# Quantum 't Hooft loops of SYM $\mathcal{N} = 4$ as instantons of $\mathrm{YM}_2$ in dual groups SU(N) and SU(N)/$\mathbf{Z}_\mathrm{N}$

A. Bassetto, S. Thambyahpillai*

*Dipartimento di Fisica "G. Galilei" and INFN, Sezione di Padova,*

*Via Marzolo 8, 35131 Padua, Italy,*

*e-mail:* bassetto,thamby@pd.infn.it



## Abstract

A relation between circular 1/2 BPS 't Hooft operators in 4d $\mathcal{N} = 4$ SYM and instantonic solutions in 2d Yang-Mills theory ($\mathrm{YM}_2$) has recently been conjectured. Localization indeed predicts that those 't Hooft operators in a theory with gauge group $G$ are captured by instanton contributions to the partition function of $\mathrm{YM}_2$, belonging to representations of the dual group ${}^LG$. This conjecture has been tested in the case $G = U(N) = {}^LG$ and for fundamental representations. In this paper we examine this conjecture in the case of the groups $G = SU(N)$ and ${}^LG = SU(N)/Z_N$ and loops in different representations. Peculiarities when groups are not self-dual and representations not "minimal" are pointed out.

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*present e-mail: thamby@particle.uni-karlsruhe.de

## I. INTRODUCTION

Electric-magnetic duality in electromagnetism [1] has been extended to non-Abelian theories and, in particular, to $\mathcal{N}=4$ super Yang-Mills (SYM $\mathcal{N}=4$), (S-duality) [2]. It is conjectured that SYM $\mathcal{N}=4$ with gauge group $G$ and coupling constant $\tau$ is equivalent to SYM $\mathcal{N}=4$ with dual gauge group ${}^LG$ [3] and dual coupling constant ${}^L\tau$ , with

$$
{}^L\tau = -\frac{1}{\tau} \tag{1}
$$

for simply laced algebras, where

$$
\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{4d}^2}, \qquad {}^L\tau = \frac{{}^L\theta}{2\pi} + \frac{4\pi i}{({}^Lg_{4d})^2}. \tag{2}
$$

The symmetry has to be understood as an operator isomorphism between the two theories [4]. Since it interchanges electric and magnetic charges, it maps a Wilson operator [5] onto a 't Hooft operator [6] and vice-versa. Conjectures have also been suggested for chiral primary operators [7], surface operators [8] and domain walls [9].

An advance has recently been made with [10] where the conjecture has been extended to correlation functions of gauge invariant operators. The set of observables in SYM $\mathcal{N}=4$ are related by the S-duality requirement

$$
\langle \Pi_i \mathcal{O}_i \rangle_{G,\tau} = \langle \Pi_i {}^L\mathcal{O}_i \rangle_{{}^LG,{}^L\tau}. \tag{3}
$$

This property is both interesting and difficult to prove since it involves strong coupling calculations. The choice has been focused on a 't Hooft operator $T({}^LR)$ in a theory with gauge group $G$, ${}^LR$ being a representation of the dual group ${}^LG$.

The expectation value of a 't Hooft loop can be computed by a path-integral where the integration is performed over all fields which have a prescribed singularity along the loop. In the weak coupling regime quantum fluctuations around the classical monopole configuration can also be obtained up to one loop order and a recipe has been provided to compute the loop perturbatively at any desired higher order.

This result has subsequently been compared with a strong coupling calculation of a Wilson loop with dual gauge group and dual coupling (see (4)).

To compute Wilson loops where some fractions of supersymmetries are preserved, one may resort to matrix models where explicit calculations are feasible, as conjectured in [11,12] and proved in [13]. A rather interesting family of contours can be obtained by coupling three of the six scalars and by restricting the contours to lie on a great $S^2$ inside $S^3$. The related 1/8 BPS loop operators are conjectured to correspond to the "zero-instanton sector" of the two-dimensional Yang-Mills theory ($YM_2$) on $S^2$ [14]. In turn this was proved long ago to be equivalent to a Gaussian matrix model with area dependent coupling $g^2A = -2g_{4d}^2$ [15], [16]. Several results which comply with this conjecture have appeared recently in [17].

