1}$. Under these assumptions the deformed action is given by⁶

$$\begin{aligned} \widehat{\mathcal{S}}_{EH_{1A}} &= \text{tr}_{*_{EH_1}}(\check{R}) = \int_{\mathcal{M}} \text{Tr} \left(\check{R} + h^2 \left(-\frac{3}{8} \Lambda^{[ab} \Lambda^{cd]} R_{ab}^{\mathcal{E}} R_{cd}^{\mathcal{E}} + s_2 \right) \check{R} + O(h^3) \right) \frac{\omega^n}{n!} \\ &= \int_{\mathcal{M}} \left(R - \frac{3}{8} h^2 X^{k'l'm'} R^{m'}_{k'} + h^2 s_2 R + O(h^3) \right) \text{vol}_M \end{aligned}$$

⁵As we use exactly the same frame (e.g. coordinate one) for both primed and unprimed indices we can consistently define primed tensors from unprimed ones and vice versa. The prime is used only as a marking for covariant derivation ∂ .

⁶The term at h vanish due to $R^k_{lab} R^l_k = 0$. The same stays true in further subsections.

Variation of the metric yields the following field equations

$$\begin{aligned}
& R^{ab} - \frac{1}{2}g^{ab}R + h^2 \left[\frac{3}{8} \left(-R^{(a} X_l^{b)kl} + \frac{1}{2}R^k{}_l X^l{}_m{}^m{}_k g^{ab} + \nabla_k \nabla^{(a} X^b)_{l}{}^{lk} - \frac{1}{2} \nabla_l \nabla^l X^a{}_k{}^{kb} \right. \right. \\
& - \left. \frac{1}{2} g^{ab} \nabla_k \nabla_l X^k{}_m{}^{ml} - 2 \nabla_k \nabla^l \left(R^{(a}{}_m Y_l^{b)mk} \right) + 2 \nabla_k \nabla_l \left(R^{km} Y^{l(a}{}_m{}^{b)} \right) \right) - \frac{1}{2} g^{ab} R s_2 \\
& \left. + R^{ab} s_2 + g^{ab} \nabla_l \nabla^l s_2 - \nabla^a \nabla^b s_2 \right] + O(h^3) = 0
\end{aligned}$$

3.2 \underline{R} and V as endomorphisms of $T\mathcal{M}$

Now, keeping unmodified $*$ -product structure given by $*_{EH_1}$, we are going to investigate another possibility of forcing correct volume form at $h = 0$. The Einstein-Hilbert action written as

$$\mathcal{S}_{EH_{1B}} = \int_{\mathcal{M}} \text{Tr} \underline{R} V \frac{\omega^n}{n!}$$

may be deformed into⁷

$$\begin{aligned}
\widehat{\mathcal{S}}_{EH_{1B}} &= \text{tr}_{*_{EH_1}} (\underline{R} *_{EH_1} V) \\
&= \int_{\mathcal{M}} \text{Tr} \left(\underline{R} V + h^2 \left(-\frac{1}{8} \Lambda^{ab} \Lambda^{cd} \left(\partial_{(a} \partial_{c)} \underline{R} \partial_{(b} \partial_{d)} V + 3 R_{[ab}^{\mathcal{E}} R_{cd]}^{\mathcal{E}} \underline{R} V \right) + s_2 \underline{R} V \right) + O(h^3) \right) \frac{\omega^n}{n!} \\
&= \int_{\mathcal{M}} \left(R - \frac{3}{8} h^2 X^{k'}{}_{l'}{}^{l'}{}_{m'} R^{m'}{}_{k'} - \frac{1}{8} h^2 \Lambda^{ab} \Lambda^{cd} \partial_b^S \partial_d^S \partial_a^S \partial_c^S R + h^2 s_2 R + O(h^3) \right) \text{vol}_M
\end{aligned}$$

Then, the field equations become

$$\begin{aligned}
& R^{ab} - \frac{1}{2}g^{ab}R + h^2 \left[\frac{3}{8} \left(-R^{(a} X_l^{b)kl} + \frac{1}{2}R^k{}_l X^l{}_m{}^m{}_k g^{ab} + \nabla_k \nabla^{(a} X^b)_{l}{}^{lk} - \frac{1}{2} \nabla_l \nabla^l X^a{}_k{}^{kb} \right. \right. \\
& - \left. \frac{1}{2} g^{ab} \nabla_k \nabla_l X^k{}_m{}^{ml} - 2 \nabla_k \nabla^l \left(R^{(a}{}_m Y_l^{b)mk} \right) + 2 \nabla_k \nabla_l \left(R^{km} Y^{l(a}{}_m{}^{b)} \right) \right) + \frac{1}{8} \left(-R^{ab} Z \right. \\
& \left. + \nabla^a \nabla^b Z - g^{ab} \nabla_l \nabla^l Z + \frac{1}{2} g^{ab} \Lambda^{jk} \Lambda^{lm} \partial_k^S \partial_m^S \partial_j^S \partial_l^S R \right) - \frac{1}{2} g^{ab} R s_2 + R^{ab} s_2 \\
& \left. + g^{ab} \nabla_l \nabla^l s_2 - \nabla^a \nabla^b s_2 \right] + O(h^3) = 0
\end{aligned}$$

3.3 $\check{\underline{R}}$ as an endomorphism of $T\mathcal{M} \otimes T\mathcal{M}$

This time, we start with the action

$$\mathcal{S}_{EH_{2A}} = \int_{\mathcal{M}} \text{Tr} \check{\underline{R}} \frac{\omega^n}{n!}$$

⁷The term with $\partial_{(a} \partial_{c)} \underline{R} \partial_{(b} \partial_{d)} V$ is integrated by parts twice, then the covariant derivatives are commuted with the trace and the torsionless property of ∂^S is used to get rid of symmetrizations.

