Improved Nguyen-Vidick Heuristic Sieve Algorithm for Shortest Vector Problem *

Xiaoyun Wang[1,2], Mingjie Liu[1], Chengliang Tian[2] and Jingguo Bi[2]

[1] Institute of Advanced Study, Tsinghua University, Beijing 100084, China
xiaoyunwang@mail.tsinghua.edu.cn,liu-mj07@mails.tsinghua.edu.cn
[2] Key Laboratory of Cryptologic Technology and Information Security, Ministry of Education, Shandong University, Jinan 250100, China
{chengliangtian,jguobi}@mail.sdu.edu.cn

Abstract. In this paper, we present an improvement of the Nguyen-Vidick heuristic sieve algorithm for shortest vector problem in general lattices, which time complexity is $\mathbf{2^{0.3836n}}$ polynomial computations, and space complexity is $\mathbf{2^{0.2557n}}$. In the new algorithm, we introduce a new sieve technique with two-level instead of the previous one-level sieve, and complete the complexity estimation by calculating the irregular spherical cap covering.

keywords: lattice, shortest vector, sieve, heuristic, sphere covering

1 Introduction

The n-dimensional lattice Λ is generated by the basis $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\} \subset \mathbb{R}^m$ which consists n linearly independent vectors.

$$\Lambda = \mathcal{L}(\mathbf{B}) = \{\mathbf{Bz} = \sum_{i=1}^{n} z_i \mathbf{b}_i : \mathbf{z} \in \mathbb{Z}^n\}.$$

The minimum distance $\lambda_1(\Lambda)$ of a lattice Λ is the length of its shortest nonzero vector:

$$\lambda_1(\Lambda) = \min_{0 \neq \mathbf{x} \in \Lambda} \|\mathbf{x}\|.$$

Here $\|\mathbf{x}\|$ is the norm of the vector $\mathbf{x}$, and in this paper, $\|.\|$ represents the Euclidean norm. The Shortest Vector Problem (SVP) consists in finding a lattice point $\mathbf{v}$ with norm $\lambda_1(\Lambda)$. The SVP$_\gamma$ which is the γ-approximation to SVP implies to find a lattice point $\mathbf{v}$ with the length $\|\mathbf{v}\| \leq \gamma\lambda_1(\Lambda)$.

SVP is a classical mathematical problem in geometry of numbers [15, 25], and is also a NP-hard problem in computational complexity theory [4]. In the past thirty years, SVP is widely used in public-key cryptanalysis and lattice-based cryptography. On one hand, the fast algorithm for searching SVP or SVP$_\gamma$ [13, 33] is the fundamental tool in public-key cryptanalysis and lattice-based cryptanalysis [26]. The most successful polynomial algorithm to search a shorter vector with approximation $2^{(n-1)/2}$ is LLL basis reduction algorithm [20] which has denied completely most Knapsack encryptions [2, 21, 32] and resulted in various weak key attacks on RSA-like cryptosystems [7, 9]. For the other hand, many cryptographic functions corresponding to SVP variants are proposed to be acted as the trapdoor one-way

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functions so that various lattice-based cryptographic schemes are easy to be constructed [3, 6, 12, 31]. There are two popular cryptographic functions which are derived from SIS (small integer solution) problem and LWE (learning with errors) problem respectively, where SIS and LWE can be reduced to SVP variant–SIVP$_\gamma$.

Today, fast searching shortest vector has become the most important focus point both on the security guarantee and cryptanalysis in lattice-based cryptography. Because SVP is NP-hard, the exact algorithm to search for SVP is not expected to be polynomial time. So far, there are essentially two different types of algorithms for exact SVP: deterministic algorithms and randomized sieve algorithms.

The first deterministic algorithm for SVP is originated from the work of Pohst [36] and Kannan [19], which is named as deterministic enumeration algorithm. Its main idea is to enumerate all lattice vectors shorter than a fixed bound $A \geq \lambda_1(\Lambda)$, with the help of the Gram-Schmidt orthogonalization of the given lattice basis. Given an LLL-reduced basis as input, the algorithm of Fincke and Pohst [11] runs in time $2^{O(n^2)}$, while the worst-case complexity of Kannan's algorithm is $n^{\frac{n}{2e}+o(n)}$ [16]. More details can refer to the survey [1]. Among the enumeration algorithms, the Schnorr-Euchner enumeration strategy [34] is the most important one used in practice, whose running time is $2^{O(n^2)}$ polynomial-time operations where the basis is either LLL-reduced or BKZ-reduced. Recently, Gama, Nguyen and Regev propose a new technique called extreme pruning in enumeration algorithm to achieve exponential speedups both in theory and in practice [14]. All enumeration algorithms we mentioned above only require a polynomial data complexity.

A completely different deterministic algorithm for SVP is based on Voronoi cell computation which originally aimed to solve Closest Vector Problem (CVP) [35]. Recently, Micciancio and Voulgaris [24] proposed an improved algorithm which is applicable to most lattice problems, including SVP, CVP and SIVP (Shortest Independent Vectors Problem). The running time is $\widetilde{O}(2^{2n+o(n)})$ polynomial-time operations, where $f = \widetilde{O}(g)$, i.e. $f(n) \leq \log^c g(n) \cdot g(n)$ for some constant c and all sufficiently large n. This is so far the best known result in lattice computational complexity in the deterministic search setting.

Another type algorithm for exact SVP is the randomized sieve algorithm, which was first proposed in 2001 by Ajtai, Kumar and Sivakumar [5] (AKS sieve algorithm). The sieve reduces upper bound of the time to $2^{O(n)}$ at the cost of $2^{O(n)}$ space. Regev [30] got the first constant estimation with time $2^{16n+o(n)}$ and space $2^{8n+o(n)}$, and further decreased to time $2^{5.90n+o(n)}$ and space $2^{2.95n+o(n)}$ by Nguyen and Vidick [28]. Micciancio and Voulgaris utilized the bound estimation of sphere packing [17], and improved both the time and space complexity to $2^{3.40n+o(n)}$ and $2^{1.97n+o(n)}$ respectively [23], and further reduced to $2^{3.199n+o(n)}$ and space $2^{1.325n+o(n)}$ by combining with ListSieve technique. By implementing the birthday attack on the sieved shorter vectors in a small ball with the radius $3.01\lambda_1(\mathit{\Lambda})$, Pujol and Stehlé give a sieve algorithm to search SVP with the time complexity $2^{2.465n+o(n)}$ [29].