From matrix models a strong coupling expression for the Wilson loop can be extracted, to be compared with the weak coupling expression of the 't Hooft loop hitherto obtained. This can eventually be used to test the S-duality conjecture

$$\langle T(^LR)\rangle_{G,\tau} = \langle W(^LR)\rangle_{^LG,^L\tau}. \tag{4}$$

An even bolder conjecture has been proposed in [18]. After retrieving the correspondence between a (supersymmetric) Wilson loop in SYM $\mathcal{N}$=4 and the zero-instanton sector of the loop in $YM_2$, the authors extended this relation to suitable 't Hooft operators. More precisely they suggested that the expectation value of the 1/2 BPS circular 't Hooft loop in representation $^LR = (m_1, ..., m_N)$ in SYM $\mathcal{N}$=4 with gauge group $G$ and with an imaginary coupling ($\theta = 0$) could be obtained from the partition function $\mathcal{Z}$ of $YM_2$ with gauge group $G$ around an unstable instanton [19] labeled by $^LR$

$$\langle T_{^LR}(\mathcal{C})\rangle_{G,\tau} = \frac{\mathcal{Z}(g; m_1, ..., m_N)}{\mathcal{Z}(g; 0, ..., 0)}, \tag{5}$$

where the configuration $(m_1, \cdots, m_N)$ is related to the boxes in the Young tableau.

Similarly, correlation functions of the 1/2 BPS 't Hooft loop with any number of 1/8 BPS Wilson loops inserted on the $S^2$ linked to the 't Hooft loop, could be computed in $YM_2$ by calculating the Wilson loop correlation functions around a fixed unstable instanton.

These suggestions are particularly intriguing since they point towards endowing those instantonic sectors with a "physical" meaning.

In fact in [18] the check was limited to the K-antisymmetric representations of the gauge group $U(N)$, which cannot be screened to give rise to sub-leading saddle points in the path integral localization (the "monopole bubbling" phenomenon [20]). Moreover the choice of $U(N)$ hid the possible occurrence of different representations $R$ and ${}^{L}R$ in the general case, $U(N)$ being self-dual.

Our purpose in this paper is to extend the analysis to the gauge group $SU(N)$ and to its dual $SU(N)/Z_N$.

In Sect.2 we develop the harmonic analysis in $SU(N)$ and $SU(N)/Z_N$ of the partition function and of a Wilson loop of an operator firstly in the $K$-fundamental representation and then in the adjoint one. We remark that the Poisson transformation, which is the bridge between the expansions in terms of characters and of unstable instantons respectively, provides us with two *different* expressions for the *same* quantity: they are representations in the different dual groups $SU(N)$ and $SU(N)/Z_N$.

In Sect.3 we test the conjecture of ref. [18] of a relation between a Wilson loop in the $K$-fundamental representation and a 't Hooft loop, obtained by singling out in the partition function the contribution of an instanton belonging to the same representation. The test was successfully performed in [18] for the group $U(N)$. The novelty in our case is that the $K$-irrep is not present in $SU(N)/Z_N$. As a consequence the test can only be exploited starting from $SU(N)/Z_N$ (in any k-sector) and ending in $SU(N)$ where the $K$-irrep is present. This example, of course, is endowed with general validity.

Then we discuss the case of the adjoint representation. In this case both $SU(N)$ and $SU(N)/Z_N$ are viable. However it turns out that the instanton contribution to the partition function, which should correspond to a 1/2 BPS 't Hooft loop in SYM $\mathcal{N} = 4$, indeed presents some extra terms (subleading in $N$) with respect to the Wilson loop in the same representation. This is a concrete realization of the possibility mentioned in [18] and there interpreted as a subleading contribution in the path-integral localization of SYM $\mathcal{N} = 4$.

Finally Sect.4 contains our conclusions together with some insight into possible future developments.

## II. THE HARMONIC ANALYSIS ON $SU(N)$ AND $SU(N)/Z_N$

It is well known that the irreducible representations (irreps) of $SU(N)/Z_N$ occur in $k$ sectors where the integer $k$ runs from 0 to $N-1$, corresponding to the $k$-th root of the identity.

The basic ingredient in computing the partition function and Wilson loop correlators in $YM_2$ is the heat kernel on a two-dimensional cylinder $\mathcal{K}(A;U_2,U_1)$ of area $A = L\tau$ ($L$=base circle, $\tau$ = length), and fixed holonomies at the boundaries $U_1$ and $U_2$. The only geometrical dependence of the kernel is on its area, thanks to the invariance of $YM_2$ under area- preserving diffeomorphisms [19]. The kernel enjoys the basic sewing property

$$\mathcal{K}(L\tau : U_2,U_1) = \int dU \mathcal{K}(Lu;U_2,U(u))\mathcal{K}(L(\tau-u));U(u),U_1). \tag{6}$$

The partition function on a sphere with area $A$ is expressed as $\mathcal{K}(A;\mathbf{1},\mathbf{1})$.

