# All Bulk and Boundary Unitary Cubic Curvature Theories in Three Dimensions

İbrahim Güllü,* Tahsin Çağrı Şişman,† and Bayram Tekin‡
*Department of Physics,*
*Middle East Technical University,*
*06531, Ankara, Turkey*
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We construct all the bulk and boundary unitary cubic curvature parity invariant gravity theories in three dimensions in (anti)-de Sitter spaces. For bulk unitarity, our construction is based on the principle that the free theory of the cubic curvature theory reduces to one of the three known unitary theories which are the cosmological Einstein-Hilbert theory, the quadratic theory of the scalar curvature or the new massive gravity (NMG). Bulk and boundary unitarity in NMG is in conflict; therefore, cubic theories that are unitary both in the bulk and on the boundary have free theories that reduce to the other two alternatives. We also study the unitarity of the Born-Infeld extensions of NMG to all orders in curvature.

**Contents**



## I. INTRODUCTION

In three dimensions, there are three parity invariant pure gravity theories that are known to be unitary in the sense of tachyon and ghost freedom at the tree level. These are the (cosmological) Einstein-Hilbert theory with no local degrees of freedom, the quadratic theory built from the curvature scalar with the Lagrangian density $R-2\Lambda_0+aR^2$ which has a single massive scalar degree of freedom [36], and the new massive gravity (NMG) defined by the action [1, 2]

$$I=\frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\left[\sigma R-2\lambda_0 m^2+\frac{1}{m^2}\left(R_{\mu\nu}^2-\frac{3}{8}R^2\right)\right], \tag{1}$$

*Electronic address: e075555@metu.edu.tr
†Electronic address: sisman@metu.edu.tr
‡Electronic address: btekin@metu.edu.tr

that provides a nonlinear extension of the Pauli-Fierz massive spin-2 theory with two degrees of freedom. Here, $\sigma = \pm 1$ or it could be set to zero to obtain a purely quadratic theory. The important point is that, with some constraints on the parameters, these three theories exhaust the list of unitary pure gravity theories in (anti)-de Sitter [(A)dS] and flat spaces in three dimensions. Therefore, if one searches for a unitary theory built from arbitrary powers of the Ricci scalar and the tensor, then the propagator of that theory should reduce to one of these unitary theories, with possibly redefined parameters (such as mass, cosmological constant etc.). In flat backgrounds, the problem is trivial: Any higher derivative (cubic and more) deformation of the above theories is allowed since the propagators are intact in this background. But, in constant curvature backgrounds, which we shall deal with in this paper, generically, all the higher derivative terms contribute to the propagators and therefore the unitarity analysis is actually quite involved. However, as we shall show in detail, tree-level unitary theories can be constructed systematically by studying their propagators with the recently developed tools in [12] and with the earlier tools of [13] for analyzing the unitarity of a higher derivative theory around (A)dS backgrounds. In fact, as an example, we will construct all the unitary theories in three dimensions that are built from at most the cubic powers of the Ricci tensor.

Several extensions of NMG have already appeared recently: In [8], cubic and quartic extensions of NMG was found using the requirement that a *simple* (essentially integrable) holographic $c$-function exists. In [9, 10], a Born-Infeld (BI) type action was defined which extends NMG up to any desired order in the curvature (and in particular reproduces the same cubic and quartic extensions of [8] with fixed parameters at each order of the curvature) and which has a holographic $c$-function. In [11], order by order extension of NMG was introduced again using the notion of a holographic $c$-function. This order by order extension also matches the curvature expansion of the Born-Infeld extended NMG [10].

It is worth to stress again, in constructing a generic unitary theory at any powers of curvature, our main principle is the following: *The propagator of the theory should reduce to the propagator of the known three unitary parity invariant theories after possible redefinitions of the parameters.* Note that this principle is merely a restatement of the unitary extension of a theory and does not assume any strong conditions such as the existence of a simple holographic $c$-function or the condition that the resulting theory can be obtained from a BI-type action.

Up to now, we have discussed bulk unitarity only. For AdS spaces, unitarity on the boundary is also an important issue because of the AdS/CFT correspondence. Out of the three bulk unitary theories, NMG always gives a nonunitary theory on the boundary [2]. The other two theories have rather wide ranges of the parameters which allow both bulk and boundary unitarity. Therefore, in AdS, if a cubic theory is unitary in the bulk and on the boundary, then its free theory reduces to either cosmological Einstein-Hilbert or the $R - 2\Lambda_0 + aR^2$ theory.

The cubic theory found before [8, 9] is a single member of the continuous family of bulk unitary theories that we shall present. Moreover, we will more directly show the region where this cubic theory is unitary. In principle, our analysis can be extended to any powers of curvature tensors and to any dimensions. We will also give two examples of arbitrary power theories: The so called Born-Infeld extension of new massive gravity and its close cousin [9], specifically we will show that their propagators reduce to that of NMG. Namely, like the cubic theory found in [8], BINMG is unitary in the bulk only.

Since NMG (1) plays an important role in the construction of cubic or higher order theories, let us recapitulate its properties. For proper ranges (which we shall discuss) of the dimensionless parameters $\sigma$, $\lambda_0$ and the dimensionful parameter $m^2$, NMG is a tree-level (bulk) unitary theory generically describing a massive spin-2 excitation with mass $M^2 = \left(-\sigma + \frac{\lambda}{2}\right) m^2$ at the linearized level around both flat and (A)dS backgrounds [1–7]. Here, the effective cosmological constant is

$\Lambda = \lambda m^2$ with $\lambda = -2\left(\sigma \pm \sqrt{1+\lambda_0}\right)$. In what follows, we will work with the mostly plus signature, assume $\kappa^2 > 0$, and our convention for the sign of the Riemann tensor follows from $\left[\nabla_\mu, \nabla_\nu\right] V^\sigma \equiv R_{\mu\nu}{}^\sigma{}_\rho V^\rho$. In flat backgrounds, unitarity analysis of this model is quite straightforward and has been carried out in several places, but in (A)dS backgrounds the analysis is somewhat more complicated: In [2], the theory was shown to be formally equivalent to the Pauli-Fierz massive gravity in (A)dS, and in [6] direct gauge-invariant canonical analysis was carried out by decomposing the spin-2 field in its irreducible parts under the rotation group.

The layout of the paper is as follows: In Section II, we start with the most general cubic action based on the Ricci tensor and the scalar, and find the equivalent quadratic action which has the same $O\left(h^2\right)$ expansion, that is the expansion in metric perturbation, as the original cubic action. In Section III, we discussed the unitarity of Born-Infeld extensions of NMG. In the Appendix, we explicitly calculate the $O\left(h^2\right)$ expansion of BINMG.