Besides the above algorithms, there is a more practical searching algorithm which is heuristic under a natural random assumption. Nguyen and Vidick [28] presented the first heuristic variant of AKS sieve [5] with time $2^{0.415n}$ and space $2^{0.2075n}$, which is so far the fastest randomized sieve algorithm. It is remarked that, Micciancio and Voulgaris [23] also described a heuristic ListSieve called Gauss Sieve, which performed fairly well in practice but the upper bound time complexity of this sieve is still unknown.

In this paper, we present an improved heuristic randomized algorithm which solves SVP with time $2^{0.3836n}$ and space $2^{0.2557n}$. The main idea of algorithm is to collect shorter vectors by two-level sieve. The estimation of the complexity is based on the computation of the irregular spherical cap covering which comes from the intersection of a spherical surface and two balls.

This paper is organized as follows: Section 2 gives some notations and preliminaries. The new algorithm is introduced in Section 3. We show the proof of the algorithm complexity in Section 4. Conclusions are given in Section 5.

2 Notations and Preliminaries

- $\omega(f(n))$ represents a function growing faster than $cf(n)$ for any $c > 0$.
- $\Theta(f(n))$ is a function as the same order as $f(n)$, when $n \to \infty$.
- Let $S^n = \{\mathbf{x} \in \mathbb{R}^n | \ \|\mathbf{x}\| = 1\}$ be the unit sphere in $\mathbb{R}^n$.
- $B_n(\mathbf{x}, r)$ denotes the n-dimensional ball centered at $\mathbf{x}$ with radius r, and is simplified as $B_n(r)$ when center is origin point.
- κ_n is the volume of the unit Euclidean n-dimensional ball.
- $B(\varphi, \mathbf{x}) = \{\mathbf{y} \mid \langle \mathbf{x}, \mathbf{y} \rangle > \cos\varphi, \mathbf{y} \in S^n, \varphi \in (0, \frac{\pi}{2})\}$ is the spherical cap with angle φ in S^n.
- $|A|$ represents its volume if A is a geometric body and its number of elements if A is a finite set.
- $C_n(\gamma R) = \{\mathbf{x} \in \mathbb{R}^n \mid \gamma R \le \|\mathbf{x}\| \le R\}$ is a spherical shell in the ball $B_n(R)$.

We note that $|S^n| = n\kappa_n$. it is well-known that

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \begin{cases} \frac{\pi^k}{k!}, & n = 2k \\ \frac{2^{2k+1}k!\pi^k}{(2k+1)!}, & n = 2k+1 \end{cases}$$

where $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$ is the gamma function.

Lemma 1. *[8] Let $\varphi \in (0, \frac{\pi}{2})$, and $\mathbf{x} \in S^n$, if $\varphi \le \arccos\frac{1}{\sqrt{n}}$, then*

$$\frac{\kappa_{n-1}}{3\cos\varphi}(\sin\varphi)^{n-1} < |B(\varphi, \mathbf{x})| < \frac{\kappa_{n-1}}{\cos\varphi}(\sin\varphi)^{n-1}.$$

Define $\Omega(\varphi) = \frac{|B(\varphi,\mathbf{x})|}{|S^n|}$, from $\sqrt{\frac{n}{2\pi}} < \frac{\kappa_{n-1}}{\kappa_n} < \sqrt{\frac{n+1}{2\pi}}$, the following Corollary holds.

Corollary 1. *[8] Let $\varphi \in (0, \frac{\pi}{2})$, if $\varphi \le \arccos\frac{1}{\sqrt{n}}$, then*

$$\frac{1}{3\sqrt{2\pi n}}\frac{1}{\cos\varphi}(\sin\varphi)^{n-1} < \Omega(\varphi) < \frac{1}{\sqrt{2\pi(n-1)}}\frac{1}{\cos\varphi}(\sin\varphi)^{n-1}.$$

For any real $s > 0$, the Gaussian function on $\mathbb{R}^n$ centered at $\mathbf{c}$ with parameter s is given as follows.

$$\forall \mathbf{x} \in \mathbb{R}^n, \rho_{\mathbf{s},\mathbf{c}}(\mathbf{x}) = e^{-\pi\|(\mathbf{x}-\mathbf{c})/s\|^2}.$$

The subscripts s and $\mathbf{c}$ are taken to be 1 and $\mathbf{0}$ (respectively) when omitted.

For any $\mathbf{c} \in \mathbb{R}^n$, real $s > 0$, and n-dimensional lattice Λ, define the discrete Gaussian distribution over Λ as:

$$\forall \mathbf{x} \in \Lambda, D_{\Lambda,s,\mathbf{c}}(\mathbf{x}) = \frac{\rho_{\mathbf{s},\mathbf{c}}(\mathbf{x})}{\rho_{\mathbf{s},\mathbf{c}}(\Lambda)},$$

where $\rho_{\mathbf{s},\mathbf{c}}(A) = \sum_{x \in A} \rho_{\mathbf{s},\mathbf{c}}(\mathbf{x})$ for any countable set A.

A function $\varepsilon(n)$ is negligible if $\varepsilon(n) < 1/n^c$ for any $c > 0$ and all sufficiently large n. Statistical distance between two distributions $\mathbf{X}$ and $\mathbf{Y}$ over a countable domain D is defined as $\frac{1}{2}\sum_{d \in D} |X(d) - Y(d)|$. We say two distributions (indexed by n) are statistically close if their statistical distance is negligible in n.

Lemma 2. *[12] Klein's randomized nearest plane algorithm is a probabilistic polynomial-time algorithm that, given a basis $\mathbf{B}$ of an n-dimensional lattice $\Lambda = \mathcal{L}(\mathbf{B})$, a parameter $s \geq \|\tilde{\mathbf{B}}\|\omega(\sqrt{\log n})$, and a center $\mathbf{c} \in \mathbb{R}^n$, outputs a sample from a distribution that is statistically close to $D_{\Lambda,s,\mathbf{c}}$.*

Similar to NV heuristic algorithm, our algorithm requires that the lattice points distribute in $C_n(\gamma_2 R)$ uniformly at any stage of the algorithm. We need to select the sample by applying Klein randomized variant of nearest plane algorithm [18] so that the initial chosen sample is indistinguishable from Gauss distribution. The distribution proof of Klein algorithm is given in [12].