Here, the Riemann tensor is treated as an endomorphism of $\mathcal{E} = T\mathcal{M} \otimes T\mathcal{M}$ whose action on $l \in T\mathcal{M} \otimes T\mathcal{M}$ yields $(\underline{R}l)^{a'b'} = R^{a'b'}{}_{c'd'}l^{c'd'}$. As a connection in \mathcal{E} we take $\partial^{\mathcal{E}} = \nabla \otimes 1 + 1 \otimes \nabla$. Its curvature is given by $R_{ab}^{\mathcal{E}} = R_{ab}^{\nabla} \otimes 1 + 1 \otimes R_{ab}^{\nabla}$, with R_{ab}^{∇} being the curvature of ∇ treated as an endomorphism of $T\mathcal{M}$. Let $*_{EH_2}$ be the corresponding $*$ -product. Thus

$$\begin{aligned} \widehat{\mathcal{S}}_{EH_2A} &= \text{tr}_{*_{EH_2}}(\underline{\check{R}}) = \int_{\mathcal{M}} \text{Tr} \left(\underline{\check{R}} + h^2 \left(-\frac{3}{8} \Lambda^{[ab} \Lambda^{cd]} R_{ab}^{\mathcal{E}} R_{cd}^{\mathcal{E}} + s_2 \right) \underline{\check{R}} + O(h^3) \right) \frac{\omega^n}{n!} \\ &= \int_{\mathcal{M}} \left(R - \frac{3}{4} h^2 \left(X^{k'}{}_{l'}{}_{m'} R^{m'}{}_{k'} + X^{k'}{}_{l'}{}_{m'} R^{l'p'}{}_{k'm'} \right) + h^2 s_2 R + O(h^3) \right) \text{vol}_M \end{aligned}$$

yielding

$$\begin{aligned} R^{ab} - \frac{1}{2} g^{ab} R + h^2 \left[\frac{3}{4} \left(-R^{(a} X_l{}^{b)kl} + \frac{1}{2} R^k{}_l X^l{}_{m'} g^{ab} + \nabla_k \nabla^{(a} X^{b)lk} - \frac{1}{2} \nabla_l \nabla^l X^a{}_{k'}{}^{kb} \right. \right. \\ \left. \left. - \frac{1}{2} g^{ab} \nabla_k \nabla_l X^k{}_{m'}{}^{ml} - 2 \nabla_k \nabla^l \left(R^{(a} Y_l{}^{b)mk} \right) + 2 \nabla_k \nabla_l \left(R^{km} Y^{l(a}{}_{m'}{}^{b)} \right) + R_{km}{}^{l(a} X^{b)mk}{}_{l'} \right. \\ \left. \left. + \frac{1}{2} \nabla_k \nabla_l X^{k(ab)l} + 2 \nabla_k \nabla_l \left(R^{mjka} Y^{bl}{}_{mj} \right) + \frac{1}{2} R^{lm}{}_{jk} X^j{}_{l'}{}^k{}_{m'} g^{ab} \right) - \frac{1}{2} g^{ab} R s_2 + R^{ab} s_2 \right. \\ \left. \left. + g^{ab} \nabla_l \nabla^l s_2 - \nabla^a \nabla^b s_2 \right] + O(h^3) = 0 \end{aligned}$$

3.4 \underline{R} and V as endomorphisms of $T\mathcal{M} \otimes T\mathcal{M}$

Analogously to section 3.2, one may keep the product $*_{EH_2}$ unchanged, but rewrite the action using V

$$\mathcal{S}_{EH_2B} = \int_{\mathcal{M}} \text{Tr} \underline{R} V \frac{\omega^n}{n!}$$

After the deformation the action takes form

$$\begin{aligned} \widehat{\mathcal{S}}_{EH_2B} &= \text{tr}_{*_{EH_2}}(\underline{R} *_{EH_2} V) \\ &= \int_{\mathcal{M}} \text{Tr} \left(\underline{R} V + h^2 \left(-\frac{1}{8} \Lambda^{ab} \Lambda^{cd} \left(\partial_{(a} \partial_{c)} \underline{R} \partial_{(b} \partial_{d)} V + 3 R_{[ab}^{\mathcal{E}} R_{cd]}^{\mathcal{E}} \underline{R} V \right) + s_2 \underline{R} V \right) + O(h^3) \right) \frac{\omega^n}{n!} \\ &= \int_{\mathcal{M}} \left(R - \frac{3}{4} h^2 \left(X^{k'}{}_{l'}{}_{m'} R^{m'}{}_{k'} + X^{k'}{}_{l'}{}_{m'} R^{l'p'}{}_{k'm'} \right) - \frac{1}{8} h^2 \Lambda^{ab} \Lambda^{cd} \partial_b^S \partial_d^S \partial_a^S \partial_c^S R + h^2 s_2 R \right. \\ &\quad \left. + O(h^3) \right) \text{vol}_M \end{aligned}$$