## II. UNITARY CUBIC THEORIES

The most general cubic curvature theory built from the Ricci tensor and the scalar is

$$I = \frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\Big[\sigma R - 2\lambda_0 m^2 + \frac{\omega}{m^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right) + \frac{\eta}{8m^2}R^2 + \frac{\alpha}{6m^4}\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} + \beta R R_{\mu\nu}^2 + \gamma R^3\right)\Big], \tag{2}$$

where $\sigma$, $\lambda_0$, $\omega$, $\eta$, $\alpha$, $\beta$ and $\gamma$ are dimensionless parameters whose signs and numerical values are arbitrary at this stage except, we normalize $\sigma^2 = 1$, and $\omega^2 = 1$ or $\omega = 0$. On the other hand, $m^2$ is of $[\text{Mass}]^2$ dimension and without loss of generality we choose $m^2 > 0$ and $\kappa^2 > 0$. In flat backgrounds, which necessarily requires $\lambda_0 = 0$, we know that for *any* $\alpha$ the theory is unitary only if $\omega\eta = 0$. For $\omega = 0$, the theory should have the "right" sign Einstein-Hilbert term with $\sigma = +1$. Furthermore, if $\eta$ is also set to zero in this case, then there is no propagating degree of freedom; while for $\eta \neq 0$ there is a spin-0 excitation with mass $m_s^2 \equiv \frac{m^2}{\eta} > 0$ in order to have a nontachyonic behavior [5, 6]. For $\eta = 0$ and $\omega \neq 0$, NMG is recovered for $\sigma = -1$ with two spin-2 degrees of freedom having mass $m_g^2 = \frac{m^2}{\omega}$ with $\omega > 0$ [1]. We will not consider the case when $\sigma = 0$. Therefore, in flat space, the already known picture at the quadratic level does not change at the cubic or higher levels. Thus, the main question is to find possible ranges of these parameters for which this theory is unitary around its *constant curvature* vacua. To answer this question, one has to find the $O\left(h_{\mu\nu}^2\right)$ action where $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ is the (A)dS vacuum (or vacua) for which $\bar{R}_{\mu\nu} = 2\lambda m^2 \bar{g}_{\mu\nu}$. One can directly compute the $O\left(h_{\mu\nu}^2\right)$ action of (2), but this is highly tedious and such a direct approach would be practically impossible for some arbitrary $R^n$ theories. Therefore, we will instead employ a technique developed in [13] which boils down to finding an equivalent quadratic action which has the same propagator and the same vacua. The procedure is quite effective and at no point one needs the complicated equations of motion. For more details and uses of this technique see [12]. Let us now first find the maximally symmetric vacuum or vacua of (2). This can be done with the help of the *equivalent quadratic* action, as we just said, but in a simpler way the vacuum can also be found from an *equivalent linear* theory. This follows from

$$\int d^3x\,\mathcal{L}\left(R, R_{\mu\nu}\right) = \int d^3x\,\mathcal{L}\left(\bar{R}, \bar{R}_{\mu\nu}\right) + \int d^3x\,\left[\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\right]_{\bar{g}_{\mu\nu}}\delta g^{\mu\nu} + \frac{1}{2}\int d^3x\,\delta g^{\alpha\beta}\left[\frac{\delta\mathcal{L}}{\delta g^{\alpha\beta}\delta g^{\mu\nu}}\right]_{\bar{g}_{\mu\nu}}\delta g^{\mu\nu} + \ldots, \tag{3}$$

where $\mathcal{L} \equiv \sqrt{-g} f\left(R, R_{\mu\nu}\right)$, and by equivalent linear action we mean an action which has the same $O\left(h^0\right)$ and $O\left(h\right)$ expansions as (3), and equivalent quadratic action has the same $O\left(h^0\right)$, $O\left(h\right)$ and $O\left(h^2\right)$ expansions as given in (3). To find the equivalent linear or quadratic actions, $f\left(R, R_{\mu\nu}\right)$ should be expanded to linear (or quadratic) order in the curvature around $\left(\bar{R}, \bar{R}_{\mu\nu}\right)$. The important point is that from the linear (or quadratic) expansion in curvature one gets all the $O\left(h\right)$ [or $O\left(h^2\right)$ ] terms of $f\left(R, R_{\mu\nu}\right)$. Therefore, the expansion in small curvature is not an approximation as far as the vacuum and the propagator of the full theory is considered. [In these expansions one has to keep in mind that $O\left(h^n\right)$ terms come from the $\sum_{i=0}^{n}\left(R-\bar{R}\right)^i$ expansions.]

We can now start our computation and find the vacua of (2). One further simplification is to consider the Lagrangian density as a function of $R^\mu_\nu$, in order not to introduce the metric or its inverse during the expansion. Therefore, we have

$$\begin{aligned} f\left(R^\mu_\nu\right) \equiv& \sigma\delta^\nu_\mu R^\mu_\nu - 2\lambda_0 m^2 + \frac{\omega}{m^2}\left(R^\mu_\nu R^\nu_\mu - \frac{3}{8}R^2\right) + \frac{\eta}{8m^2}\left(\delta^\nu_\mu R^\mu_\nu\right)^2 \\ &+ \frac{\alpha}{6m^4}\left[R^\mu_\rho R^\rho_\nu R^\nu_\mu + \beta\left(\delta^\gamma_\lambda R^\lambda_\gamma\right)\left(R^\mu_\nu R^\nu_\mu\right) + \gamma\left(\delta^\nu_\mu R^\mu_\nu\right)^3\right]. \end{aligned} \tag{4}$$

Then, expanding $f\left(R^\mu_\nu\right)$ to the first order around the yet to be found background $\left(\bar{R}^\mu_\nu = 2\lambda m^2 \delta^\mu_\nu\right)$ with the assumption of small fluctuations [that is $\left(R^\alpha_\beta - \bar{R}^\alpha_\beta\right)$ being small] as

$$f\left(R^\mu_\nu\right) = f\left(\bar{R}^\mu_\nu\right) + \left[\frac{\partial f}{\partial R^\alpha_\beta}\right]_{\left(\bar{R}^\mu_\nu\right)}\left(R^\alpha_\beta - \bar{R}^\alpha_\beta\right) + O\left[\left(R^\alpha_\beta - \bar{R}^\alpha_\beta\right)^2\right], \tag{5}$$

one obtains the equivalent linear Lagrangian density $g_{\text{lin-equal}}\left(R^\mu_\nu\right)$ after dropping the quadratic order as

$$\begin{aligned} g_{\text{lin-equal}}\left(R^\mu_\nu\right) =& \left[-2\lambda_0 + \frac{3\lambda^2}{2}\left(\omega - 3\eta\right) - 8\alpha\lambda^3\left(1+3\beta+9\gamma\right)\right] m^2 \\ &+ \left[\sigma - \frac{\lambda}{2}\left(\omega - 3\eta\right) + 2\alpha\lambda^2\left(1+3\beta+9\gamma\right)\right] R. \end{aligned} \tag{6}$$

Therefore, the equivalent linear action becomes

$$\begin{aligned} I_{\text{lin-equal}} =& \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left[\sigma - \frac{\lambda}{2}\left(\omega - 3\eta\right) + 2\alpha\lambda^2\left(1+3\beta+9\gamma\right)\right] \\ &\times\left[R - \frac{\left[4\lambda_0 - 3\left(\omega-3\eta\right)\lambda^2 + 16\alpha\lambda^3\left(1+3\beta+9\gamma\right)\right]}{\left[2\sigma - \lambda\left(\omega-3\eta\right) + 4\alpha\lambda^2\left(1+3\beta+9\gamma\right)\right]} m^2\right]. \end{aligned} \tag{7}$$

Let us stress again that (7) and (2) have the same $O\left(h^0\right)$ and $O\left(h\right)$ expansions. Since $O\left(h\right)$ expansion of (7) evaluated at $\bar{g}_{\mu\nu}$ just gives the equations of motion, that is the Einstein tensor evaluated in the vacuum in this case, we can easily read the vacuum, by comparing it to $\sqrt{-g}\left(R - 2\lambda m^2\right)$ and find

$$2\lambda = \frac{4\lambda_0 - 3\left(\omega-3\eta\right)\lambda^2 + 16\alpha\lambda^3\left(1+3\beta+9\gamma\right)}{2\sigma - \lambda\left(\omega-3\eta\right) + 4\alpha\lambda^2\left(1+3\beta+9\gamma\right)} \Rightarrow 4\sigma\lambda + \lambda^2\left(\omega-3\eta\right) - 8\alpha\lambda^3\left(1+3\beta+9\gamma\right) = 4\lambda_0, \tag{8}$$

which has always at least one real root for *generic* values of the parameters: Therefore, unlike the NMG case which requires $\lambda_0 \geq -1$ for (A)dS to be the vacuum, for any $\lambda_0$ , (2) has a maximally symmetric vacuum. At this stage, no restriction exists on the ranges of the parameters, but as we will see now, unitarity of the theory will constrain some of these parameters.

