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A quantum isomonodromy equation and its application to $\mathcal{N} = 2$ $SU(N)$ gauge theories

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Abstract

We give an explicit differential equation which is expected to determine the instanton partition function in the presence of the full surface operator in $\mathcal{N} = 2$ $SU(N)$ gauge theory. The differential equation arises as a quantization of a certain Hamiltonian system of isomonodromy type discovered by Fuji, Suzuki and Tsuda.

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1 Introduction

In [1], Alday and Tachikawa formulated a combinatorial formula[1] for the instanton partition function Z_{inst} in the presence of the full surface operator in $\mathcal{N} = 2$ $SU(N)$ gauge theory, based on a result [2] about the affine Laumon space. Furthermore, they observed an interesting relation between Z_{inst} for $SU(2)$ theory and KZ equation with affine $SL(2)$ symmetry as an extension of the AGT relation [6]. A similar relation to Virasoro CFT including the irregular singularities was then examined in [3][4]. For $SU(N)$ case, the relation to affine $SL(N)$ conformal blocks was studied in [5].

[1] We will recapitulate it in section 2, eq.(7).

In this note, we will study the differential equation satisfied by $Z_{\rm inst}$ from a point of view slightly different from KZ equations or W_N-algebras. Our basic strategy is to use the isomonodromy equations. It is known that some aspects of AGT relation have a natural interpretation [4][7][8] through the isomonodromy systems (or Painlevé equations for $SL(2)$ cases). That is, the $2d$ CFT's (=quantum isomonodromy systems) can be viewed as non-autonomous and quantum deformation of the Hitchin systems [9][10] (=Seiberg-Witten theory) arising through the Ω-deformation [11].

At the classical level, the relation between the Seiberg-Witten theory and isomonodromy equation is directly recognized by looking at the curves. For instance, the $SU(2)$ $N_f = 4$ Seiberg-Witten curve in Gaiotto form [12]

$$x^2 = (\frac{\mu_1}{z} + \frac{\mu_2}{z-1} + \frac{\mu_3}{z-t})^2 - \frac{\kappa z + u}{z(z-1)(z-t)}, \quad \kappa = \mu_0^2 - (\sum_{i=1}^{3} \mu_i)^2, \tag{1}$$

coincide with the Hamiltonian of the sixth Painlevé equation

$$H_{\rm VI} = q(q-1)(q-t)p^2 - 2\{\mu_1(q-1)(q-1) + \mu_2 q(q-t) + \mu_3 q(q-1)\}p - \kappa q, \tag{2}$$

by the following variable change

$$p = x + \frac{\mu_1}{z} + \frac{\mu_2}{z-1} + \frac{\mu_3}{z-t}, \quad q = z, \quad H_{\rm VI} = u. \tag{3}$$

Similar relations for degenerate cases were considered in [13].

We want to generalize this kind of correspondence to $SU(N)$ $N_f = 2N$ cases at the quantum level. The first problem is to look for the suitable isomonodromy system with higher rank symmetries. Fortunately, a nice candidate appeared in recent work [15][17]. The system, which we call Fuji-Suzuki-Tsuda (FST) equation[2], can be described as an isomonodromy defor-

[2]This equation (called $P_{\rm VI}$-chain in [15]) was first considered by Tsuda in 2008, as a similarity reduction of his 'UC-hierarchy' (certain generalization of KP hierarchy). Independently, in the context of the Drinfeld-Sokolov hierarchy, it was obtained by Fuji-Suzuki [14] in case of $N = 3$ and generalized in [17]. Though our description in section 3 will follow the notation of [17], the isomonodromic picture considered here is naturally understood from the UC-hierarchy. Tsuda's construction contains also the Garnier type extension with spectral type $(N-1,1), \cdots, (N-1,1), (1^N), (1^N)$ (see [16]).

