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Hamiltonian Analysis of Non-Relativistic Covariant RFDiff Hořava-Lifshitz Gravity

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ABSTRACT: We perform the Hamiltonian analysis of non-relativistic covariant Hořava-Lifshitz gravity in the formulation presented recently in arXiv:1009.4885. We argue that the resulting Hamiltonian structure is in agreement with the original construction of non-relativistic covariant Hořava-Lifshitz gravity presented in arXiv:1007.2410. Then we extend this construction to the case of RFDiff invariant Hořava-Lifshitz theory. We find well behaved Hamiltonian system with the number of the first and the second class constraints that ensure the correct number of physical degrees of freedom of gravity.

KEYWORDS: Hořava-Lifshitz gravity.

Contents

1. Introduction and Summary

In 2009 Petr Hořava formulated new proposal of quantum theory of gravity that is power counting renormalizable [1, 2, 3]. This theory is now known as Hořava-Lifshitz gravity (HL gravity). It was also expected that this theory reduces do General Relativity in the infrared (IR) limit. HL theory was studied from different point of view due to the fact that this is a new and intriguing formulation of gravity as a theory with reduced amount of symmetries that leads to remarkable new phenomena [1].

The HL gravity is based on an idea that the Lorentz symmetry is restored in IR limit of given theory and can be absent at high energy regime of given theory. Explicitly, Hořava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

$$\mathbf{x}' = l\mathbf{x} \ , \quad t' = l^z t \ . \tag{1.1}$$

In $(D+1)$ dimensional space-time in order to have power counting renormalizable theory requires that $z \geq D$. It turns out however that the symmetry group of given theory is reduced from the full diffeomorphism invariance of General Relativity to the foliation preserving diffeomorphism

$$x'^i = x^i + \zeta^i(t,\mathbf{x}) \ , \quad t' = t + f(t) \ . \tag{1.2}$$

Due to the fact that the diffeomorphism is restricted (1.2) one more degree of freedom appears that is a spin-0 graviton. It turns out that the existence of this mode could be dangerous since it has to decouple in the IR regime, in order to be consistent with observations. Unfortunately, it seems that this might not be the case. It was shown that the spin-0 mode is not stable in the original version of the HL theory [1] as well as in the Sotiriou, Visser and Weinfurtner (SVW) generalization [8]. Note that in both of these two versions, it was all assumed the projectability condition that means that the lapse function N depends on t only. This presumption has a fundamental consequence for the formulation

[1] For review and extensive list of references, see [4, 5, 6].

of the theory since there is no local form of the Hamiltonian constraint but the only global one. However we would like to stress that these instabilities are all found in the Minkowski background. Recently, it was found that the de Sitter spacetime is stable in the SVW setup [9, 10]. Then we can presume that this background is legitimate background.

On the other hand there is the second version of HL gravity where the projectability condition is not imposed so that $N = N(\mathbf{x}, t)$. Properties of given theory were extensively studied in [11, 12, 13, 14, 15, 14, 17, 18, 19, 20, 21, 22]. It was shown recently in [14] that so called healthy extended version of given theory could really be an interesting candidate for the quantum theory of reality without ghosts and without strong coupling problem despite its unusual Hamiltonian structure [17, 18].

Recently Hořava and Malby-Thompson in [25] proposed very interesting way how to eliminate the spin-0 graviton. They considered the projectable version of HL gravity together with extension of the foliation preserving diffeomorphism to include a local $U(1)$ symmetry. The resulting theory is then called as non-relativistic covariant theory of gravity [2]. It was argued there [25] that the presence of this new symmetry forces the coupling constant λ to be equal to one, however this result was questioned in [26] (see also [30]) where an alternative formulation of non-relativistic general covariant theory of gravity was presented. Further, it was shown in [25, 26] that the presence of this new symmetry implies that the spin-0 graviton becomes non-propagating and the spectrum of the linear fluctuations around the background solution coincides with the fluctuation spectrum of General Relativity.

This new proposal of non-relativistic general covariant HL gravity is very interesting and it certainly deserves further study. In this paper we present the Hamiltonian analysis of the formulation of non-relativistic covariant HL gravity given in [26]. We argue that resulting Hamiltonian and constraint structure has the same form as in [25] even if they differ in explicit form since they are derived from different Lagrangians. This fact shows that these two Lagrangian formulations of non-relativistic covariant HL gravities are equivalent on the level of the Hamiltonian formalism as well.

Despite the fact that non-relativistic covariant HL gravity seems to solve the problem of the scalar graviton and the content of the physical degrees of freedom is the same as in General Relativity there is still one additional first class constraint which is the global Hamiltonian constraint. The meaning of this constraint should be investigated further as was nicely discussed on page 30 in [25]. In order to find version of non-relativistic covariant HL gravity without global Hamiltonian constraint we recall that there exists formulation of the HL gravity with reduced symmetry group known as *restricted-foliation-preserving Diff* (RFDiff) HL gravity [14, 23]. This is the theory that is invariant under following symmetries

$$t' = t + \delta t \ , \delta t = \text{const} \ , \quad x'^i = x^i + \zeta^i(\mathbf{x}, t) \ . \tag{1.3}$$

The characteristic property of given theory is that in its simplest version [23] based on the detailed balance construction [1, 2, 3] there is no reason to introduce the lapse function

[2]This theory was also studied in [27, 28, 29, 30].

N [3]. Then we introduce $U(1)$ symmetry as in [25] or its alternative version given in [26]. Finally we proceed to the Hamiltonian formulation of given theory and we find there is no global Hamiltonian constraint due to the absence of the lapse function N in the action. We further determine all constraints in given theory and we show that the number of the first class and the second class constraints implies that the physical phase space has the same dimensions as in case of General Relativity. On the other hand we show that the presence of the second class constraints implies that the symplectic structure of given theory that is determined by corresponding Dirac brackets between physical degrees of freedom is rather complicated due to the fact that Dirac brackets generally depend on phase space variables.