The field equations are given by

$$\begin{aligned}
& R^{ab} - \frac{1}{2}g^{ab}R + h^2 \left[\frac{3}{4} \left(-R^{(a} X_l^{b)kl} + \frac{1}{2}R^k_l X^l_m{}^m{}_k g^{ab} + \nabla_k \nabla^{(a} X^{b)l}{}_k - \frac{1}{2}\nabla_l \nabla^l X^a{}_k{}^{kb} \right. \right. \\
& - \frac{1}{2}g^{ab} \nabla_k \nabla_l X^k{}_m{}^{ml} - 2\nabla_k \nabla^l \left(R^{(a}{}_m Y_l^{b)mk} \right) + 2\nabla_k \nabla_l \left(R^{km} Y_l^{(a}{}_m{}^{b)} \right) + R_{km}{}^{l(a} X^{b)mk}{}_l \\
& + \frac{1}{2}\nabla_k \nabla_l X^{k(ab)l} + 2\nabla_k \nabla_l \left(R^{mjka} Y^{bl}{}_{mj} \right) + \frac{1}{2}R^{lm}{}_{jk} X^j{}_l{}^k{}_m g^{ab} \left. \right) + \frac{1}{8} \left(-R^{ab} Z \right. \\
& + \nabla^a \nabla^b Z - g^{ab} \nabla_l \nabla^l Z + \frac{1}{2}g^{ab} \Lambda^{jk} \Lambda^{lm} \partial_k^S \partial_m^S \partial_j^S \partial_l^S R \left. \right) - \frac{1}{2}g^{ab} R s_2 + R^{ab} s_2 \\
& \left. + g^{ab} \nabla_l \nabla^l s_2 - \nabla^a \nabla^b s_2 \right] + O(h^3) = 0
\end{aligned}$$

3.5 Corrections to the metric

In all considered cases field equations are of the form $G_{ab} = W_{ab} + O(h^3)$, where $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is the Einstein tensor, and the term W_{ab} is of h^2 order i.e. $W_{ab} = h^2 \overset{(2)}{W}_{ab} + O(h^3)$. Let us investigate how W_{ab} influences the metric. For this purpose one can rewrite g_{ab} as a formal power series with respect to h

$$g_{ab} = \overset{(0)}{g}_{ab} + h \overset{(1)}{g}_{ab} + h^2 \overset{(2)}{g}_{ab} + \dots$$

Coefficients of Levi-Civita connection corresponding to g_{ab} can be written as

$$\Gamma^a{}_{bc} = \overset{(0)}{\Gamma}^a{}_{bc} + h \overset{(1)}{\Gamma}^a{}_{bc} + h^2 \overset{(2)}{\Gamma}^a{}_{bc} + \dots$$

One can quite easily calculate that

$$\overset{(0)}{\Gamma}^a{}_{bc} = \frac{1}{2}g^{ak} \left(\frac{\partial g_{kb}}{\partial x^c} + \frac{\partial g_{kc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^k} \right), \quad (7a)$$

$$\overset{(1)}{\Gamma}^a{}_{bc} = \frac{1}{2}g^{ak} \left(\overset{(0)}{\nabla}_c g_{kb} + \overset{(0)}{\nabla}_b g_{kc} - \overset{(0)}{\nabla}_k g_{bc} \right), \quad (7b)$$

$$\overset{(2)}{\Gamma}^a{}_{bc} = \frac{1}{2}g^{ak} \left(\overset{(0)}{\nabla}_c g_{kb} + \overset{(0)}{\nabla}_b g_{kc} - \overset{(0)}{\nabla}_k g_{bc} \right) - \overset{(0)}{g}^{ak} g_{kl} \overset{(1)}{\Gamma}^l{}_{bc}, \quad (7c)$$

where $\overset{(0)}{\nabla}$ denotes Levi-Civita connection of metric $\overset{(0)}{g}_{ab}$. Observe that $\overset{(1)}{\Gamma}^a{}_{bc}$ and $\overset{(2)}{\Gamma}^a{}_{bc}$ are tensorial objects. Hence, for the Riemann tensor

$$R^a{}_{bcd} = \overset{(0)}{R}^a{}_{bcd} + h \overset{(1)}{R}^a{}_{bcd} + h^2 \overset{(2)}{R}^a{}_{bcd} + \dots$$

one obtains

$$\overset{(1)}{R}^a{}_{bcd} = 2 \overset{(0)}{\nabla}_{[c} \overset{(1)}{\Gamma}^a{}_{d]b}, \quad (8a)$$

$$\overset{(2)}{R}^a{}_{bcd} = 2 \overset{(0)}{\nabla}_{[c} \overset{(2)}{\Gamma}^a{}_{d]b} + 2 \overset{(1)}{\Gamma}^a{}_{k[c} \overset{(1)}{\Gamma}^k{}_{d]b}, \quad (8b)$$

and $R_{bcd}^{(o)}$ is the Riemann tensor of metric $g_{ab}^{(o)}$. Substituting above relations into field equations and analyzing terms at h^0 and h^1 one calculates that

$$R_{ab}^{(o)} = 0, \quad (9a)$$

$$g^{(o)kl} \left(\nabla_k^{(o)} \nabla_a^{(o)} g_{bl}^{(1)} + \nabla_k^{(o)} \nabla_b^{(o)} g_{al}^{(1)} - \nabla_k^{(o)} \nabla_l^{(o)} g_{ab}^{(1)} - \nabla_a^{(o)} \nabla_b^{(o)} g_{kl}^{(1)} \right) = 0, \quad (9b)$$

where $R_{ab}^{(o)}$ is zeroth order term in power series expansion of R_{ab} and also Ricci tensor of $g_{ab}^{(o)}$. For h^2 the following relation can be derived

$$\begin{aligned} g^{(o)kl} \left(\nabla_k^{(o)} \nabla_a^{(o)} g_{bl}^{(2)} + \nabla_k^{(o)} \nabla_b^{(o)} g_{al}^{(2)} - \nabla_k^{(o)} \nabla_l^{(o)} g_{ab}^{(2)} - \nabla_a^{(o)} \nabla_b^{(o)} g_{kl}^{(2)} \right) = \\ = 2W_{ab} - \frac{1}{n-1} g_{ab}^{(o)} W - 4\Gamma_{l[k}^{(1)k} \Gamma_{b]a}^{(1)l} + 4g^{(o)r^k} \nabla_{[r}^{(o)} \left(\Gamma_{b]a}^{(1)l} g_{kl}^{(1)} \right), \end{aligned} \quad (9c)$$

where $W = g^{(o)rs} W_{rs}$. The term $W_{ab}^{(2)}$ is given by the following formulae⁸:

