Thermal magnetized D-branes on $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ in the generalized TFD approach

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1 November 2010

Abstract

We construct the D-brane states at finite temperature in thermal equilibrium in the $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ spacetime in the presence of cold (unthermalized) Kalb-Ramond (KR) and $U(1)$ gauge potential background. To this end, we first generalize the Thermo Field Dynamics (TFD) to wrapped closed strings. This generalization is consistent with the spatial translation invariance on the string world-sheet. Next, we determine the thermal string vacuum and define the entropy operator. From these data we calculate the entropy of the closed string and the free energy. Finally, we define the thermal D-brane states in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ in the presence of cold constant KR field and $U(1)$ gauge potential as the boundary states of the thermal closed string and compute their entropy.

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1 Introduction

The understanding of D-branes boundary states in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ spacetime is very important for fundamental reasons as well as for their applications. The D-branes in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ are of intrinsic importance for the String Theory since the background $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ with fluxes represent one of the few backgrounds on which the strings can be defined perturbatively and the D-brane boundary conditions can be solved exactly. The magnetized D-brane systems in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ can be used to engineer D-brane configurations that can support chiral gauge theories in four dimensions [1]. The D-branes wrapped around the cycles of $\mathbb{T}^{d-p-1}$ can support gravitons in $\mathbb{R}^{1,p}$ through the Kaluza-Klein mechanism which is useful to make contact with the brane-world scenarios. Important results have been obtained along these lines, from which one could cite: the construction of various extensions of the standard model with three generations of chiral fermions from configurations of magnetic D9-branes on orbifolds [2, 3, 4, 5, 6, 7, 8, 9]; the moduli stabilization with fluxes [10, 11]; the effect of the instantons and the Kähler metric on the chiral multiplets [12, 13, 14, 15]. These results establish relationships between the four dimensional field theory and string theory phenomenology[1]. (For more details on how the extensions of the Standard Model can be obtained from magnetic D-branes and various examples see [1, 16] and the references therein.) The wrapping of the D-branes on six cycles of the ten dimensional spacetime represents a key ingredient in these examples. Magnetized maximal D-brane states on tori were constructed recently in [17, 18, 19] following the method from [20]. (A detailed analysis of the topological structure of the boundary states on $U(N)$ gauge bundles was done in [18] and the T-duality was analysed in [21].)

The existence of the microscopic structure of the magnetized D-branes suggests that they should have an intrinsic thermodynamics created by the closed string excitations that condensate on the world-volume. In this paper, we are going to make the idea of the thermodynamics of the magnetized D-brane more precise by constructing the thermal D-branes on $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ and calculating their entropy under the hypothesis of the local thermodynamical equilibrium. The technique to construct thermal D-brane boundary states from D-branes at zero temperature was developed in [23, 24, 25, 26, 27] and its various aspects were reviewed and discussed in [28, 29, 30, 31, 32]. It is based on the Thermo Field Dynamics (TFD) approach to field theory at finite temperature which was previously applied to string theory and string field theory in [33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] and more recently in [46, 47, 48, 49, 50]. In this setting, the thermal D-branes are defined by boundary conditions on the thermal closed string which, on its turn, is obtained by applying the Bogoliubov operator of all string oscillators to the closed string at zero temperature[2]. The D-brane equilibrium thermodynamical functions are defined as the expectation values of the corresponding thermal string operators in the thermal D-brane state. The statistical properties of the strings in the presence of D-branes have been discussed in the literature in the path integral formulation in [53, 54, 55, 56]. However, the TFD approach to the thermodynamics of D-branes has the advantage that explicit

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[1]Very important results have been obtained in the low energy description of magnetized D-branes, too, in which the gravitational effects can be studied at the expense of neglecting the quantum structure, but here we will not address the D-branes from this point of view.

[2]The thermal D-branes in the open string channel and their connection with the entangled string states have been discussed recently in [51, 52].

thermal D-brane states can be constructed in a similar way to the zero temperature theory and statistical observables can be computed as expectation values of the corresponding operators in these states. This can be used, for example, to identify the sectors where the symmetries are broken due to thermal effects (see, for example, [26]).

In order to obtain the thermal D-brane states in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$, the general method previously developed for D-branes in $\mathbb{R}^{1,d-1}$ must be modified because of the non-trivial zero modes that arise from wrapping the closed string on the torus. The zero mode, or topological sector introduces a temperature dependent factor in the thermal string vacuum [3]. We show that the thermalization of the winding modes is different from that of the string oscillators and leads to products of the Jacobi theta functions in the partition function, unlike the products of Bose-Einstein distributions for the oscillating modes. Consequently, the topological thermal string vacuum and the topological entropy operator in the $\mathbb{T}^{d-p-1}$ subspace are no longer related to the corresponding objects at zero temperature by Bogoliubov-like operators. However, we show that the mapping can be put in an operatorial form by using the formal (Heisenberg) algebra of creation and annihilation operators for the winding number and the center of mass momentum along the $\mathbb{T}^{d-p-1}$ compact directions.

The intrinsic thermodynamics of the magnetic D-brane can be calculated without any matter field. However, the background fields in the magnetized brane models are necessary in order to stabilize the moduli and to cancel out the tadpole anomalies. Therefore, we consider the general background with the constant Kalb-Ramond (KR) and the constant $U(1)$ gauge potential. We make the simplifying assumption that the KR and $U(1)$ fields are cold, that is, the average values of the corresponding string states are taken to be the same as at zero temperature. The most general case when the fluxes are thermalized as a consequence of string thermalization will be studied elsewhere.

The paper is organized as follows. In Section 2 we firstly review the wrapped magnetized bosonic boundary states in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ at zero temperature following [17]. The topological sector of these states, i. e. the sector corresponding to the momentum and number operators along the cycles of the torus, is given at a fixed pair of cycles (i, j), i. e. the states belong to the subspace $\mathcal{H}_{(i,j)}$ of the Hilbert space. We generalize this solution to the full Hilbert space by including all cycles of the torus. Since we are interested in the physical degrees of freedom only, we choose to work in the light-cone gauge with the light-cone coordinates from $\mathbb{R}^{1,p}$. In Section 3 we construct the physical thermal vacuum and the Bogoliubov operators for the bosonic string. The non-trivial zero modes are a consequence of the wrapping of the closed string around $\mathbb{T}^{d-p-1}$ and of the interaction between the string and the KR background. We show that the topological terms introduce a temperature dependent factor in the thermal vacuum but leave the form of the Bogoliubov operators of the string oscillators unchanged. However, the thermal parameter is modified with a term proportional to the $\mathbb{T}^{d-p-1}$ typical radius. Next, we generalize the TFD method to bosonic string fields on $\mathbb{T}^{d-p-1}$ consistently with the spatial translation invariance on

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[3]The understanding of the winding modes of strings and branes is of utmost importance in the string cosmology. It has been shown that the string winding modes lead to a positive Hagedorn temperature and that they determine the energy of a gas of string and its pressure [57]. On the other hand, the brane winding modes lead to three un-compact space-like dimensions by determining the hierarchy in the size of the extra-dimension without the cosmological horizon problem [58].

the string world-sheet in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$. The mapping between the thermal vacuum and the vacuum at zero temperature is given in the operatorial form. In the $\mathbb{R}^{1,p}$ sector, the mapping is reduced to a product of the Bogoliubov operators. In the topological $\mathbb{T}^{d-p-1}$ sector we obtain a new operator which we write in the form of a product of creation and annihilation operators for the winding number and the center of mass momentum along the compact directions. By using this construction, we are able to define the entropy operator and calculate the entropy and the free energy of the closed string in both oscillator and topological sectors. In Section 4 we construct the thermal magnetized D-brane states by applying the modified generalized TFD formalism obtained in Section 3 and calculate the D-brane entropy. The last section is devoted to conclusions.

2 Magnetized D-branes at $T = 0$

Consider the bosonic closed string in $\mathbb{R}^{1,p} \times \mathbb{T}^{d-p-1}$ in the presence of the $U(1)$ field A^μ and the constant Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$, and let $G_{\mu\nu} = \{\eta_{ab}, G_{ij}\}$ be the metric on the target space. Here, η_{ab} is the Minkowski metric on $\mathbb{R}^{1,p}$ and G_{ij} is the internal metric on the torus $\mathbb{T}^{d-p-1}$. Also, we consider for simplicity the factorization of the gauge field and KR field in to $A^\mu = \{A^a, A^i\}$ and $B_{\mu\nu} = \{B_{ab}, B_{ij}\}$, respectively. We are using the following index notation: $\mu, \nu = \overline{0, d}$ are the target space indices, $a, b = \overline{0, p}$ are $\mathbb{R}^{1,p}$ indices and $i, j = \overline{p+1, d-1}$ are $\mathbb{T}^{d-p-1}$ indices, respectively. We denote the bosonic string coordinates by $X^\mu(\tau, \sigma)$. Also, since we are interested in the physical degrees of freedom of the string, we fix the light-cone gauge $X^+ = \alpha' p_+ \tau$ in the target space. In the light-cone gauge, $a, b = \overline{2, p}$ are $\mathbb{R}^{p-1}$ indices and $\mu, \nu = \overline{p+1, d-1}$ label $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ spacetime objects.