The kernel $\mathcal{K}$ can be expanded as a series of the characters $\chi_R$ of all the irreps, according to the equation

$$\mathcal{K}(A;U_2,U_1) = \sum_R \chi_R^\dagger(U_2)\chi_R(U_1)\exp\left[\frac{-g^2A}{4}C^R\right], \tag{7}$$

$C^R \equiv C_2(R)$ being the quadratic Casimir operator of the R-representation.

The generalization of the above construction to $SU(N)/Z_N$ is simple: following [21] and [22], we project the final state of the heat kernel onto k-states

$$\begin{aligned}\mathcal{K}_k(A;U_2,U_1) &= \sum_{z\in Z_N} z^k\mathcal{K}(A;zU_2,U_1),\\ &= \sum_{n=0}^{N-1}\sum_R e^{\frac{2\pi in}{N}(k-m^R)}\chi_R^\dagger(U_2)\chi_R(U_1)\exp\left[\frac{-g^2A}{4}C^R\right],\end{aligned} \tag{8}$$

where $z=\exp(2\pi i\frac{n}{N})$, $n=0,...,N-1.$ and $m^{(R)}=\sum_{q=1}^{N-1}m_q^{(R)}$ is the total number of boxes of the Young tableau.

Choosing $U_1 = U_2 = \mathbf{1}$, the partition function in the k-sector takes the expression

$$\mathcal{Z}_k(A) = N \sum_R (d_R)^2 \exp\left[\frac{-g^2 A}{4} C^R\right] \delta_{[N]}(k - m^{(R)}), \tag{9}$$

where $d_R$ is the dimension of the R-irrep and $\delta_{[N]}$ is the $N$-periodic delta-function.

Averaging over $k$, the $SU(N)$ partition function is immediately recovered. The presence of the periodic $\delta$-function constraint marks the difference between $SU(N)/Z_N$ and $SU(N)$.

Introducing the explicit expression for the characters [23] enables us to write eq.(9) explicitly in terms of a new set of indices $\{l_i\} = (l_1, \ldots, l_N)$, $l_i = m_i + N - i$ (see the Appendix). By recalling the relations

$$\begin{aligned} C_2(R) &= \sum_{i=1}^N \Big(l_i - \frac{l}{N}\Big)^2 - \frac{N}{12}(N^2 - 1) \\ d_R &= \Delta(l_1, ..., l_N), \qquad l = \sum_{i=1}^N l_i, \end{aligned} \tag{10}$$

where $\Delta$ is the Vandermonde determinant , we get [22]

$$\begin{aligned} \mathcal{Z}_k(A) &= \frac{(2\pi)^{N-1}}{(N-1)!\sqrt{\pi}} \sum_{l_i=-\infty}^{+\infty} \int_0^{2\pi} d\alpha \, e^{-\left(\alpha - \frac{2\pi}{N} l\right)^2} \delta_{[N]}\Big(k - l + \frac{N(N-1)}{2}\Big) \\ \times &\quad \exp\left[-\frac{g^2 A}{4} C_2(l_i)\right] \Delta^2(l_1, ..., l_N). \end{aligned} \tag{11}$$

The dual representation in this context is realized by means of a Poisson transformation

$$\begin{aligned} &\sum_{l_i=-\infty}^{+\infty} F(l_1, \ldots, l_N) = \sum_{n_i=-\infty}^{+\infty} \tilde{F}(n_1, \ldots, n_N), \\ &\tilde{F}(n_1, \ldots, n_N) = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N F(z_1, \ldots, z_N) \exp\Big[2\pi i(z_1 n_1 + \ldots + z_N n_N)\Big]. \end{aligned} \tag{12}$$

In order to perform this multiple Fourier transform, we remember that the transformation of a product is turned into a convolution; moreover we recall the result

$$\begin{aligned} &\int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp\Big[i(z_1 p_1 + \ldots + z_N p_N)\Big] \Delta(\{z_i\}) \exp\Big(-\frac{g^2 A}{8} \sum_{q=1}^N z_q^2\Big) = \\ &\left[\frac{4i}{g^2 A}\right]^{\frac{N(N-1)}{2}} \left[\frac{8\pi}{g^2 A}\right]^{\frac{N}{2}} \Delta(\{p_i\}) \exp\Big(-\frac{2}{g^2 A} \sum_{q=1}^N p_q^2\Big). \end{aligned} \tag{13}$$