– for $\widehat{\mathcal{S}}_{EH_{1A}}$

$$\begin{aligned} W_{ab}^{(2)} = -\frac{3}{8} \left(\nabla_k^{(o)} \nabla_{(a}^{(o)} X_{b)l}^{(o)}{}^{lk} - \frac{1}{2} \nabla_l^{(o)} \nabla^l X_{ak}^{(o)}{}^k{}_b - \frac{1}{2} g_{ab}^{(o)} \nabla_k^{(o)} \nabla_l^{(o)} X_m^{(o)k}{}^{ml} \right) \\ - g_{ab}^{(o)} \nabla_l^{(o)} \nabla^l s_2 + \nabla_a^{(o)} \nabla_b s_2, \end{aligned} \quad (9d)$$

– for $\widehat{\mathcal{S}}_{EH_{1B}}$

$$\begin{aligned} W_{ab}^{(2)} = -\frac{3}{8} \left(\nabla_k^{(o)} \nabla_{(a}^{(o)} X_{b)l}^{(o)}{}^{lk} - \frac{1}{2} \nabla_l^{(o)} \nabla^l X_{ak}^{(o)}{}^k{}_b - \frac{1}{2} g_{ab}^{(o)} \nabla_k^{(o)} \nabla_l^{(o)} X_m^{(o)k}{}^{ml} \right) \\ - \frac{1}{8} \left(\nabla_a^{(o)} \nabla_b^{(o)} Z - g_{ab}^{(o)} \nabla_l^{(o)} \nabla^l Z \right) - g_{ab}^{(o)} \nabla_l^{(o)} \nabla^l s_2 + \nabla_a^{(o)} \nabla_b s_2, \end{aligned} \quad (9e)$$

– for $\widehat{\mathcal{S}}_{EH_{2A}}$

$$\begin{aligned} W_{ab}^{(2)} = -\frac{3}{4} \left(\nabla_k^{(o)} \nabla_{(a}^{(o)} X_{b)l}^{(o)}{}^{lk} - \frac{1}{2} \nabla_l^{(o)} \nabla^l X_{ak}^{(o)}{}^k{}_b - \frac{1}{2} g_{ab}^{(o)} \nabla_k^{(o)} \nabla_l^{(o)} X_m^{(o)k}{}^{ml} + R_{km}^{(o)l}{}_{(a} X_{b)l}^{(o)mk} \right. \\ \left. + \frac{1}{2} \nabla_k^{(o)} \nabla_l^{(o)} X_{(ab)}^{(o)k}{}^l + 2 \nabla_k^{(o)} \nabla_l^{(o)} \left(R^{mj^k}{}_{(a} Y_{b)l}^{(o)mj} \right) + \frac{1}{2} R^{lm}{}_{jk} X_l^{(o)jk}{}^{(o)ab} \right) \\ - g_{ab}^{(o)} \nabla_l^{(o)} \nabla^l s_2 + \nabla_a^{(o)} \nabla_b s_2, \end{aligned} \quad (9f)$$

⁸In equations (9d–9g) indices are manipulated by means of metric $g_{ab}^{(o)}$.

– for $\widehat{\mathcal{S}}_{EH_{2B}}$

$$\begin{aligned}
W_{ab} = & -\frac{3}{4} \left(\nabla_k^{(0)} \nabla_l^{(0)} X_{(a}^{(0)} X_{b)l}^{(0)lk} - \frac{1}{2} \nabla_l^{(0)} \nabla^l X_{ak}^{(0)k} - \frac{1}{2} g_{ab} \nabla_k^{(0)} \nabla_l X_m^{(0)kml} + R_{km}^{(0)l} X_{(a}^{(0)} X_{b)l}^{(0)mk} \right. \\
& + \frac{1}{2} \nabla_k^{(0)} \nabla_l X_{(ab)}^{(0)k}{}^l + 2 \nabla_k^{(0)} \nabla_l \left(R^{(0)mjk} X_{(a}^{(0)} Y_{b)l}^{(0)mj} \right) + \frac{1}{2} R^{lm} X_{jk}^{(0)j} X_l^{(0)k} g_{ab} \left. \right) \quad (9g) \\
& - \frac{1}{8} \left(\nabla_a^{(0)} \nabla_b^{(0)} Z - g_{ab} \nabla_l \nabla^l Z \right) - g_{ab} \nabla_l \nabla^l s_2 + \nabla_a \nabla_b s_2,
\end{aligned}$$

where $X^{(0)ijkl}$, $Y^{(0)ijkl}$, Z are zeroth order terms in power series expansion of X_{ijkl} , Y_{ijkl} and Z , which can be expressed by means of $R_{jkl}^{(0)}$ and $g_{ab}^{(0)}$.

