Inflation of Bianchi type-VII_0 Universe with Dirac Field in Einstein-Cartan theory

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Abstract

We discuss the Bianchi type-VII_0 cosmology with Dirac field in Einstein-Cartan theory. We obtain the equations of Dirac field and gravitational field in Einstein-Cartan theory. We find a Bianchi type-VII_0 inflationary solution.

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1 Introduction

The many cosmological models are based upon the homogeneous and isotropic Robertson-Walker(R-W) spacetime. The field equation is simpler when we use R-W metric. In practice, no compelling reason exists, apart from simplicity, why we must decide on an isotropic condition from Universe's beginning. The inflationary epoch occurs very early, when Universe as only 10^{-34}s old. At that time the energy density was of the order of $10^{77} g/cm^3$. In the epoch the effective energy density $\frac{8\pi G}{c^4}S^2$ produced by the matter spin could be non-negligible compared to $10^{77} g/cm^3$ [1, 2, 3, 5]. Therefore the geometry theory of the early universe should also include the effect of the spin. It is therefore interesting to discuss early cosmology in the Einstein-Cartan theory. The Einstein-Cartan field equations are non-self-consistent with R-W metric [1, 2, 3, 4]. Therefore, an anisotropic spacetime metric is needed. Bianchi-type spacetime metrics are possible choices. It is interesting to consider the viability of the Bianchi type inflationary models in the Einstein-Cartan theory [10].

In this paper, we will investigate the Bianchi-type VII_0 spacetime with Dirac field in the Einstein-Cartan theory. This paper is organized as follows: In section 2, we construct the Dirac field equation and gravitational field equation. In section 3, we solve field equations in Bianchi-type VII_0 spacetime and obtain an inflation solution of universe. Finally, a summary is presented.

2 The field equations in the Einstein-Cartan theory

The action of Dirac field in Riemann-Cartan space[6, 7, 8, 9]:

$$S_D = \int L_D(\Psi, \Psi^+, \widetilde{\Gamma^{\lambda}_{\mu\nu}}, g_{\mu\nu})\sqrt{g}d^4x = \frac{\hbar c}{2}\int \left([(\Psi^+\gamma^\mu\widetilde{\nabla_\mu}\Psi) - (\widetilde{\nabla_\mu}\Psi^+\gamma^\mu\Psi)] + \frac{2mc}{\hbar}\Psi^+\Psi \right)\sqrt{g}d^4x \tag{1}$$

Where m is rest mass of Dirac particle. Hereafter, the tilde denotes the Riemann-Cartan geometrical objects. The usual notation, without any additional marks, is used for the Riemannian objects(torsion-independent). In Riemann-Cartan spacetime, the covariant derivative is

$$\widetilde{\nabla_\mu}\Psi = \nabla_\mu\Psi - \frac{1}{4}\gamma^\rho T^{\lambda}_{\ .\rho\mu}\gamma_\lambda\Psi \tag{2}$$

In Riemann spacetime, covariant derivative is:

$$\nabla_\mu\Psi = \partial_\mu\Psi + \Gamma_\mu\Psi \tag{3}$$

Where

$$\Gamma_\mu = \frac{1}{4}\gamma^\rho \gamma_{\rho;\mu} \tag{4}$$

the ∇_μ and ";" denote covariant derivative in Riemann spacetime. $T^\lambda{}_{.\rho\mu}$ is complex of torsion tensor

$$T^\lambda{}_{.\rho\mu} \equiv Q^\lambda{}_{.\rho\mu} + Q_{\rho\mu}{}^{.\lambda} + Q_{\mu\rho}{}^{.\lambda} \tag{5}$$

In above quation, $Q^\lambda{}_{.\rho\mu}$ is the torsion tensor. The γ_μ are Dirac matrices in Riemann spacetime. They have the relation:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g^{\mu\nu} I \tag{6}$$

Where I is the unit matrix.

For a flat anisotropic Bianchi type-VII_0 spacetime

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)(dy^2 + dz^2) \tag{7}$$

If $a(t) = b(t)$, Eq. (7) returns to the spatially flat Robertson-Walker metric. The relation of γ^μ and $\widetilde{\gamma}_\mu$ are:

$$\gamma^0 = \widetilde{\gamma}_0,\ \gamma^1 = \frac{-\widetilde{\gamma_1}}{a(t)},\ \gamma^2 = \frac{-\widetilde{\gamma_2}}{b(t)},\ \gamma^3 = \frac{-\widetilde{\gamma_3}}{b(t)} \tag{8}$$

Where $\widetilde{\gamma_\mu}$ are Dirac matrices in Minkowski spacetime. They are

$$\widetilde{\gamma_0} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \widetilde{\gamma_j} = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3 \tag{9}$$