The classical dynamics of the bosonic string fields can be derived from the following action

$$S_0 = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\eta^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}\right) \partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1}$$

where $\alpha, \beta = 0, 1$, $\sigma^\alpha = (\tau, \sigma)$ and $\epsilon^{\alpha\beta}$ is the Kronecker symbol in two dimensions with $\epsilon^{01} = 1$. The corresponding equations of motion and closed string boundary conditions have the form

$$\partial^\alpha \partial_\alpha X^\mu = 0, \tag{2}$$

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma) + 2\pi R^i m^i \delta^{\mu i}, \tag{3}$$

for all $\mu = \overline{2, d-1}$ and all $i = \overline{p+1, d-1}$. Here, R^i's are the torus radii in the corresponding directions. The solutions of the equations (2) with the boundary conditions (3) have the following Fourier expansion

$$X^a(\tau, \sigma) = x^a + 2\alpha'^a \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left[\hat{a}_n^a e^{-2in(\tau-\sigma)} + \hat{b}_n^a e^{-2in(\tau+\sigma)}\right], \tag{4}$$

$$X^j(\tau, \sigma) = x^j + \sqrt{\alpha'} \left[2\hat{m}^j \sigma + 2G^{jk}\left(\hat{n}_k - B_{kl}\hat{m}^l\right)\tau\right] \tag{5}$$

$$+ i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left[\hat{a}_n^j e^{-2in(\tau-\sigma)} + \hat{b}_n^j e^{-2in(\tau+\sigma)}\right], \tag{6}$$

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where $\hat{n}^j$ is the center-of-mass number operator and $\hat{m}^j \in \mathbb{Z}$ is the winding number operator in the compact direction j, respectively. The eigenvalues of both kinds of operators are entire numbers. In what follows we are going to normalize the bosonic string operators to the oscillator operators

$$\hat{a}_m^\mu = \sqrt{m}\hat{\alpha}_m^\mu \quad , \quad \hat{a}_{-m}^\mu = \sqrt{m}\hat{\alpha}_m^{\mu\dagger}, \tag{7}$$

$$\hat{b}_m^\mu = \sqrt{m}\hat{\beta}_m^\mu \quad , \quad \hat{b}_{-m}^\mu = \sqrt{m}\hat{\beta}_m^{\mu\dagger}, \tag{8}$$

for all $m > 0$. The physical Hilbert space is the tensor product of the Hilbert spaces of all string modes factorized by the level matching condition

$$G_{\mu\nu}\left[\frac{\hat{n}^\mu\hat{n}^\nu}{\sqrt{\alpha'}} + \sum_n n\left(\hat{\alpha}_n^{\mu\dagger}\hat{\alpha}_n^\nu - \hat{\beta}_n^{\mu\dagger}\hat{\beta}_n^\nu\right)\right]|\Psi_{phys}\rangle = 0, \tag{9}$$

where the operator $\hat{n}^\mu$ is zero along the non-compact directions. The units have been chosen such that the square of the winding number quantize the string energy in multiples of $\varepsilon_s = l_s^{-1} = (\sqrt{\alpha'})^{-1}$. The total Hamiltonian is the sum of $\hat{H}^\mathbb{R}$ and $\hat{H}^\mathbb{T}$. If we choose the reference frame of the center-of-mass in $\mathbb{R}^{p-1}$, the two Hamiltonians are

$$\hat{H}^\mathbb{R} = \frac{1}{2}\sum_{n\neq 0} n\left(\hat{\alpha}_n^{a\dagger}\hat{\alpha}_n^a + \hat{\beta}_n^{a\dagger}\hat{\beta}_n^a\right), \tag{10}$$

$$\hat{H}^\mathbb{T} = \frac{1}{2\sqrt{\alpha'}}\left[G_{ij}\hat{m}^i\hat{m}^j + \left(\hat{n}_i - B_{ik}\hat{m}^k\right)G^{ij}\left(\hat{n}_j - B_{jl}\hat{m}^l\right)\right] + \frac{1}{2}G_{ij}\sum_{n\neq 0} n\left(\hat{\alpha}_n^{i\dagger}\hat{\alpha}_n^j + \hat{\beta}_n^{i\dagger}\hat{\beta}_n^j\right). \tag{11}$$

As observed in [19], in order to obtain the correct number of degrees of freedom of the supersymmetric boundary state in the compact directions, one must take $\hat{H}^\mathbb{T}/2$ in the total Hamiltonian.

The bosonic magnetized D-brane boundary state $|B\rangle$ is defined by the boundary conditions in the closed string Hilbert space. The zero mode boundary conditions are given by the following equations

$$\hat{p}^a|B\rangle = 0 \quad , \quad \left(\hat{n}_i - 2\pi\alpha'qF_{ij}\hat{m}^j\right)|B\rangle = 0, \tag{12}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The string fields couple with the gauge invariant combination of the $U(1)$ field and the KR field

$$\mathcal{B}_{\mu\nu} = (B - 2\pi\alpha'qF)_{\mu\nu}. \tag{13}$$

The coupling determines the oscillator boundary conditions in terms of creation and annihilation operators

$$\left[(\mathbf{1} + \mathcal{B})_{ab}\hat{\alpha}_n^b + (\mathbf{1} - \mathcal{B})_{ab}^T\hat{\beta}_n^{b\dagger}\right]|B\rangle = 0, \tag{14}$$

$$\left[(\mathbf{1} + \mathcal{B})_{ab}\hat{\alpha}_n^{b\dagger} + (\mathbf{1} - \mathcal{B})_{ab}^T\hat{\beta}_n^b\right]|B\rangle = 0, \tag{15}$$

$$\left(\mathcal{E}_{ij}\hat{\beta}_n^j + \mathcal{E}_{ij}^T\hat{\alpha}_n^{j\dagger}\right)|B\rangle = 0, \tag{16}$$

$$\left(\mathcal{E}_{ij}\hat{\beta}_n^{j\dagger} + \mathcal{E}_{ij}^T\hat{\alpha}_n^j\right)|B\rangle = 0, \tag{17}$$

where $n > 0$ is the mode index and $\mathcal{E}_{ij} = (G - \mathcal{B})_{ij}$. The solution to the equations (14)-(15) can be factorized as:

$$|B\rangle = |B\rangle_{osc}^{\mathbb{R}} \otimes |B\rangle_{osc}^{\mathbb{T}} \otimes |B\rangle_{top}^{\mathbb{T}} . \tag{18}$$

The states $|B\rangle_{osc}^{\mathbb{R}}$ and $|B\rangle_{osc}^{\mathbb{T}}$ are generated by oscillators only, and their form is already known [59, 60]

$$|B\rangle_{osc}^{\mathbb{R}} = N_{p-1}(\mathcal{B}) \left(\prod_{n=1}^{\infty} e^{-\hat{\alpha}_n^{a\dagger} M_{ab} \hat{\beta}_n^{b\dagger}} \right) |0\rangle_{osc} , \tag{19}$$

$$|B\rangle_{osc}^{\mathbb{T}} = N_{d-p-1}(\mathcal{B}) \left(\prod_{n=1}^{\infty} e^{-\hat{\alpha}_n^{i\dagger} G_{ik} (\mathcal{E}^{-1})^{kl} (\mathcal{E}^T)_{lj} \hat{\beta}_n^{j\dagger}} \right) |0\rangle_{osc} , \tag{20}$$

where the constants $N_{p-1}(\mathcal{B})$ and $N_{d-p-1}(\mathcal{B})$ depend on the background fields and brane tension and can be computed from the cilinder diagram interpreted in the open and closed string sectors. The topological boundary state $|B\rangle_{top}^{\mathbb{T}}$ is a solution of the second equation from (12). As can be easily checked, it has the follwing form

$$|B\rangle_{top}^{\mathbb{T}} = \prod_{i=p+1}^{d-1} \sum_{n_i, m_i} \delta_{n_i - 2\pi\alpha' q F_i^{k_i} m_{k_i}} |n_i\rangle |m_i\rangle . \tag{21}$$

In what follows, we are going to use the following normalization of momentum and wrapping number eigenstates

$$\langle p^a | p'^b \rangle = 2\pi \delta^{ab} \delta(p - p'), \quad \langle n_i | n'_j \rangle = \left(2\pi\sqrt{\alpha'} \right)^{\frac{1}{2}} \delta_{n_i, n'_j}, \quad \langle m^i | m'^j \rangle = \left(2\pi\sqrt{\alpha'} \right)^{\frac{1}{2}} \delta_{m^i, m'^j} . \tag{22}$$

Note that the state given in (21) generalizes the one given in [17] that satisfies the equation (12) only in two directions of the torus. The maximal flat D-brane boundary state is obtained for $p = d - 1$.

3   Thermodynamics of closed string in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$

The boundary states from the equation (18) describe condensates of the string modes in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$. Our main goal is to calculate the thermal magnetized D-branes and study their thermodynamical properties. To this end, we need to generalize the method from [23, 24, 26] which has been used to study the thermal D-branes in flat spacetime to $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$. This method already generalizes the TFD to closed string boundary states and it is the only one that provides the explicit construction of the thermal vacuum states and thermal boundary states [4].

In the TFD approach, one studies the thermodynamical properties of the D-branes in the flat spacetime by allowing the system to interact with a thermal reservoir. This interaction, called *thermalization*, is described in terms of the Bogoliubov operator that acts on the total string Hilbert space which is the tensor product of the closed string

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[4]Note that in the path integral treatment of the strings at finite temperature the thermal vacuum and the D-brane states are not explicitly constructed.

Hilbert space and the Hilbert space of an identical copy of the original string denoted by a tilde. The tilde string describes the reservoir degrees of freedom that interact, one by one, with the string degrees of freedom [22]. The Bogoliubov operator maps the total Hilbert space to the thermal Hilbert space which has a Fock space structure, that is, the thermal states can be obtained by acting with thermal creation and annihilation operators on the physical thermal vacuum $|0\,(\beta_T)\rangle\rangle$ [22]. However, in the case of the thermal magnetized D-brane in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$, there is an extra topological sector of the string Hilbert space as reviewed in the previous section. There is no TFD prescription for the thermalization of the topological modes and their inclusion in to the thermal vacuum state. Therefore, our main goal in this section is to generalize the TFD method to the string in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ and to determine the thermal operator that performs the thermalization of the topological sector. We will calculate the physical thermal vacuum of the closed string in the presence of the constant $U(1)$ field and the Kalb-Ramond field from the principles of the TFD.