Let us outline our results and suggest possible extension of this work. We perform the Hamiltonian analysis of the theory suggested in [26] and we show that its Hamiltonian structure is equivalent to the Hamiltonian structure found in paper [25]. We also suggest an alternative formulation of non-relativistic covariant HL gravity that is based RFDiff HL gravity. We show that the resulting theory has consistent Hamiltonian formulation with the same content of the local constraints as in case of non-relativistic covariant HL gravity but without global Hamiltonian constraint. On the other hand we should stress that we are not able to solve explicitly the second class constraints with respect to physical degrees of freedom in the full generality. We are also not able to determine corresponding Dirac brackets. Then it would be clearly desirable to find exact results at least for some special situations. It would be also interesting to find exact solutions of the equations of motion of non-relativistic general covariant RFDiff HL gravity. We hope to return to these problems in near future.

The organization of this paper is as follows. In the next section (2) we introduce the non-relativistic general covariant HL gravity in the formulation firstly presented in [26]. Then in section (3) we perform its Hamiltonian analysis. In section (4) we introduce the non-relativistic general covariant RFDiff-invariant HL gravity. Then we perform its Hamiltonian analysis and shows that the resulting theory correctly describes physical degrees of freedom of $D+1$ dimensional gravity.

2. Non-Relativistic Covariant HL Gravity

We begin this section with the introduction of basic notation, for detailed treatment of $D+1$ formalism, see [24].

Let us consider $D+1$ dimensional manifold $\mathcal{M}$ with the coordinates x^μ , $\mu=0,\dots,D$ and where $x^\mu=(t,\mathbf{x})$, $\mathbf{x}=(x^1,\dots,x^D)$. We presume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-,+,\dots,+)$. Suppose that $\mathcal{M}$ can be foliated by a family of space-like surfaces Σ_t defined by $t=x^0$. Let $g_{ij}, i,j=1,\dots,D$ denotes the

[3] More general form of RFDiff HL gravity was considered in [14] where the action contains time and space derivatives of the lapse function N according to general principles of effective field theory construction. However the presence of such terms has no impact on the Hamiltonian structure of given theory simply from the fact that the momentum conjugate to N is not primary constraint of the theory and hence the Hamiltonian constraint is absent. In order to make our analysis transparent we consider the simplest version of RFDiff HL gravity keeping in mind that it can be easily extended to its more general versions.

metric on Σ_t with inverse g^{ij} so that $g_{ij}g^{jk} = \delta_i^k$. We further introduce the operator ∇_i that is covariant derivative defined with the metric g_{ij}. We introduce the future-pointing unit normal vector n^μ to the surface Σ_t. In ADM variables we have $n^0 = \sqrt{-\hat{g}^{00}}, n^i = -\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables we write the components of the metric $\hat{g}_{\mu\nu}$ as

$$\begin{aligned} &\hat{g}_{00} = -N^2 + N_i g^{ij} N_j \ , \quad \hat{g}_{0i} = N_i \ , \quad \hat{g}_{ij} = g_{ij} \ , \\ &\hat{g}^{00} = -\frac{1}{N^2} \ , \quad \hat{g}^{0i} = \frac{N^i}{N^2} \ , \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2} \ . \end{aligned} \tag{2.1}$$

Let us now consider the general form of Hořava-Lifshitz action

$$S = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N \left[K_{ij} \mathcal{G}^{ijkl} K_{kl} - \mathcal{V}(g) \right] \ , \tag{2.2}$$

where K_{ij} denotes the extrinsic derivative

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) \ . \tag{2.3}$$

Further the generalized De Witt metric $\mathcal{G}^{ijkl}$ is defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl} \ , \tag{2.4}$$

where λ is a real constant that in case of General Relativity is equal to one. Finally $\mathcal{V}(g)$ is general function of g_{ij} and its covariant derivative. Note also that we consider *projectable* version of HL gravity where $N = N(t)$.

The action (2.2) is invariant under foliation preserving diffeomorphism

$$t' - t = f(t) \ , \quad x'^i - x^i = \xi^i(t, \mathbf{x}) \ . \tag{2.5}$$

Following [25] we introduce $U(1)$ transformation with parameter $\alpha(\mathbf{x}, t)$ under which g_{ij}, N_i and N transform as

$$\delta_\alpha N = 0 \ , \quad , \delta_\alpha g_{ij}(\mathbf{x}, t) = 0 \ , \quad \delta_\alpha N_i(\mathbf{x}, t) = N(t) \nabla_i \alpha(\mathbf{x}, t) \ . \tag{2.6}$$

As was shown in [25] the action (2.2) is not invariant under the transformation (2.6) at least for $D \neq 2$. Then the general procedure how to find an invariant action was formulated in [25]. It is based on an introducing of the scalar field ν that transforms under (2.6) as

$$\delta_\alpha \nu(t, \mathbf{x}) = \alpha(t, \mathbf{x}) \ . \tag{2.7}$$

Then it turns out that the action invariant under (2.6) can be written in the form

$$S = \frac{1}{\kappa^2} \int dt d^D \mathbf{x} \sqrt{g} N((K_{ij} + \nabla_i \nabla_j \nu) \mathcal{G}^{ijkl}(K_{kl} + \nabla_k \nabla_l \nu) - \mathcal{V}(g)) \tag{2.8}$$

or in even more suggestive form by introducing

$$\hat{N}_i = N_i - N \nabla_i \nu \ , \quad \hat{K}_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i \hat{N}_j - \nabla_j \hat{N}_i) \tag{2.9}$$

so that

$$S = \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}N\left[\hat{K}_{ij}\mathcal{G}^{ijkl}\hat{K}_{kl} - \mathcal{V}(g)\right] \ . \tag{2.10}$$