Where I is 2×2 unit matrix, σ_j is pauli matrix. The total action of gravitational and Dirac field is:

$$S = \int \{\frac{R}{2} + L_D\}\sqrt{g} d^4x \tag{10}$$

Independently varying $g_{\mu\nu}$, $\widetilde{\Gamma}^\lambda_{\mu\nu}$, Ψ and Ψ^+ from above action, one get:

$$\frac{\delta S}{\delta g^{\mu\nu}} \equiv \widetilde{G_{\mu\nu}} - \frac{\hbar c}{4}[\widetilde{\nabla}_\nu \Psi^+ \gamma_\mu \Psi - \Psi^+ \gamma_\mu \widetilde{\nabla}_\nu \Psi + \widetilde{\nabla}_\mu \Psi^+ \gamma_\nu \Psi - \Psi^+ \gamma_\nu \widetilde{\nabla}_\mu \Psi] = 0 \tag{11}$$

$$\frac{\delta S}{\delta \widetilde{\Gamma}_\mu{}^{.\nu\lambda}} \equiv Q^\mu{}_{.\nu\lambda} - (S^\mu{}_{.\nu\lambda} + \delta^\mu{}_{[\nu} S_{\lambda]}) = 0 \tag{12}$$

$$\frac{\delta S}{\delta \Psi^+} \equiv \gamma^\mu(\widetilde{\nabla_\mu} - Q_\mu)\Psi + \frac{mc}{\hbar}\Psi = 0 \tag{13}$$

$$\frac{\delta S}{\delta \Psi} \equiv (\widetilde{\nabla_\mu} - Q_\mu)\Psi^+ \gamma^\mu - \frac{mc}{\hbar}\Psi^+ = 0 \tag{14}$$

Correspondingly the Einstein equation is

$$\widetilde{G}_{\mu\nu} = \widetilde{T}_{\mu\nu} = \frac{\hbar c}{4}[\widetilde{\nabla}_\nu \Psi^+ \gamma_\mu \Psi - \Psi^+ \gamma_\mu \widetilde{\nabla}_\nu \Psi + \widetilde{\nabla}_\mu \Psi^+ \gamma_\nu \Psi - \Psi^+ \gamma_\nu \widetilde{\nabla}_\mu \Psi] \tag{15}$$

and the cartan type equation is

$$Q^\mu_{\ .\nu\lambda} = (S^\mu_{\ .\nu\lambda} + \delta^\mu_{\ [\nu} S_{\lambda]}) \tag{16}$$

The antisymmetric tension of spin density is:

$$\begin{gathered} S_{\lambda\mu\nu} = S_{[\lambda\mu\nu]} \equiv \Psi^+ \gamma_{[\lambda} \gamma_\mu \gamma_{\nu]} \Psi \\ S^\mu = \Psi^+ \gamma^\mu \gamma_5 \Psi \end{gathered} \tag{17}$$

The equations of Dirac field are:

$$\gamma^\mu (\widetilde{\nabla_\mu} - Q_\mu) \Psi + \frac{mc}{\hbar} \Psi = 0 \tag{18}$$

$$(\widetilde{\nabla_\mu} - Q_\mu) \Psi^+ \gamma^\mu - \frac{mc}{\hbar} \Psi^+ = 0 \tag{19}$$

Applying the relation of Einstein-Cartan affine connection $\widetilde{\Gamma}^\lambda_{\mu\nu}$ and christoffel symbol $\Gamma^\lambda_{\mu\nu}$:

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - [S^\lambda_{\mu\nu} + 2 S^\lambda_{(\mu\nu)}] \tag{20}$$

One get

$$\widetilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{\hbar^2 c^2}{16} S^\kappa S_\kappa g_{\mu\nu} - \frac{\hbar^2 c^2}{8} S_\mu S_\nu \tag{21}$$

Where $G_{\mu\nu}$ is Einstein tensor in Riemann spacetime.

$$\widetilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{\hbar^2 c^2}{8} (S^\kappa S_\kappa g_{\mu\nu} - S_\mu S_\nu) \tag{22}$$

Where $T_{\mu\nu}$ is the energy-momentum tensor of Dirac field in Riemann spacetime. It is

$$T_{\mu\nu} = \frac{\hbar c}{4}[\nabla_\nu \Psi^+ \gamma_\mu \Psi - \Psi^+ \gamma_\mu \nabla_\nu \Psi + \nabla_\mu \Psi^+ \gamma_\nu \Psi - \Psi^+ \gamma_\nu \nabla_\mu \Psi] \tag{23}$$

Substituting above equations into Eq. (15), we can obtain

$$G_{\mu\nu} = T_{\mu\nu} + \frac{3}{16} \hbar^2 c^2 (S^\kappa S_\kappa g_{\mu\nu}) \tag{24}$$

Finally, the Dirac Eq. (18) can be rewritten as:

$$\gamma^\mu \nabla_\mu \Psi - \frac{3}{8} \hbar c S^\mu \gamma_\mu \gamma_5 \Psi + \frac{mc}{\hbar} \Psi = 0 \tag{25}$$

From the Bianchi identities $\nabla_\nu G^{\mu\nu} = 0$ one can obtain the Riemann-Cartan generalization of the relativistic conservation of angular momentum and energy-momentum in Bianchi type-VII_0 Universe.