Let us now find the equivalent quadratic action by expanding $f\left(R^{\mu}_{\nu}\right)$ up to second order in the curvature:

$$g_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)\equiv f\left(\bar{R}^{\mu}_{\nu}\right)+\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}}\left(R^{\alpha}_{\beta}-\bar{R}^{\alpha}_{\beta}\right)+\frac{1}{2}\left[\frac{\partial^{2}f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}}\left(R^{\alpha}_{\beta}-\bar{R}^{\alpha}_{\beta}\right)\left(R^{\rho}_{\sigma}-\bar{R}^{\rho}_{\sigma}\right),\quad(9)$$

where

$$\begin{aligned}f\left(\bar{R}^{\mu}_{\nu}\right)&=\left[6\sigma\lambda-2\lambda_{0}-\frac{3\lambda^{2}}{2}\left(\omega-3\eta\right)+4\alpha\lambda^{3}\left(1+3\beta+9\gamma\right)\right]m^{2},\\\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}}&=\left[\sigma-\frac{\lambda}{2}\left(\omega-3\eta\right)+2\alpha\lambda^{2}\left(1+3\beta+9\gamma\right)\right]\delta^{\beta}_{\alpha},\\\left[\frac{\partial^{2}f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}}&=\frac{2}{m^{2}}\left\{\left[\omega+\alpha\lambda\left(1+\beta\right)\right]\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha}-\frac{3}{8}\left[\omega-\frac{1}{3}\eta-\frac{8\alpha\lambda}{9}\left(2\beta+9\gamma\right)\right]\delta^{\sigma}_{\rho}\delta^{\beta}_{\alpha}\right\}.\end{aligned}\quad(10)$$

Then, collecting all these we get the equivalent quadratic Lagrangian density

$$\begin{aligned}g_{\text{quad-equal}}\left(R^{\mu}_{\nu}\right)=&\left[-2\lambda_{0}+4\alpha\lambda^{3}\left(1+3\beta+9\gamma\right)\right]m^{2}+\left[\sigma-2\alpha\lambda^{2}\left(1+3\beta+9\gamma\right)\right]R\\&+\frac{1}{m^{2}}\left[\omega+\alpha\lambda\left(1+\beta\right)\right]R^{2}_{\mu\nu}-\frac{3}{8m^{2}}\left[\omega-\frac{1}{3}\eta-\frac{8\alpha\lambda}{9}\left(2\beta+9\gamma\right)\right]R^{2},\end{aligned}\quad(11)$$

whose $O\left(h^{2}\right)$, $O\left(h\right)$ and $O\left(h^{0}\right)$ expansions match the same expansions of (2). At this stage, it is clear that there are three different ways for the general cubic theory (2) to be unitary: Its equivalent quadratic action (11) can be, with redefined parameters, equal to the cosmological Einstein-Hilbert theory or $R+aR^{2}$ theory or NMG. [Again, we exclude the case for which Einstein-Hilbert term drops out.] First, it pays to rewrite the equivalent quadratic action as

$$I_{\text{quad-equal}}=\frac{1}{\kappa^{2}}\int d^{3}x\sqrt{-g}\left[\tilde{\sigma}R-2\tilde{\lambda}_{0}m^{2}+\frac{\tilde{\omega}}{m^{2}}\left(R^{2}_{\mu\nu}-\frac{3}{8}R^{2}\right)+\frac{\tilde{\eta}}{8m^{2}}R^{2}\right],\quad(12)$$

where

$$\begin{aligned}&\tilde{\sigma}\equiv\sigma-2\alpha\lambda^{2}\left(1+3\beta+9\gamma\right),\qquad\tilde{\lambda}_{0}\equiv\lambda_{0}-2\alpha\lambda^{3}\left(1+3\beta+9\gamma\right),\\&\tilde{\omega}\equiv\omega+\alpha\lambda\left(1+\beta\right),\qquad\tilde{\eta}\equiv\eta+\frac{\alpha\lambda}{3}\left(9+25\beta+72\gamma\right).\end{aligned}\quad(13)$$

Here, it is worth restating that $\lambda$ appearing in the redefined parameters is the vacuum of (2) satisfying (8) which can also be directly obtained by computing the vacuum of (12) which reads from the somewhat simpler looking expression

$$\tilde{\sigma}\lambda+\frac{1}{4}\left(\tilde{\omega}-3\tilde{\eta}\right)\lambda^{2}=\tilde{\lambda}_{0}.\quad(14)$$

Canonical analysis of (12) have shown that there are generically three, *not necessarily unitary*, degrees of freedom with the masses [6]:

$$m^{2}_{s}=\left[\frac{\tilde{\sigma}}{\tilde{\eta}}-\frac{3}{2}\lambda\left(1-\frac{\tilde{\omega}}{3\tilde{\eta}}\right)\right]m^{2}\qquad\text{helicity-0 mode,}\quad(15)$$

$$m^{2}_{g}=\left[-\frac{\tilde{\sigma}}{\tilde{\omega}}+\frac{1}{2}\lambda-\frac{3}{2}\lambda\frac{\tilde{\eta}}{\tilde{\omega}}\right]m^{2}\qquad\text{helicity-}\pm2\text{ modes.}\quad(16)$$

For (12) to be unitary, the necessary but *not sufficient* condition is $\tilde{\omega}\tilde{\eta}=0$ which again exhausts all three unitary theories. Among these theories, NMG, for which $\tilde{\eta}=0$, seems to be the most interesting one with spin-2 excitations (scalar mode decouples), therefore we start with it. But, NMG in (A)dS is not unitary by default: There are constraints on the parameters which we discuss below. Since the parameters appear in certain combinations let us define $\xi \equiv 2\alpha\left(1+3\beta+9\gamma\right)$ and $\chi \equiv \alpha\left(1+\beta\right)$, then the effective parameters (13) become

$$\tilde{\sigma} \equiv \sigma - \lambda^2 \xi, \qquad \tilde{\lambda}_0 \equiv \lambda_0 - \lambda^3 \xi,$$

$$\tilde{\omega} \equiv \omega + \lambda\chi, \qquad \tilde{\eta} \equiv \eta + \frac{\lambda}{3}\left(\chi + 4\xi\right). \tag{17}$$