However from this analysis it is clear that ν has a character of the Stückelberg field and hence the symmetry (2.6) is trivial. The novelty of the analysis [25] in the formulation [26] is in the introduction of the additional term into action

$$S_{\nu,k} = \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}\mathcal{G}(g_{ij})(\mathcal{A} - a) \ , \tag{2.11}$$

where

$$a = \dot{\nu} - N^i\nabla_i\nu + \frac{N}{2}\nabla^i\nabla_i\nu \ . \tag{2.12}$$

In the original work [25] the function $\mathcal{G}(g)$ was equal to $R - \Omega$ where R is $D-$dimensional curvature and Ω is constant. Note that in principle it is possible to consider more general form of $\mathcal{G}$ as was suggested in [26]. Further, a transforms under α variation as

$$a'(t,\mathbf{x}) = a(t,\mathbf{x}) + \dot{\alpha}(t,\mathbf{x}) - N^i(t,\mathbf{x})\nabla_i\alpha(t,\mathbf{x}) \ . \tag{2.13}$$

Now when we presume that $\mathcal{A}$ transforms under α variation as

$$\mathcal{A}'(t,\mathbf{x}) = \mathcal{A}(t,\mathbf{x}) + \dot{\alpha}(t,\mathbf{x}) - N^i(t,\mathbf{x})\nabla_i\alpha(t,\mathbf{x}) \tag{2.14}$$

we immediately find that (2.11) is invariant under $\alpha-$variation. Say differently, $\mathcal{A}$ can be interpreted as the gauge field that has to be introduced when we gauge the α transformation [25]. More precisely, it is clear that the action (2.10) is invariant under general $\alpha(t,\mathbf{x})$ however as we argued this is trivial Stückelberg extension with no impact on physical content of given theory. On the other hand let us presume that we want to construct more interesting modification of given theory when we add (2.11) without $\mathcal{A}$ to the original HL action. Now this term is invariant under $\alpha-$variation on condition that α obeys the equation

$$\dot{\alpha}(t,\mathbf{x}) - N^i(t,\mathbf{x})\nabla_i\alpha(t,\mathbf{x}) = 0 \ . \tag{2.15}$$

that means that α is covariantly constant [25] and hence should be interpreted as a parameter of a global symmetry. Gauging this symmetry means that we relax this condition and also introduce the gauge field $\mathcal{A}$ that transforms as (2.14).

It is clear from the analysis given above that the non-relativistic covariant HL gravity is invariant under (2.6) for arbitrary λ as was firstly stressed in [26]. Then it was argued that there is no scalar graviton in the perturbative spectrum about the flat background that makes this action very attractive since it solves the main issue of HL gravity.

3. Hamiltonian Formalism For Non-relativistic Covariant HL Gravity

For reader's convenience we again write non-relativistic covariant HL action

$$\begin{aligned} S &= \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}N(\hat{K}_{ij}\mathcal{G}^{ijkl}\hat{K}_{kl} - \mathcal{V}(g)) + \\ &+ \frac{1}{\kappa^2}\int dtd^D\mathbf{x}\sqrt{g}\mathcal{G}(R)(\mathcal{A} - a) \ , \end{aligned} \tag{3.1}$$

where we now restrict to the case when $\mathcal{G}$ depends g_{ij} through the $D-$ dimensional curvature $R(g_{ij})$. From (3.1) we find the conjugate momenta

$$\begin{aligned}
\pi^{ij} &= \frac{1}{\kappa^2}\sqrt{g}\mathcal{G}^{ijkl}\hat{K}_{kl} \ , \quad p_N \approx 0 \ , \quad p^i \approx 0 \ , \\
p_{\mathcal{A}} &\approx 0 \ , \quad p_\nu = -\frac{1}{\kappa^2}\sqrt{g}\mathcal{G}
\end{aligned} \tag{3.2}$$

that imply the $3+D$ primary constraints

$$p_N \approx 0 \ , \quad p^i(\mathbf{x}) \approx 0 \ , \quad \Phi_1(\mathbf{x}) : p_{\mathcal{A}}(\mathbf{x}) \approx 0 \ , \quad \Phi_2(\mathbf{x}) : p_\nu(\mathbf{x}) + \frac{1}{\kappa^2}\sqrt{g}\mathcal{G}(\mathbf{x}) \approx 0 \ . \tag{3.3}$$

Then following standard procedure we determine the Hamiltonian in the form

$$\begin{aligned}
H &= \int d^D\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i + v^A\Phi_A + v_N p_N + v_i p^i) - \\
&- \frac{1}{\kappa^2}\int d^D\mathbf{x}\sqrt{g}\mathcal{G}(R)(\mathcal{A} - N^i\nabla_i\nu + \frac{N}{2}\nabla^i\nabla_i\nu) \ ,
\end{aligned} \tag{3.4}$$

where $v_N, v_i, v^A, A = 1,2$ are Lagrange multipliers related to corresponding primary constraints and where

$$\begin{aligned}
\mathcal{H}_T &= \frac{\kappa^2}{\sqrt{g}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{\kappa^2}\sqrt{g}\mathcal{V}(g) - \frac{2}{\kappa^2}\nu\nabla_i\nabla_j\pi^{ij} \ , \\
\mathcal{H}_i &= -2g_{il}\nabla_k\pi^{kl} \ .
\end{aligned} \tag{3.5}$$