3 Algorithm

Section 3.1 gives a brief description of Nguyen-Vidick heuristic sieve algorithm (NV algorithm). Then in section 3.2, under the same natural heuristic assumption as NV algorithm, we present a new algorithm with two-level sieve, which can output the shortest vector in time $2^{0.3836n}$.

3.1 Nguyen and Vidick's heuristic sieve algorithm

The randomized algorithm proposed by Ajtai, Kummar and Sivakumar[5] is denoted as the AKS sieve. The main idea of the algorithm is as follows: sample $2^{O(n)}$ lattice vectors in a ball $B_n(R)$ for $R = 2^{O(n)}\lambda_1$, then implement a partition and a sieve method to search enough shorter vectors within $B_n(\gamma R)$, for $\gamma < 1$ without losing many vectors. By a polynomial iterations, R is close to λ_1, while the short vectors left are enough to get the shortest vector. Until now, the perturbation technique in sampling procedure is necessary to prove the successful probability of finding the shortest vector in all randomized algorithms. But the effect of perturbation in practice is unclear. In [28], Nguyen and Vidick presented a fast heuristic algorithm (called NV algorithm) which collects the short vectors by directly sieving the chosen lattice vectors instead of sieving the lattice vectors derived from perturbed points. The NV algorithm has $2^{0.415n}$ time and $2^{0.2075n}$ space complexity.

In every sieve iteration of NV algorithm, the input is a set S of lattice vectors whose maximal norm are R which can be regarded as having the random distribution in $C_n(\gamma R)$. The main purpose of NV algorithm is to randomly select a subset C of S as the center points which are located in $C_n(\gamma R)$. C has enough points so that, for any vector $\mathbf{a}$ in S, there is at least a point $\mathbf{c} \in C$ such that $\mathbf{a} - \mathbf{c}$ has the length shorter than γR. $\mathbf{a} - \mathbf{c}$ is an output shorter vector in the iteration. In every iteration, the sieve captures a new set of vectors within the ball $B_n(\gamma R)$ without losing many vectors by selecting available γ and the size of C, i.e. the upper norm bound of the set shrinks by γ. Then after a polynomial iterations, the shortest vector will be included in the sieved short vectors, and can be found by searching. The core of NV algorithm is the sieve in Algorithm 1.

The main part of the data complexity is decided by the upper size of the point centers C which should guarantee that after polynomial number of iterations the set S is not empty. The estimation of $|C|$ is based on a natural assumption, and the experiments shows the assumption is rational.

Algorithm 1 The NV sieve

Input: An subset $S \subseteq B_n(R)$ of vectors in a lattice L, sieve factors $\sqrt{\frac{2}{3}} < \gamma < 1$.
Output: A subset $S' \subseteq B_n(\gamma R) \cap L$.

$R \leftarrow \max_{\mathbf{v} \in S} \|\mathbf{v}\|$.
$C \leftarrow \emptyset, S' \leftarrow \emptyset$
for $\mathbf{v} \in S$ do
  if $\|\mathbf{v}\| \leq \gamma R$ then
    $S' \leftarrow S' \cup \{\mathbf{v}\}$.
  else
    if $\exists \mathbf{c} \in C, \|\mathbf{v} - \mathbf{c}\| \leq \gamma R$ then
      $S' \leftarrow S' \cup \{\mathbf{v} - \mathbf{c}\}$.
    else
      $C \longleftarrow C \cup \{\mathbf{v}\}$
    end if
  end if
end for

Heuristic Assumption: At any stage in the Algorithm, the vectors in $S \cap C_n(\gamma R)$ are uniformly distributed in $C_n(\gamma R) = \{\mathbf{x} \in \mathbb{R}^n : \gamma R \leq \|\mathbf{x}\| \leq R\}$.

3.2 New Sieve Algorithm

In NV algorithm, the time complexity is the square of space complexity. In order to achieve some balance between time and space, we try a new sieve which in fact is a two-level sieve. (Algorithm 3). Our algorithm also includes polynomial iterations, and each iteration consists of two level sieves. At first level, we partition the lattice points in the spherical shell $C_n(\gamma_2 R)$ into different big balls rather than small ones in Algorithm 1 (See Fig.1). In the second level, we cover every spherical cap (intersection of the big ball and $C_n(\gamma_2 R)$) using small balls which are centered at a lattice point in the same spherical cap. By comparing all the lattice points in the spherical cap with the small ball centers, we can get some shorter vectors. Merging all the short vectors calculated in all spherical caps, we obtain the required short lattice vectors to the next iteration. It is clear that, the first level sieve needs less number of big balls which cuts the comparing time. At the second level, each shorter vector is obtained by pair-wise subtracting among the lattice points in a spherical cap, which means that more data are required to get enough shorter vectors. In particular, the Heuristic Assumption in NV sieve guarantees the uniform distribution in $S \cap C_n(\gamma R)$ which also supports our algorithm.

The frame of our algorithm is given in Algorithm 2. Instead of the shrink factor γ in NV sieve, γ_1, γ_2 are two pivotal input parameters in this algorithm, which determine N, as well as the complexity and efficiency of the algorithm. In step 1-5, we generate N lattice vectors within a proper length just as NV sieve. Step 6-11 are the key parts of our algorithm which reduce the norm of lattice vectors in S by a factor γ_2 without decreasing the size of S much. This new sieve is different from any known sieve in which we put larger lattice vectors in

Fig. 1.

different big balls firstly, then do sieve again in separate big balls. The details will be given in Algorithm 3. This main loop repeats until the set S is empty. And the shortest vector in S_0 outputs. The size of S decreases in two ways: first in Algorithm 3 the vector used as center vector is removed from S; second, in step 10 the appearance of zero vector vanishes some vectors. In order to present the efficiency of the algorithm we will estimate the number of vectors lost in the process below in details.

Algorithm 2 Finding short lattice vectors based on sieving

Input: An LLL-reduced basis $B = [b_1, \cdots, b_n]$ of a lattice L, sieve factors γ_1, γ_2 such that $\sqrt{\frac{2}{3}} < \gamma_2 < 1 < \gamma_1 < \sqrt{2}\gamma_2$, and a number N.
Output: A short non-zero vector of L.
1: $S \leftarrow \emptyset$.
2: for $j = 1$ to N do
3: $\quad S \leftarrow S \bigcup$ sampling(B) using Klein's algorithm
4: end for
5: Remove all zero vectors from S.
6: $S_0 \leftarrow S$
7: repeat
8: $\quad S_0 \leftarrow S$
9: $\quad S \leftarrow$ latticesieve(S, γ_1, γ_2) using algorithm 3.
10: $\quad$ Remove all zero vectors from S.
11: Until $S = \emptyset$
12: Compute $\mathbf{v_0} \in S_0$ such that $\|\mathbf{v_0}\| = \min\{\|\mathbf{v}\|, \mathbf{v} \in S_0\}$
13: return $\mathbf{v_0}$

Under the heuristic assumption, the collisions in step 10 of algorithm 2 are negligible until $\sqrt{|B_n(R) \cap L|} \leq |S \cap C_n(\gamma_2 R)|$ which means the upper bound of norm R very close to the shortest length in lattice.