Thus, in all cases equations which describe deformed metric are of the same structure. At h^0 one is dealing with arbitrary Ricci-flat metric $g_{ab}^{(0)}$. The h^1 correction $g_{ab}^{(1)}$ can be understood as a classical⁹ (undeformed) first order perturbation of $g_{ab}^{(0)}$, governed by the linear homogeneous equations (9b). Noncommutativity appears for the first time at h^2 . Correction $g_{ab}^{(2)}$ is given by linear, inhomogeneous equations (9c). The inhomogeneity consists of two parts – the one describing interaction with first order perturbation $g_{ab}^{(1)}$, and the other given by W_{ab} , with purely noncommutative origin. Discarding first order classical perturbation by putting $g_{ab}^{(1)} = 0$, we are able to point out special solution of (9c) for actions $\widehat{\mathcal{S}}_{EH_{1A}}$ and $\widehat{\mathcal{S}}_{EH_{1B}}$. It reads

$$g_{ab}^{(2)} = -\frac{3}{8} X_{ak}^{(0)k}{}_b - \frac{1}{n-1} \left(s_2 - \frac{3}{16} X_{mk}^{(0)km} \right) g_{ab}^{(0)} \quad (10)$$

for $\widehat{\mathcal{S}}_{EH_{1A}}$, and

$$g_{ab}^{(2)} = -\frac{3}{8} X_{ak}^{(0)k}{}_b - \frac{1}{n-1} \left(s_2 - \frac{1}{8} Z - \frac{3}{16} X_{mk}^{(0)km} \right) g_{ab}^{(0)} \quad (11)$$

for $\widehat{\mathcal{S}}_{EH_{1B}}$. (Here, like in (9d) – (9g), indices at X are manipulated by $g_{ab}^{(0)}$). Let us observe that the difference between above solutions and arbitrary other solution of (9c) with $g_{ab}^{(1)} = 0$ must be a solution of homogeneous variant of (9c). Thus, such a difference may be interpreted as a classical perturbation of metric $g_{ab}^{(0)}$. For this reason one can regard (10) and (11) as the solutions carrying full information about considered noncommutativity at h^2 .

4 Palatini action

Now, let us switch to the Palatini formalism with the connection and the tetrad field as separate dynamical variables. Thus, one is dealing with the vector bundle \mathcal{L} for which

⁹Arguments leading to (9b) is essentially identical to standard calculations concerning small perturbations of classical vacuum relativity, e.g. in *shortwave formalism* ([29] §35.13).

$SO(3,1)$ transformations preserve the canonical form of the Lorentzian metric η_{AB} . The bundle \mathcal{L} is equipped with some metric-compatible connection $\partial^{\mathcal{L}}$. Its local coefficients are denoted as $\underline{\Gamma}^{AB}_i$ and are antisymmetric in AB . The corresponding curvature is given by \underline{R}^A_{Bij} . The bundle \mathcal{E} is taken to be $\mathcal{L} \otimes T\mathcal{M}$. The tetrad field $\theta^A_{b'}$ induces the metric $g_{a'b'} = \theta^A_{a'} \eta_{AB} \theta^B_{b'}$ and the metric connection ∇ in $T\mathcal{M}$ (not necessarily torsionless). Local coefficients of ∇ can be computed from the expression $\Gamma^{i'}_{j'k} = \theta_A^{i'} \underline{\Gamma}^A_{Bk} \theta^B_{j'} + \theta_A^{i'} \frac{\partial}{\partial x^k} \theta^A_{j'}$. The curvature tensors are related by $R^{i'}_{j'kl} = \theta_A^{i'} \underline{R}^A_{Bkl} \theta^B_{j'}$. As a connection in \mathcal{E} we choose $\partial^{\mathcal{E}} = \partial^{\mathcal{L}} \otimes 1 + 1 \otimes \nabla$. These data encode the $*$ -product $*_P$.

We are going to make use of the following two endomorphisms of \mathcal{E} : $\underline{R}^A_B{}^{a'}_{b'}$ (defined by the curvature of $\partial^{\mathcal{L}}$ and the tetrad which raises index a'), and $\Theta^A_B{}^{a'}_{b'} = \theta^{Aa'} \theta_{Bb'}$. As a starting point one may take the following version of Palatini action¹⁰

$$\mathcal{S}_P = \int_{\mathcal{M}} \text{Tr} \check{\underline{R}} \Theta \frac{\omega^n}{n!}$$

The deformation procedure yields particularly simple expression due to $\partial_i \Theta = 0$, $\text{Tr}(R_{ab}^{\mathcal{E}} \{ \underline{R}, \Theta \}) = 0$ and $\text{Tr}(R_{ab}^{\mathcal{E}} R_{cd}^{\mathcal{E}} \{ \underline{R}, \Theta \}) = 0$.

$$\widehat{\mathcal{S}}_P = \text{tr}_{*P}(\check{\underline{R}} * \Theta) = \int_{\mathcal{M}} \text{Tr} \left(\underline{R} \Theta + h^2 s_2 \check{\underline{R}} \Theta + O(h^3) \right) \text{vol}_M$$

The variation with respect to $\delta\theta$ leads to the equations

$$(1 + h^2 s_2) \left(R_{ab} - \frac{1}{2} g_{ab} R \right) + O(h^3) = 0 \quad (12a)$$

clearly equivalent (up to h^2) to the condition $R_{ab} = 0$. The variation of the connection field $\delta \underline{\Gamma}$ produces¹¹

$$(1 + h^2 s_2) Q^a_{bc} = \frac{h^2}{n-1} \delta^a_{[b} \frac{\partial s_2}{\partial x^c]} + O(h^3) \quad (12b)$$

where $Q^a_{bc} = \Gamma^a_{cb} - \Gamma^a_{bc}$ is the torsion tensor of the connection ∇ . Thus, one obtains the theory with vanishing Ricci tensor and nonvanishing torsion generated by the scalar s_2 . A quick calculation shows that the trace-free part of Q^c_{ab} is equal to zero. Equation (12b) means that for

$$Q^a_{bc} = Q^{(\theta)a}_{bc} + h Q^{(1)a}_{bc} + h^2 Q^{(2)a}_{bc} + \dots$$

¹⁰The function v modifying \underline{R} is taken with respect to the metric g_{ab} induced by the tetrad. Obviously vol_M and the volume form given by the determinant of θ coincide in such case.