### A. Reducing the cubic theory to NMG in (A)dS

Setting $\tilde{\eta}=0$, the equivalent quadratic action (12) reduces to NMG with $m_g^2 = \left(-\frac{\tilde{\sigma}}{\tilde{\omega}}+\frac{1}{2}\lambda\right)m^2$ where $\lambda = -\frac{2}{\tilde{\omega}}\left(\tilde{\sigma} \pm \sqrt{\tilde{\sigma}^2+\tilde{\omega}\tilde{\lambda}_0}\right)$ which requires $\tilde{\sigma}^2+\tilde{\omega}\tilde{\lambda}_0 \geq 0$. The theory is unitary if $\frac{m^2}{\tilde{\omega}}\left(\lambda m^2 - 2\tilde{\sigma}\frac{m^2}{\tilde{\omega}}\right) > 0$ which comes from the ghost freedom requirement of [2] and reduces to $\tilde{\omega}\lambda - 2\tilde{\sigma} > 0$ in our notation. This requirement can be seen by rewriting NMG in the form of a massive Pauli-Fierz theory at the linearized level. In de Sitter case ($\lambda > 0$), there is also the Higuchi bound [14] $m_g^2 \geq \lambda m^2$ which becomes $\frac{2\tilde{\sigma}}{\tilde{\omega}} + \lambda \leq 0$, and in anti-de Sitter case ($\lambda < 0$), there is the Breitenlohner-Freedman (BF) bound [15] $m_g^2 \geq \lambda m^2$ which is exactly like the Higuchi bound for this three-dimensional case. [Strictly speaking BF bound was derived for massive scalar field in AdS, but it works for massive spin-2 field as well [16]] In this setting, unitarity analysis of (12) for $\tilde{\eta}=0$ is the same as NMG with an essential difference: $\tilde{\sigma}$ and $\tilde{\omega}$ are not in general $\pm 1$. However, as implied by the unitarity constraints, unitary regions can be classified according to the signs of $\tilde{\sigma}$ and $\tilde{\omega}$ just like in the case of NMG. Since the unitarity regions of NMG in (A)dS were studied in detail in [2], we will not repeat the analysis here, but simply give an example in AdS ($\lambda < 0$). Choose $\tilde{\sigma} < 0$ and $\tilde{\omega} > 0$: BF bound is automatically satisfied, so the unique constraint on the vacuum of the theory is $\lambda > \frac{2\tilde{\sigma}}{\tilde{\omega}}$ with $\lambda = -\frac{2}{\tilde{\omega}}\left(\tilde{\sigma} + \sqrt{\tilde{\sigma}^2+\tilde{\omega}\tilde{\lambda}_0}\right)$ which can be achieved if the parameters of the theory satisfy the inequality

$$0 < \tilde{\lambda}_0 < \frac{3\tilde{\sigma}^2}{\tilde{\omega}}. \tag{18}$$

This is a rather weak condition on the parameters, therefore there is a continuum of unitary theories.

1. Choose $\sigma = -1$ and $\omega = 1$: For the sake of simplicity, let us further assume $\eta = 0$ which fixes $\xi = -\frac{\chi}{4}$ that yields $\gamma = -\frac{25\beta+9}{72}$ in terms of the original parameters of the theory (we discuss $\eta \neq 0$ cases below). Then, for $\lambda_0 < 0$ there is no unitary theory, but for $\lambda_0 > 0$ the theory is unitary if the following conditions are met:

$$\left(\chi \leq \frac{1}{4}, \quad \text{and} \quad 0 < \lambda_0 < \frac{-1+\left(1-4\chi\right)^{3/2}+6\chi}{2\chi^2}\right) \quad \text{or} \quad \left(\chi > \frac{1}{4}, \quad \text{and} \quad 0 < \lambda_0 < \frac{1}{\chi}\right). \tag{19}$$

For example, consider the $\chi = 0$ case, it is unitary for $0 < \lambda_0 < 3$ with the same vacuum as NMG, $\lambda = 2\left(1-\sqrt{1+\lambda_0}\right)$. In fact, NMG with $\alpha = 0$ is a member of this family, since

$\chi = \alpha(1+\beta)$. But, $\beta = -1$ gives a cubic order extension which is probably the simplest unitary one parameter extension of NMG with the action

$$I = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\Big[-R - 2\lambda_0 m^2 + \frac{1}{m^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right) + \frac{\alpha}{6m^4}\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} - RR_{\mu\nu}^2 + \frac{2}{9}R^3\right)\Big], \tag{20}$$

with an *arbitrary* $\alpha$. The other one parameter extension of NMG introduced in [8] is also a member of $\tilde{\eta} = 0$ and $\eta = 0$ family of unitary theories, for this case one chooses $\beta = -9/8$ which then fixes $\gamma = 17/64$ yielding an action

$$I = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\Big[-R - 2\lambda_0 m^2 + \frac{1}{m^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right) - \frac{4\chi}{3m^4}\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} - \frac{9}{8}RR_{\mu\nu}^2 + \frac{17}{64}R^3\right)\Big], \tag{21}$$

whose unitarity region is given in (19). [In fact, original sign choice for $\sigma$ is $+1$ in [8].] Note that for $\chi = -1/2$, (21) reduces to the cubic order expansion of BINMG which is unitary for $0 < \lambda_0 < -8 + 6\sqrt{3}$.
Let us also give an example for $\eta \neq 0$. For simplicity choose $\xi = 0$ which yields $\eta = -\frac{\lambda\chi}{3}$, then choosing $\lambda_0 = 1$ yields the unitarity region $-3 < \chi < 1$ for the theory

$$I = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\Big[-R - 2m^2 + \frac{1}{m^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right) - \frac{\lambda\chi}{24m^2}R^2, + \frac{\chi}{6(1+\beta)m^4}\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} + \beta RR_{\mu\nu}^2 - \frac{1+3\beta}{9}R^3\right)\Big], \tag{22}$$

where $\beta$ is arbitrary, and $\lambda$ is the vacuum of the theory. Let us stress that the propagator of this theory is exactly like NMG with redefined parameters.

2. Choose $\sigma = -1$ and $\omega = -1$: Then, $\eta = 0$ theory is unitary if $\lambda_0 > 0$ ($\lambda_0 < 0$ is ruled out) and

$$\chi < 0 \quad \text{and} \quad -\frac{1}{\chi} < \lambda_0 < \frac{1 + (1-4\chi)^{3/2} - 6\chi}{2\chi^2}. \tag{23}$$

For $\eta \neq 0$ and with the choice $\xi = 0$, the unitary region is $\lambda_0 > 0$ and $-\frac{3(\lambda_0+3)}{2\lambda_0^2} < \chi < -\frac{1}{\lambda_0}$.

3. Choose $\sigma = 1$ and $\omega = 1$: Then, $\eta = 0$ theory has no unitary region. For $\eta \neq 0$, certain $\xi$ theories such as $\xi = 1$ have unitary regions.

4. Choose $\sigma = 1$ and $\omega = -1$: Then, $\eta = 0$ theory is unitary if

$$-\frac{1}{4} < \chi < 0 \quad \text{and} \quad \frac{1}{\chi} < \lambda_0 < \frac{1 + 6\chi + (1+4\chi)^{3/2}}{2\chi^2}. \tag{24}$$

For $\eta \neq 0$ and with choice $\xi = 1$, the unitary region is $-2 < \chi < 0$ for $\lambda_0 = 1$.

The above discussion reveals just a sample unitary cubic theories. The other branches for various sign choices of $\tilde{\sigma}$, $\tilde{\omega}$, $\sigma$, $\omega$ and existence or non-existence of $\eta$ can be studied both in AdS and dS.