Two levels of center points are used in our sieve. Let C_1 be the set of centers of big balls with radius $\gamma_1 R$ in the first level, where $\gamma_1 > 1$. Since we can not get the short vectors in a big ball by subtracting its center directly, the second level covering are needed. $C_2^{\mathbf{c}}$ consists of the

Algorithm 3 The lattice sieve

Input: An subset $S \subseteq B_n(R)$ of vectors in a lattice L, sieve factors $\sqrt{\frac{2}{3}} < \gamma_2 < 1 < \gamma_1 < \sqrt{2}\gamma_2$.

Output: A subset $S' \subseteq B_n(\gamma_2 R) \cap L$.

$R \leftarrow \max_{\mathbf{v}\in S} \|\mathbf{v}\|$.
$C_1 \leftarrow \emptyset, C_2 \leftarrow \{\emptyset\}, S' \leftarrow \emptyset$
for $\mathbf{v} \in S$ do
  if $\|\mathbf{v}\| \leq \gamma_2 R$ then
    $S' \leftarrow S' \cup \{\mathbf{v}\}$.
  else
    if $\exists \mathbf{c} \in C_1, \|\mathbf{v} - \mathbf{c}\| \leq \gamma_1 R$ then
      if $\exists \mathbf{c}' \in C_2^{\mathbf{c}}, \|\mathbf{c}' - \mathbf{v}\| \leq \gamma_2 R$ \ $C_2^{\mathbf{c}}$ is initialized as empty set \
        $S' \leftarrow S' \cup \{\mathbf{v} - \mathbf{c}'\}$.
      else
        $C_2^{\mathbf{c}} \longleftarrow C_2^{\mathbf{c}} \cup \{\mathbf{v}\}$
      end if
    else
      $C_1 \longleftarrow C_1 \cup \{\mathbf{v}\}, C_2 \longleftarrow C_2 \cup \{C_2^{\mathbf{c}} = \{v\}\}$
    end if
  end if
end for

centers of small balls in the second level that cover a big ball with center $\mathbf{c}$. It is clear that, $C_2^{\mathbf{c}}$ is selected in the regular spherical cap $C_n(\gamma_2 R) \cap B_n(\mathbf{c}, \gamma_1 R)$. All $C_2^{\mathbf{c}}$ are merging into one set C_2, i.e., $C_2 = \bigcup_{c\in C_1} C_2^{\mathbf{c}}$. Denote the expected number of lattice vectors in C_1 as N_{C_1} and the expected size of every $C_2^{\mathbf{c}}$ as $N_{C_2^{\mathbf{c}}}$. To estimate N_{C_1}, we have to calculate the fraction of the spherical cap $C_n(\gamma_2 R) \cap B_n(\mathbf{c}, \gamma_1 R)$ in $C_n(\gamma_2 R)$. The right part of Fig. 1 illustrates the covering of first level, and O_b denotes a center $\mathbf{c}$ of a big ball. $N_{C_2^{\mathbf{c}}}$ is the number of small balls centered in $C_2^{\mathbf{c}}$ with radius $\gamma_2 R$, which cover the spherical cap $C_n(\gamma_2 R) \cap B_n(\mathbf{c}, \gamma_1 R)$ with probability close to 1. The region of $C_n(\gamma_2 R) \cap B_n(\mathbf{c}, \gamma_1 R) \cap B_n(\mathbf{c}', \gamma_2 R)$, $\mathbf{c}' \in C_2^{\mathbf{c}}$ is a regular or irregular spherical cap whose volume determines the number of $N_{C_2^{\mathbf{c}}}$. The left part of Fig. 2 shows the second covering. O_b denotes a center $\mathbf{c}$ of the first-level big ball, and O_s is a center $\mathbf{c}'$ of second-level small ball. We give the estimations for N_{C_1} and $N_{C_2^{\mathbf{c}}}$ in the following theorems.

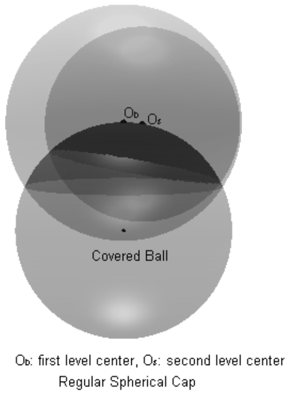
The purpose of every iteration is to compress a large number of lattice points in $B_n(R)$ to $B_n(\gamma_2 R)$ without losing many points. So we just consider the covering of the spherical shell $C_n(\gamma_2 R)$. Applying LLL reduced input basis and the Klein's sample algorithm with proper parameter, we can choose the initial R smaller than $2^{O(n)}\lambda_1$. After every two-level sieve of the algorithm, the upper bound of the norm shrinks by γ_2. If the number of sampled vectors is not less than $poly(n) N_{C_1} N_{C_2^{\mathbf{c}}}$, then after a polynomial iterations, we expect the shortest vector left.

The upper bound of C_1 and $C_2^{\mathbf{c}}$ are given in Theorem 1 and Theorem 2 respectively.