mation of the $N \times N$ Fuchsian connection on $\mathbb{P}^1$ with regular singularity at $z = 0, 1, t, \infty$

$$D = \partial_z - \mathcal{A}, \quad \mathcal{A} = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}, \tag{4}$$

with the following spectral type (= eigenvalue multiplicity of the residue matrices): A_0 and $A_\infty = -A_0 - A_1 - A_t$ are of type (1^N) while A_1 and A_t are of type $(N-1, 1)$ [3]. We will recall the explicit form of the isomonodromy equation in section 3. Here, we look at the classical spectral curve in order to see its relation to the gauge theory.[4] The curve for eq.(4) is given by

$$\begin{aligned} \det(v - z\mathcal{A}) &\propto \det\Big((z-1)(z-t)(v - A_0) - z(z-t)A_1 - z(z-1)A_t\Big) \\ &= \{(z-1)(z-t)\}^{N-1} f(z, v) = 0, \end{aligned} \tag{5}$$

where $v = z\partial_z$ is considered as a commuting variable. The factorization in the second line is due to the fact that A_1, A_t are of rank one. Hence $f(z, v)$ is of bi-degree $(2, N)$ in variables (z, v) and has the form

$$f(z, v) = t\prod_{i=1}^{N}(v - m_i) + z\Big\{\sum_{i=0}^{N-1} u_i v^i - (1+t)v^N\Big\} + z^2\prod_{i=1}^{N}(v - \tilde{m}_i), \tag{6}$$

where $A_0 \sim \mathrm{diag}(m_1, \cdots, m_N)$, $A_\infty \sim -\mathrm{diag}(\tilde{m}_1, \cdots, \tilde{m}_N)$. This is the desired form as the Seiberg-Witten curve for $SU(N)$ with $N_f = 2N$.

In the next section, we formulate our main conjecture that a differential equation (12) determines the instanton partition function Z_{inst}. The isomonodromic origin of our equation (12) is discussed in section 3.

2 Main conjecture

The instanton partition function $Z_{\mathrm{inst}} = Z_{\mathrm{inst}}(y; a, m, \tilde{m})$ in the presence of the full surface operator in $SU(N)$ $N_f = 2N$ superconformal gauge theory is

[3] Then, without loss of generality, one can assume that the eigenvalues of A_1 are $(0, \cdots, 0, c)$ and similar for A_t.

[4] Another indication comes from the special solutions. In both theories, generalized hypergeometric series ${}_N F_{N-1}$ appears [18][19], [20][21][22].

a function of N-variables $y = (y_1, \cdots, y_N)$ depending on $3N-1$ parameters $m = (m_1, \cdots, m_N)$, $\tilde{m} = (\tilde{m}_1, \cdots, \tilde{m}_N)$ and $a = (a_1, \cdots, a_N)$, $\sum_{i=1}^{N} a_i = 0$.

Let us recall the combinatorial formula for Z_{inst} following [1][5]

$$Z_{\text{inst}} = \sum_{\lambda} Z(\lambda) \prod_{i=1}^{N} y_i^{k_i(\lambda)}. \tag{7}$$

Here the sum is taken over all N-tuples $\lambda = (\lambda^1, \cdots, \lambda^N)$ of partitions $\lambda^i = (\lambda^i_1 \geq \lambda^i_2 \geq \lambda^i_3 \geq \cdots \geq \lambda^i_{\ell_i} > 0)$. The indices of $\lambda^i = (\lambda^i_j)$ will be extended to $\mathbb{Z}$ by $\lambda^i = \lambda^{i+N}$ and $\lambda^i_j = 0$ ($j \leq 0$ or $j > \ell_i$). We put $|\lambda| = \sum_{i=1}^{N} |\lambda^i| = \sum_{i=1}^{N} \sum_{j \geq 1} \lambda^i_j$. The exponents $k_i(\lambda)$ are given by

$$k_i(\lambda) = \sum_{j \geq 1} \lambda^{i-j+1}_j. \tag{8}$$

The coefficients $Z(\lambda)$ are defined as

$$Z(\lambda) = \frac{n_f(a, \lambda, m) n_{\tilde{f}}(a, \lambda, \tilde{m})}{n_v(a, \lambda)}, \tag{9}$$