Note that N and p_N do not depend on $\mathbf{x}$. Now the requirement of the preservation of the primary constraints $p_N \approx 0, p_i(\mathbf{x}) \approx 0, \Phi_1(\mathbf{x}) \approx 0$ implies following secondary ones

$$\begin{aligned}
\partial_t\Phi_1 &= \{\Phi_1, H\} = -\frac{1}{\kappa^2}\sqrt{g}\ \mathcal{G} \equiv -\Phi_1^{II} \approx 0 \ , \\
\partial_t p_N &= \{p_N, H\} = -\int d^D\mathbf{x}\mathcal{H}_T + \frac{1}{2}\int d^D\mathbf{x}\Phi_1^{II}\nabla^i\nabla_i\nu \approx \\
&\approx -\int d^D\mathbf{x}\mathcal{H}_T \approx 0 \\
\partial_t p_i &= \{p_i, H\} = -\mathcal{H}_i - \Phi_1^{II} \approx -\mathcal{H}_i \approx 0 \ .
\end{aligned} \tag{3.6}$$

Now using following formulas

$$\begin{aligned}
\left\{R(\mathbf{x}), \pi^{ij}(\mathbf{y})\right\} &= -R^{ij}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) + \nabla^i\nabla^j\delta(\mathbf{x}-\mathbf{y}) - g^{ij}\nabla_k\nabla^k\delta(\mathbf{x}-\mathbf{y}) \ , \\
\nabla^i\nabla^j\mathcal{G}_{ijkl}\pi^{kl} \ - g^{ij}\nabla_m\nabla^m\mathcal{G}_{ijkl}\pi^{kl} &= \nabla_k(\nabla_l\pi^{kl}) + \frac{1-\lambda}{\lambda D - 1}\nabla_i\nabla^i\pi
\end{aligned} \tag{3.7}$$

we find that the time derivative of Φ_2 is equal to

$$\partial_t \Phi_2 = \{\Phi_2, H\} \approx -2N\frac{d\mathcal{G}}{dR}\left(R^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1-\lambda}{(\lambda D-1)}\nabla_k\nabla^k\pi\right) = 2N\frac{d\mathcal{G}}{dR}\Phi_2^{II} \ , \tag{3.8}$$

where

$$\Phi_2^{II} = -R_{ij}\pi^{ji} + \frac{\lambda}{D\lambda - 1}R\pi + \frac{1-\lambda}{(\lambda D-1)}\nabla_k\nabla^k\pi \equiv M_{ij}(g(\mathbf{x}))\pi^{ji}(\mathbf{x}) \ , \tag{3.9}$$

where generally $M_{ij}(g(\mathbf{x}))$ is a differential operator acting on π^{ij} that it reduces to ordinary multiplicative operator in case $\lambda = 1$. Note that in the calculation of (3.8) we used following result

$$\{p_\nu, H\} = -N\nabla^i\mathcal{H}_i + \frac{1}{\kappa^2}\nabla_i(\sqrt{g}N^i\mathcal{G}) + \frac{N}{2\kappa^2}\nabla^i\nabla_i(\sqrt{g}\mathcal{G}) \approx 0 \ , \tag{3.10}$$

where in the final step we used the fact that the result is proportional to the constraints $\mathcal{H}_i$ and $\Phi_1^{II} \approx 0$. In the same way we find that

$$\begin{aligned} &-2\left\{\sqrt{g}\mathcal{G}, \int d^D\mathbf{x} N\nu\nabla_i\nabla_j\pi^{ij}(\mathbf{x}))\right\} \approx \\ &\approx \left\{\int d^D\mathbf{x} N\nabla_i\nu\mathcal{H}^i, \sqrt{g}\mathcal{G}\right\} \approx \sqrt{g}\partial_i\mathcal{G}\nabla^i\nu \approx 0 \ . \end{aligned} \tag{3.11}$$

Let us review constraints that we derived at this stage. We have following set of secondary constraints $\Phi_1^{II} \approx 0$, $\Phi_2^{II} \approx 0$, $\mathcal{H}_i \approx 0$ and one global $\mathbf{T} = \int d^D\mathbf{x}\mathcal{H}_T \approx 0$. Note also that $p_\nu = \Phi_1 - \Phi_1^{II} \approx 0$ that according to (3.10) is the first class constraint. Then the total Hamiltonian takes the form

$$H_T = \int d^D\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i + v^{\mathcal{A}}p_{\mathcal{A}} + v_N p_N + v_i p^i + v^\nu p_\nu + v_{II}^1\Phi_1^{II} + v_{II}^2\Phi_2^{II}) \ , \tag{3.12}$$

where $v_N, v_i, v^{\mathcal{A}}, v_{II}^1, v_{II}^2$ are corresponding Lagrange multipliers. Note that we included the expression $(\mathcal{A} - N^i\nabla_i\nu + \frac{N}{2}\nabla^i\nabla_i\nu)$ into definition of the Lagrange multiplier v_{II}^1.