3.1   Thermal vacuum of closed string

In the TFD, the thermal vacuum of a general system at the thermodynamical equilibrium is defined by postulating that for any observable $\widehat{A}$ the statistical average calculated with the canonical ensemble is the vacuum expectation value of $\widehat{A}$ in the thermal vacuum

$$\left\langle \widehat{A} \right\rangle = Z^{-1}\,(\beta_T)\, Tr\left[\widehat{A} e^{-\beta_T \widehat{H}}\right] = \left\langle\!\left\langle 0\,(\beta_T)\left|\widehat{A}\right|0\,(\beta_T)\right\rangle\!\right\rangle, \tag{23}$$

where $\beta_T = 1/k_B T$ [5].

The thermal string vacuum in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ can be obtained by generalizing this general TFD as follows. Since in the general definition (23) the trace is taken over all physical states of the system under consideration, in the case of string theory the relation (23) must be modified in order to produce a thermal vacuum from physical states only. The modification consists in imposing the constraint that the trace be taken over the states that are invariant under the translation of the worldsheet along the σ direction (the level matching condition). This condition is necessary and sufficient in the case of the bosonic string and the GS superstring [23, 26, 27, 48, 50]. It follows that, in order to construct the thermal vacuum from physical states, the relation (23) should be generalized to the following relations

$$\left\langle \widehat{A} \right\rangle = Z^{-1}\,(\beta_T)\,\sqrt{\alpha'}\int_0^1 d\lambda Tr\left[e^{-\beta_T \hat{H}(p^a,N,M,N_{osc})+2\pi i\lambda \hat{P}}\widehat{A}\right]$$

$$= \sqrt{\alpha'}\int d^{p-2}p \int_0^1 d\lambda \left\langle\!\left\langle 0\,(\beta_T,\lambda,p^a)\left|\delta\left(\hat{P}=0\right)\widehat{A}\right|0\,(\beta_T,\lambda,p^a)\right\rangle\!\right\rangle. \tag{24}$$

The action of the string momentum $\hat{P} = \hat{L}_0^l - \hat{L}_0^r$ on the states is defined by the relation (9). The trace over $\hat{P}$ involves the integral over the real Lagrange multiplier λ and the sum over the indices that label vectors from the physical subspace of the total Hilbert space [26, 46]. The same integral should be taken in the r.h.s. of the relation (24) where the thermal

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[5]If the Lorentz invariant product $\beta_T P^0$ is used to define the canonical ensemble, then the trace from (23) should be replaced by $\int dp^+ Tr[\hat{A}\exp(-\frac{\beta_T}{p^{\mp}}\widehat{H})]$ (see e.g. [63, 64]).

string vacuum dependes on the variable λ. The string Hamiltonian $\hat{H}\left(p^a, N, M, N_{osc}\right)$ is given by the sum of the two operators from (10) and (11). The string partition function is the trace of the identity operator

$$Z\left(\beta_T\right) = \sqrt{\alpha'} \int d^{p-2} p \int_0^1 d\lambda Tr\left[e^{-\beta_T \hat{H}\left(p^a, N, M, N_{osc}\right) + 2\pi i \lambda \hat{P}}\right]. \tag{25}$$

The thermal string vacuum $\left|0\left(\beta_T, \lambda, p^a\right)\right\rangle\rangle$ belongs to the total Hilbert space

$$\mathcal{H}_{phys}^{tot} = \mathcal{H}_{phys} \otimes \widetilde{\mathcal{H}}_{phys}. \tag{26}$$

After the projection onto the physical Hilbert subspace, one is left with the physical thermal vacuum $\left|0\left(\beta_T, p^a\right)\right\rangle\rangle$. The projections of $\mathcal{H}$ onto $\mathcal{H}_{phys}$ and of $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}_{phys}$ are given by the level matching condition for the string and the tilde string, respectively. In order to simplify the notation, we have introduced the following multi-indices: N and M represent all compact linear momenta and all winding numbers, respectively, while N_{osc} is the number of all left- and right-moving string oscillators along all directions from $\mathbb{R}^{p-1}$ and $\mathbb{T}^{d-p-1}$

$$|N\rangle = \bigotimes_{i=1}^{d-p-1} |n_i\rangle, |M\rangle = \bigotimes_{i=1}^{d-p-1} \left|m^i\right\rangle, |N_{osc}\rangle = \bigotimes_{n_1, n_2, \ldots = 1}^{\infty} |n_1, n_2, \ldots\rangle, \tag{27}$$

where

$$\widehat{n}_i |n_i\rangle = n_i |n_i\rangle, \quad \widehat{m}^i \left|m^i\right\rangle = m^i \left|m^i\right\rangle. \tag{28}$$

The string states are tensor products of the form

$$|N, M, N_{osc}\rangle = |N\rangle |M\rangle |N_{osc}\rangle. \tag{29}$$

The state $\left|0\left(\beta_T, p^a\right)\right\rangle\rangle$ can be obtained from the second equality of (23). In order to compute the integral over λ from the trace, one has to choose an explicit form of the metric. In what follows, we choose $\mathbb{T}^{d-p-1}$ to be flat with the metric $G_{\mu\nu} = \left(\delta_{ab}, \frac{R^2}{\alpha'}\delta_{ij}\right)$, where $R^i = R$ is the typical compactification radius. Without loss of generality, we can work in $\mathbb{R}^{p-1}$ in the reference frame of the string center of mass in which $\left|0\left(\beta_T, p^a\right)\right\rangle\rangle|_{c.m.} = \left|0\left(\beta_T\right)\right\rangle\rangle$. After a simple algebra, the relation (23) can be cast in to the following form

$$\left\langle\left\langle 0\left(\beta_T\right)\left|\delta\left(\widehat{P} = 0\right)\widehat{A}\right| 0\left(\beta_T\right)\right\rangle\right\rangle = Z^{-1}\left(\beta_T\right)\sqrt{\alpha'}$$

$$\times \sum_{N, M, N_{osc}} e^{-\beta_T \left[E\left(N, M, N_{osc}\right)\right]} \int_0^1 d\lambda e^{2\pi i \lambda P} A_{N, M, N_{osc}; N, M, N_{osc}}, \tag{30}$$

The notation $A_{N, M, N_{osc}; N, M, N_{osc}}$ stands for the matrix elements of $\hat{A}$ in the corresponding string states The eigenvalues of the hamiltonian from (30) have the following form

$$E\left(N, M, N_{osc}\right) = \frac{R^2}{2(\alpha')^{\frac{3}{2}}}\left[(m)^2 + (n - Bm)^2\right] + \frac{1}{2}\sum_{n>0}\left(E_{l,n}^{\mathbb{R}} + E_{r,n}^{\mathbb{R}}\right) + \frac{R^2}{2\alpha'}\sum_{n>0}\left(E_{l,n}^{\mathbb{T}} + E_{r,n}^{\mathbb{T}}\right), \tag{31}$$

where the energy of the left- and right-moving oscillators is $E_* = nN_{*,n}$ for $* = l, r$. The energy of zero modes is given by the first term in the above relation. The relation (30) defines the physical thermal vacuum and it must hold for any observable $\hat{A}$.

The thermal vacuum can be obtained from the equation (30) as follows. To begin with, we note that the integral over λ represents the integral form of the multivariable δ-function and can be written in terms of the orthonormal functions

$$\Psi_E(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-2\pi i E\lambda}, \quad \int_0^1 d\lambda \Psi_{E'}^*(\lambda)\Psi_E(\lambda) = \frac{1}{2\pi}\delta_{E',E}. \tag{32}$$

When these relations are applied to the left- and right-moving modes, they lead to the generalized level matching condition

$$E_{osc}^{\mathbb{R},r} = E_{osc}^{\mathbb{R},l}, \tag{33}$$

$$E_{osc}^{\mathbb{T},r} = E_{osc}^{\mathbb{T},l} + \frac{1}{\sqrt{\alpha'}}N^2, \tag{34}$$

where $N^2 = \sum_n n^2 = \sum_{n^i} \delta_{ij} n^i n^j$. The above relations should appear in the matrix elements of $\hat{A}$ from the r. h. s. of (30) after the integration over λ. Next, we decompose the thermal vacuum in the Fock basis tensored with the compact momentum and widing number basis of the total physical space [22],

$$|0(\beta_T)\rangle\rangle = \sum_{N,M,N_{osc}} [f(\beta_T)]_{N,M,N_{osc}} |N, M, N_{osc}\rangle |N, \widetilde{M, N_{osc}}\rangle, \tag{35}$$

where $[f(\beta_T)]_{N,M,N_{osc}}$ are complex coefficients. By substituting (35) into (30), we can show that the thermal vacuum has the following form

$$|0(\beta_T)\rangle\rangle = Z^{-\frac{1}{2}}(\beta_T) \sum_{N,M,N_{osc}} \exp\left[-\frac{\beta_T}{2}E(N, M, N_{osc})\right] |N, M, N_{osc}\rangle |N, \widetilde{M, N_{osc}}\rangle. \tag{36}$$