where

$$\begin{aligned} n_f(a, \lambda, m) &= n_{\text{bif}}(m, a, (\phi)^N, \lambda, 0), \\ n_{\tilde{f}}(a, \lambda, \tilde{m}) &= n_{\text{bif}}(a, \tilde{m}, \lambda, (\phi)^N, 0), \\ n_v(a, \lambda) &= n_{\text{bif}}(a, a, \lambda, \lambda, 0), \\ n_{\text{bif}}(a, b, \lambda, \mu, x) &= \prod_{t=1}^{|\lambda|+|\mu|} (w_t - x), \end{aligned} \tag{10}$$

and finally, the weights $w_t = w_t(a, b, \lambda, \mu)$[5] are determined by the formula [2]

[5] $a, b \in \mathbb{C}^N$, and λ, μ are N-tuples of partitions. The constraints $\sum_{i=1}^{N} a_i = 0$, $\sum_{i=1}^{N} b_i = 0$ will be considered after the substitutions (10).

$$
\begin{aligned}
\chi(a,b,\lambda,\mu) &= \sum_{t=1}^{|\lambda|+|\mu|} e^{w_t} \\
&= \sum_{k=1}^{N}\sum_{l'\geq 1} e^{a_k - b_{k-l'} + \epsilon_2(\lfloor \frac{l'-k}{N}\rfloor - \lfloor \frac{-k}{N}\rfloor)} \sum_{s=1}^{\mu_{l'}^{k-l'}} e^{\epsilon_1 s} \\
&- \sum_{k=1}^{N}\sum_{l\geq 1}\sum_{l'\geq 1} e^{a_{k-l+1} - b_{k-l'} + \epsilon_2(\lfloor \frac{l'-k}{N}\rfloor - \lfloor \frac{l-k-1}{N}\rfloor)} (e^{\epsilon_1 \mu_{l'}^{k-l'}} - 1) \sum_{s=1}^{\lambda_l^{k-l+1}} e^{\epsilon_1 (s - \lambda_l^{k-l+1})} \\
&+ \sum_{k=1}^{N}\sum_{l\geq 1}\sum_{l'\geq 1} e^{a_{k-l+1} - b_{k-l'+1} + \epsilon_2(\lfloor \frac{l'-k-1}{N}\rfloor - \lfloor \frac{l-k-1}{N}\rfloor)} (e^{\epsilon_1 \mu_{l'}^{k-l'+1}} - 1) \sum_{s=1}^{\lambda_l^{k-l+1}} e^{\epsilon_1 (s - \lambda_l^{k-l+1})} \\
&+ \sum_{k=1}^{N}\sum_{l\geq 1} e^{a_{k-l+1} - b_k + \epsilon_2(\lfloor \frac{-k}{N}\rfloor - \lfloor \frac{l-k-1}{N}\rfloor)} \sum_{s=1}^{\lambda_l^{k-l+1}} e^{\epsilon_1 (s - \lambda_l^{k-l+1})},
\end{aligned}
\tag{11}
$$

where $\lfloor x \rfloor$ is the largest integer such that $\lfloor x \rfloor \leq x$. The formula (11) is applicable to periodic parameters $a_{i+N} = a_i$, $b_{i+N} = b_i$. However, another convention is possible, where all terms $\epsilon_2(\lfloor \cdots \rfloor - \lfloor \cdots \rfloor)$ in the exponentials in eq.(11) are taken away in exchange for assuming quasi-periodicity $a_{i+N} = a_i + \epsilon_2$, $b_{i+N} = b_i + \epsilon_2$. In what follows, we will adopt the second option.