As the final step we analyze the stability of the secondary constraints. Let us begin with the constraint $\mathcal{H}_i$. It is convenient to extend these constraints by appropriate combinations of additional constraints $p_\nu \approx 0, p_{\mathcal{A}} \approx 0$ so that

$$\mathcal{H}_i = -2g_{ik}\nabla_l\pi^{kl} + \partial_i\mathcal{A}p_{\mathcal{A}} + \partial_i\nu p_\nu \ . \tag{3.13}$$

Then $\mathbf{T}_S(N^i) = \int d^D\mathbf{x}N^i\mathcal{H}_i$ is generator of the spatial diffeomorphism that is clearly preserved during the time evolution of the system since the Hamiltonian is invariant under spatial diffeomorphism. Further, $p_\nu \approx 0$ is preserved during the time evolution of the system according to (3.10). On the other hand the time evolution of the constraint $\Phi_1^{II} \approx 0$ is equal to

$$\partial_t \Phi_1^{II} = \left\{\Phi_1^{II}, H_T\right\} \approx \int d^D\mathbf{x} \left(N\sqrt{g}\frac{d\mathcal{G}}{dR}\Phi_2^{II}(\mathbf{x}) + v_{II}^2(\mathbf{x})\left\{\Phi_1^{II}, \Phi_2^{II}(\mathbf{x})\right\}\right) \approx$$
$$\approx \int d^D\mathbf{x} v_{II}^2(\mathbf{x})\left\{\Phi_1^{II}, \Phi_2^{II}(\mathbf{x})\right\} = 0 \ . \tag{3.14}$$

Since

$$\left\{\Phi_1^{II}(\mathbf{x}), \Phi_2^{II}(\mathbf{y})\right\} = M_{ij}(\mathbf{y})\left(\left\{\Phi_1^{II}(\mathbf{x}), \pi^{ji}(\mathbf{y})\right\}\right) \approx$$
$$\approx M_{ij}(\mathbf{y})\left(\sqrt{g}\frac{\delta\mathcal{G}}{\delta R}\frac{\delta R(\mathbf{x})}{\delta g_{ij}(\mathbf{y})}\right) \neq 0 \ . \tag{3.15}$$

we find that the equation (3.14) implies that $v_{II}^2 = 0$. In the same way the requirement of the preservation of the constraint Φ_2^{II} implies

$$\partial_t \Phi_2 = N\left\{\Phi_2, \mathbf{T}\right\} + \int d^D\mathbf{x} v_{II}^1(\mathbf{x})\left\{\Phi_2, \Phi_1(\mathbf{x})\right\} = 0 \tag{3.16}$$

that due to the fact that $\{\Phi_2, \mathbf{T}\} \neq 0$ and (3.15) allows to determine v_{II}^1 as a function of the canonical variables. In other words, Φ_1^{II} and Φ_2^{II} are the second class constraints.

We see that the requirement of the preservation of the secondary constraints does not imply additional constraints so that we obtained following constraint structure. We have first class constraints $\mathcal{H}_i \approx 0, p_\nu \approx 0, p_i \approx 0, p_{\mathcal{A}} \approx 0$ together with two global first class constraints $p_N \approx 0, \mathbf{T} \approx 0$. Then we have two second class constraints Φ_1^{II}, Φ_2^{II}. The detailed discussion of these constraints will be given in the next section.

In this section we performed the Hamiltonian analysis of non-relativistic covariant HL gravity in the formulation presented in [26] and we showed that it leads to the same structure of the constraints as in the original proposal [25]. Note that our analysis is valid for general λ with agreement with [26]. Further, as was shown in [25] the number of physical degrees of freedom is the same as in the General Relativity even if the constraint structures of these two theories are different. On the other hand the non-relativistic covariant HL gravity has an additional global Hamiltonian constraint. However when we consider RFDiff invariant HL gravity as the starting point for $U(1)$ extension of HL Gravity we find theory with the same content of physical degrees of freedom as in non-relativistic covariant HL gravity with additional important difference which is an absence of the global Hamiltonian constraint.

4. Non-Relativistic Covariant RFDiff HL Gravity

RFDiff invariant Hořava-Lifshitz gravity was introduced in [14] and further studied in [23]. This is the version of the Hořava-Lifshitz gravity that is not invariant under foliation preserving diffeomorphism but only under reduced set of diffeomorphism

$$t' = t + \delta t \ , \quad \delta t = \mathrm{const} \ , \quad x'^i = x^i + \xi^i(t, \mathbf{x}) \tag{4.1}$$

As was argued in [23] the simplest form of RFDiff invariant Hořava-Lifshitz gravity takes the form

$$S = \frac{1}{\kappa^2} \int dt d^D\mathbf{x} \sqrt{g} (\tilde{K}_{ij} \mathcal{G}^{ijkl} \tilde{K}_{kl} - \mathcal{V}(g)) \ , \tag{4.2}$$

where

$$\tilde{K}_{ij} = \frac{1}{2}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) \ . \tag{4.3}$$

Note that this action differs from HL gravity action (2.2) by absence of the lapse N and by replacement of the extrinsic curvature K_{ij} with $\tilde{K}_{ij}$ given above. This action is invariant under RFDiff symmetries (4.1) that is reduced with respect to foliation preserving diffeomorphism.

In order to find the $U(1)$ extension of given theory we introduce the field ν and replace N_i with $\hat{N}_i$ as

$$\hat{N}_i = N_i - \nabla_i \nu \ . \tag{4.4}$$

Then it is again easy to see that the action is invariant under transformation

$$N'_i(t, \mathbf{x}) = N_i(t, \mathbf{x}) + \nabla_i \alpha(t, \mathbf{x}) \ , \quad \nu'(t, \mathbf{x}) = \nu(t, \mathbf{x}) + \alpha(t, \mathbf{x}) \ . \tag{4.5}$$

Clearly this replacement is as trivial as the one performed in the projectable version of Hořava-Lifshitz gravity. Then following the same procedure as in section (2) we find the action in the form

$$S_{RFD} = \frac{1}{\kappa^2} \int dt d^D\mathbf{x} \sqrt{g} (\hat{K}_{ij} \mathcal{G}^{ijkl} \hat{K}_{kl} - \mathcal{V}(g) + \mathcal{G}(R)(\mathcal{A} - a)) \ , \tag{4.6}$$

where

$$\hat{K}_{ij} = \frac{1}{2}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i + \nabla_i \nabla_j \nu + \nabla_j \nabla_i \nu) \ . \tag{4.7}$$

Clearly this action is invariant under (4.1) and under (4.5). Further $\mathcal{A}$ and a transform as scalar under (4.1)

$$\mathcal{A}'(t', \mathbf{x}') = \mathcal{A}(t, \mathbf{x}) \ , \quad a'(t', \mathbf{x}') = a(t, \mathbf{x}) \tag{4.8}$$

Note that the action (4.6) can be derived from non-relativistic covariant HL action by setting $N = 1$ and hence one can expect that these theories describe the same local physics. However the my difference between these two formulations emerges when we perform the Hamiltonian analysis of the action (4.6).