Taking these relations into account, eq. (11) becomes

$$\mathcal{Z}_k(A) = \sum_{n=0}^{N-1} \exp\Big[\frac{2\pi i n k}{N}\Big] \mathcal{Z}^{(n)}(A), \tag{14}$$

where

$$\mathcal{Z}^{(n)}(A) = (-1)^{n(N-1)}\, \mathcal{C}(A,N) \sum_{n_q=-\infty}^{+\infty} \delta(n - \sum_{q=1}^{N} n_q) \exp\Big[-\frac{4\pi^2}{g^2A}\sum_{q=1}^{N}(n_q - \frac{n}{N})^2\Big]\zeta_n(\{n_q\}), \tag{15}$$

with

$$\zeta_n(\{n_q\}) = \int_{-\infty}^{+\infty} dz_1 \dots dz_N \exp\Big[-\frac{1}{2}\sum_{q=1}^{N} z_q^2\Big]\Delta(\{\sqrt{\frac{g^2A}{2}}z_q + 2\pi n_q\})\Delta(\{\sqrt{\frac{g^2A}{2}}z_q - 2\pi n_q\})$$

$$= \int_{-\infty}^{+\infty} dz_1 \dots dz_N \exp\Big[-\frac{1}{2}\sum_{q=1}^{N} z_q^2\Big]\Delta^2(\{\sqrt{\frac{g^2A}{2}}z_q - 2\pi i n_q\}) \tag{16}$$

and $\mathcal{C}(A,N)$ an unessential normalization factor [22]. $\mathcal{Z}^{(n)}$ is clearly invariant under a common translation $\{n_q\} \to \{n_q - h\},\ h \in \mathbf{Z} :\ \mathcal{Z}^{(n)} = \mathcal{Z}^{(n+hN)}$.

The classical instanton action $\mathcal{S} = \Big[\frac{4\pi^2}{g^2A}\sum_{q=1}^{N}(n_q - \frac{n}{N})^2\Big]$ can be nicely compared to the Casimir expression in the exponential of eq.(9). One can already remark that the factor $\frac{4\pi^2}{g^2A}$ here corresponds to the factor $\frac{g^2A}{4}$ there, as suggested by duality.

The duality can most easily be realized by taking the average over the sectors $k$, firstly in (9):

$$\frac{1}{N}\sum_{k=0}^{N-1}\mathcal{Z}_k = \sum_R (d_R)^2 \exp\Big[\frac{-g^2A}{4}C_2(R)\Big], \tag{17}$$

as expected in $SU(N)$ (the $\delta$-constraint on $m^{(R)}$ has disappeared) and then in (14):

$$\frac{1}{N}\sum_{k=0}^{N-1}\mathcal{Z}_k = \mathcal{Z}^{(0)}, \qquad mod\, N, \tag{18}$$

as expected in $SU(N)/Z_N$.

The dual relation can easily be obtained from eq.(14)

$$\sum_{n=0}^{N-1}\mathcal{Z}^{(n)} = \mathcal{Z}_0, \qquad mod\, N. \tag{19}$$

The expressions (18) and (19) are indeed symmetric under the interchange of the two sets of integers $\{m_q\}$ and $\{n_q\}$.

The next step to be performed is to obtain the expression for a Wilson loop average in the $k$-sector of $SU(N)/Z_N$. We begin by considering the simplest case of a Wilson loop in the K-fundamental representation. Moreover we choose a regular non self-intersecting loop placed on the equator of our sphere $S^2$

$$\mathcal{W}_k(\frac{A}{2},\frac{A}{2}) = \frac{1}{\mathcal{Z}_k}\sum_{z\in Z_N} z^k \int dU \mathcal{K}(\frac{A}{2}; z\cdot\mathbf{1}, U)\, \frac{1}{d_K} Tr_K[U]\, \mathcal{K}(\frac{A}{2}; U, \mathbf{1}). \tag{20}$$