Theorem 1. *Let n be a non-negative integer, and $\frac{1}{2} < \gamma_2 < 1 < \gamma_1 < \sqrt{2}\gamma_2$,*

$$N_{C_1} = c_{\mathcal{H}_1}^n \lceil 3\sqrt{2\pi} n^{\frac{3}{2}} \rceil,$$

where $c_{\mathcal{H}_1} = \frac{1}{\gamma_1 \sqrt{1 - \frac{\gamma_1^2}{4}}}$. S is a subset of $C_n(\gamma_2 R) = \{\mathbf{x} \in \mathbb{R}^n \mid \gamma_2 R \leq \|\mathbf{x}\| \leq R\}$ of cardinality N whose points are picked independently at random with uniform distribution. If $N_{C_1} < N <$



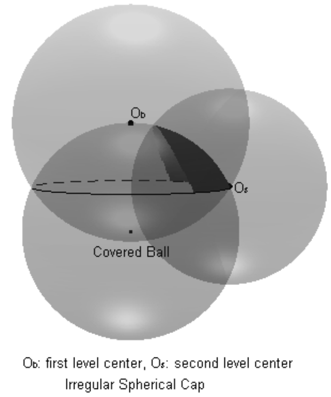


Fig. 2. Second Level Covering in the New Algorithm

2^n, *then for any subset* $C \subseteq S$ *of size at least* N_{C_1} *whose points are picked independently at random with uniform distribution, with overwhelming probability, for all* $\mathbf{v} \in S$, *there exists a* $\mathbf{c} \in C$ *such that* $\|\mathbf{v} - \mathbf{c}\| \leq \gamma_1 R$.

Theorem 2. *Let* n *be a non-negative integer,* $\sqrt{\frac{2}{3}} < \gamma_2 < 1 < \gamma_1 < \sqrt{2}\gamma_2$,

$$N_{C_2^{\mathbf{c}}} = c\left(\frac{c_{\mathcal{H}_2}}{d_{\min}}\right)^n \lceil n^{\frac{3}{2}} \rceil,$$

where $c_{\mathcal{H}_2} = \frac{\gamma_1}{\gamma_2}\sqrt{1 - \frac{\gamma_1^2}{4\gamma_2^2}}, d_{\min} = \gamma_2\sqrt{1 - \frac{\gamma_2^2 c_{\mathcal{H}_1}^2}{4}}$, *c is a positive constant unrelated to* n. S *is a subset of* $\{\mathbf{x} \in C_n(\gamma_2, R) \mid \|\mathbf{x} - \mathbf{c}_1\| \leq \gamma_1 R\}$ *of cardinality* N *whose points are picked independently at random with uniform distribution. If* $N_{C_2^{\mathbf{c}}} < N < 2^n$, *then for any subset* $C \subseteq S$ *of size at least* $N_{C_2^{\mathbf{c}}}$ *whose points are picked independently at random with uniform distribution, with overwhelming probability, for all* $\mathbf{v} \in S$, *there exists a* $\mathbf{c} \in C$ *such that* $\|\mathbf{v} - \mathbf{c}\| \leq \gamma_2 R$.

4 Proof of the Complexity

In this section, we give the proofs and the complexity estimation of our algorithm.

Since R has no effect on the conclusion of Theorem 1 and Theorem 2, we prove following lemma for unit ball. Let $\Omega_n(\gamma_1)$ be the minimum fraction of $C_n(\gamma_2)$ that is covered by a ball of radius γ_1 centered in a point of $C_n(\gamma_2)$.

Lemma 3. $\sqrt{\frac{2}{3}} < \gamma_2 < 1 < \gamma_1 < \sqrt{2}\gamma_2$,

$$\frac{1}{3\sqrt{2\pi n}}\frac{1}{\cos\theta_2}(\sin\theta_2)^{n-1} < \Omega(\gamma_1) < \frac{1}{\sqrt{2\pi(n-1)}}\frac{1}{\cos\theta_1}(\sin\theta_1)^{n-1},$$

where $\theta_1 = \arccos(1 - \frac{\gamma_1^2}{2\gamma_2^2})$, $\theta_2 = \arccos(1 - \frac{\gamma_1^2}{2})$.

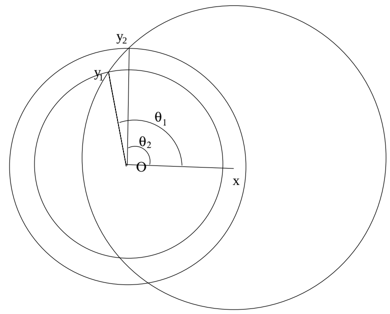


Fig. 3.

Proof. Let $\mathbf{x} \in C_n(\gamma_2)$, $\|\mathbf{x}\| = \alpha_1$ where $\gamma_2 \le \alpha_1 \le 1$. $\mathbf{y}_1$ and $\mathbf{y}_2$ are two points in the spherical cap $C_n(\gamma_2)$ which are at distance γ_1 from $\mathbf{x}$, and $\|\mathbf{y}_1\| = \gamma_2$, and $\|\mathbf{y}_2\| = 1$. Denote the angle of vertices $\mathbf{x}$, O and $\mathbf{y}_1$ as θ_1 and θ_2 is the angle of vertices $\mathbf{x}$, O and $\mathbf{y}_2$. We have $\cos\theta_1 = \frac{\alpha_1^2+\gamma_2^2-\gamma_1^2}{2\alpha_1\gamma_2}$, $\cos\theta_2 = \frac{\alpha_1^2+1-\gamma_1^2}{2\alpha_1}$. From $\gamma_1^2 > \alpha_1^2 - \gamma_2$ and $\gamma_2 < 1$, we know that $\cos\theta_1 < \cos\theta_2$, which means $\theta_1 > \theta_2$. Then $B(\theta_2, \mathbf{x}) \subset \Omega_n(\gamma_1) \subset B(\theta_1, \mathbf{x})$. By Corollary 1, we have

$$\frac{1}{3\sqrt{2\pi n}}\frac{1}{\cos\theta_2}(\sin\theta_2)^{n-1} < \Omega_n(\gamma_1) < \frac{1}{\sqrt{2\pi(n-1)}}\frac{1}{\cos\theta_1}(\sin\theta_1)^{n-1}.$$

Furthermore, both $\cos\theta_1$ and $\cos\theta_2$ increases with α_1. So the lower bound is given by $\alpha_1 = 1$, where $\theta_2 = \arccos(1 - \frac{\gamma_1^2}{2})$. When $\alpha_1 = \gamma_2$, $\theta_1 = \arccos(1 - \frac{\gamma_1^2}{2\gamma_2^2})$, we get the upper bound for $\Omega_n(\gamma_1)$. □

Remark 1. It is noted that Lemma 3 is similar to lemma 4.2 in [28]. The difference is that we generalize the formula to that of reflecting the exact expressions of angles θ_1, θ_2 with parameters γ_1 and γ_2, which are important in the main complexity estimation of Theorem 1 and Theorem 2.

Using lemma 3, we can give the proof of Theorem 1.