As in previous section we find the primary constraints

$$p_i(\mathbf{x}) \approx 0 \ , \quad \Phi_1 : p_{\mathcal{A}}(\mathbf{x}) \approx 0 \ , \quad \Phi_2 : p_\nu(\mathbf{x}) + \frac{1}{\kappa^2} \sqrt{g} \mathcal{G}(\mathbf{x}) \tag{4.9}$$

and the relation between $\hat{K}_{ij}$ and conjugate momenta π^{ij}

$$\hat{K}_{ij} = \frac{1}{\sqrt{g}} \mathcal{G}_{ijkl} \pi^{kl} \ . \tag{4.10}$$

Then it is easy to find the total Hamiltonian in the form

$$\begin{aligned} H &= \int d^D\mathbf{x}(\mathcal{H}_T + N^i\mathcal{H}_i + v^i\Phi_i + v_N p_N + v_i p^i) - \\ &- \frac{1}{\kappa^2}\int d^D\mathbf{x}\sqrt{g}\mathcal{G}(R)(\mathcal{A} - N^i\nabla_i\nu + \frac{1}{2}\nabla^i\nabla_i\nu) \ , \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \mathcal{H}_T &= \frac{\kappa^2}{\sqrt{g}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{\kappa^2}\sqrt{g}\mathcal{V}(g) - \frac{2}{\kappa^2}\nu\nabla_i\nabla_j\pi^{ji} \ , \\ \mathcal{H}_i &= -2g_{il}\nabla_k\pi^{kl} \ . \end{aligned} \tag{4.12}$$

The requirement of the preservation of the primary constraints $p_i(\mathbf{x}) \approx 0, \Phi_1(\mathbf{x}) \approx 0$ implies following secondary ones

$$\begin{aligned} \partial_t\Phi_1 &= \{\Phi_1, H\} = -\frac{1}{\kappa^2}\sqrt{g}\mathcal{G}(R^{(D)}) \equiv -\Phi_1^{II} \approx 0 \ , \\ \partial_t p_i &= \{p_i, H\} = -\mathcal{H}_i - \Phi_1^{II} \approx -\mathcal{H}_i \approx 0 \ . \end{aligned} \tag{4.13}$$

In case of the preservation of the constraint Φ_2 we proceed as in previous section and we find

$$\partial_t\Phi_2 = \{\Phi_2, H\} \approx -2\frac{d\mathcal{G}}{dR}\left(R^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1-\lambda}{(\lambda D-1)}\nabla_k\nabla^k\pi\right) = \frac{d\mathcal{G}}{dR}\Phi_2^{II} \ , \tag{4.14}$$

where

$$\Phi_2^{II} = -R_{ij}\pi^{ji} + \frac{\lambda}{D\lambda - 1}R\pi + \frac{1-\lambda}{(\lambda D-1)}\nabla_k\nabla^k\pi \equiv M_{ij}(g(\mathbf{x}))\pi^{ji}(\mathbf{x}) \ , \tag{4.15}$$

and where generally $M_{ij}(g(\mathbf{x}))$ is a differential operator acting on π^{ij} that it reduces to ordinary multiplicative operator in case $\lambda = 1$. Note also that $p_\nu \approx 0$ is the first class constraint.

Following general analysis of constraints systems we introduce the total Hamiltonian in the form

$$H_T = \int d^D\mathbf{x}(\mathcal{H}_T + N^i\mathcal{H}_i + v^{\mathcal{A}}p_{\mathcal{A}} + v^\nu p_\nu + v_i p^i + v_{II}^1\Phi_1^{II} + v_{II}^2\Phi_2^{II}) \ . \tag{4.16}$$

As the final step we should perform the analysis of the secondary constraints. However this was done in previous section so that we do not repeat it here.

Let us now discuss the second class constraints Φ_1^{II},Φ_2^{II}. According to standard analysis these constraints have to vanish strongly and allow to solve for two phase space variables as a functions of remaining physical phase space variables that span the reduced phase space. However solving these constraints in full generality is very difficult. On the other hand it is easy to see that in linearized approximation these constraints can be solved as $h=0$, $\pi=0$ where h is the trace part of the metric fluctuation and π is its conjugate momenta.

Even if we cannot solve these constraints explicitly in general case we can still determine the number of physical degrees of freedom. To do this note that there are $D(D+1)$ gravity phase space variables g_{ij},π^{ij}, $2D$ variables N_i,p^i, 2 variables $\mathcal{A},p_{\mathcal{A}}$ and 2 variables ν,p_ν. In summary the total number of degrees of freedom is $N_{D.o.f}=D^2+3D+4$. On the other hand we have D first class constraints $\mathcal{H}_i\approx 0$, D first class constraints $p_i\approx 0$, 2 first class constraints $p_\nu\approx,p_{\mathcal{A}}\approx 0$ and two second class constraints Φ_1^{II},Φ_2^{II}. Then we have $N_{f.c.c}=2D+2$ first class constraints and $N_{s.c.c.}=2$ second class constraints. Then the number of physical degrees of freedom is [31]

$$N_{D.o.f.}-2N_{f.c.c}-N_{s.c.c.}=D^2-D-2 \qquad (4.17)$$

that exactly corresponds to the number of the phase space physical degrees of freedom of $D+1$ dimensional gravity. For example for $D=3$ the equation (4.17) is equal to 4 which is the number of phase space degrees of freedom of massless graviton.