In the $k$-sector the loop exhibits the expected $\delta_{[N]}$ constraint on the total number of boxes $m^{(S)}$ of the Young tableau corresponding to the representation $S$ of the group element that has been twisted:

$$\mathcal{W}_k(\frac{A}{2},\frac{A}{2}) = \frac{1}{\mathcal{Z}_k}\frac{1}{d_K}\sum_{R,S} d_R d_S \exp\Big[-\frac{g^2 A}{8}(C^R + C^S)\Big] \int dU \chi^\dagger_S[U] Tr_K[U] \chi_R[U] \delta_{[N]}(k - m^{(S)}). \tag{21}$$

By making the expression of the characters explicit, after integrating over the group variables, taking suitable invariance under permutations into account and invariance of the Vandermonde determinants under constant translations in their arguments, a calculation (partially sketched in the Appendix) leads to

$$\begin{aligned}\mathcal{W}_k = {} & \frac{1}{\mathcal{Z}_k}\sum_{l_i=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\beta \int_{-\infty}^{+\infty} dl\, e^{i\beta(l-\sum_i l_i)}\delta_{[N]}\Big(k - l + \frac{N(N-1)}{2}\Big) \\ & \int_0^{2\pi} d\alpha\, e^{-(\alpha-\frac{2\pi l}{N})^2} \exp\Big[-\frac{g^2 A}{8}\Big(2C(l_i) - 2\sum_{j=1}^{K} l_j + \frac{K}{N}(N + 2l - K)\Big)\Big] \\ & \Delta(l_1 - 1, l_2 - 1, \cdots, l_K - 1, l_{K+1}, \cdots, l_N)\Delta(l_1, l_2, \cdots, l_N),\end{aligned} \tag{22}$$

Before undertaking the Poisson transformation it is useful to factorize the $\delta_{[N]}$-constraint using its exponential representation $\delta_{[N]}(q) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\frac{2\pi i\, p\, q}{N}}$. Then, by repeating the procedure used for $\mathcal{Z}_k$, a long but straightforward calculation leads to

$$\mathcal{W}_k(\frac{A}{2},\frac{A}{2}) = \frac{1}{\mathcal{Z}_k} e^{\frac{g^2AK^2}{16N}} \sum_{\{n_i\}} \frac{1}{N} \sum_{n=0}^{N-1} \delta(n - \sum_i n_i) e^{\frac{2\pi ink}{N}}$$
$$e^{i\pi \sum_{j=1}^K \left(n_j - \frac{n}{N}\right)} \int_{-\infty}^{+\infty} dy_1 \cdots dy_N \Pi_{i<j} \Big[4\pi^2 n_{ij}^2 - y_{ij}^2\Big]$$
$$\exp\Big[-\frac{4\pi^2}{g^2A}\sum_j \left(n_j - \frac{n}{N}\right)^2\Big] e^{-\frac{i}{2}\sum_{j=1}^K y_j} e^{-\frac{1}{g^2A}\sum_j y_j^2}, \tag{23}$$

where $n_{ij} \equiv n_i - n_j, y_{ij} \equiv y_i - y_j$.

Both the expressions for $\mathcal{Z}_k$ and $\mathcal{W}_k$ we have hitherto obtained, are representations in the $k$-sector of $SU(N)/Z_N$ in the character expansion and of $SU(N)$ in the corresponding dual instanton expansion. It is straightforward to switch to $SU(N) \to SU(N)/Z_N$, averaging over $k$

$$\mathcal{Z}(A) = \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{Z}_k(A) = \mathcal{Z}^{(0)}(A) = \mathcal{C}(A,N) \sum_{n_q=-\infty}^{+\infty} \delta_{[N]}(\sum_{q=1}^N n_q) \exp\Big[-\frac{4\pi^2}{g^2A}\sum_{q=1}^N n_q^2\Big]\zeta_n(\{n_q\}) \tag{24}$$

and

$$\mathcal{W}(\frac{A}{2},\frac{A}{2}) = \frac{1}{\mathcal{Z}} e^{\frac{g^2AK^2}{16N}} \sum_{\{n_i\}} \delta_{[N]}(\sum_{i=1}^N n_i) e^{i\pi\sum_{j=1}^K n_j} \int_{-\infty}^{+\infty} dy_1 \cdots dy_N \Pi_{i<j}\Big[4\pi^2 n_{ij}^2 - y_{ij}^2\Big]$$
$$\exp\Big[-\frac{4\pi^2}{g^2A}\sum_j n_j^2\Big] e^{-\frac{i}{2}\sum_{j=1}^K y_j} e^{-\frac{1}{g^2A}\sum_j y_j^2}, \tag{25}$$

corresponding to eqs.(14) and (23) respectively.