As compared with the above complicated formulae for Z_{inst}, the differential equation we propose is rather simple and defined as

$$
\begin{aligned}
\mathcal{D}\mathcal{Z}(y) = \Big\{ &(\Delta_N + \sum_{i=1}^{N} u_i \vartheta_i) + (\prod_{j=1}^{N} y_j)(\Delta_N + \sum_{i=1}^{N} v_i \vartheta_i + \sum_{i=1}^{N} r_i s_i) \\
&+ \sum_{i=1}^{N} (y_i + y_i y_{i+1} + \cdots + \prod_{j=0}^{N-2} y_{i+j})(\vartheta_{i-1} - \vartheta_i + r_i)(\vartheta_{i-1} - \vartheta_i + s_i) \Big\} \mathcal{Z} = 0,
\end{aligned}
\tag{12}
$$

where $y_{i+N} = y_i$, $\vartheta_i = y_i \dfrac{\partial}{\partial y_i}$, $\Delta_N = \dfrac{1}{2}\sum_{i=1}^{N} (\vartheta_i - \vartheta_{i+1})^2$. The parameters

u_i, v_i, r_i, s_i $(1 \le i \le N)$ will be set as

$$\begin{aligned} u_i &= \frac{a_{i+1} - a_i}{\epsilon_1}, \quad v_i = \frac{a_{i+1} - a_i + m_{i+1} - m_{i+2} + \tilde{m}_i - \tilde{m}_{i+1}}{\epsilon_1}, \\ r_i &= \frac{a_i - m_{i+1} - \epsilon_1}{\epsilon_1}, \quad s_i = \frac{a_i - \tilde{m}_i}{\epsilon_1}, \end{aligned} \tag{13}$$

where $x_{i+N} = x_i + \epsilon_2$ for $x = a, m, \tilde{m}$ (while u_i, v_i, r_i, s_i are periodic).

The main claim in this note is the following

Conjecture. *The instanton partition function Z_{inst} is characterized as the unique formal power series solution of the form $\mathcal{Z} = 1 + \mathcal{O}(y)$ for the differential equation (12) with parameters (13).*

We have checked this conjecture for $N \le 5$ up to total degree 5 in y-variables (in some cases by specializing the parameters to numerical values).

Under a degeneration limit $y_i \to \varepsilon^2 y_i$, $m_i(\tilde{m}_i) \to \varepsilon^{-1}\Lambda$, $(\varepsilon \to 0)$, the differential equation (12) reduces to the Toda equation

$$\Big(\Delta_N + \sum_{i=1}^{N} u_i \vartheta_i + \frac{\Lambda^2}{\epsilon_1^2} \sum_{i=1}^{N} y_i\Big) \mathcal{Z} = 0, \tag{14}$$

whose relation to $\mathcal{N} = 2$ $SU(N)$ pure gauge theory has already been established by Braverman-Etingof [24] (see also [25][26][27]).

3 Origin of the differential equation

In this section, we explain an isomonodromic origin of our differential equation (12). As already mentioned in the introduction, the equation (12) is a quantization of the Fuji-Suzuki-Tsuda (FST) equation [14][15][17]. The FST equation can be written as a Hamiltonian system for $2(N-1)$ variables $q = (q_1, \cdots, q_{N-1})$ and $p = (p_1 \cdots, p_{N-1})$

$$t(t-1)\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad t(t-1)\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{15}$$

with parameters η and $\alpha = (\alpha_0, \cdots, \alpha_{2N-1})$, $\sum_{j=0}^{2N-1} \alpha_j = 1$. The Hamiltonian $H = H(q, p, t; \eta, \alpha)$ is given by[6]

$$\begin{aligned} H = & \sum_{i=1}^{N-1} H_{\mathrm{VI}}(q_i, p_i; a_i, b_i, c_i, d_i) \\ & + \sum_{1 \le i < j \le N-1} (q_i - 1)(q_j - t)\{(q_i p_i + \alpha_{2i-1})p_j + p_i(q_j p_j + \alpha_{2j-1})\}, \end{aligned} \tag{16}$$

where H_{VI} is the Hamiltonian of sixth Painlevé equation

$$H_{\mathrm{VI}}(q, p; a, b, c, d) = q(q-1)(q-t)p^2 - \{aq(q-1) + bq(q-t) + c(q-1)(q-t)\}p + dq, \tag{17}$$

and $a_i = \sum_{j=i}^{N-1} \alpha_{2j}$, $b_i = \sum_{j=0}^{i-1} \alpha_{2j}$, $c_i = \sum_{j=0}^{N-1} \alpha_{2j+1} - \alpha_{2i-1} - \eta$, $d_i = \eta \alpha_{2i-1}$.