The partition function can be obtained from (25) and the normalization of the thermal vacuum to unity. In an arbitrary inertial frame, $Z(\beta_T)$ can be factorized as

$$Z(\beta_T) = Z_0(\beta_T) Z_{top}(\beta_T) Z_{osc}(\beta_T), \tag{37}$$

where

$$Z_0(\beta_T) = \int d^{p-2}k \left\langle k \left| \exp\left(-\frac{\beta_T}{2}\alpha\hat{p}^2\right) \right| k \right\rangle, \tag{38}$$

$$Z_{top}(\beta_T) = \sum_{N,M} \left\langle N, M \left| \exp\left\{-\beta_T \frac{R^2}{2(\alpha')^{\frac{3}{2}}}\left[(\hat{m})^2 + (\hat{n} - B\hat{m})^2\right]\right\} \right| N, M \right\rangle, \tag{39}$$

$$\begin{aligned}
Z_{osc}(\beta_T) = &\sum_{N_{osc}} \left\langle N_{osc} \left| \exp\left[-\frac{\beta_T}{2}\sum_{n>0} n\left(\hat{N}_{l,n} + \hat{N}_{r,n}\right)\right] \right| N_{osc} \right\rangle \\
&+ \int_0^1 d\lambda \sum_N \sum_{N_{osc}} \left\langle N_{osc} \left| \exp\left\{\pi i\lambda\left[\sum_{n>0} n\left(\hat{N}_{l,n} - \hat{N}_{r,n}\right) + \frac{R^2}{(\alpha')^{\frac{3}{2}}}\hat{N}^2\right]\right\} \right| N_{osc} \right\rangle,
\end{aligned} \tag{40}$$

where $N_{osc} = (N_{osc}^{\mathbb{R}}, N_{osc}^{\mathbb{T}})$, $\hat{N}_{l,n} = \hat{N}_{l,n}^{\mathbb{R}} + \frac{R^2}{\alpha'}\hat{N}_{l,n}^{\mathbb{T}}$ and $\hat{N}_{r,n} = \hat{N}_{r,n}^{\mathbb{R}} + \frac{R^2}{\alpha'}\hat{N}_{r,n}^{\mathbb{T}}$. The factor $Z_0(\beta_T)$ is a gaussian integral over the momenta of the center of mass in the $\mathbb{R}^{p-1}$ subspace. In the center of mass reference frame $Z_0(\beta_T)$ does not contribute to the partition function. The integral over λ from $Z_{osc}(\beta_T)$ can be easily computed by using the orthonormal functions from (32) and the result is the δ-function

$$\delta_{E_r, E_l + \frac{R^2 N^2}{2(\alpha')^{\frac{3}{2}}}}.$$

The sums over $N = (N_{l,n}^{\mathbb{R}}, N_{r,n}^{\mathbb{R}}, N_{l,n}^{\mathbb{T}} N_{r,n}^{\mathbb{T}})$ reduce to sums over the left-moving oscillator numbers by noting that the equations (33) and (34) are equivalent to

$$N_{osc}^{l,j} = N_{osc}^{r,,j}, \tag{41}$$

$$N_{osc}^{l,j} = N_{osc}^{r,j} + \frac{12R^2 N^2}{(d-p-1)(\alpha')^{\frac{3}{2}}}, \tag{42}$$

for $d > p+1$. For the maximal flat subspace $d = p+1$ it follows that $N = 0$ since there are no compact directions at all. Then it is easy to see that $Z_{osc}(\beta_T)$ has the following form

$$Z_{osc}(\beta_T) = \prod_{n=1}^{\infty} \left(1 - e^{-\beta_T n}\right)^{1-p} \left(1 - e^{-\frac{\beta_T R^2}{2\alpha'}n}\right)^{p+1-d}. \tag{43}$$

The topological partition function $Z_{top}(\beta_T)$ can be calculated, too, and one can show that it is given by the relation

$$Z_{top}(\beta_T) \sim \sum_{M} \prod_{j=p+1}^{d-1} \vartheta\left(-\frac{i\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}} B_k^j m^k; \frac{i\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}} \left[\frac{2(d-p)+22}{d-p-1}\right]\right)$$

$$\times \exp\left\{-\frac{\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}} \left[m_j^2 + \left(B_{jm}m^k\right)^2\right]\right\}, \tag{44}$$

where $\vartheta(z;\tau)$ is the Jacobi's theta function. Note that $\operatorname{Im}\tau > 0$ in either $d = 10$ or $d = 26$ for $d > p+1$ as required by the definition of the ϑ-function. By collecting the above results and by taking into account the normalization relations for the momenta (22), one can see that the thermal vacuum is given by the relation

$$|\, 0\,(\beta_T)\rangle\rangle = (2\pi)^{\frac{3-d}{2}}(\alpha')^{\frac{3-2p}{4}} R^{\frac{p+1-d}{2}} \beta_T^{\frac{p-2}{2}} \prod_{n=1}^{\infty} \left[\left(1 - e^{-\beta_T n}\right)^{1-p} \left(1 - e^{-\frac{\beta_T R^2}{2\alpha'}n}\right)^{p+1-d}\right]$$

$$\times \left\{\sum_{M} \prod_{j=p+1}^{d-1} \vartheta\left(-\frac{i\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}} B_k^j m^k; \frac{i\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}} \left[\frac{2(d-p)+22}{d-p-1}\right]\right)\right.$$

$$\left.\times \exp\left\{-\frac{\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}} \left[m_j^2 + \left(B_{jm}m^k\right)^2\right]\right\}\right\}^{-\frac{1}{2}}$$

$$\times \sum_{N,M,N_{osc}} \exp\left[-\frac{\beta_T}{2} E\left(N, M, N_{osc}\right)\right] |N, M, N_{osc}\rangle |\widetilde{N, M, N_{osc}}\rangle. \tag{45}$$

Some comments are in order now. One way to interpret the relation (43) is by recalling that the energy of the n-th mode is $\varepsilon_n = n$ in flat spacetime. Then one can see that the oscillators from $\mathbb{T}^{d-p-1}$ have the energy

$$\varepsilon'_n = \frac{R^2 n}{2\alpha'}, \quad \forall n > 0. \tag{46}$$

Another interpretation of the relation (43) is that the string modes on the torus behave as they would be at an effective temperature $T' = 2\alpha'T/R^2$ [6]. Of course that this is not true, since the full string should be at the thermodynamical equilibrium. However, this remark is important since the first major problem of determining the thermalization of the the string modes in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$, namely the thermalization of the string oscillators, is solved by the relation (45). Indeed, from it one can see that the mapping of the string modes to the finite temperature is realized by Bogoliubov operators. Along the directions of the $\mathbb{R}^{p-1}$, the Bogoliubov operators are the same as in the flat spacetime

$$\hat{G}(\beta_T) = \sum_{n=1}^{\infty}\sum_{\mu=2}^{d-1} \hat{G}_n^\mu(\beta_T), \qquad \hat{G}_n^\mu(\beta_T) = \hat{G}_{l,n}^\mu(\beta_T) + \hat{G}_{r,n}^\mu(\beta_T), \tag{47}$$

$$\hat{G}_{l,n}^\mu(\beta_T) = -i\theta_n(\beta_T)\left(\hat{\alpha}_n^{\mu\dagger}\widehat{\tilde{\alpha}}_n^{\mu\dagger} - \hat{\alpha}_n^\mu\widehat{\tilde{\alpha}}_n^\mu\right), \quad \hat{G}_{r,n}^\mu(\beta_T) = -i\theta_n(\beta_T)\left(\hat{\beta}_n^{\mu\dagger}\widehat{\tilde{\beta}}_n^{\mu\dagger} - \hat{\beta}_n^\mu\widehat{\tilde{\beta}}_n^\mu\right). \tag{48}$$

The function $\theta_n(\beta_T)$ depends on the temperature and on the oscillator frequency. Thus, it is the same in $\mathbb{R}^{p-1}$ and $\mathbb{T}^{d-p-1}$ and is given by the following equation

$$\cosh\theta_n(\beta_T) = \left(1 - e^{-\beta_T n}\right)^{-1/2}. \tag{49}$$

However, for the $\mathbb{T}^{d-p-1}$ submanifold, the argument of $\theta_n(\beta_T)$ should be replaced by $\theta_n(\beta_T R^2/2\alpha')$. The form of the function $\theta_n(\beta_T)$ given by the relation (49) remains unchanged under the multiplication of the temperature variable by $R^2/2\alpha'$. It follows that, in order to thermalize the string modes in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$, one should apply the same method as in the flat spacetime with the only difference of the rescaled temperature in the compact subspace.

The second important problem of the thermal string in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ is the thermalization of the topological states, which is a process that is not defined by the TFD method. Indeed, in the TFD approach, the thermalization relies on the Fock space structure of the oscillator Hilbert space while the topological Hilbert subspace does not have this structure. From (45), we can see that the topological sector is not mapped to the finite temperature by a Bogoliubov operator. In fact, one can factorize the thermal vacuum as

$$|0(\beta_T)\rangle\rangle \sim \beta_T^{\frac{p-2}{T^2}} |0(\beta_T)\rangle\rangle_{top} \sum_{N_{osc}} e^{-\frac{\beta_T}{2}E(N_{osc})} |N_{osc}\rangle \widetilde{|N_{osc}\rangle}. \tag{50}$$

In order to find the map between $|0\rangle\rangle_{top}$ and $|0(\beta_T)\rangle\rangle_{top}$ we introduce the following algebra $\left\{\hat{n}_i, \hat{n}_i^+, \hat{n}_i^-\right\}$

$$\left[\hat{n}_i^+, \hat{n}_j^-\right] = 0, \quad \left[\hat{n}_i, \hat{n}_j^+\right] = \hat{n}_i^+\delta_{ij}, \quad \left[\hat{n}_i, \hat{n}_j^-\right] = -\hat{n}_i^-\delta_{ij}, \tag{51}$$

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[6]However, since one expects that the whole string be at thermal equilibrium, this statement should not be taken literally.