## III. THE CONJECTURE

As discussed in the Introduction, the average value of a 1/2 BPS t'Hooft circular loop winding on a large circle on $S^2$ in SYM $\mathcal{N}$=4 with gauge group $G$ in the representation ${}^LR = (m_1, \cdots, m_N)$ can be obtained from the contribution to the partition function $\mathcal{Z}$ of $YM_2$ of an unstable instanton labeled by ${}^LR$ (see eq.(5)). In turn this should be dual to the "zero instanton" contribution to the average value of a Wilson loop in the representation ${}^LR$ (in the character expansion), winding over a large circle of $S^2$ of $YM_2$ ( [18]).

We are now in the position to discuss this conjecture when the groups considered are $SU(N)$ and its dual $SU(N)/Z_N$. Let us first start from the $k$-sector of $SU(N)/Z_N$ in the character expansion, moving to the dual instanton expansion of $SU(N)$. The "zero instanton" contribution of $\mathcal{Z}_k$ and of $\mathcal{W}_k$ in the $K$-fundamental representation are easily derived from eqs.(14) and (23)

$$\mathcal{Z}_k^{[0]} = \int dz_1, \cdots, dz_N \exp\Big[-\frac{1}{2}\sum_q z_q^2\Big]\Delta^2(\{z_q\}), \tag{26}$$

where the nomalization has been suitably modified, and

$$\mathcal{W}_k^{[0]}(\frac{A}{2},\frac{A}{2}) = \frac{1}{\mathcal{Z}_k^{[0]}} e^{\frac{g^2AK^2}{16N}} \int_{-\infty}^{+\infty} dy_1\cdots dy_N \Pi_{i<j}\Big[y_{ij}^2\Big] e^{-\frac{i}{2}\sum_{j=1}^K y_j} e^{-\frac{1}{g^2A}\sum_j y_j^2}. \tag{27}$$

We remark that neither $\mathcal{Z}_k^{[0]}$ nor $\mathcal{W}_k^{[0]}$ actually depend on the sector $k$ we are considering.

According to the conjecture, we should now calculate the instanton contribution to the partition function $\mathcal{Z}_k$ corresponding to the same $K$-fundamental representation $\{n_q\} = (1,\cdots,1,0,\cdots,0)$ with the first $K$-elements being unity, and permutations thereof.

Inserting this configuration in eqs.(14),(15) and (16), we get

$$\mathcal{Z}_k^{(K)} = e^{\frac{2\pi ikK}{N}}(-1)^{K(N-1)} e^{\frac{4\pi^2K^2}{g^2AN}} \int_{-\infty}^{+\infty} dz_1\cdots dz_N \Pi_{i<j}\Big[z_{ij}^2\Big] e^{-2\pi i\sqrt{\frac{2}{g^2A}}\sum_{j=1}^K z_j}\; e^{-\frac{1}{2}\sum_j z_j^2}, \tag{28}$$

where permutations have been taken into account.

The change of variables $y_i = \sqrt{\frac{g^2A}{2}} z_i$ in eq.(27) would lead to a perfect agreement with eq.(28) under the interchange $\frac{8\pi^2}{g^2A} \leftrightarrow \frac{g^2A}{8}$, were it not for the phase factor in (28). The occurrence of a similar factor was also noticed in [18]. Here the extra $k$-dependence could be disposed of by choosing the sector $k=0$ [1].

The other option ($SU(N) \to SU(N)/Z_N$) is not viable. As a matter of fact the presence of the constraint $\delta_{[N]}(\sum_{q=1}^N n_q)$ in $\mathcal{Z}^{(0)}$ makes the representation $\{n_q\} = (1,\cdots,1,0,\cdots,0)$ for the 't Hooft loop impossible, as it is not shared by the group $SU(N)/Z_N$.

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[1]We notice that an analogous phase factor $e^{\frac{2\pi nk}{N}}$ is present in $\mathcal{W}_k$ (see (23)); however it is washed out by the choice $n=0$ there.

In conclusion, when the group is not self-dual, one ought to choose a representation ${}^L R$ for the 't Hooft loop among the ones available in ${}^L G$. Then no problem ensues for the Wilson loop in the ${}^L R$- representation since the conjecture always requires its contribution to the zero-instanton sector (see eqs.(23) and (25)).