¹¹The variation gives

$$w(\Gamma^c_{ab} - \Gamma^c_{ba}) = \delta^c_a \left(\frac{\partial w}{\partial x^b} - \Gamma^d_{bd} w \right) - \delta^c_b \left(\frac{\partial w}{\partial x^a} - \Gamma^d_{ad} w \right)$$

with the tensor density $w = \sqrt{-g}(1 + h^2 s_2)$. Contraction of this relation enables expressing Γ^d_{ad} in terms of Γ^d_{da} , leading in turn to (12b).

one has $Q^a_{bc} = Q^a_{bc} = 0$ and

$$Q^a_{bc} = \frac{1}{n-1} \delta^a_{[b} \frac{\partial s_2}{\partial x^c]}.$$

Connection coefficients for ∇ are given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ak} \left(\frac{\partial g_{bk}}{\partial x^c} + \frac{\partial g_{ck}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^k} + Q_{bkc} + Q_{ckb} - Q_{kbc} \right).$$

Hence, Γ^a_{bc} and Γ^a_{bc} are still expressed by relations (7a) and (7b). For Γ^a_{bc} one computes that

$$\Gamma^a_{bc} = \frac{1}{2} g^{ak} \left(\nabla_c g_{kb} + \nabla_b g_{kc} - \nabla_k g_{bc} \right) - g^{ak} g_{kl} \Gamma^l_{bc} + \frac{1}{2(n-1)} \left(\delta^a_c \frac{\partial s_2}{\partial x^b} - g_{bc} g^{ak} \frac{\partial s_2}{\partial x^k} \right). \quad (13a)$$

Corrections to Riemann tensor are again given by (8). Substituting them to $R_{ab} = 0$ we obtain that for h^0 and h^1 relations (9a) and (9b) remain valid. However, equations for g_{ab} take the following form

$$\begin{aligned} g^{kl} \left(\nabla_k \nabla_a g_{bl} + \nabla_k \nabla_b g_{al} - \nabla_k \nabla_l g_{ab} - \nabla_a \nabla_b g_{kl} \right) = \\ = 2 \nabla_a \nabla_b s_2 + \frac{1}{n-1} g_{ab} g^{kl} \nabla_k \nabla_l s_2 - 4 \Gamma^k_{l[k} \Gamma^l_{b]a} + 4 g^{rk} \nabla_{[r} \left(\Gamma^l_{b]a} g_{kl} \right). \end{aligned} \quad (13b)$$

Like in the case of Einstein-Hilbert action, one can easily guess special solution of (13b) by requiring that $g^{(1)}_{ab} = 0$, i.e. that classical first order perturbation vanish. It reads

$$g_{ab}^{(2)} = -\frac{1}{n-1} s_2 g_{ab}^{(0)} \quad (14)$$

For such case the correction $\Gamma^a_{bc}^{(1)}$ is equal to zero, and $\Gamma^a_{bc}^{(2)}$ is given by

$$\Gamma^a_{bc}^{(2)} = -\frac{1}{2(n-1)} \delta^a_b \frac{\partial s_2}{\partial x^c}. \quad (15)$$

Repeating arguments of the previous section, one can observe that arbitrary other solution of (13b), with $g^{(1)}_{ab} = 0$, differs from (14) by a classical perturbation of $g_{ab}^{(0)}$.

5 Relation to the theory of Seiberg-Witten map

Let us explain how proposed models can be understood in terms of the theory of Seiberg-Witten map. This becomes quite straightforward when one combines results of [1] with the property (3). Indeed, what [1] states is that Seiberg-Witten map is an local isomorphism of $*$ -product algebras, while the relation (3) says that the trace functional is invariant on such isomorphisms.

More precisely, suppose that one prescribes to each frame e in \mathcal{E} $*$ -product isomorphism $M_{\langle e \rangle}$ which transforms the initial global product $*$ to the local one $*_S$. (Recall that $*_S$ is

nothing but matrix multiplication with commutative product of entries replaced by Fedosov $*$ -product of functions). Thus

$$M_{\langle e \rangle}(F_{\langle e \rangle} * G_{\langle e \rangle}) = M_{\langle e \rangle}(F_{\langle e \rangle}) *_S M_{\langle e \rangle}(G_{\langle e \rangle})$$

where $F_{\langle e \rangle}, G_{\langle e \rangle}$ are matrices representing endomorphisms F and G in the frame e . It turns out ([1] section 3.1) that if we switch to different frame $\tilde{e} = eg^{-1}$ then $M_{\langle e \rangle}$ and $M_{\langle \tilde{e} \rangle}$ are related by

$$M_{\langle \tilde{e} \rangle}(F_{\langle \tilde{e} \rangle}) = \hat{g}_{\langle e \rangle}(g, \Gamma^{\mathcal{E}}) *_S M_{\langle e \rangle}(F_{\langle e \rangle}) *_S \hat{g}_{\langle e \rangle}^{-1}(g, \Gamma^{\mathcal{E}}) \quad (16)$$

with $\hat{g}_{\langle e \rangle}(g, \Gamma^{\mathcal{E}}) = g + O(h)$ dependent both on g and connection one-forms $\Gamma_i^{\mathcal{E}}$ in the frame e and their derivatives. Moreover, if we combine two gauge transformations, then \hat{g} fulfills “consistency conditions” (compare [27, 28]) given by

$$\hat{g}_{\langle e \rangle}(g'g, \Gamma^{\mathcal{E}}) = \hat{g}_{\langle \tilde{e} \rangle}(g', g\Gamma^{\mathcal{E}}g^{-1} + gdg^{-1}) *_S \hat{g}_{\langle e \rangle}(g, \Gamma^{\mathcal{E}}). \quad (17)$$