A quantization of the system is given by the Schrödinger equation[7]

$$\left\{ t(t-1)\frac{\partial}{\partial t} - H(q_i, \frac{\partial}{\partial q_i}) \right\} \tilde{\Psi}(q_1, \cdots, q_{N-1}, t) = 0. \tag{18}$$

To connect this equation with Z_{inst}, we make a gauge transformation $\tilde{\Psi}(q, t) = y_1^{k_1} \cdots y_N^{k_N} \Psi(y)$, together with the variables change

$$\begin{aligned} & y_1 = q_1, \quad y_2 = \frac{q_2}{q_1}, \quad \cdots, \quad y_{N-1} = \frac{q_{N-1}}{q_{N-2}}, \quad y_N = \frac{t}{q_{N-1}}, \\ & q_i \frac{\partial}{\partial q_i} = \vartheta_i - \vartheta_{i+1} \quad (i = 1, \cdots, N-1), \quad t\frac{\partial}{\partial t} = \vartheta_N, \end{aligned} \tag{19}$$

($\vartheta_i = y_i \frac{\partial}{\partial y_i}$). Then the eq.(18) takes the form (in the following, we will

[6]This is a kind of coupled system of the Painlevé VI. Another equation of such type first found by Sasano [23] will also be important for some superconformal gauge theories.

[7]Here we will not consider the problem of operator ordering seriously since the ambiguities can be absorbed by shifts of parameters.

concentrate on $N=3$ case)

$$\begin{aligned}
&\mathcal{D}'\Psi(y) = \{\Delta_3 + u_1\vartheta_1 + u_2\vartheta_2 + u_3\vartheta_3 + M_0 \\
&+y_1(\vartheta_{31}+s_1)(\vartheta_{12}+r_1') + y_2(\vartheta_{12}+s_2)(\vartheta_{23}+r_2') + y_3(\vartheta_{23}+s_3)(\vartheta_{31}+r_3') \\
&+y_2y_3(\vartheta_{12}+s_2)(\vartheta_{31}+r_3') + y_3y_1(\vartheta_{23}+s_3)(\vartheta_{12}+r_1') + y_1y_2(\vartheta_{31}+s_1)(\vartheta_{23}+r_2') \\
&+y_1y_2y_3(\Delta_3 + v_1\vartheta_1 + v_2\vartheta_2 + v_3\vartheta_3 + M_1)\}\Psi(y) = 0,
\end{aligned} \tag{20}$$

where $\vartheta_{ij} = \vartheta_i - \vartheta_j$, and $r_i', s_i, u_i, v_i, M_0, M_1$ are some constants depending only on the parameters η, α and $k_1, \cdots, k_N$ (their precise expressions are not necessary). We can and will choose k_N so that $M_0 = 0$. Then, the equation (20) have a unique formal series solution of the form

$$\Psi(y) = \sum_{i,j,k=0}^{\infty} c_{ijk} y_1^i y_2^j y_3^k. \quad (c_{000} = 1) \tag{21}$$

The coefficients c_{ij0} are written in terms of the very-well-poised, balanced hypergeometric series

$$F(a_0; a_1, \cdots, a_5) = \sum_{k=0}^{\infty} \frac{(a_0+2k)}{a_0} \prod_{i=0}^{5} \frac{(a_i)_k}{(a_0+1-a_i)_k}, \tag{22}$$

as

$$c_{ij0} = \frac{(r_1'-j)_i (r_2')_j (-s_1)_i (-s_2)_j}{i!j!(u_1+1-j)_i (u_2+1)_j} F(u_1-j; -i, -j, u_1-r_1'+1, u_1-s_2, -u_2-j), \tag{23}$$

where $(x)_i = \Gamma(x+i)/\Gamma(x)$ is the Pochhammer symbol (see Appendix A for the proof). Similarly, c_{0ij} and c_{j0i} are given by cyclic shifts of parameters $x_i \to x_{i+1 \pmod 3}$ $(x = r, s, u)$.