Proof. By lemma 3, we have

$$\Omega_n(\gamma_1) > \frac{1}{3\sqrt{2\pi n}}\frac{1}{\cos\theta_2}(\sin\theta_2)^{n-1} > \frac{1}{3\sqrt{2\pi n}}(\sin\theta_2)^{n-1} > \frac{1}{3\sqrt{2\pi n}}c_{\mathcal{H}_1}^{-n}.$$

The expected proportion of $C_n(\gamma_2)$ that is not covered by N_{C_1} balls of radius γ_1 centered at randomly chosen points of $C_n(\gamma_2)$ is $(1-\Omega_n(\gamma_1))^{N_{C_1}}$. So,

$$N_{C_1}\log(1-\Omega_\mathrm{n}(\gamma_1)) \le N_{C_1}(-\Omega_n(\gamma_1)) < c_{\mathcal{H}}^n\lceil 3\sqrt{2\pi}n^{\frac{3}{2}}\rceil \cdot \frac{-1}{3\sqrt{2\pi n}}c_{\mathcal{H}}^{-n} \le -n < -\log N,$$

which implies

$$(1-\Omega_n(\gamma))^{N_C} < e^{-n} < \frac{1}{N}.$$

Therefore, the expected number of uncovered points is smaller than 1. In other words, any point in $C(\gamma_2)$ is covered by a ball of radius γ_1 with successful probability $1 - e^{-n}$. □

Without loss of generality, we denote the center of one big ball centered at $C_n(\gamma_2)$ as $(\alpha_1, 0, \ldots, 0)$, where $\gamma_2 \le \alpha_1 \le 1$. The region of the regular spherical cap $B_n(c, \gamma_1) \cap C_n(\gamma_2)$ is denoted as M, i.e.,

$$M = \{(x_1, x_2, x_3, \ldots, x_n) \in C_n(\gamma_2) \mid (x_1 - \alpha_1)^2 + x_2^2 + \ldots + x_n^2 < \gamma_1^2\},$$

where $\gamma_2 \le \alpha_1 \le 1$. To discuss the covering of the M by the small balls $B'_n(\mathbf{c}', \gamma_2)$, $c' \in C_2^c$, we need to calculate the minimum fraction of the spherical cap $B'_n(\mathbf{c}', \gamma_2) \cap B_n(\mathbf{c}, \gamma_1) \cap C_n(\gamma_2)$ which is expressed as $\Omega_n(\gamma_1, \gamma_2)$.

We denote $B'_n(\mathbf{c}', \gamma_2) \cap B_n(\mathbf{c}, \gamma_1) \cap C_n(\gamma_2)$ as H. Before estimate the proportion of spherical cap H in M, we need to clarify its location. From Fig. 2, we know that, when the small ball completely fall into the big ball, H is a regular spherical cap, otherwise it is an irregular spherical cap. Especially, if $\mathbf{c}'$ slips along the sphere of the big ball (See right part of Fig. 2), the fraction of H in M is minimal. We noted that, only a little part of H is regular, and most centers $\mathbf{c}'$ are close to surface of big balls. So, we only compute the volume of minimum H, i.e., $\mathbf{c}'$ is located at the sphere of a big ball $B_n(\mathbf{c}, \gamma_1)$.

Lemma 4. *Let* $\sqrt{\frac{2}{3}} < \gamma_2 < 1 < \gamma_1 < \sqrt{2}\gamma_2$, *we have* $\Omega_n(\gamma_1, \gamma_2) \ge c\frac{d_{\min}^{n-2}}{2\pi n}$, *where* $d_{\min} = \gamma_2\sqrt{1 - \frac{\gamma_2^2 c_{\mathcal{H}_1}^2}{4}}$, $c_{\mathcal{H}_1} = \frac{1}{\gamma_1\sqrt{1-\frac{\gamma_1^2}{4}}}$, *and* c *is a positive constant.*

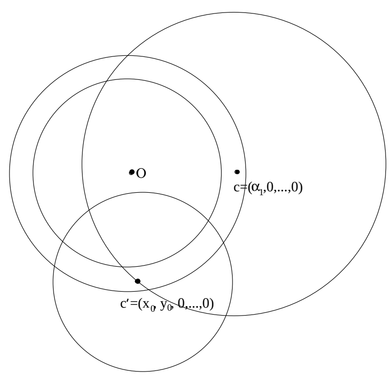


Fig. 4.

Proof. To estimate the covering of the regular spherical cap M by irregular spherical cap, because γ_2 is selected close to 1 in our algorithm, we just estimate the proportion on the sphere covering rather than the shell covering.

Without loss of generality, we assume the center of $B_n(c, \gamma_1)$ as $(\alpha_1, 0, \ldots, 0)$, and the center $B'_n(c', \gamma_2)$ as $(x_0, y_0, 0, \ldots, 0)$ where $x_0 > 0$, $y_0 > 0$. According to the above description, the irregular spherical cap $B'_n(\mathbf{c}', \gamma_2) \cap B_n(\mathbf{c}, \gamma_1) \cap C_n(\gamma_2)$ with the minimum volume is expressed as:

$$\begin{cases} x_1^2 + x_2^2 + \ldots + x_n^2 = 1 \\ (x_1 - \alpha_1)^2 + x_2^2 + \ldots + x_n^2 < \gamma_1^2 \\ (x_1 - x_0)^2 + (x_2 - y_0)^2 + \ldots + x_n^2 < \gamma_2^2 \end{cases},$$

where $\gamma_2 \le \alpha_1 \le 1$, $(x_0-\alpha_1)^2+y_0^2=\gamma_1^2$, and $\gamma_2 \le x_0^2+y_0^2 \le 1$. In order to calculate this surface integral we project the target region to the hyperplane orthogonal to x_1, then this integral is changed to multiple integral. To simplify the expression, denote $A=x_0^2+y_0^2$ and $B=(A+1-\gamma_2^2)/2$. Let

$$D_1=\left\{(x_2,x_3,\ldots,x_n)\in\mathbb{R}^{n-1} \mid x_2^2+x_3^2+\ldots+x_n^2<1-\left(\frac{\alpha_1^2-\gamma_1^2+1}{2\alpha_1}\right)^2\right\}.$$

$$D_2^1=\left\{(x_2,x_3,\ldots,x_n)\mid \frac{A}{x_0^2}(x_2-\frac{By_0}{A})^2+x_3^2+\ldots+x_n^2<1-\frac{B^2}{x_0^2}(1-\frac{y_0^2}{A})\cap x_2<\frac{B}{y_0}\right\},$$

$$D_2^2=\left\{(x_2,x_3,\ldots,x_n)\mid x_2^2+x_3^2+\ldots+x_n^2<1\cap x_2\ge\frac{B}{y_0}\right\},\ D_2=D_2^1\cup D_2^2.$$