Thus, M and \hat{g} behave exactly like Seiberg-Witten map [25]. Indeed, if M is set up with Fedosov’s techniques of generating $*$ -product isomorphisms, then one can compute¹² M and \hat{g} , and for the case of $*_S$ given by Moyal product $*_T$ obtain results which are well-known expressions for Seiberg-Witten map ([1] section 4).

We are going to rewrite investigated actions in terms of Seiberg-Witten map. Let $M(F) = \hat{F}$, as it is justified by relations (17) and (16). Also, let us separately distinguish Moyal case of $*_S = *_T$, for which ∂^S is flat, one works in Darboux coordinates with coefficients of ∂^S equal to zero, and the trace functional tr_{*_T} is given by the integral (4). Due to property (3), actions considered in this paper can be locally (for endomorphisms with support small enough to be covered by a single frame in \mathcal{E} , and – in Moyal case – by a single Darboux coordinates) rewritten as follows.

	arbitrary $*_S$	$*_S = *_T$
$\hat{\mathcal{S}}_{EH_{1A}} =$	$\text{tr}_{*_S}(\hat{\underline{R}})$	$\int_{\mathbb{R}^{2n}} \text{Tr}(\hat{\underline{R}}) d^{2n}x$
$\hat{\mathcal{S}}_{EH_{1B}} =$	$\text{tr}_{*_S}(\hat{\underline{R}} *_S \hat{V})$	$\int_{\mathbb{R}^{2n}} \text{Tr}(\hat{\underline{R}} *_T \hat{V}) d^{2n}x$
$\hat{\mathcal{S}}_{EH_{2A}} =$	$\text{tr}_{*_S}(\hat{\underline{\underline{R}}})$	$\int_{\mathbb{R}^{2n}} \text{Tr}(\hat{\underline{\underline{R}}}) d^{2n}x$
$\hat{\mathcal{S}}_{EH_{2B}} =$	$\text{tr}_{*_S}(\hat{\underline{\underline{R}}} *_S \hat{V})$	$\int_{\mathbb{R}^{2n}} \text{Tr}(\hat{\underline{\underline{R}}} *_T \hat{V}) d^{2n}x$
$\hat{\mathcal{S}}_P =$	$\text{tr}_{*_S}(\hat{\underline{R}} *_S \hat{\Theta})$	$\int_{\mathbb{R}^{2n}} \text{Tr}(\hat{\underline{R}} *_T \hat{\Theta}) d^{2n}x$

¹²Fedosov construction enables *computation* of Seiberg-Witten map, up to arbitrary order in h , by its recursive techniques. This situation is rather different from the usual framework, where the Seiberg-Witten equations must be *solved*.

Let us observe that such setting clarifies how considered models are related to the spacetime noncommutativity described by $*_S$. Indeed, due to (2), above mentioned local versions of action functionals are invariant with respect to gauge transformations (16) realized by means of $*_S$. Thus, one is able to reasonably claim that models considered in this paper correspond to noncommutativity of spacetime generated by Fedosov product of functions $*_S$.

6 Discussion

We have obtained number of nonequivalent geometric deformations of vacuum Einstein relativity. They have been analyzed at \hbar^2 order, starting from the action functional, through field equations, up to corrections to the metric which have been explicitly given for the case of $\widehat{\mathcal{S}}_{EH_{1A}}$, $\widehat{\mathcal{S}}_{EH_{1B}}$ and $\widehat{\mathcal{S}}_P$. Using results of [1], we have pointed out the relation between proposed models, the theory of Seiberg-Witten map, and the noncommutativity of the spacetime described by Fedosov $*$ -product generated by symplectic form ω and symplectic connection ∂^S .

The multiplicity of models arises as a consequence of ambiguity in translating traditional action functionals to the language of traces of endomorphisms of some bundle. From the gauge simplicity point of view, actions $\widehat{\mathcal{S}}_{EH_{1A}}$ and $\widehat{\mathcal{S}}_{EH_{1B}}$ seem to be most straightforward as they correspond to the natural $GL(2n, \mathbb{R})$ gauging. On the other hand, action $\widehat{\mathcal{S}}_P$ produces especially simple expressions for deformed field equations and for corrections to the metric.

The fundamental problem related to presented models concerns incompatibility of the volume forms – metric and symplectic ones. Both proposed solutions (rescaling one of the endomorphisms and multiplication by V) seem to be a bit unnatural. One can suspect that this problem is related to fixing symplectic structure as a nondynamical background. (Notice however, that Fedosov construction provides natural framework for the variation of the symplectic data). Moreover, it could turn out that some refinements to the Fedosov theory should be made, to put the metric into the internal structure of the deformation quantization procedure. Such considerations are hoped to be covered in author's subsequent work.

Due to the symmetries of the Riemann tensor, in all considered cases imaginary terms at \hbar^1 have vanished. It must be however stressed, that we have no clear evidence that the same stays true for other odd powers of \hbar . Thus, some further analysis of the reality of proposed actions should be performed. This suggests deeper investigation of the structure of the trace functional, which seems to be rather hard task (but not hopeless, as it can be inferred from Fedosov's results [30, 31] on relating $\text{tr}_*(1)$ to integrals of characteristic classes of \mathcal{TM} and $\text{End}(\mathcal{E})$). On the other hand, construction of some appropriate involution operator in

the Fedosov algebra may be useful¹³ and it is also matter of author's further interest.