Although classifying all the unitary theories of the form of (2) for all parameter choices is a tedious job, it is relatively easy to find the unitary regions if some parameters are fixed as in the cubic extension of NMG given in [8] and as in the case of BINMG [9, 10]. In [8], existence of a holographic $c$-function in a specific form is the main theme, so in this AdS/CFT based context $\lambda_0$ is set to be negative $\lambda_0 \equiv -\frac{1}{\ell^2}$ and $c$-function in the considered form can only exist, if $\beta = -9/8$ and $\gamma = 17/64$ with an arbitrary $\alpha$. Also, $\sigma = +1$ is preferred, while $\omega$ is allowed to be both $\pm 1$. Then, the equivalent quadratic action becomes

$$I_{\text{quad-equal}} = \frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\left[\left(1-\frac{\alpha\lambda^2}{32}\right)R-\left(2\lambda_0-\frac{\alpha\lambda^3}{16}\right)m^2+\frac{1}{m^2}\left(\omega-\frac{\alpha\lambda}{8}\right)\left(R_{\mu\nu}^2-\frac{3}{8}R^2\right)\right], \tag{25}$$

where the vacua of the theory satisfies $4\lambda + \lambda^2\omega - \frac{\alpha}{8}\lambda^3 = 4\lambda_0$. The unitarity condition and the Higuchi/BF bounds in terms of the original parameters of the theory become $\lambda\omega - 2 - \frac{\alpha\lambda^2}{16} > 0$ and $\lambda + \frac{32-\alpha\lambda^2}{2(8\omega-\alpha\lambda)} \leq 0$, respectively. With this setting, the theory is unitary in AdS if $\omega = +1$ and $\alpha < \frac{8}{\lambda_0^2}\left(3\lambda_0 - 8 - (4-\lambda_0)^{3/2}\right)$; or if $\omega = -1$, there are some constraints on $\alpha$ which are not particularly illuminating to write. For cubic order of BINMG, $\alpha$ is further set to be 4, but there is no unitary region for $\sigma = +1$. On the other hand, for $\sigma = -1$ and $\omega = 1$, cubic order of BINMG is unitary in dS if $-2 < \lambda_0 < 0$ and unitary in AdS if $0 < \lambda_0 < \left(-8 + 6\sqrt{3}\right)$.

The above analysis shows that for nontrivial $\chi$ (or $\alpha$, $\beta$ in terms of original parameters), there is generically a continuous family of unitary theories, and the cubic theory of [8–10] is just an example of this family. Just like in the NMG case, there are some special points which need further attention. For example, at $m_g^2 = \lambda m^2$ a new *scalar* gauge invariance of the form $\delta_\zeta h_{\mu\nu} = \lambda m^2 \bar{g}_{\mu\nu}\zeta$ arises, and one has a *partially massless* theory with a single degree of freedom [17–19]. [Note that for the Pauli-Fierz spin-2 theory in (A)dS which is not a diffeomorphism invariant theory, at the partially massless point the new gauge invariance is of the form $\delta_\zeta h_{\mu\nu} = \nabla_\mu\nabla_\nu\zeta + \lambda m^2\bar{g}_{\mu\nu}\zeta$, but the higher derivative theories that we are dealing here are diffeomorphism invariant, and therefore, $\nabla_\mu\nabla_\nu\zeta$ part is simply part of the diffeomorphism invariance, and should not be counted as a new gauge symmetry.] The theory defined by (2) has unitary partially massless *regions* (in contrast to a point in NMG) for $\chi\sigma < \frac{1}{12}$ and $\omega\lambda_0 > -\frac{4}{3}$ (we have assumed $\eta = 0$) with

$$\lambda_\pm = \frac{2}{\omega}\left(-2\sigma \pm \sqrt{4 + 3\omega\lambda_0}\right). \tag{26}$$

Another special point is $\lambda - 2\frac{\tilde{\sigma}}{\tilde{\omega}} = 0$ where $m_g^2 = 0$ for which the linearized theory reduces to the Proca theory for massive spin-1 field which can be seen by first writing the equivalent quadratic action in the form of Pauli-Fierz action by use of an auxiliary field say $f_{\mu\nu}$, and then by integrating out the metric perturbation $h_{\mu\nu}$ which then yields a massive spin-1 field with mass $\left(-8\frac{\tilde{\sigma}}{\tilde{\omega}}m^2\right)$. The details of this procedure has been given in [2]. An overall $\frac{\tilde{\omega}}{m^2}$ appears in the Lagrangian; therefore, for ghost freedom $\tilde{\omega} > 0$, and hence $\tilde{\sigma} < 0$ is required for nontachyonic mass in the region $\sigma\chi \geq -\frac{1}{4}$ and $\omega\lambda_0 \leq 4$ (we have assumed $\eta = 0$) with $\lambda_+ = 4\sigma + 2\sqrt{4-\lambda_0}$ for $\omega = 1$ and $\lambda_- = -4\sigma + 2\sqrt{4+\lambda_0}$ for $\omega = -1$ in both dS and AdS. [NMG is unitary only in AdS for this spin-1 limit.]

In the above analysis, we required that the $O\left(h^2\right)$ theory of (2) reduce to $O\left(h^2\right)$ of NMG with redefined parameters. Next, we discuss the remaining two possibilities.

### B. Reducing the cubic theory to Einstein's theory in (A)dS

Pure Einstein's theory in three dimensions is locally trivial. Namely, there is no propagating degree of freedom; but in any case it is a unitary theory, and therefore the cubic theory should be allowed to have the same $O\left(h^2\right)$ form as Einstein's theory around (A)dS. This follows from (12) by setting the coefficients of $R^2$ and $R_{\mu\nu}^2$ to zero. One then obtains

$$I = \frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\left(\tilde{\sigma}R - 2\tilde{\lambda}_0 m^2\right), \tag{27}$$

where

$$\tilde{\sigma} \equiv \sigma + \frac{\lambda}{4}\left(3\eta - \omega\right), \qquad \tilde{\lambda}_0 \equiv \lambda_0 + \frac{\lambda^2}{4}\left(3\eta - \omega\right), \tag{28}$$

with the vacuum $\lambda = \frac{\tilde{\lambda}_0}{\tilde{\sigma}}$ which reduces to $\lambda = \sigma\lambda_0$ (assume $\lambda_0 \neq 0$). Then, $\beta$ and $\gamma$ can be determined in terms of other parameters in (2) as

$$\beta = -\left(1 + \frac{\omega}{\sigma\alpha\lambda_0}\right), \qquad \gamma = \frac{2}{9} - \frac{3\eta - 25\omega}{72\sigma\alpha\lambda_0}. \tag{29}$$

For unitarity, we should impose the right sign Einstein-Hilbert theory that is $\sigma\left[1 + \frac{\lambda_0}{4}\left(3\eta - \omega\right)\right] > 0$. Therefore, any cubic theory satisfying this constraint will be unitary, yet with no local degrees of freedom at the linearized level. As a simple example, consider $\omega = 0$, $\eta = 0$, then one should have $\beta = -1$ and $\gamma = 2/9$, and $\sigma = +1$ is required to have a unitary theory with the action

$$I = \frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\left[R - 2\lambda_0 m^2 + \frac{\alpha}{6m^4}\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} - RR_{\mu\nu}^2 + \frac{2}{9}R^3\right)\right]. \tag{30}$$

As in Sec.II A, the cubic theory with arbitrary $\alpha$ and with choices $\beta = -1$ and $\gamma = 2/9$ turned out to be special. Actually, $\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} - RR_{\mu\nu}^2 + \frac{2}{9}R^3\right)$ is the unique cubic curvature combination that does not effect the free theory in both flat and (A)dS backgrounds. Let us give another interesting example in the case for $\omega \neq 0$ for which the cubic theory

$$\begin{aligned} I = \frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\Bigg\{&\sigma R - 2\lambda_0 m^2 + \frac{\omega}{m^2}\left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right) \\ &+ \frac{\alpha}{6m^4}\left[R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} - \left(1 + \frac{\omega}{\sigma\alpha\lambda_0}\right)RR_{\mu\nu}^2 + \left(\frac{2}{9} + \frac{25\omega}{72\alpha\sigma\lambda_0}\right)R^3\right]\Bigg\}, \end{aligned} \tag{31}$$

has the same $O\left(h^0\right)$, $O\left(h\right)$ and $O\left(h^2\right)$ expansions as (27). Although this theory involves two massive excitations in flat space; in (A)dS, there is *no* propagating degree of freedom. Unitary regions of (31) is given in Table I. In the $\eta \neq 0$ case, $\beta$ and $\gamma$ are determined as $\beta = -1$, $\gamma = \frac{2}{9} - \frac{\eta}{24\alpha\lambda}$. To have a unitary theory in AdS, $\eta < -\frac{4}{3\lambda_0}$ constraint should be satisfied; while in dS one has $\eta > -\frac{4}{3\lambda_0}$.