Let $R_1=\sqrt{1-\left(\frac{\alpha_1^2-\gamma_1^2+1}{2\alpha_1}\right)^2}$, $R_2=\sqrt{1-\frac{B^2}{x_0^2}\left(1-\frac{y_0^2}{A}\right)}$. The integral region is denoted as D which is the intersection of D_1 and D_2. By the equation $x_1^2+x_2^2+\ldots+x_n^2=1$, we have

$$x_1=\pm\sqrt{1-(x_2^2+\ldots+x_n^2)},\quad \frac{\partial x_1}{\partial x_i}=\frac{\mp x_i}{\sqrt{1-(x_2^2+\ldots+x_n^2)}}.$$

Now we can calculate the volume of the target region by computing

$$\begin{aligned}Q&=\int\int\cdots\int_D\sqrt{1+\left(\frac{\partial x_1}{\partial x_2}\right)^2+\ldots+\left(\frac{\partial x_1}{\partial x_n}\right)^2}\mathrm{d}x_2\mathrm{d}x_3\ldots\mathrm{d}x_n\\&=\int\int\cdots\int_D\frac{1}{\sqrt{1-(x_2^2+x_3^2+\ldots+x_n^2)}}\mathrm{d}x_2\mathrm{d}x_3\ldots\mathrm{d}x_n.\end{aligned}$$

We analysis the region D and first compute x_2 to simplify the above multiple integral. The upper bound and lower bound of x_2 is $\sqrt{R_1^2-(x_3^2+\ldots+x_n^2)}$ and $\frac{By_0}{A}-\frac{x_0}{\sqrt{A}}\sqrt{R_2^2-(x_3^2+\ldots+x_n^2)}$ respectively, while the region of $(x_3,\ldots,x_n)$ is a ball of dimension $n-2$ with radius

$$d=\sqrt{R_1^2-\left(\frac{1}{y_0}\left(B-x_0\left(\frac{\alpha_1^2-\gamma_1^2+1}{2\alpha_1}\right)\right)\right)^2}.$$

Therefore Q is expressed as,

$$\begin{aligned}Q&=\int_{\sum_{i=3}^n x_i^2<d}\cdots\int\left(\int_{\frac{By_0}{A}-\frac{x_0}{\sqrt{A}}\sqrt{R_2^2-\sum_{i=3}^n x_i^2}}^{\sqrt{R_1^2-\sum_{i=3}^n x_i^2}}\frac{1}{\sqrt{1-\sum_{i=3}^n x_i^2-x_2^2}}\mathrm{d}x_2\right)\mathrm{d}x_3\ldots\mathrm{d}x_n\\&=\int_{\sum_{i=3}^n x_i^2<d}\cdots\int\left(\arcsin\frac{\sqrt{R_1^2-\sum_{i=3}^n x_i^2}}{\sqrt{1-\sum_{i=3}^n x_i^2}}-\arcsin\frac{\frac{By_0}{A}-\frac{x_0}{\sqrt{A}}\sqrt{R_2^2-\sum_{i=3}^n x_i^2}}{\sqrt{1-\sum_{i=3}^n x_i^2}}\right)\mathrm{d}x_3\ldots\mathrm{d}x_n.\end{aligned}$$

Let

$$\begin{cases}x_3=t\cos\varphi_1\\x_4=t\sin\varphi_1\cos\varphi_2\\\qquad\vdots\\x_{n-1}=t\sin\varphi_1\ldots\sin\varphi_{n-4}\cos\varphi_{n-3}\\x_n=t\sin\varphi_1\ldots\sin\varphi_{n-4}\sin\varphi_{n-3}\end{cases},$$

then $0 \le t \le d, 0 \le \varphi_k \le \pi, k = 1, \dots, n-4, 0 \le \varphi_{n-3} \le 2\pi$. Furthermore, we get,

$$\frac{\partial(x_3, x_4, \dots, x_n)}{\partial(t, \varphi_1, \dots \varphi_{n-3})} = t^{n-3} \sin\varphi_{n-4} \dots (\sin\varphi_2)^{n-5} (\sin\varphi_1)^{n-4}.$$

So,

$$\begin{aligned}
Q = &\int_0^d \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} t^{n-3} \sin\varphi_{n-4} \dots (\sin\varphi_2)^{n-5} (\sin\varphi_1)^{n-4} \Bigg(\arcsin \frac{\sqrt{R_1^2 - t^2}}{\sqrt{1-t^2}} \\
&\quad - \arcsin \frac{\frac{By_0}{A} - \frac{x_0}{\sqrt{A}}\sqrt{R_2^2 - t^2}}{\sqrt{1-t^2}} \Bigg) \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_{n-3} \mathrm{d}t \\
=&2\pi \int_0^d t^{n-3} \left(\arcsin \frac{\sqrt{R_1^2 - t^2}}{\sqrt{1-t^2}} - \arcsin \frac{\frac{By_0}{A} - \frac{x_0}{\sqrt{A}}\sqrt{R_2^2 - t^2}}{\sqrt{1-t^2}} \right) \mathrm{d}t \prod_{k=1}^{k=n-4} \int_0^{\pi} \sin^k \varphi \mathrm{d}\varphi.
\end{aligned}$$

From $\int_0^{\pi} \sin^k \varphi \mathrm{d}\varphi = 2\int_0^{\pi/2} \sin^k \varphi \mathrm{d}\varphi = \sqrt{\pi} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)}$, we obtain

$$Q = \frac{2\pi^{(n-2)/2} \int_0^d t^{n-3} \left(\arcsin \frac{\sqrt{R_1^2 - t^2}}{\sqrt{1-t^2}} - \arcsin \frac{\frac{By_0}{A} - \frac{x_0}{\sqrt{A}}\sqrt{R_2^2 - t^2}}{\sqrt{1-t^2}} \right) \mathrm{d}t}{\Gamma(\frac{n-2}{2})},$$

and

$$\Omega_{\mathrm{n}}(\gamma_1, \gamma_2) = \frac{Q}{|S^n|} = \frac{n-2}{2\pi} \int_0^d t^{n-3} \left(\arcsin \frac{\sqrt{R_1^2 - t^2}}{\sqrt{1-t^2}} - \arcsin \frac{\frac{By_0}{A} - \frac{x_0}{\sqrt{A}}\sqrt{R_2^2 - t^2}}{\sqrt{1-t^2}} \right) \mathrm{d}t.$$