The situation changes if we consider the correlator of the 1/2 BPS 't Hooft loop with one (or more) 1/8 BPS Wilson loops inserted on the $S^2$ linked to the 't Hooft loop. In $YM_2$ the conjecture suggests we compute the Wilson loop around a fixed unstable instanton. In the $SU(N)/Z_N \leftrightarrow SU(N)$ case we are considering, it amounts to selecting in the expansions (23) or (25) a given instanton configuration. The novelty is that such a configuration can only be chosen among the representations available in ${}^L G$. For instance the choice $(1, \cdots, 1, 0, \cdots, 0)$, corresponding to a fundamental representation, would be possible in eq.(23), but forbidden in eq.(25).

At this point some comments concerning other irreps are in order.

Suppose we consider a 't Hooft loop in the adjoint representation. The total number of boxes in the Young tableau being $N$ in this case, we can equally well consider $SU(N)$ or $SU(N)/Z_N$. Going back to eqs.(5) and (24), the Young tableau of the adjoint representation has the configuration $\{n_q\} = (2, 1, \cdots, 1, 0)$, which is equivalent $mod\, N$ to $(1, 0, \cdots, 0, -1)$, its highest weight. We get

$$\mathcal{Z}_{adj} = \frac{1}{\mathcal{Z}^{[0]}} \int_{-\infty}^{+\infty} dz_1, \cdots, dz_N \exp\Big[-\frac{1}{2}\sum_{q=1}^{N} z_q^2\Big] \exp\Big[\frac{2\sqrt{2}\pi i}{\sqrt{g^2 A}} z_{1N}\Big] \Delta^2(\{z_q\}). \qquad (29)$$

Taking invariance under permutations into account, it becomes

$$\begin{aligned}\mathcal{Z}_{adj} &= \frac{1}{\mathcal{Z}^{[0]}}\Big(1+\frac{1}{N}\Big) \int_{-\infty}^{+\infty} dz_1, \cdots, dz_N \Delta^2(\{z_q\}) \exp\Big[-\frac{1}{2}\sum_{q=1}^{N} z_q^2\Big] \times \\ &\Big[\frac{\sum_{r,s=1}^{N} \exp(2\pi i\sqrt{2/g^2 A} z_{rs}) - 1}{N^2-1} - \frac{1}{N+1}\Big] \\ = \frac{1}{\mathcal{Z}^{[0]}}&\Big[(1+\frac{1}{N}) \int \mathcal{D}F \exp(-\frac{1}{2} Tr F^2) \frac{1}{N^2-1}\Big(|Tr[\exp(2\pi i\sqrt{\frac{2}{g^2 A}} F)]|^2 - 1\Big)\Big] - \frac{1}{N}. \end{aligned} \qquad (30)$$

Here $F$ is a traceless hermitian matrix.

In ref. [22] the "zero instanton" contribution to the Wilson loop in the adjoint representation has been computed in the $k$-sector of $SU(N)/Z_N$

$$\mathcal{W}_k^{[0]}(\frac{A}{2},\frac{A}{2}) = \frac{1}{N+1}\Bigg[1+\frac{N}{\mathcal{Z}^{[0]}}\int_{-\infty}^{+\infty} dz_1\ldots dz_N \exp\Big[-\frac{1}{2}\sum_{q=1}^{N} z_q^2\Big]\times$$
$$\exp\Big[\frac{i}{2}\sqrt{\frac{g^2A}{2}}z_{12}\Big]\Delta^2(z_1,\ldots,z_N)\Bigg]. \tag{31}$$

The dependence on $k$ has disappeared. Eventually the expression above turns into the matrix integral [24]

$$\mathcal{W}_{adj}^{[0]} = \frac{1}{\mathcal{Z}^{[0]}}\int \mathcal{D}F \exp(-\frac{1}{2}TrF^2)\frac{1}{N^2-1}(|Tr[\exp\frac{ig}{2}\sqrt{\frac{A}{2}}F]|^2-1). \tag{32}$$

Comparing eqs.(30) and (32), we notice the expected duality relation $\frac{g^2A}{8}\leftrightarrow\frac{8\pi^2}{g^2A}$, but also the occurrence in (30) of extra terms, possibly related to the mentioned "monopole bubbling" [20].

## IV. CONCLUSIONS

We have extended the conjecture of ref. [18] concerning a 1/2 BPS 't Hooft loop in the group $U(N)$, to the more general case of a group which is not self-dual. We have concretely examined the choice $SU(N) \leftrightarrow SU(N)/Z_N$. The duality mapping is performed in our treatment by a Poisson transformation between an expansion in terms of characters and the one in terms of instantons.