with

$$\left(\hat{n}_i^+\right)^\dagger = \hat{n}_i^-, \quad \left(\hat{n}_i^-\right)^\dagger = \hat{n}_i^+. \tag{52}$$

The action of the above operators on the states $\{|n_i\rangle\}$ is given by the following relations

$$\hat{n}_i |n_i\rangle = n_i |n_i\rangle, \tag{53}$$
$$\hat{n}_i^+ |n_i\rangle = |n_i + 1\rangle, \tag{54}$$
$$\hat{n}_i^- |n_i\rangle = |n_i - 1\rangle. \tag{55}$$

The operators $\hat{n}_i^+$ and $\hat{n}_i^-$ create and annihilate quanta of the momentum of the center of mass along the compact direction i with arbitrary eigenvalues $n_i \in \mathbb{Z}$. An identical algebra $\{\hat{m}_i, \hat{m}_i^+, \hat{m}_i^-\}$ can be introduce to describe the increase and the decrease of the winding number, with the same unbounded eigenvalues [7]. Then the topological vacuum can be written as

$$|0\left(\beta_T\right)\rangle\rangle_{top} = \hat{\Omega}\left(\beta_T\right)|0\rangle\rangle_{top}, \tag{56}$$

where

$$\hat{\Omega}\left(\beta_T\right) = Z_{top}^{-\frac{1}{2}}(\beta_T) \sum_{N,M} \prod_{i=p+1}^{d-1} \exp\left[-\frac{\beta_T}{2}E_0(N,M)\right] \left[\hat{n}_i^\pm \widehat{\widetilde{n}}_i^\pm\right]^{|n_i|} \left[\hat{m}_i^\pm \widehat{\widetilde{m}}_i^\pm\right]^{|m_i|}, \tag{57}$$

where for negative values of n_i and m_i in the multi-indices N and M, respectively, the operators are $\hat{n}_i^-$, etc. The operator $\hat{\Omega}\left(\beta_T\right)$ maps the topological vacuum at zero temperature to

$$
\begin{aligned}
|0\left(\beta_T\right)\rangle\rangle_{top} \sim & \left\{\sum_M \prod_{j=p+1}^{d-1} \vartheta\left(-\frac{i\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}}B_k^j m^k; \frac{i\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}}\left[\frac{2(d-p)+22}{d-p-1}\right]\right)\right. \\
& \times \left. \exp\left\{-\frac{\beta_T R^2}{2\pi(\alpha')^{\frac{3}{2}}}\left[m_j^2 + \left(B_{jm}m^k\right)^2\right]\right\}\right\}^{-\frac{1}{2}} \\
& \times \sum_{N,M} \exp\left[-\frac{\beta_T}{2}E_0\left(N,M\right)\right]|N,M\rangle\,|\widetilde{N,M}\rangle,
\end{aligned} \tag{58}
$$

where the zero mode energy is

$$E_0\left(N,M\right) = \frac{R^2}{2(\alpha')^{\frac{3}{2}}}\left[(m)^2 + (n - Bm)^2\right]. \tag{59}$$

Thus, the solution to the thermalization of the topological sector of the string is the relation (58) which has been obtained from the basic postulate of the TFD given by the equation (24). It represents the generalization of the TFD thermal vacuum to the topological sector of the closed strings and defines the thermal states for the center of mass of momenta and the winding states in $\mathbb{T}^{d-p-1}$, respectively. Also, we have given the explicit form of the map $\hat{\Omega}\left(\beta_T\right)$ betweeen the topological vacuum at zero temperature and at finite temperature in terms of the algebra (51). The operator $\hat{\Omega}\left(\beta_T\right)$ represents the generalization of the thermal Bogoliubov operator $\hat{G}(\beta_T)$ to the topological sector.

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[7] A concrete representation of the algebra $\{\hat{n}_i, \hat{n}_i^+, \hat{n}_i^-\}$ can be given in terms of the wave functions $\langle\sigma|n_j\rangle = \varphi_{n_j}(\sigma) = \exp(in_j\sigma)$ with $\hat{n}_j = -i\partial_\sigma, \hat{n}_j^+ = \exp(i\sigma)$ and $\hat{n}_j^- = \exp(-i\sigma)$.

3.2 Thermodynamics of closed thermal string

The thermodynamic variables of the closed thermal string can be calculated from the partition function derived in the previous subsection. However, that calculation cannot be applied to the thermal D-branes since in the derivation of (43) and (44) only the thermal vacuum has been used. Another possibility is to calculate the thermodynamic variables from the entropy. From the TFD relation (30), the entropy of string is given by the expectation value of the entropy operator in the thermal vacuum

$$\frac{1}{k_B}S(\beta_T) = \left\langle\left\langle 0\,(\beta_T)\left|\hat{K}(\beta_T)\right|0\,(\beta_T)\right\rangle\right\rangle, \tag{60}$$

where $|0\,(\beta_T)\rangle\rangle$ is given by the equation (45). The analysis from the previous subsection has shown that, in the reference frame of the center of mass, the thermal string degrees of freedom can be split in to an oscillator factor and a topological factor

$$\hat{K}(\beta_T) = \hat{K}_{osc}(\beta_T) + \hat{K}_{top}(\beta_T). \tag{61}$$

According to the TFD method [22], the entropy operator corresponding to the string oscillators $\hat{K}_{osc}(\beta_T)$ is related to the Bogoliubov operator (48) by the relation

$$e^{-i\hat{G}(\beta_T)} = e^{-\frac{1}{2}\hat{K}_{osc}(\beta_T)}\exp\left(\sum_{n,\mu}\hat{\alpha}_n^{\mu\dagger}\widehat{\tilde{\alpha}}_n^{\mu\dagger} + \hat{\beta}_n^{\mu\dagger}\widehat{\tilde{\beta}}_n^{\mu\dagger}\right). \tag{62}$$

The explicit form of $\hat{K}_{osc}(\beta_T)$ can be obtained from the above equation and it is given by the following relations

$$\hat{K}_{osc}(\beta_T) = \hat{K}_{osc}^{\mathbb{R}}(\beta_T) + \hat{K}_{osc}^{\mathbb{T}}(\beta_T), \tag{63}$$

$$\hat{K}_{osc}^{\mathbb{R}}(\beta_T) = -\sum_{a=2}^{p}\sum_{n=0}^{\infty}\left\{\left(\hat{N}_{l,n}^a + \hat{N}_{r,n}^a\right)\ln\left[\frac{e^{-\beta_T n}Z_n(\beta_T)}{e^{-\beta_T n}Z_n(\beta_T) + 1}\right] - \ln[e^{-\beta_T n}Z_n(\beta_T) + 1]\right\}, \tag{64}$$

$$\hat{K}_{osc}^{\mathbb{T}}(\beta_T) = -\frac{R^2}{2\alpha'}\sum_{j=p+1}^{d-1}\sum_{n=0}^{\infty}\left\{\left(\hat{N}_{l,n}^j + \hat{N}_{r,n}^j\right)\ln\left[\frac{e^{-\frac{\beta_T R^2}{2\alpha'}n}Z_n(\frac{\beta_T R^2}{2\alpha'})}{e^{-\frac{\beta_T R^2}{2\alpha'}n}Z_n(\frac{\beta_T R^2}{2\alpha'}) + 1}\right] - \ln[e^{-\frac{\beta_T R^2}{2\alpha'}n}Z_n(\frac{\beta_T R^2}{2\alpha'}) + 1]\right\}, \tag{65}$$

where $Z_n(\beta_T)$ is the partition function of a single oscillator. Note that the equation (63) defines the entropy of the closed string oscillators only. For the reservoir, a similar operator can be constructed by replacing the string operators with the corresponding tilde operators. However, since we are interested in the entropy of the string, it is not necessary to compute the entropy of the reservoir. By plugging (64) and (65) into (60) and after some algebra, one can show that the entropy of the closed string oscillators has the following form

$$S_{osc}(\beta_T) = (p-1)k_B\sum_{n=1}^{\infty}\left[\beta_T n e^{-\beta_T n}Z_n(\beta_T) - \ln Z_n(\beta_T)\right]$$

$$+ (p+1-d)k_B\frac{R^2}{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{\beta_T R^2 n e^{-\frac{\beta_T R^2}{2\alpha'}n}}{2\alpha'}Z_n\left(\frac{\beta_T R^2}{2\alpha'}\right) - \ln Z_n\left(\frac{\beta_T R^2}{2\alpha'}\right)\right] \tag{66}$$

where N_n is the eigenvalue of the number operator of the oscillator of frequency n. The factor of two is from the contributions of the left- and right-moving sectors, respectively. We note that at the critical radius $R_0 = \sqrt{2\alpha'}$ the equation (66) reproduces the entropy of $d-2$ transversal massless scalar fields.