On the other hand, corresponding coefficients of the instanton partition function

$$Z_{\text{inst}}(y) = \sum_{i,j,k=0}^{\infty} c_{ijk}^L y_1^i y_2^j y_3^k, \tag{24}$$

are obtained (at least for the first several terms) as

$$
\begin{aligned}
c^L_{ij0} &= \sum_{k=0}^{\min(i,j)} Z(\{i,k\},\{j-k\},\phi) \\
&= (-1)^{i+j} \frac{(\frac{\epsilon_1-a_1+m_2}{\epsilon_1})_i (\frac{\epsilon_1-a_2+m_3}{\epsilon_1})_j (\frac{-a_1+\tilde{m}_1}{\epsilon_1})_i (\frac{-a_2+\tilde{m}_2}{\epsilon_1})_j}{i!j!(\frac{\epsilon_1-\epsilon_1 j-a_1+a_2}{\epsilon_1})_i (\frac{\epsilon_1-a_2+a_3}{\epsilon_1})_j} \\
&\quad \times F(\tfrac{-\epsilon_1 j-a_1+a_2}{\epsilon_1}; -i, -j, \tfrac{-\epsilon_1 j+a_2-a_3}{\epsilon_1}, \tfrac{\epsilon_1-a_1+m_3}{\epsilon_1}, \tfrac{-a_1+\tilde{m}_2}{\epsilon_1}),
\end{aligned} \tag{25}
$$

together with similar formulas for c^L_{0ij}, c^L_{j0i} obtained by shifts $x_1 \to x_2 \to x_3 \to x_1 + \epsilon_2$ $(x = a, m, \tilde{m})$. Comparing the coefficients c_{ij0} and c^L_{ij0} etc., we find that they almost coincide if we put

$$
\begin{aligned}
&r'_1 = \frac{a_2 - m_3}{\epsilon_1}, \quad r'_2 = \frac{a_3 - (m_1+\epsilon_2)}{\epsilon_1}, \quad r'_3 = \frac{a_1 - m_2}{\epsilon_1}, \\
&s_1 = \frac{a_1 - \tilde{m}_1}{\epsilon_1}, \quad s_2 = \frac{a_2 - \tilde{m}_2}{\epsilon_1}, \quad s_3 = \frac{a_3 - \tilde{m}_3}{\epsilon_1}, \\
&u_1 = \frac{a_2 - a_1}{\epsilon_1}, \quad u_2 = \frac{a_3 - a_2}{\epsilon_1}, \quad u_3 = \frac{(a_1+\epsilon_2) - a_3}{\epsilon_1}.
\end{aligned} \tag{26}
$$

In fact, under this parameter identification, the ratio of the coefficients c_{ij0}/c^L_{ij0} etc. are simply given by

$$
\frac{c_{ijk}}{c^L_{ijk}} = (r'_1)_{i-j}(r'_2)_{j-k}(r'_3)_{k-i}. \tag{27}
$$

Moreover, this relation (27) is satisfied also for $(ijk) = (111), (211), (121), (112)$ and further (as far as we checked), by putting

$$
\begin{aligned}
&M_1 = (r'_3 - 1)s_1 + (r'_1 - 1)s_2 + (r'_2 - 1)s_3, \\
&v_1 = r'_1 - r'_3 - s_1 + s_2 - u_1, \quad v_2 = r'_2 - r'_1 - s_2 + s_3 - u_2, \\
&v_3 = r'_2 - r'_1 - s_3 + s_1 - u_3 - \tfrac{\epsilon_2}{\epsilon_1}.
\end{aligned} \tag{28}
$$