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References

- [1] M. Dobrski *Seiberg-Witten equations from Fedosov deformation quantization of endomorphism bundle*, arXiv:0904.4409
- [2] X. Calmet, A. Kobakhidze *Noncommutative general relativity*, Phys. Rev. D 72 045010 (2005), hep-th/0506157
- [3] X. Calmet, A. Kobakhidze *Second order noncommutative corrections to gravity*, Phys. Rev. D 74 047702 (2006), hep-th/0605275
- [4] P. Mukherjee, A. Saha *Note on the noncommutative correction to gravity*, Phys. Rev. D 74 027702 (2006), hep-th/0605287
- [5] A.H. Chamseddine *Deforming Einstein's Gravity*, Phys. Lett. B 504, 33 (2001), hep-th/0009153
- [6] A.H. Chamseddine *An invariant action for noncommutative gravity in four-dimensions*, J.Math.Phys. 44, 2534 (2003), hep-th/0202137
- [7] M. A. Cardella, D. Zanon *Noncommutative deformation of four dimensional Einstein gravity*, Class. Quant. Grav. 20, L95-L104 (2003), hep-th/0212071
- [8] M. Chaichian, A. Tureanua, G. Zet *Corrections to Schwarzschild solution in noncommutative gauge theory of gravity* Phys. Lett. B 660, 573 (2008), arXiv:0710.2075
- [9] A.H. Chamseddine *SL(2,C) gravity with a complex vierbein and its noncommutative extension*, Phys. Rev. D 69, 024015 (2004), hep-th/0309166
- [10] H. Garcia-Compean, O. Obregon, C. Ramirez, M. Sabido *Noncommutative self-dual gravity*, Phys. Rev. D 68, 044015 (2003), hep-th/0302180
- [11] S. Marculescu, F. Ruiz Ruiz *Seiberg-Witten maps for SO(1,3) gauge invariance and deformations of gravity*, Phys. Rev. D 79, 025004 (2009), arXiv:0808.2066
- [12] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, J. Wess *A gravity theory on noncommutative spaces* Class. Quant. Grav. 22 (2005) 3511, hep-th/0504183

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- [13] P. Aschieri, M. Dimitrijević, F. Meyer, J. Wess *Noncommutative geometry and gravity*, Class. Quant. Grav. 23 (2006) 1883-1912, [hep-th/0510059](#)
- [14] F. Meyer, *Noncommutative spaces and gravity*, [hep-th/0510188](#)
- [15] H. Steinacker *Emergent gravity from noncommutative gauge theory*, JHEP 12 (2007) 049, [arXiv:0708.2426](#)
- [16] H. Steinacker *Emergent gravity and noncommutative branes from Yang-Mills matrix models*, Nucl. Phys. B 810, 1 (2009), [arXiv:0806.2032](#)
- [17] B. V. Fedosov, *Deformation quantization and index theory*, Akademie Verlag, Berlin 1996
- [18] C. Emmrich, A. Weinstein *The differential geometry of Fedosov's quantization in Lie Theory and Geometry. In Honor of Bertram Kostant*, Birkhäuser 1994, [hep-th/9311094](#)
- [19] D. R. Farkas *A ring-theorist's description of Fedosov quantization*, Lett. Math. Phys. 51, 161 (2000), [math/0004071](#)
- [20] M. Gadella, M. A. del Olmo, J. Tosiek *Geometrical origin of the *-product in the Fedosov formalism*, J. Geom. Phys. 55, 316 (2005), [hep-th/0405157](#)
- [21] M. Gadella, M. A. del Olmo, J. Tosiek *Quantization on a 2-dimensional phase space with a constant curvature tensor*, Annals Phys. 307 (2003) 272-307, [hep-ph/0306117](#)
- [22] J. Tosiek *The Fedosov deformation quantization for some induced symplectic connection*, [arXiv:0907.4911](#)
- [23] I. Gelfand, V. Retakh, M. Shubin *Fedosov Manifolds*, Adv. Math. 136, 104 (1998), [dg-ga/9707024](#)
- [24] P. Bieliavsky, M. Cahen, S. Gutt, J. Rawnsley, L. Schwachhöfer *Symplectic connections*, Int. J. Geom. Meth. Mod. Phys. 3, 375 (2006), [math/0511194](#)
- [25] N. Seiberg, E. Witten *String theory and noncommutative geometry*, JHEP 09 (1999) 032, [hep-th/9908142](#)
- [26] <http://www.exact.es>
- [27] B. Jurčo, L. Möller S. Schraml, P. Schupp, J. Wess *Construction of non-abelian gauge theories on noncommutative spaces*, Eur. Phys. J. C 21, 383 (2001),
- [28] P. Schupp *Noncommutative field theory*, Fortschr. Phys. 54, No. 2 - 3, 165 - 174 (2006)
- [29] C. Misner, K. Thorne, J. A. Wheeler *Gravitation*, W.H. Freeman and Company, San Francisco 1973
- [30] B.V. Fedosov *On the trace density in deformation quantization in Deformation quantization: proceedings of the meeting of theoretical physicists and mathematicians, Strasbourg, May 31-June 2, 2001*, de Gruyter, Berlin 2002, MPIM2001-75

- [31] B. V. Fedosov *The Atiyah-Bott-Patodi Method in Deformation Quantization*, Comm. Math. Phys. 209, 691 (2000)