In summary the Hamiltonian of non-relativistic general covariant RFDiff HL gravity gives the appropriate number of physical degrees of freedom of gravitational theory without introducing global Hamiltonian constraint. There is also another interesting aspect of given theory which is its non-trivial symplectic structure. In fact, let us denote the constraints $\Phi_{1,2}^{II}$ as Φ_A^{II} where $A,B=I,II$ so that the Poisson bracket between constraints can be written as

$$\left\{\Phi_A^{II}(\mathbf{x}),\Phi_B^{II}(\mathbf{y})\right\}=\triangle_{AB}(\mathbf{x},\mathbf{y}) \ . \qquad (4.18)$$

From the structure of these constraints we find that the matrix $\triangle_{AB}$ has following structure

$$\triangle_{AB}(\mathbf{x},\mathbf{y})=\left(\begin{array}{cc} 0 & * \\ * & * \end{array}\right) \ , \qquad (4.19)$$

where $*$ means non-zero elements. Then the inverse matrix $(\triangle^{-1})_{AB}$ has the form

$$(\triangle^{-1})^{AB}=\left(\begin{array}{cc} * & * \\ * & 0 \end{array}\right) \ . \qquad (4.20)$$

Now we observe that

$$\begin{aligned}
&\left\{g_{ij}(\mathbf{x}),\Phi_1^{II}(\mathbf{y})\right\}=0 \ , \left\{g_{ij}(\mathbf{x}),\Phi_2^{II}(\mathbf{y})\right\}\neq 0 \ , \\
&\left\{\pi^{ij}(\mathbf{x}),\Phi_1^{II}(\mathbf{y})\right\}\neq 0 \ , \left\{\pi^{ij}(\mathbf{x}),\Phi_2^{II}(\mathbf{y})\right\}\neq 0 \ .
\end{aligned} \qquad (4.21)$$

Then we find that the Dirac brackets between canonical variables take the form

$$\begin{aligned}
& \{g_{ij}(\mathbf{x}), g_{kl}(\mathbf{y})\}_D = -\int d\mathbf{z}d\mathbf{z}' \{g_{ij}(\mathbf{x}), \Phi^{II}_A(\mathbf{z})\} (\triangle^{-1})^{AB}(\mathbf{z},\mathbf{z}') \{\Phi^{II}_B(\mathbf{z}'), g_{kl}(\mathbf{y})\} = 0 \ , \\
& \left\{\pi^{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\right\}_D = \\
= & -\int d\mathbf{z}d\mathbf{z}' \{\pi^{ij}(\mathbf{x}), \Phi^{II}_A(\mathbf{z})\} (\triangle^{-1})^{AB}(\mathbf{z},\mathbf{z}') \left\{\Phi^{II}_B(\mathbf{z}'), \pi^{kl}(\mathbf{y})\right\} = \Omega^{ijkl}(\mathbf{x},\mathbf{y}) \ , \\
& \left\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\right\}_D = \left\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\right\} - \\
- & \int d\mathbf{z}d\mathbf{z}' \{g_{ij}(\mathbf{x}), \Phi^{II}_A(\mathbf{z})\} (\triangle^{-1})^{AB}(\mathbf{z},\mathbf{z}') \left\{\Phi^{II}_B(\mathbf{z}'), \pi^{kl}(\mathbf{y})\right\} = \Omega^{kl}_{ij}(\mathbf{x},\mathbf{y}) \ ,
\end{aligned} \tag{4.22}$$

where the matrix Ω depends on phase-space variables according to (4.20) and (4.21). Hence the non-relativistic covariant RFDiff HL gravity has well defined Hamiltonian formulation with symplectic structure that generally depends on phase space variables. Note however that in case of the linearized approximation one can choose the constraints in such a way that the Dirac bracket coincides with the Poisson bracket. Explicitly, in linearized approximation the second class constraints can be chosen as $h = 0, \pi = 0$ as follows from the analysis given above. These constraints have vanishing Poisson brackets with remaining dynamical variables and consequently the Dirac brackets between physical phase space variables coincide with Poisson brackets.

As the final remark we again emphasize the important point that the Hamiltonian of non-relativistic covariant RFDiff HL gravity does not vanish on constraint surface. This is the similar situation as in case of the Hamiltonian of the healthy extended HL gravity [17, 18] which is however in sharp contrast with the Hamiltonian of General Relativity. As we argued in these papers this fact has a strong impact on the definition of observables in healthy extended Hořava- Lifshitz gravity or in any theory of gravity where the Hamiltonian is not given as linear combination of constraints. Since the discussion presented in [17, 18] can be applied in case of the non-relativistic covariant RFDiff HL gravity as well we are not going to repeat it here. Instead we recommend these papers to reader that is interested in these problems.

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References

[1] P. Horava, *"Quantum Gravity at a Lifshitz Point,"* Phys. Rev. D **79** (2009) 084008 [arXiv:0901.3775 [hep-th]].

[2] P. Horava, *"Membranes at Quantum Criticality,"* JHEP **0903** (2009) 020 [arXiv:0812.4287 [hep-th]].