The contribution of the topological sector to the entropy of string (60) cannot be calculated in the same way as the entropy of the oscillators because there is no TFD definition for the topological entropy operator of the closed string in $\mathbb{T}^{d-p-1}$. Let us construct this operator. According to the TFD principles expressed by the relation (60), the topological entropy is defined as

$$\frac{1}{k_B} S_{top}(\beta_T) = \left\langle\!\!\left\langle 0\,(\beta_T) \left| \hat{K}_{top}(\beta_T) \right| 0\,(\beta_T) \right\rangle\!\!\right\rangle. \tag{67}$$

This definition must be consistent with the derivation of the entropy from the partition function which is given by the thermodynamical relation

$$S_{top}(\beta_T) = -\frac{\partial}{\partial T}\left[k_B T \ln Z_{top}(\beta_T)\right]. \tag{68}$$

By using (44) in the above relation, one can see that the entropy operator in the topological sector of the closed string has the form

$$\hat{K}_{top}(\beta_T) = (2\pi R)^{p+1-d}\left[\frac{\beta_T}{2}\hat{H}_{top}\exp\left(\frac{\beta_T}{2}\hat{H}_{top}\right) + (2\pi R)^{(p+1-d)}Z_{top}(\beta_T)\ln Z_{top}(\beta_T)\exp\left(\beta_T\hat{H}_{top}\right)\right], \tag{69}$$

where

$$\hat{H}_{top} = \hat{H}_0 + \frac{R^2}{2(\alpha')^{\frac{3}{2}}}\hat{N}^2, \tag{70}$$

and $\hat{H}_0$ represents the topological part of the Hamiltonian with the eigenvalues $E_0\,(N,M)$ given by the relation (59). The vacuum expectation value of the operator $\hat{K}_{top}(\beta_T)$ reproduces correctly the topological entropy

$$S_{top}(\beta_T) = k_B \ln Z_{top}(\beta_T) + \frac{k_B\beta_T}{2}Z_{top}^{-1}(\beta_T)\sum_{N,M}E_{top}(N,M)e^{-\frac{\beta_T}{2}E_{top}(N,M)}, \tag{71}$$

where

$$E_{top}(N,M) = \frac{R^2}{2(\alpha')^{\frac{3}{2}}}\left[(m)^2 + (n-Bm)^2\right] + \frac{R^2}{2(\alpha')^{\frac{3}{2}}}n^2, \tag{72}$$

are the eigenvalues of the operator $\hat{H}_{top}$.

The free energy of the thermal closed string can be computed as the vacuum expectation value in the thermal vacuum of the operator

$$\hat{F}(\beta_T) = \hat{H} - \frac{1}{\beta_T}\hat{K}(\beta_T), \tag{73}$$

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where $S = S_{osc}+S_{top}$ and $\hat{H} = \hat{H}^{\mathbb{R}}+\hat{H}^{\mathbb{T}}$ defined by the equations (10) and (11), respectively. Calculation of the vacuum expectation value of $\hat{F}$ in the thermal vacuum gives

$$F(\beta_T) = \beta_T^{-1}\left[(p-1)\sum_{n=1}^{\infty}\ln Z_n\left(\beta_T\right) + (d-p-1)\frac{R^2}{2\alpha'}\sum_{n=1}^{\infty}\ln Z_n\left(\frac{\beta_T R^2}{2\alpha'}\right) + \ln Z_{top}\left(\beta_T\right).\right]$$
(74)

Again, we see that at the critical radius $\sqrt{2\alpha'}$ the free energy is the sum of the free energy of $d-2$ massless scalar fields and the topological free energy of the zero modes. For the maximal flat subspace $d = p+1$, i. e. in the absence of the compact directions, the free energy is obtained from the oscillators since in the last term all n_i and m_i are zero.

4 Thermal magnetized D-branes

In this section, we are going to derive the magnetized thermal D-brane states in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ by applying the generalized TFD formalism derived in the previous section. The D-brane boundary states at finite temperature in flat spacetime were defined in [23] as the states from the thermal closed string Hilbert space that satisfy the same Dirichlet and Neumann boundary conditions as the ones at zero temperature. This ammounts to imposing two sets of boundary conditions: one for the string degrees of freedom and the other for the reservoir degrees of freedom. The same definition can be apply in $\mathbb{R}^{p-1}\times\mathbb{T}^{d-p-1}$ and the thermal magnetized D-brane states can be calculated. From the thermal boundary state we can calculate the entropy and the free energy of the magnetized D-brane at finite temperature as the expectation value of the entropy operator and the free energy operator, respectively, in the thermal magnetized D-brane state.

4.1 Thermal magnetized D-branes states

The thermalization of the closed string is described by the operators $\hat{\Omega}\left(\beta_T\right)$ and $\hat{G}(\beta_T)$ which map the total system from zero temperature to the finite temperature which implies that the generic string operators $\hat{O}$ transform to $\hat{O}(\beta_T)$ as

$$\hat{O} \longrightarrow \hat{O}(\beta_T) = \left[\hat{\Omega}\left(\beta_T\right)\otimes e^{i\hat{G}(\beta_T)}\right]\hat{O}\left[e^{-i\hat{G}(\beta_T)}\otimes\hat{\Omega}^{-1}\left(\beta_T\right)\right].$$
(75)

In the flat spacetime, the thermal D-brane boundary states have been defined by two sets of boundary conditions obtained from the total lagrangian at finite temperature and imposed on the Hilbert space and on the tilde Hilbert space, respectively [23]. We can generalize this definition to the magnetized D-branes in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ and look for the solutions

$|B(\beta_T)\rangle\rangle$ of the following equations

$$\hat{p}^a(\beta_T)\,|B(\beta_T)\rangle\rangle = 0, \tag{76}$$

$$\left(\hat{n}_i - 2\pi\alpha' q F_{ij}\hat{m}^j\right)(\beta_T)\,|B(\beta_T)\rangle\rangle = 0, \tag{77}$$

$$\left[(\mathbf{1}+\mathcal{B})_{ab}\,\hat{\alpha}_n^b(\beta_T) + (\mathbf{1}-\mathcal{B})_{ab}^T\,\hat{\beta}_n^{b\dagger}(\beta_T)\right](\beta_T)\,|B(\beta_T)\rangle\rangle = 0, \tag{78}$$

$$\left[(\mathbf{1}+\mathcal{B})_{ab}\,\hat{\alpha}_n^{b\dagger}(\beta_T) + (\mathbf{1}-\mathcal{B})_{ab}^T\,\hat{\beta}_n^{b}(\beta_T)\right](\beta_T)\,|B(\beta_T)\rangle\rangle = 0, \tag{79}$$

$$\left(\mathcal{E}_{ij}\hat{\beta}_n^j(\beta_T) + \mathcal{E}_{ij}^T\hat{\alpha}_n^{j\dagger}(\beta_T)\right)|B(\beta_T)\rangle\rangle = 0, \tag{80}$$

$$\left(\mathcal{E}_{ij}\hat{\beta}_n^{j\dagger}(\beta_T) + \mathcal{E}_{ij}^T\hat{\alpha}_n^{j}(\beta_T)\right)|B(\beta_T)\rangle\rangle = 0, \tag{81}$$

for all $n > 0$. Here, the operators at finite temperature have been obtained by applying the map (75) to the boundary operators from the equations (12) and (14)-(17). Similar equations should be written for the tilde operators. In order to solve the above set of equations, we firstly note that the solution can be factorized as

$$|B(\beta_T)\rangle\rangle = |B(\beta_T)\rangle\rangle_{top}\,|B(\beta_T)\rangle\rangle_{osc}. \tag{82}$$

Then one can easily show that $|B(\beta_T)\rangle\rangle_{top}$ and $|B(\beta_T)\rangle\rangle_{osc}$ can be written in terms of zero temperature D-branes as

$$|B(\beta_T)\rangle\rangle_{top} = \hat{\Omega}\,(\beta_T)\,|B\rangle\rangle_{top}^{\mathbb{T}} = \hat{\Omega}\,(\beta_T)\,|B\rangle_{top}^{\mathbb{T}}\,\widetilde{|B\rangle}_{top}^{\mathbb{T}}, \tag{83}$$

$$|B(\beta_T)\rangle\rangle_{osc} = e^{i\hat{G}(\beta_T)}\,|B\rangle\rangle_{osc} = e^{i\hat{G}(\beta_T)}\,|B\rangle_{osc}\,\widetilde{|B\rangle}_{osc}, \tag{84}$$

where $|B\rangle\rangle_{top}$ and $|B\rangle\,B\rangle_{osc} = |B\rangle\rangle_{osc}^{\mathbb{R}} \otimes |B\rangle\rangle_{osc}^{\mathbb{T}}$ belong to the corresponding total Hilbert spaces at zero temperature and represent two copies of the states given in the equations (21) and (19) and (20), respectively. The relations (83) and (84) represent the mapping of the magnetized D-brane states from the total Hilbert space at zero temperature to the thermal magnetized D-brane space from the Hilbert space of the thermal closed string. Then after some algebra, the solution to the system (76)-(81) is found to be

$$|B(\beta_T)\rangle\rangle = N_{p-1}^2(\mathcal{B})\,N_{d-p-1}^2(\mathcal{B})\prod_{i=p+1}^{d-1}\sum_{n_i}\delta_{n_i-2\pi\alpha' q F_i^{k_i}m_{k_i}}\left[\hat{n}_i^{\pm}\widehat{\tilde{n}}_i^{\pm}\right]^{|n_i|}(\beta_T)\left[\hat{m}_i^{\pm}\widehat{\tilde{m}}_i^{\pm}\right]^{|m_i|}(\beta_T)$$

$$\otimes\left(\prod_{n=1}^{\infty}\exp\left\{-\left[\hat{\alpha}_n^{a\dagger}(\beta_T)M_{ab}\hat{\beta}_n^{b\dagger}(\beta_T) + \widehat{\tilde{\alpha}}_n^{a\dagger}(\beta_T)\tilde{M}_{ab}\widehat{\tilde{\beta}}_n^{b\dagger}(\beta_T)\right]\right\}\right)|0(\beta_T)\rangle\rangle_{osc}^{\mathbb{R}}$$

$$\otimes\left(\prod_{n=1}^{\infty}\exp\left\{-\frac{R^2}{\alpha'}\left[\hat{\alpha}_n^{i\dagger}(\frac{\beta_T R^2}{2\alpha'})\mathcal{S}_{ij}\hat{\beta}_n^{j\dagger}(\frac{\beta_T R^2}{2\alpha'}) + \widehat{\tilde{\alpha}}_n^{i\dagger}(\frac{\beta_T R^2}{2\alpha'})\mathcal{S}_{ij}\widehat{\tilde{\beta}}_n^{j\dagger}(\frac{\beta_T R^2}{2\alpha'})\right]\right\}\right)\left|0(\frac{\beta_T R^2}{2\alpha'})\right\rangle\rangle_{osc}^{\mathbb{T}}, \tag{85}$$

where $\mathcal{S}_{ij} = (\mathcal{E}^{-1})_i^l\,(\mathcal{E}^T)_{lj}$. The above relation represents the thermal magnetized D-brane. It shows that the magnetized D-brane at finite temperature is a vector from the thermal total Hilbert space. In the absence of the topological sector and with $d = p+1$ it reduces to the already known D-branes at finite temperature [23][24]. Note that the background fields

from (85) are not thermalized, i. e. the average values of the corresponding string states are taken to be the same as at zero temperature. Therefore, one can take $M_{ab} = \tilde{M}_{ab}$ and $\mathcal{E}_{ij} = \tilde{\mathcal{E}}_{ij}$. As the rest of thermal string states, there is no simple interpretation of $|B(\beta_T)\rangle\rangle$ in terms of magnetized D-branes at zero temperature.