### C. Reducing the cubic theory to $R - 2\Lambda_0 + aR^2$ theory in (A)dS

The third and the final option of how (2) can be unitary is that it has the same propagator as the $R - 2\Lambda_0 + aR^2$ theory. For this to happen, the coefficient of $R_{\mu\nu}^2$ in the equivalent quadratic

| | | $\boldsymbol{\lambda_0}$ | $\boldsymbol{\sigma\left(1-\frac{\omega\lambda_0}{4}\right)>0}$ | $\boldsymbol{\omega}$ | **Unitary Region** |
|---|---|---|---|---|---|
| **AdS** | $\boldsymbol{\sigma=-1}$ | $\lambda_0>0$ | $\omega\lambda_0>4$ | $+1$ | $\lambda_0>4$ |
| $\boldsymbol{\sigma\lambda_0<0}$ | $\boldsymbol{\sigma=+1}$ | $\lambda_0<0$ | $\omega\lambda_0<4$ | $-1$ | $-4<\lambda_0<0$ |
| | | | | $+1$ | $\lambda_0<0$ |
| **dS** | $\boldsymbol{\sigma=-1}$ | $\lambda_0<0$ | $\omega\lambda_0>4$ | $-1$ | $\lambda_0<-4$ |
| $\boldsymbol{\sigma\lambda_0>0}$ | $\boldsymbol{\sigma=+1}$ | $\lambda_0>0$ | $\omega\lambda_0<4$ | $-1$ | $\lambda_0>0$ |
| | | | | $+1$ | $0<\lambda_0<4$ |

Table I: Unitary regions for $\omega\neq 0$ and $\eta=0$.

Lagrangian density (12) should be set to zero. Therefore, this determines $\beta$ to be $\beta=-1-\frac{\omega}{\alpha\lambda}$. Then, after using the vacuum equation $4\sigma\lambda+\lambda^2\left(25\omega-3\eta\right)+8\alpha\lambda^3\left(2-9\gamma\right)=4\lambda_0$, or in a slightly more efficient form $4\sigma\lambda+\lambda^2\left(\omega-3\eta\right)-4\xi\lambda^3=4\lambda_0$, the equivalent quadratic action can be reduced to

$$I=\frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left\{\frac{1}{2}\left[4\sigma\lambda+\lambda^2\left(\omega-3\eta\right)-8\lambda_0\right]m^2+\frac{4\lambda_0-\lambda^2\left(\omega-3\eta\right)}{4\lambda}R+\frac{\sigma\lambda-\lambda_0}{6\lambda^2m^2}R^2\right\}. \quad (32)$$

This theory is not unitary for generic values of the parameters. One-particle amplitude [5] and the canonical analyses [6] of the action

$$I=\int d^3x\sqrt{-g}\left[\frac{1}{\kappa}\left(R-2\Lambda_0\right)+aR^2\right], \quad (33)$$

show that it describes a single massive excitation with mass $m_s^2=\frac{1}{8\kappa a}-\frac{3\Lambda}{2}$ where $\Lambda$ is determined by $\Lambda-\Lambda_0-6a\kappa\Lambda^2=0$. For unitarity $a>0$ is required for both AdS and dS, and for dS $m_s^2>0$ and for AdS we have the BF bound $m_s^2\geq\Lambda$. Therefore, the mass of the scalar excitation described by (32) is

$$m_s^2=\frac{3\lambda\left[12\lambda_0-8\sigma\lambda-\lambda^2\left(\omega-3\eta\right)\right]}{16\left(\sigma\lambda-\lambda_0\right)}m^2. \quad (34)$$

The analysis of the unitary regions follows similar to Sec.II A above. We will not repeat the analysis in its full detail, but just give some examples of the regions where the cubic theory

$$\begin{aligned}I=\frac{1}{\kappa^2}\int d^3x\,\sqrt{-g}\Bigg\{&\sigma R-2\lambda_0m^2+\frac{\omega}{m^2}\left(R_{\mu\nu}^2-\frac{3}{8}R^2\right)+\frac{\eta}{8m^2}R^2\\&+\frac{\alpha}{6m^4}\left[R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu}-\left(1+\frac{\omega}{\sigma\alpha\lambda_0}\right)RR_{\mu\nu}^2+\gamma R^3\right]\Bigg\}\end{aligned} \quad (35)$$

that reduces to (32) is unitary or nonunitary. For concreteness, consider $\eta=0$ and $\omega=+1$, then for $\sigma=-1$, the theory is not unitary in dS. For $\sigma=+1$, the theory is unitary if $\xi>\frac{1}{16}$ and $\frac{1}{4\xi}<\lambda_0<\frac{1+72\xi+\left(1+48\xi\right)^{3/2}}{864\xi^2}$. In AdS, for $\sigma=+1$, the unitary region is $\xi<0$ and $\lambda_0<\frac{1}{4\xi}$. For $\sigma=-1$, for any value of $\lambda_0$ there is a unitary region for $\xi<0$. The analysis for $\eta\neq 0$ can also be done in the same lines.

### D. Central charge and boundary unitarity

In all the above analysis, we have considered bulk unitarity only. For the applications of AdS/CFT, boundary unitarity is also relevant. From the detailed work of [2], we know that for

NMG bulk and boundary unitarity are in conflict. This conflict is not resolved in the cubic order extension [8], or the infinite order extension of NMG [9–11, 24]. The bulk and boundary unitarity conflict follows from the requirement that a positive central charge is not allowed for NMG in the region where NMG is bulk unitary. Therefore, it would be quite interesting to find both bulk and boundary unitary higher derivative theories. As we will see in this section, there are many such theories. First, recall that the central charge of a generic three-dimensional higher curvature gravity theory can be found by using [29–32]

$$c = \frac{8\pi}{\sqrt{|\lambda|}m}\left[g_{\mu\nu}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}\right]_{\bar{R}_{\mu\nu}}, \tag{36}$$

where the coefficient in front was put to conform to the normalization of Brown-Henneaux [33]. It is easy to see that the central charge of a generic higher derivative theory can be computed directly from the equivalent quadratic action, since $\left[\frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}\right]_{\bar{R}_{\mu\nu}}$ is the first order term in the Taylor series expansion of the full Lagrangian around its constant curvature vacuum. This simple observation leads to a remarkable conclusion in the light of the discussion above: *Any higher curvature theory that reduces to NMG cannot be unitary both in the bulk and on the boundary.* This explains why an extension of NMG, be it cubic or any power, that has a free theory like NMG will not have unitarity on the boundary and in the bulk, and hence perhaps will not be relevant to AdS/CFT. But, *any higher curvature theory that has the same free theory as the cosmological Einstein theory* will be unitary both in the bulk and on the boundary. The theories constructed in Sec.II B have the central charge to be

$$c = \frac{24\pi\tilde{\sigma}}{\sqrt{|\lambda|}\kappa^2 m}. \tag{37}$$

Both bulk and boundary unitarity requires $\tilde{\sigma} > 0$. But, these are not the only theories that are unitary everywhere: Let us now consider the higher curvature theories that have the same free theory as the $\sigma R - 2\Lambda_0 + aR^2$ that we discussed in Sec.II C. The central charge of (12) with $\tilde{\omega} = 0$ can be computed as

$$c = \frac{24\pi}{\sqrt{|\lambda|}\kappa^2 m}\left(\tilde{\sigma} + \frac{3\tilde{\eta}\lambda}{2}\right). \tag{38}$$