In order to remove the factor in eq.(27), we make a "gauge transformation" (in momentum space) defined by

$$
\mathcal{D}' \to \mathcal{D} = V^{-1}\mathcal{D}'V, \quad V = (r'_1)_{\vartheta_{12}}(r'_2)_{\vartheta_{23}}(r'_3)_{\vartheta_{31}}. \tag{29}
$$

Under this transformation, the Euler derivatives remain invariant $\vartheta_i \to V^{-1}\vartheta_i V = \vartheta_i$. On the other hand, by using the relation $\vartheta_i y_j = y_j(\vartheta_i + \delta_{ij})$, the multiplication operators y_i are transformed as

$$\begin{aligned} &y_1 \to V^{-1}y_1V = y_1\frac{(r_1')_{\vartheta_{12}}(r_2')_{\vartheta_{23}}(r_3')_{\vartheta_{31}}}{(r_1')_{\vartheta_{12}+1}(r_2')_{\vartheta_{23}}(r_3')_{\vartheta_{31}-1}} = y_1\frac{\vartheta_{31}+r_3'-1}{\vartheta_{12}+r_1'}, \\ &y_2 \to y_2\frac{\vartheta_{12}+r_1'-1}{\vartheta_{23}+r_2'}, \quad y_3 \to y_3\frac{\vartheta_{23}+r_2'-1}{\vartheta_{31}+r_3'}, \end{aligned} \tag{30}$$

and hence

$$y_1y_2 \to y_1\frac{\vartheta_{31}+r_3'-1}{\vartheta_{12}+r_1'}y_2\frac{\vartheta_{12}+r_1'-1}{\vartheta_{23}+r_2'} = y_1y_2\frac{\vartheta_{31}+r_3'-1}{\vartheta_{23}+r_2'}. \tag{31}$$

Then the differential operator $\mathcal{D}'$ in eq.(20) is transformed into

$$\begin{aligned} &\mathcal{D} = V^{-1}\mathcal{D}'V = \Delta_3 + u_1\vartheta_1 + u_2\vartheta_2 + u_3\vartheta_3 \\ &+y_1(\vartheta_{31}+s_1)(\vartheta_{31}+r_1) + y_2(\vartheta_{12}+s_2)(\vartheta_{12}+r_2) + y_3(\vartheta_{23}+s_3)(\vartheta_{23}+r_3) \\ &+y_2y_3(\vartheta_{12}+s_2)(\vartheta_{12}+r_2) + y_3y_1(\vartheta_{23}+s_3)(\vartheta_{23}+r_3) + y_1y_2(\vartheta_{31}+s_1)(\vartheta_{31}+r_1) \\ &+y_1y_2y_3(\Delta_3 + v_1\vartheta_1 + v_2\vartheta_2 + v_3\vartheta_3 + M_1), \end{aligned} \tag{32}$$

where $(r_1, r_2, r_3) = (r_3'-1, r_1'-1, r_2'-1)$. Thus we arrived at the eq.(12) for $N = 3$ case.

4 Summary and discussions

In this note, we formulated an explicit differential equation (12) which is expected to determine the instanton partition function Z_{inst} in the presence of the full surface operator in $\mathcal{N} = 2$, $SU(N)$ gauge theory with $N_f = 2N$. The differential equation is derived as a quantization of the FST equation of isomonodromy type.

In [1][5], it was claimed that the partition function Z_{inst} is the conformal block of the affine Lie algebra SL_N (with the insertion of the K-operators). It is known [28][29] that the KZ equation satisfied by the conformal blocks can be interpreted as quantization of a typical isomonodromy system, the

Schlesinger equation. Hence, it is quite natural to expect a direct relation between the formulation of [1][5] and the isomonodromy approach here. For instance, the specialization of the primary fields V_χ, $\chi = \kappa\Lambda_1, \kappa\Lambda_{N-1}$ in [5] for the simple punctures agrees with the choice of the spectral type $(N-1, 1)$. More precise relations between these two formulations, in particular the understanding of the mysterious K-operators, will be an important future problem.