[3] P. Horava, *"Quantum Criticality and Yang-Mills Gauge Theory,"* arXiv:0811.2217 [hep-th].

[4] A. Padilla, *"The good, the bad and the ugly .... of Horava gravity,"* arXiv:1009.4074 [hep-th].

[5] S. Mukohyama, *"Horava-Lifshitz Cosmology: A Review,"* arXiv:1007.5199 [hep-th].

[6] S. Weinfurtner, T. P. Sotiriou and M. Visser, *"Projectable Horava-Lifshitz gravity in a nutshell,"* J. Phys. Conf. Ser. **222**, 012054 (2010) [arXiv:1002.0308 [gr-qc]].

[7] T. P. Sotiriou, *"Horava-Lifshitz gravity: a status report,"* arXiv:1010.3218 [hep-th]. Huang:2010rq

[8] T. P. Sotiriou, M. Visser and S. Weinfurtner, *"Quantum gravity without Lorentz invariance,"* JHEP **0910** (2009) 033 [arXiv:0905.2798 [hep-th]].

[9] Y. Huang, A. Wang and Q. Wu, *"Stability of the de Sitter spacetime in Horava-Lifshitz theory,"* Mod. Phys. Lett. A **25** (2010) 2267 [arXiv:1003.2003 [hep-th]].

[10] A. Wang and Q. Wu, *"Stability of spin-0 graviton and strong coupling in Horava-Lifshitz theory of gravity,"* arXiv:1009.0268 [hep-th].

[11] D. Blas, O. Pujolas and S. Sibiryakov, *"On the Extra Mode and Inconsistency of Horava Gravity,"* JHEP **0910** (2009) 029 [arXiv:0906.3046 [hep-th]].

[12] D. Blas, O. Pujolas and S. Sibiryakov, *"A healthy extension of Horava gravity,"* arXiv:0909.3525 [hep-th].

[13] D. Blas, O. Pujolas and S. Sibiryakov, *"Comment on 'Strong coupling in extended Horava-Lifshitz gravity',"* arXiv:0912.0550 [hep-th].

[14] D. Blas, O. Pujolas and S. Sibiryakov, *"Models of non-relativistic quantum gravity: the good, the bad and the healthy,"* arXiv:1007.3503 [hep-th].

[15] M. Li and Y. Pang, *"A Trouble with Hořava-Lifshitz Gravity,"* JHEP **0908** (2009) 015 [arXiv:0905.2751 [hep-th]].

[16] M. Henneaux, A. Kleinschmidt and G. L. Gomez, *"A dynamical inconsistency of Horava gravity,"* arXiv:0912.0399 [hep-th].

[17] J. Kluson, *"Note About Hamiltonian Formalism of Modified $F(R)$ Hořava-Lifshitz Gravities and Their Healthy Extension,"* Phys. Rev. D **82** (2010) 044004 [arXiv:1002.4859 [hep-th]].

[18] J. Kluson, *"Note About Hamiltonian Formalism of Healthy Extended Horava-Lifshitz Gravity,"* JHEP **1007** (2010) 038 [arXiv:1004.3428 [hep-th]].

[19] J. Bellorin and A. Restuccia, *"Closure of the algebra of constraints for a non-projectable Hořava model,"* arXiv:1010.5531 [hep-th].

[20] J. Bellorin and A. Restuccia, *"On the consistency of the Horava Theory,"* arXiv:1004.0055 [hep-th].

[21] A. Kobakhidze, *"On the infrared limit of Horava's gravity with the global Hamiltonian constraint,"* Phys. Rev. D **82** (2010) 064011 [arXiv:0906.5401 [hep-th]].

[22] J. M. Pons and P. Talavera, *"Remarks on the consistency of minimal deviations from General Relativity,"* Phys. Rev. D **82** (2010) 044011 [arXiv:1003.3811 [gr-qc]].

[23] J. Kluson, *"Horava-Lifshitz Gravity And Ghost Condensation,"* arXiv:1008.5297 [hep-th].

[24] E. Gourgoulhon, *"3+1 Formalism and Bases of Numerical Relativity,"* arXiv:gr-qc/0703035.

[25] P. Horava and C. M. Melby-Thompson, *"General Covariance in Quantum Gravity at a Lifshitz Point,"* Phys. Rev. D **82** (2010) 064027 [arXiv:1007.2410 [hep-th]].

[26] A. M. da Silva, *"An Alternative Approach for General Covariant Horava-Lifshitz Gravity and Matter Coupling,"* arXiv:1009.4885 [hep-th].

[27] J. Greenwald, V. H. Satheeshkumar and A. Wang, *"Black holes, compact objects and solar system tests in non-relativistic general covariant theory of gravity,"* arXiv:1010.3794 [hep-th].

[28] J. Alexandre and P. Pasipoularides, *"Spherically symmetric solutions in Covariant Horava-Lifshitz Gravity,"* arXiv:1010.3634 [hep-th].

[29] A. Wang and Y. Wu, *"Cosmology in nonrelativistic general covariant theory of gravity,"* arXiv:1009.2089 [hep-th].

[30] Y. Huang and A. Wang, *"Nonrelativistic general covariant theory of gravity with a running constant λ,"* arXiv:1011.0739 [hep-th].

[31] M. Henneaux and C. Teitelboim, *"Quantization of gauge systems," Princeton, USA: Univ. Pr. (1992) 520 p*

[32] J. Govaerts, *"The quantum geometer's universe: Particles, interactions and topology,"* arXiv:hep-th/0207276.

[33] J. Govaerts, *"Hamiltonian Quantization And Constrained Dynamics," Leuven, Belgium: Univ. Pr. (1991) 371 p. (Leuven notes in mathematical and theoretical physics, B4)*