4.2 Thermodynamics of magnetized D-branes

The thermodynamic properties of the magnetized D-branes at finite temperature can be derived from their entropy. The entropy and the free energy of the thermal D-branes are defined as the expectation values of $\hat{K}$ and $\hat{F}$ operators from the equations (61) and (73), respectively, in the state $|B(\beta_T)\rangle\rangle$

$$\frac{1}{k_B} S_D(\beta_T) = \left\langle\!\left\langle B\left(\beta_T\right)\left|\hat{K}(\beta_T)\right| B\left(\beta_T\right)\right\rangle\!\right\rangle. \tag{86}$$

Then one can write

$$S_D(\beta_T) = S_D(\beta_T)_{osc} + S_D(\beta_T)_{top}, \tag{87}$$

$$S_D(\beta_T)_{osc} = S_D(\beta_T)_{osc}^{\mathbb{R}} + S_D(\beta_T)_{osc}^{\mathbb{T}}, \tag{88}$$

The computations are somewhat lengthy but straightforward. The main technical detail is that the Bogoliubov transformations can be linearized [22]

$$\hat{\alpha}_n^a = \sqrt{Z_n\left(\beta_T\right)}\hat{\alpha}_n^a\left(\beta_T\right) + \sqrt{Z_n\left(\beta_T\right) - 1}\widehat{\tilde{\alpha}}_n^{a\dagger}\left(\beta_T\right), \tag{89}$$

$$\hat{\alpha}_n^{a\dagger} = \sqrt{Z_n\left(\beta_T\right)}\hat{\alpha}_n^{a\dagger}\left(\beta_T\right) + \sqrt{Z_n\left(\beta_T\right) - 1}\widehat{\tilde{\alpha}}_n^a\left(\beta_T\right), \tag{90}$$

with similar relations holding for the $\hat{\beta}$ operators and tilde operators. Note that in $\mathbb{T}^{d-p-1}$, β_T should be replaced by $\beta_T R^2/2\alpha'$. The contribution of the oscillators from $\mathbb{R}^{p-1}$ to the entropy of the magnetized D-brane is

$$S_D(\beta_T)_{osc}^{\mathbb{R}} = -2V_{D,top}^{\mathbb{T}} V_{D,osc}^{\mathbb{T}} k_B \times$$

$$\left\{\sum_{n=1}^{\infty}\sum_{a=2}^{p}\left[\prod_{m\neq n=1}^{\infty}\prod_{c\neq a=2}^{p}\prod_{d=2}^{p}\left[\sum_{k_{cd}^m=0}^{\infty}(M_{cd})^{2k_{cd}^m}\right]\right]\right.$$

$$\times \left[2Z_n\left(\beta_T\right) - 1\right]\ln\left(\frac{e^{-\beta_T n}Z_n\left(\beta_T\right)}{e^{-\beta_T n}Z_n\left(\beta_T\right) + 1}\right)\left[\sum_{s_{ad}^n=0}^{\infty}s_{ad}^n\left(M_{cd}\right)^{2s_{ad}^n}\right]$$

$$\left. +(p-1)V_{D,osc}^{\mathbb{R}}\sum_{n=1}^{\infty}\left\{\left[Z_n\left(\beta_T\right) - 1\right]\ln\left(\frac{e^{-\beta_T n}Z_n\left(\beta_T\right)}{e^{-\beta_T n}Z_n\left(\beta_T\right) + 1}\right) + \ln\left[e^{-\beta_T n}Z_n\left(\beta_T\right) + 1\right]\right\}\right\}. \tag{91}$$

Here, we have denoted by $V_{D,top}^{\mathbb{T}}$, $V_{D,osc}^{\mathbb{T}}$ and $V_{D,osc}^{\mathbb{R}}$ the norms of $|B\left(\beta_T\right)\rangle\rangle_{top}^{\mathbb{T}}$, $|B\left(\beta_T\right)\rangle\rangle_{osc}^{\mathbb{T}}$ and $|B\left(\beta_T\right)\rangle\rangle_{osc}^{\mathbb{R}}$, respectively. Since the Bogoliubov operator is unitary, the norm of the oscillator boundary states is the square of the of the norm of the corresponding boundary states at zero temperature. The terms $(M_{cd})^{2k_{cd}^m}$ represent the matrix elements of $M_{ab} =$

$\tilde{M}_{ab}$ at the corresponding power. The overall factor of 2 in front of all terms is a result of adding the entropy of the left-moving and the right-moving oscillators. The entropy of the oscillators in $\mathbb{T}^{d-p-1}$ can be calculated in the same way as $S_D(\beta_T)_{osc}^{\mathbb{R}}$ and the result is formally the same

$$
S_D(\beta_T)_{osc}^{\mathbb{T}} = -2V_{D,top}^{\mathbb{R}}V_{D,osc}^{\mathbb{R}}k_B \times
$$
$$
\left\{ \sum_{n=1}^{\infty} \sum_{i=p+1}^{d-1} \left[\prod_{m \neq n=1}^{\infty} \prod_{l \neq i=p+1}^{d-1} \prod_{r=p+1}^{d-1} \left[\sum_{k_{lr}^m=0}^{\infty} (\mathcal{S}_{lr})^{2k_{lr}^m} \right] \right] \right.
$$
$$
\times \left[2Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right) - 1 \right] \ln \left(\frac{e^{-\frac{\beta_T R^2}{2\alpha'} n} Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right)}{e^{-\frac{\beta_T R^2}{2\alpha'} n} Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right) + 1} \right) \left[\sum_{t_{ir}^n=0}^{\infty} t_{ir}^n (\mathcal{S}_{lr})^{2t_{lr}^m} \right]
$$
$$
+ (d-p-1)V_{D,osc}^{\mathbb{T}} \sum_{n=1}^{\infty} \left[Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right) - 1 \right] \ln \left(\frac{e^{-\frac{\beta_T R^2}{2\alpha'} n} Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right)}{e^{-\frac{\beta_T R^2}{2\alpha'} n} Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right) + 1} \right)
$$
$$
+ (d-p-1)V_{D,osc}^{\mathbb{T}} \sum_{n=1}^{\infty} \ln \left[e^{-\frac{\beta_T R^2}{2\alpha'} n} Z_n \left(\frac{\beta_T R^2}{2\alpha'} \right) + 1 \right] \right\}. \tag{92}
$$

The contribution of the topological sector to the entropy can be calculated from the relations (69) and (85) and the result is

$$
S_D(\beta_T)_{top} = R^{p+1-d} \alpha'^{\frac{d-p-1}{2}} V_{D,osc}^{\mathbb{R}} V_{D,osc}^{\mathbb{T}} k_B Z_{top}^{-1}(\beta_T) \times
$$
$$
\left\{ \sum_{N,M}^{\infty} \sum_{K,S}^{p+1} (2\pi R)^{p+1-d} Z_{top}(\beta_T) \ln Z_{top}(\beta_T) \exp[\beta_T E_{top}(N \pm K, M \pm S)] \right.
$$
$$
\left. + \frac{\beta_T}{2} E_{top}(N \pm K, M \pm S) \exp\left[\frac{\beta_T}{2} E_{top}(N \pm K, M \pm S) \right] \right\}. \tag{93}
$$

The sum of the entropies (91), (92) and (93) represents the entropy of the magnetized D-brane at finite temperature in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$. It can be used to derive the rest of the thermodynamic potentials of the D-brane in the usual fashion since the thermal D-brane is at the thermodynamical equilibrium. This is left as an exercise to the reader.

5   Conclusions

In this paper, we have constructed the thermal magnetized D-brane boundary states in $\mathbb{R}^{p-1} \times \mathbb{T}^{d-p-1}$ and derived their entropy. This represents a generalization of the previous results from [23, 24] and [27] where the thermal bosonic and GS D-branes were constructed and their entropy was calculated in the flat spacetime. In order to obtained the thermal boundary states, we have generalized the TFD formalism to include the zero-mode sector of the closed strings in $\mathbb{T}^{d-p-1}$ and we have used this generalization to obtain the entropy and the free energy of the closed string at finite temperature. The generalization of the TFD formalism is an interesting result by itself, since it extends the method of the canonical

thermalization to non-trivial boundary conditions and generalizes the previous studies from [61, 62] that establish the form of the Bogoliubov operator for a scalar and spinor field on $\mathbb{T}^{d-p-1}$ but without the winding conditions which are specific to the bosonic string fields. With these boundary conditions, the map from the total Hilbert space at zero temperature to the Hilbert space at finite temperature must be generalized to the direct product given in $\hat{\Omega}(\beta_T) \otimes e^{i\hat{G}(\beta_T)}$. The operator $\hat{\Omega}(\beta_T)$ takes the states from the zero mode sector to finite temperature and allows one to extend the TFD method to the topological sector.