The novelty in the case of groups which are not self-dual lies in the circumstance that not all representations are shared by them. For instance it is well known that the spinorial representations of $SU(2)$ are not shared by its dual partner $SU(2)/Z_2$. As a consequence the choice of the representation for the 't Hooft loop should be made among those allowed.

In the example $SU(N) \leftrightarrow SU(N)/Z_N$ we have discussed, if we want a 't Hooft loop belonging to one of the fundamental irreps of $SU(N)$, we ought to start from any k-sector of $SU(N)/Z_N$, landing, after the Poisson transformation, in $SU(N)$. No problem arises with the Wilson loop, as only its zero-instanton contribution is required by the conjecture. The situation is different when correlators of the 1/2 BPS 't Hooft loop with one (or more) Wilson loop are considered. In this case the representation of the 't Hooft loop has to be compatible with the instanton configuration of the Wilson loop(s).

We have also briefly discussed the adjoint irrep, which belongs to both $SU(N)$ and $SU(N)/Z_N$. Here we have concretely realized that this choice in the partition function for the 't Hooft loop involves subleading corrections, as expected on general grounds [18].

It would be nice in the future to be able to extend the conjecture beyond the 1/2 BPS 't Hooft loop. As a preliminary requirement we need to thoroughly understand more general configurations of a 't Hooft loop in SYM $\mathcal{N}$=4, in particular their contribution as saddle points in the localization of the path-integral [13].

From the mathematical side one should perhaps understand in a more general and systematic way the connection between a formulation of duality in terms of algebras and of groups. We remark that previous treatments were mostly based on a relation between algebras exchanging their highest weights under the duality transformation [3], [4], [10]. Here the conjecture forces us to choose their group counterparts where duality operates in the form of an integral Poisson transformation.

## V. APPENDIX

Let us introduce for $SU(N)$ the usual variables

$$\hat{l}_q = m_q + N - q, \qquad\qquad q = 1, \cdots, N-1, \tag{33}$$

which give rise to a strongly monotonous sequence $\hat{l}_1 > \hat{l}_2 > \cdots, \hat{l}_{N-1} > 0$ [22]. Then, with the twofold purpose of extending the range of the $\hat{l}_q$'s to negative integers and of gaining the symmetry over permutations of a full set of $N$ indices, we introduce the obvious equality

$$\sqrt{\pi} = \int_0^{2\pi} d\alpha \sum_{\hat{l}_N=-\infty}^{+\infty} e^{-(\alpha - \frac{2\pi}{N}\sum_{j=1}^{N-1}\hat{l}_j - 2\pi\hat{l}_N)^2}, \tag{34}$$

where $\hat{l}_N$ is a dummy quantity. Now we extend the representation indices by defining the new set

$$l_q = \hat{l}_q + \hat{l}_N, \qquad q = 1, \cdots, N-1,$$
$$l_N = \hat{l}_N, \tag{35}$$

which appears in eq.(10) and those following.

In terms of these indices eq.(21) takes the form

$$\mathcal{W}_k = \frac{1}{\sqrt{\pi}\, Z_k} \sum_{l_i^R, l_i^S = -\infty}^{+\infty} \exp\Big[-\frac{g^2 A}{8}(C^R + C^S)\Big] \delta_{[N]}\Big(k - l^{(S)} + \frac{N(N-1)}{2}\Big)$$
$$\Delta(l_1^R, \cdots, l_N^R)\Delta(l_1^S, \cdots, l_N^S) \int_0^{2\pi} d\alpha_1\, d\alpha_2 e^{(-\alpha_1 - \frac{2\pi}{N} l^R)^2} e^{(-\alpha_2 - \frac{2\pi}{N} l^S)^2}$$
$$\int_0^{2\pi} d\theta_1 \cdots d\theta_N \;\delta_P(\sum_j \theta_j)\; e^{i \sum_{q=1}^K \theta_q}\, \Pi_{r=1}^N e^{i l_r^R \theta_r}\; \Pi_{s=1}^N e^{-i l_s^S \theta_s}, \tag{36}$$

where $\delta_P = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in \sum_{j=1}^N \theta_j}$ is a periodic $\delta$-distribution, as required for $SU(N)$.

By making the expression of the characters explicit, by taking invariance under permutations into account, and after integrating over the group variables eq.(22) is eventually recovered.

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