For unitarity $\tilde{\eta} > 0$, and in AdS since $\lambda < 0$ we should have $\tilde{\sigma} > -\frac{3\tilde{\eta}\lambda}{2}$ to have $c > 0$. We should check if this constraint is consistent with the other constraint (the BF bound) $m_s^2 \geq \lambda m^2$ with $m_s^2 = \left(\frac{\tilde{\sigma}}{\tilde{\eta}} - \frac{3\lambda}{2}\right)m^2$ and the existence of a negative $\lambda$ satisfying the vacuum equation $\tilde{\sigma}\lambda - \frac{3\tilde{\eta}}{4}\lambda^2 = \tilde{\lambda}_0$. One can find families of theories satisfying these bounds, let us give a simple example for which we take $\eta = 0$ and $\omega = 1$, then the action (35) is bulk and boundary unitary for

$$\begin{aligned} &\sigma = +1 \quad \text{and} \quad \xi < 0 \quad \text{and} \quad \frac{24\xi - (1-16\xi)^{3/2} - 1}{32\xi^2} < \lambda_0 < \frac{1}{4\xi}, \\ &\sigma = -1 \quad \text{and} \quad -\frac{1}{16} < \xi < 0 \quad \text{and} \quad -\frac{24\xi + (1+16\xi)^{3/2} + 1}{32\xi^2} < \lambda_0 < -\frac{1}{4\xi}, \end{aligned} \tag{39}$$

where $\xi$ was defined just before Sec.II A.

To summarize, if a higher curvature theory is required to be unitary both in the bulk and on the boundary, then it should have the same free theory as either the cosmological Einstein-Hilbert theory, or the $R - 2\Lambda_0 + aR^2$ theory with the constraints satisfying the bounds discussed above.

## III. UNITARITY OF BINMG

Up to now, we have constructed all the unitary cubic curvature theories in (A)dS. The procedure can be carried on to quartic or more powers of curvature, but here let us give two examples of Born-Infeld gravities which in principle include infinite powers of curvature. Our first example is the Born-Infeld extension of NMG was introduced in [9] with the action

$$I_{\text{BINMG}} = -\frac{4m^2}{\kappa^2}\int d^3x \left[\sqrt{-\det\left(g + \frac{\sigma}{m^2}G\right)} - \left(1 - \frac{\lambda_0}{2}\right)\sqrt{-\det g}\right], \tag{40}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and $\sigma = \pm 1$. This particular form of the action was chosen to reproduce the cosmological Einstein-Hilbert action at the first order in the curvature expansion and the NMG in the second order expansion. These two conditions are actually met by another BI-type action that we shall discuss below which constitute our second example. On the other hand, the cubic and fourth order extensions of NMG given in [8] which was constructed with the help of a holographic $c$-function matches the same orders of (40). Certain aspects of BINMG such as its central charge [10, 24], $c$-functions [10], classical solutions [24–27] have been studied. We will study the unitarity of BINMG with two different methods: First, with the help of an equivalent quadratic action that we have employed above, and secondly we will explicitly calculate the second order expansion in metric perturbation $h_{\mu\nu}$ with the methods developed in [12]. These two methods obviously will give the same answer, but it is worth checking that the equivalent quadratic action method works with the help of the second more direct method for this infinite order theories. This more direct method is highly involved in terms of computation; therefore, we put it in the Appendix.

Let us analyze the BINMG action by finding its equivalent quadratic action: To do that we have to expand the determinant in terms of traces which was done in [28]

$$I_{\text{BINMG}} = -\frac{4m^2}{\kappa^2}\int d^3x\sqrt{-g}\left[\sqrt{1 - \frac{\sigma}{2m^2}\left(R + \frac{\sigma}{m^2}K - \frac{1}{12m^4}S\right)} - \left(1 - \frac{\lambda_0}{2}\right)\right], \tag{41}$$

where $K$ and $S$ are defined as

$$K \equiv R_{\mu\nu}^2 - \frac{1}{2}R^2, \qquad S \equiv 8R^{\mu\nu}R_{\mu\alpha}R^{\alpha}{}_{\nu} - 6RR_{\mu\nu}^2 + R^3. \tag{42}$$

The *unique* vacuum of (41) by directly studying the equations of motion was found in [10, 24] as

$$\lambda = \sigma\lambda_0\left(1 - \frac{\lambda_0}{4}\right), \quad \lambda_0 < 2. \tag{43}$$

In the spirit of the current work, let us verify this result by finding the equivalent linear action which circumvents the use of equations of motion. Let us define

$$\begin{aligned} f\left(R^\mu_\nu\right) \equiv \Bigg(1 - \frac{\sigma}{2m^2}\Bigg\{\delta^\nu_\mu R^\mu_\nu + \frac{\sigma}{m^2}\left[R^\mu_\nu R^\nu_\mu - \frac{1}{2}\left(\delta^\nu_\mu R^\mu_\nu\right)^2\right] \\ -\frac{1}{12m^4}\left[8R^\mu_\rho R^\rho_\nu R^\nu_\mu - 6R^\mu_\nu R^\nu_\mu\left(\delta^\gamma_\lambda R^\lambda_\gamma\right) + \left(\delta^\nu_\mu R^\mu_\nu\right)^3\right]\Bigg\}\Bigg)^{1/2} - \left(1 - \frac{\lambda_0}{2}\right), \end{aligned} \tag{44}$$

which assumes, as above, that $R^\mu_\nu$ is the independent variable. Expanding $f\left(R^\mu_\nu\right)$ around its constant curvature background $\left(\bar{R}^\mu_\nu = 2\lambda m^2\delta^\mu_\nu\right)$ to the first order in $\left(R^\beta_\alpha - \bar{R}^\beta_\alpha\right)$ as (5) one can find the equivalent linear Lagrangian density. For this one needs

$$f\left(\bar{R}^\mu_\nu\right) = (1 - \sigma\lambda)^{3/2} - \left(1 - \frac{\lambda_0}{2}\right), \tag{45}$$

which requires $\sigma\lambda \leq 1$,

$$\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} = -\frac{\sigma}{4m^2\left(1-\sigma\lambda\right)^{3/2}} \left[\delta^{\beta}_{\alpha} + \frac{\sigma}{m^2}\left(2\bar{R}^{\beta}_{\alpha} - \bar{R}\delta^{\beta}_{\alpha}\right) - \frac{1}{12m^4}\left(24\bar{R}^{\beta}_{\lambda}\bar{R}^{\lambda}_{\alpha} - 12\bar{R}^{\beta}_{\alpha}\bar{R} - 6\bar{R}^{\gamma}_{\lambda}\bar{R}^{\lambda}_{\gamma}\delta^{\beta}_{\alpha} + 3\bar{R}^2\delta^{\beta}_{\alpha}\right)\right] \tag{46}$$

which requires $\sigma\lambda \neq 1$, then

$$\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} = -\frac{\sigma\delta^{\beta}_{\alpha}}{4m^2}\sqrt{1-\sigma\lambda}. \tag{47}$$

With these results, the equivalent linear action for BINMG becomes

$$I_{\text{lin-equal}} = \frac{\sigma\sqrt{1-\sigma\lambda}}{\kappa^2}\int d^3x\,\sqrt{-g}\left\{R - 4\sigma m^2\left[1+\frac{\sigma}{2}\lambda + \frac{1}{2\sqrt{1-\sigma\lambda}}\left(\lambda_0 - 2\right)\right]\right\}, \tag{48}$$

where one can read the effective cosmological constant as

$$\lambda = 2\sigma\left[1+\frac{\sigma}{2}\lambda + \frac{1}{2\sqrt{1-\sigma\lambda}}\left(\lambda_0-2\right)\right] \Rightarrow 2\sqrt{1-\sigma\lambda} = 2-\lambda_0, \tag{49}$$

which requires $\lambda_0 < 2$, and after taking the square of the equation, one obtains (43).