For the future directions it is an interesting subject to study the generalization of the results from this paper to the thermalized magnetized D-branes obtained from the supersymmetric magnetized D-branes at finite temperature. While the construction of the thermal D-branes from supersymmetric D-branes has been carried out in [27] in the flat spacetime and with no background fields in the GS approach, the thermalization of the supersymmetric D-branes in the RNS formalism is still an open problem even in the flat spacetime.

The analysis of the thermodynamical properties of the thermal magnetized D-branes in detail is a very challenging problem. Due to the presence of infinite string modes and of the multiplication by the norm of the D-brane state and the normalization constants $N_{p-1}(\mathcal{B})$ and $N_{d-p-1}(\mathcal{B})$, several quantities are divergent and should be renormalized before extracting any physical information from them. This is a general problem of the boundary state description of the D-branes at zero or finite temperature. Nevertheless, interesting information could be obtained at a finite energy scale where only a finite number of string modes contribute. This truncation could be helpful to the analysis of the thermodynamics of the magnetized D-branes as a function of radius which is an interesting problem, too, and which requires a carefully treatment of the T-duality of the compact directions at finite temperature.

Acknowledgments

References

[1] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, Phys. Rept. **445**, 1 (2007) [arXiv:hep-th/0610327].

[2] R. Blumenhagen, B. Kors and D. Lust, JHEP **0102**, 030 (2001) [arXiv:hep-th/0012156].

[3] M. Larosa, arXiv:hep-th/0212109.

[4] R. Blumenhagen, D. Lust and T. R. Taylor, Nucl. Phys. B **663**, 319 (2003) [arXiv:hep-th/0303016].

[5] J. F. G. Cascales and A. M. Uranga, arXiv:hep-th/0311250.

[6] L. Gorlich, arXiv:hep-th/0401040.

[7] I. Antoniadis and S. Dimopoulos, Nucl. Phys. B **715**, 120 (2005) [arXiv:hep-th/0411032].

[8] I. Antoniadis, A. Kumar and B. Panda, Nucl. Phys. B **795**, 69 (2008) [arXiv:0709.2799 [hep-th]].

[9] C. M. Chen, T. Li and D. V. Nanopoulos, Nucl. Phys. B **732**, 224 (2006) [arXiv:hep-th/0509059].

[10] M. Bianchi and E. Trevigne, JHEP **0508**, 034 (2005) [arXiv:hep-th/0502147].

[11] I. Antoniadis, A. Kumar and T. Maillard, arXiv:hep-th/0505260.

[12] M. Billo, M. Frau, I. Pesando, P. Di Vecchia, A. Lerda and R. Marotta, JHEP **0712**, 051 (2007) [arXiv:0709.0245 [hep-th]].

[13] M. Billo, arXiv:0804.0755 [hep-th].

[14] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, JHEP **0903**, 029 (2009) [arXiv:0810.5509 [hep-th]].

[15] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, Fortsch. Phys. **57**, 718 (2009) [arXiv:0901.4458 [hep-th]].

[16] I. Antoniadis, arXiv:0710.4267 [hep-th].

[17] P. Di Vecchia, A. Liccardo, R. Marotta, F. Pezzella and I. Pesando, arXiv:hep-th/0601067.

[18] D. Duo, R. Russo and S. Sciuto, JHEP **0712**, 042 (2007) [arXiv:0709.1805 [hep-th]].

[19] P. Di Vecchia, A. Liccardo, R. Marotta, I. Pesando and F. Pezzella, JHEP **0711**, 100 (2007) [arXiv:0709.4149 [hep-th]].

[20] I. Pesando, arXiv:hep-th/0505052.

[21] I. Pesando, arXiv:0910.2576 [hep-th].

[22] H. Umezawa, Advanced field theory: Micro, macro, and thermal physics, *New York, USA: AIP (1993)* .

[23] I. V. Vancea, Phys. Lett. B **487**, 175 (2000) [arXiv:hep-th/0006228].

[24] M. C. B. Abdalla, A. L. Gadelha and I. V. Vancea, Phys. Rev. D **64**, 086005 (2001) [arXiv:hep-th/0104068].

[25] M. C. B. Abdalla, A. L. Gadelha and I. V. Vancea, Phys. Rev. D **66**, 065005 (2002) [arXiv:hep-th/0203222].

[26] I. V. Vancea, Phys. Rev. D **74**, 086002 (2006) [arXiv:hep-th/0607167].

[27] I. V. Vancea, Int. J. Mod. Phys. A **23**, 4485 (2008) [arXiv:0712.1569 [hep-th]].

[28] M. C. B. Abdalla, A. L. Gadelha and I. V. Vancea, Int. J. Mod. Phys. A **18**, 2109 (2003) [arXiv:hep-th/0301249].

[29] M. C. B. Abdalla, A. L. Gadelha and I. V. Vancea, arXiv:hep-th/0308114.

[30] M. C. B. Abdalla, A. L. Gadelha and I. V. Vancea, Nucl. Phys. Proc. Suppl. **127** (2004) 92.

[31] M. C. B. Abdalla, A. L. Gadelha and I. V. Vancea, Phys. Lett. A **273**, 235 (2000) [arXiv:hep-th/0003209].

[32] H. Belich, E. L. Graca, M. A. Santos and I. V. Vancea, JHEP **0702**, 037 (2007) [arXiv:hep-th/0610271].

[33] Y. Leblanc, Phys. Rev. D **36**, 1780 (1987).

[34] Y. Leblanc, Phys. Rev. D **37** 1547 (1988).

[35] Y. Leblanc, Phys. Rev. D **38**, 3087 (1988).

[36] Y. Leblanc, Phys. Rev. D **39** 1139 (1989).

[37] Y. Leblanc, M. Knecht and J. C. Wallet, Phys. Lett. B **237** 357 (1990).

[38] Y. Leblanc, Phys. Rev. Lett. **64** 831 (1990).

[39] H. Fujisaki, K. Nakagawa and I. Shirai, Prog. Theor. Phys. **81**, 570 (1989).

[40] H. Fujisaki and K. Nakagawa, Prog. Theor. Phys. **82**, 236 (1989).

[41] H. Fujisaki and K. Nakagawa, Prog. Theor. Phys. **82**, 1017 (1989).

[42] H. Fujisaki and K. Nakagawa, Europhys. Lett. **14**, 737 (1991).

[43] H. Fujisaki and K. Nakagawa, Europhys. Lett. **20**, 677 (1992).

[44] H. Fujisaki, Europhys. Lett. **28**, 623 (1994).

[45] H. Fujisaki, Europhys. Lett. **39**, 479 (1997).

[46] D. L. Nedel, M. C. B. Abdalla and A. L. Gadelha, Phys. Lett. B **598**, 121 (2004).

[47] M. C. B. Abdalla, A. L. Gadelha and D. L. Nedel, JHEP **0510**, 063 (2005).

[48] M. C. B. Abdalla, A. L. Gadelha and D. L. Nedel, Phys. Lett. B **613**, 213 (2005).

[49] M. C. B. Abdalla, A. L. Gadelha and D. L. Nedel, PoS **WC2004**, 020 (2004).

[50] M. C. B. Abdalla, A. L. Gadelha and D. L. Nedel, PoS **WC2004**, 032 (2004).

[51] M. B. Cantcheff, Eur. Phys. J. C **55**, 517 (2008) [arXiv:0710.3186 [hep-th]].

[52] M. B. Cantcheff, Phys. Rev. D **80**, 046001 (2009) [arXiv:0906.3049 [hep-th]].

[53] M. A. Vazquez-Mozo, Phys. Lett. B **388**, 494 (1996) [arXiv:hep-th/9607052].

[54] J. Ambjorn, Yu. Makeenko, G. W. Semenoff and R. J. Szabo, Phys. Rev. D **60**, 106009 (1999) [arXiv:hep-th/9906134].

[55] J. Ambjorn, Y. M. Makeenko, G. W. Semenoff and R. J. Szabo, JHEP **0302**, 026 (2003) [arXiv:hep-th/0012092].

[56] K. Hotta, Prog. Theor. Phys. **112**, 653 (2004) [arXiv:hep-th/0403078].

[57] S. Alexander, R. H. Brandenberger and D. A. Easson, Phys. Rev. D **62**, 103509 (2000) [arXiv:hep-th/0005212].

[58] R. H. Brandenberger and C. Vafa, Nucl. Phys. B **316**, 391 (1989).

[59] P. Di Vecchia and A. Liccardo, NATO Adv. Study Inst. Ser. C. Math. Phys. Sci. **556**, 1 (2000) [arXiv:hep-th/9912161].

[60] P. Di Vecchia and A. Liccardo, arXiv:hep-th/9912275.

[61] J. C. da Silva, F. C. Khanna, A. Matos Neto and A. E. Santana, arXiv:hep-th/0208183.

[62] H. Queiroz, J. C. da Silva, F. C. Khanna, J. M. C. Malbouisson, M. Revzen and A. E. Santana, Annals Phys. **317**, 220 (2005) [Erratum-ibid. **321**, 1274 (2006)] [arXiv:hep-th/0411228].

[63] E. Alvarez and M. A. R. Osorio, Phys. Rev. D **36**, 1175 (1987).

[64] M. C. B. Abdalla, A. L. Gadelha and D. L. Nedel, Phys. Lett. A **334**, 123 (2005) [arXiv:hep-th/0409116].

[65] E. Kiritsis, String theory in a nutshell, *Princeton, USA: Univ. Pr. (2007).*