3 Inflationary Solution in Bianchi type-VII_0 Universe

In Bianchi type-VII_0 spacetime, Einstein field Eq. (24) becomes

$$2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} = T_0^0 - \frac{3}{16}\hbar^2 c^2 S^2 \tag{26}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} = T_1^1 + \frac{3}{16}\hbar^2 c^2 S^2 \tag{27}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = T_2^2 + \frac{3}{16}\hbar^2 c^2 S^2 \tag{28}$$

$$\frac{\dot{a}}{a} - \frac{\dot{b}}{b} = \frac{const}{ab^2} \tag{29}$$

where $\dot{a}$ denotes differentiation with respect to time. S^2 is spin energy density, where $S^2 = \frac{1}{2}S_{\mu\nu}S^{\mu\nu}$.

In this model, the hubble parameters $H_i (i = 1, 2)$ are $H_1 = \frac{\dot{a}}{a}$ and $H_2 = \frac{\dot{b}}{b}$, the shear is $\sigma_i = H_i - \frac{\dot{u}}{u}$, where $u^3 = ab^2$. The anisotropic energy density:

$$\sigma^2 \equiv \frac{1}{2}(\sigma_1^2 + 2\sigma_2^2) = \frac{\sigma_0^2}{u^6} \tag{30}$$

where σ_0 is constant.

The conservation of angular momentum can describe following form:

$$S(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}) + \dot{S} = 0 \tag{31}$$

Integrating the above expression, we have

$$Su^3 = S_0 \tag{32}$$

where S_0 is constant. Eq. (26)-Eq. (28) can become following form:

$$3(\frac{\dot{u}}{u})^2 + \frac{3}{16}\hbar^2 c^2 S^2 - \sigma^2 = T_0^0 \tag{33}$$

$$-\frac{2\ddot{u}}{u} - (\frac{\dot{u}}{u})^2 + \frac{3}{16}\hbar^2 c^2 - \sigma^2 = -T_1^1 = -T_2^2 \tag{34}$$

Dirac Eq. (25) becomes

$$\gamma_0\dot{\Psi}+\frac{3}{2}\frac{\dot{u}}{u}\gamma_0\Psi+\frac{3}{8}S^{\mu}\gamma_\mu\gamma_5\Psi-\frac{mc}{\hbar}\Psi=0 \tag{35}$$

For the homogeneous and isotropic Robertson-Walker metric, authors had obtained the Dirac equation in Einstein-Cartan theory [11]. In fact, the Dirac equation Eq.(35) obtained in anisotropic spacetime metric is more suitable. Eq.(35) will return to the form of Dirac equation in [11] when $a(t)=b(t)$. From Eq. (33) and Eq. (34), we obtain

$$\frac{\ddot{u}}{u}=-\frac{(T_0^0-3T_i^i)}{6}+\frac{1}{3}(\frac{3}{16}S^2-\sigma^2) \tag{36}$$

when $\frac{3}{16}S^2-\sigma^2>0$, the universe may be avoid singularity. We assume $T_0^0=\rho_\Lambda(\rho_\Lambda$ is the vacuum energy density) in inflation epoch.

From Eq.(33), we can obtain an inflationary solution:

$$log(Z+\sqrt{Z^2+B^2})=3\sqrt{A}t \tag{37}$$

where $A=\frac{\rho_\Lambda}{3}$, $B=\frac{(\frac{3}{16}S_0^2-\sigma_0^2)}{3}$, $Z=u^3=ab^2$.

4 Summary

We have investigated the Bianchi type-VII_0 cosmology with Dirac field in Einstein-Cartan theory. The inflationary epoch occurs very early, when universe as only 10^{-34}s old. At that time the energy density was of the order of $10^{77}\mathrm{g}cm^{-3}$. In the epoch the effective energy density $\frac{8\pi G}{c^4}S^2$ produced by the matter spin could be non-negligible compared to $10^{77}\mathrm{g}cm^{-3}$ Therefore the geometry theory of early universe should also include the effect of the spin. It is therefore interesting to discuss early cosmology in the Einstein-Cartan theory. We obtain the equations of Dirac field and gravitational field in the Bianchi type-VII_0 spacetime. We find an analytical inflationary solution.

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