Though we have considered the isomonodromy deformation of an operator of the form (4), it can also be formulated by a scalar differential operator

$$L = \partial_z^N + u_2 \partial_z^{N-2} + \cdots + u_N. \tag{33}$$

Then the relation to W_N-algebras is also naturally expected (see [30][31] and references therein).

For the present, our understanding of the relation between $4d$ gauge theory and isomonodromy equation is still extrinsic. In [32] it was noted that the linear action of the loop operators (monodromy of surface operators) on the chiral partition function is independent of the gauge coupling. This observation may be a key ingredient for more conceptual understanding of the isomonodromic nature of gauge theories and the AGT relation.

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Appendix A : Proof of eq.(23)

From the equation (20), the coefficients $c_{i,j,0}$ in (21) are determined by

$$\begin{aligned} &c_{i-1,j,0}(i-1-j+r_1)(i-1-s_1)+c_{i-1,j-1,0}(j-1+r_2)(i-1-s_1) \\ &-c_{i,j-1,0}(j-1+r_2)(1+i-j+s_2)-c_{i,j,0}\,(i^2-ij+j^2+iu_1+ju_2) = 0. \end{aligned} \tag{A.1}$$

Plugging (23) into this equation and rewriting the parameters as $a_0 = u_1 - j,\ a_1 = -i,\ a_2 = -j,\ a_3 = -u_2 - j,\ a_4 = u_1 - r_1' + 1,\ a_5 = u_1 - s_2$, the relation we should prove reduces to

$$\begin{aligned}&(a_0a_1-a_1^2-a_2a_3)F-(a_0-a_1)a_1F^{a_1}+\tfrac{(1+a_0)a_2a_3(1+a_0-a_1-a_4)(1+a_0-a_1-a_5)}{(1+a_0-a_1)(1+a_0-a_4)(1+a_0-a_5)}F^{a_0a_2}\\&+\tfrac{(1+a_0)a_1a_2a_3}{(1+a_0-a_4)(1+a_0-a_5)}F^{a_0a_1a_2a_3}=0,\end{aligned} \tag{A.2}$$

where $F = F(a_0; a_1, a_2, a_3, a_4, a_5)$ and $F^{a_0a_2} = F|_{a_0\to a_0+1, a_2\to a_2+1}$ etc. Expanding the series F in each term, (A.2) can be written as

$$\sum_{k=0}^{\infty} \varphi_k \omega_k = 0, \tag{A.3}$$

where $\varphi_k = \prod_{i=0}^{k} \frac{(a_i)_k}{(1+a_0-a_i)_k}$ and

$$\begin{aligned}\omega_k &= (a_0a_1-a_1^2-a_2a_3)(a_0+2k)-(a_0-a_1+k)(a_1+k)(a_0+2k)\\&+\tfrac{(1+a_0-a_1-a_4)(1+a_0-a_1-a_5)(a_0+k)(a_2+k)(a_3+k)(1+a_0+2k)}{(1+a_0-a_1+k)(1+a_0-a_4+k)(1+a_0-a_5+k)}\\&+\tfrac{(a_0+k)(a_1+k)(a_2+k)(a_3+k)(1+a_0+2k)}{(1+a_0-a_4+k)(1+a_0-a_5+k)}.\end{aligned} \tag{A.4}$$

Then the equation (A.3) follows from an identity

$$\varphi_k\omega_k = \varphi_{k+1}u_{k+1} - \varphi_k u_k, \quad u_k = k(k+a_0-a_2)(k+a_0-a_3), \tag{A.5}$$

since the infinite sum is terminating: $\varphi_k = 0$ for $k > \min(i, j)$. $\square$

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