When $t \in [0, d]$, the difference of the two anti-trigonometric function is bounded and positive which is independent of n. More precisely, denote

$$f(t) = \left(\arcsin \frac{\sqrt{R_1^2 - t^2}}{\sqrt{1-t^2}} - \arcsin \frac{\frac{By_0}{A} - \frac{x_0}{\sqrt{A}}\sqrt{R_2^2 - t^2}}{\sqrt{1-t^2}} \right),$$

which is a decreasing function with $t \in [0, d]$. We have

$$\Omega_n(\gamma_1, \gamma_2) \ge \frac{n-2}{2\pi} \int_0^{d-\varepsilon} t^{n-3} f(t) \mathrm{d}t \ge \frac{d^{n-2}}{2\pi} (1-\varepsilon)^{n-2} f(d-\varepsilon).$$

And since the omitted part which the integral region is from $d-\varepsilon$ to d is negligible compared to that from 0 to $d-\varepsilon$, our estimate is tight. Let $\varepsilon = \frac{d}{n}$, using Taylor series to estimate $f(d-\varepsilon)$, we have $f(d-\varepsilon) = \Theta(\frac{1}{n})$. Also $(1-\frac{1}{n})^{n-2} \ge (1-\frac{1}{n})^n \approx e^{-1}$ when n is sufficient large. Based on the above discussion, $\Omega_n(\gamma_1, \gamma_2) \ge \frac{cd^{n-2}}{2\pi n}$.

Next, given γ_1 and γ_2, we compute the minimum d with the variables α_1, x_0, y_0. Because x_0,y_0 satisfy the equation $(x_0 - \alpha_1)^2 + y_0^2 = \gamma_1^2$, let $\alpha_2 = \sqrt{x_0^2 + y_0^2}$, then

$$x_0 = \frac{\alpha_2^2 + \alpha_1^2 - \gamma_1^2}{2\alpha_1}, \quad y_0 = \sqrt{\alpha_2^2 - \left(\frac{\alpha_2^2 + \alpha_1^2 - \gamma_1^2}{2\alpha_1} \right)^2}.$$

So d can be regarded as a function with two variables α_1 and α_2, and $\gamma_2 \leq \alpha_1 \leq 1$, $\gamma_2 \leq \alpha_2 \leq 1$. By calculating the partial derivative $\frac{\partial d(\alpha_1,\alpha_2)}{\partial \alpha_2}$, from $\sqrt{\frac{2}{3}} < \gamma_2 \leq \alpha_1 < 1 < \gamma_1 < \sqrt{2}\gamma_2$, it can be proven that, d is a decreasing function with α_2. Let $\alpha_2 = 1$, we get

$$d = \gamma_2 \sqrt{1 - \frac{\gamma_2^2}{4T^2}}, \quad T = \sqrt{1 - \left(\frac{1 + \alpha_1^2 - \gamma_1^2}{2\alpha_1}\right)^2}.$$

It is obvious that d decreases with α_1. Let $\alpha_1 = 1$, we achieve the minimum d.

$$d_{\min} = \gamma_2 \sqrt{1 - \frac{\gamma_2^2 c_{\mathcal{H}_1}^2}{4}}, \quad c_{\mathcal{H}_1} = \frac{1}{T} = \frac{1}{\gamma_1 \sqrt{1 - \frac{\gamma_1^2}{4}}}.$$

The proof of Lemma 4 is completed. □

Now we prove Theorem 2.

Proof. Combing the Lemma 3 and Lemma 4, we get

$$\frac{\Omega_n(\gamma_1, \gamma_2)}{\Omega_n(\gamma_1)} \geq \frac{c}{\sqrt{2\pi n}} \left(1 - \frac{\gamma_1^2}{2\gamma_2^2}\right) \left(\frac{d_{\min}}{c_{\mathcal{H}_2}}\right)^n,$$

which reflects the fraction of M covered by a small ball with radius γ_2 centered in M.

Similar to Theorem 1, it is easy to know the center points C_2^c of the second level in every big ball are less than $c' n^{\frac{3}{2}} (\frac{c_{\mathcal{H}_2}}{d_{\min}})^n$. □

Theorem 3. *The time complexity of our algorithm is* $\max\{N_{C_1}^2 N_{C_2^c}, N_{C_1} N_{C_2^c}^2\}$*, while the space complexity is* $N_{C_1} N_{C_2^c}$*. When* $\gamma_2 \longrightarrow 1, \gamma_1 = 1.0927$*, we get the optimal time complexity* $2^{0.3836n}$*, and the space complexity* $2^{0.2557n}$*.*

Proof. The total number of point centers in C_2 is about $N_{C_1} N_{C_2^c}$. If sampling $poly(n) N_{C_1} N_{C_2^c}$ vectors, after a polynomial iterations, we expect the vector left is enough to include the shortest vector. So the space complexity is $\text{poly}(n) N_{C_1} N_{C_2^c}$.

The initial sampling size S is $poly(n) N_{C_1} N_{C_2^c}$. In each iteration, steps 3-17 in algorithm 3 repeat $N_{C_1} N_{C_2^c}$ times, every repeat time complexity is about $N_{C_1} + N_{C_2^c}$. So the total time complexity is $\max\{N_{C_1}^2 N_{C_2^c}, N_{C_1} N_{C_2^c}^2\}$ polynomial computations.

Because N_{C_1} only depends on γ_1, and $N_{C_2^c}$ decreases with γ_2, we obtain the minimum time complexity by selecting $\gamma_2 \longrightarrow 1$ and $N_{C_1} = N_{C_2^c}$ which leads to $\gamma_1 = 1.0927$ and $N_{C_1} = N_{C_2^c} = 2^{0.1278n}$. □

5 Conclusion

In this paper, we describe a new algorithm of heuristic sieve with $2^{0.3836n}$ polynomial time operations and $2^{0.2557n}$ lattice points, which solves the SVP. Although our algorithm decreases the index of the time complexity from 0.415 to 0.3836, the polynomial part of the time complexity increases to $n^{4.5}$ instead of n^3 in NV algorithm. So our algorithm performs better than NV algorithm for large n.

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