Expansion of $f\left(R^{\mu}_{\nu}\right)$ around the constant curvature background by using (9) with the assumption of small fluctuations about the background requires the quantity

$$\left[\frac{\partial^2 f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} = -\frac{1}{2m^4\sqrt{1-\sigma\lambda}}\left(\delta^{\beta}_{\rho}\delta^{\sigma}_{\alpha} - \frac{3}{8}\delta^{\beta}_{\alpha}\delta^{\sigma}_{\rho}\right). \tag{50}$$

Using this and (45), (47); one obtains the equivalent quadratic action as

$$I_{O(R^2)} = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left[\tilde{\sigma}R - 2m^2\tilde{\lambda}_0 + \frac{\tilde{\omega}}{m^2}\left(R^2_{\mu\nu} - \frac{3}{8}R^2\right)\right], \tag{51}$$

where, for $\sigma\lambda < 1$,

$$\tilde{\sigma} = \frac{\left(\sigma - \frac{\lambda}{2}\right)}{\sqrt{1-\sigma\lambda}}, \qquad \tilde{\lambda}_0 = \lambda_0 - 2 + \frac{1}{\sqrt{1-\sigma\lambda}}\left(2 - \sigma\lambda - \frac{\lambda^2}{4}\right), \qquad \tilde{\omega} = \frac{1}{\sqrt{1-\sigma\lambda}}. \tag{52}$$

Remarkably, the equivalent quadratic action turned out to be NMG with redefined parameters. Namely, the effect of all the terms beyond $O\left(R^2\right)$ simply change the parameters of the $O\left(R^2\right)$ expansion of the action which was NMG by construction. Let us stress again that this equivalent quadratic action has the same free theory (that is the propagator), same vacuum and same central charge as BINMG. Vacuum of BINMG in terms of the redefined parameters is

$$\tilde{\omega}\lambda^2 + 4\tilde{\sigma}\lambda - 4\tilde{\lambda}_0 = 0. \tag{53}$$

From the discussion in Sec.II A, we know that NMG is unitary under two conditions $\tilde{\omega}\lambda - 2\tilde{\sigma} > 0$ and $\frac{2\tilde{\sigma}}{\tilde{\omega}} + \lambda \leq 0$. Now, the question is whether these conditions are satisfied together with the BINMG condition $\lambda_0 < 2$ or not. A simple analysis shows that BINMG is unitary only for $\sigma = -1$ in AdS for $0 < \lambda_0 < 2$, and in dS for $\lambda_0 < 0$. Therefore, this analysis answers the question raised in [10] about the unitarity of the $\sigma = +1$ theory in the negative. This is true for bulk unitarity, for boundary unitarity recall the central charge from [10, 24], or just compute it from the equivalent action (51) as

$$c = \frac{3\ell}{2G_3}\left(\tilde{\sigma} - \frac{\tilde{\omega}\lambda}{2}\right) = \frac{3\sigma\ell}{4G_3}\left(2 - \lambda_0\right). \tag{54}$$

Since in AdS $0 < \lambda_0 < 2$, and $\sigma = -1$, the theory is not unitary on the boundary just like NMG, or the cubic extension of NMG. The $\sigma = +1$ theory is unitary on the boundary, but as we have just seen it is not unitary in the bulk. This is an expected result, because the free theory of BINMG is the same as the free theory of NMG with redefined parameters, and there is the obvious conflict between the bulk unitarity condition $\tilde{\omega}\lambda - 2\tilde{\sigma} > 0$ and the boundary unitarity condition $2\tilde{\sigma} - \tilde{\omega}\lambda > 0$.

We mentioned that there was a second BI-type action that reproduces NMG in the curvature expansion. The action of this theory reads [9]

$$I = -\frac{4m^2}{\kappa^2}\int d^3x \left\{\sqrt{-\det\left[g_{\mu\nu} + \frac{\sigma}{m^2}\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right)\right]} - \left(1 - \frac{\lambda_0}{2}\right)\sqrt{-\det g}\right\}, \tag{55}$$

which, by use of

$$\det A = \frac{1}{6}\left[(\mathrm{Tr}A)^3 - 3\mathrm{Tr}A\mathrm{Tr}\left(A^2\right) + 2\mathrm{Tr}\left(A^3\right)\right], \tag{56}$$

becomes

$$I = -\frac{4m^2}{\kappa^2}\int d^3x\sqrt{-g}$$
$$\times\left\{\sqrt{1 + \frac{1}{2m^2}\left[R - \frac{1}{m^2}\left(R_{\mu\nu}^2 - \frac{1}{2}R^2\right) + \frac{2}{3m^4}\left(R^{\mu\nu}R_\nu{}^\alpha R_{\alpha\mu} - \frac{5}{4}RR_{\mu\nu}^2 + \frac{23}{72}R^3\right)\right]} - \left(1 - \frac{\lambda_0}{2}\right)\right\}. \tag{57}$$

Quite interestingly, this action reduces to NMG at $O\left(h^2\right)$ with the same redefined parameters as the BINMG. Therefore, at the free level, these two theories cannot be distinguished.

## IV. CONCLUSION

We have found all the unitary cubic curvature theories in three dimensions around constant curvature backgrounds. Without any further constraint, we have shown that unitarity in the bulk and on the boundary allows a large family of solutions as opposed to the cubic curvature theories that have appeared in the literature before, which allowed only bulk or boundary unitarity. The theories we have found should be studied in the context of AdS/CFT. We have also studied the unitarity of two Born-Infeld extensions of NMG which turned out to be unitary in the bulk only. Besides the parity violating extension with the addition of a Chern-Simons term and/or carrying out the unitarity analysis to $O\left(R^4\right)$, a quite physically relevant extension of our work is to find the unitary cubic curvature theories in four dimensions, which is currently under construction.

## V. ACKNOWLEDGMENTS

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## APPENDIX: $O\left(h^2\right)$ ACTION OF BINMG

In this Appendix, we calculate explicitly $O\left(h\right)$ and $O\left(h^2\right)$ expansions of the BINMG action. First of all, let us find the constant curvature vacuum of (40) by explicitly calculating the first order action in the metric perturbation. In [12], it was shown that $O\left(h\right)$ of the generic BI-type action

$$I = \frac{2}{\kappa\alpha}\int d^D x\left[\sqrt{-\det\left(g_{\mu\nu}+A_{\mu\nu}\right)} - \left(\alpha\Lambda_0+1\right)\sqrt{-\det g}\right], \tag{58}$$

where $A_{\mu\nu}$ is in the form $A_{\mu\nu} = \alpha\left(R_{\mu\nu}+\beta\tilde{R}_{\mu\nu}\right)+O\left(R^2\right)$ with the definition $\tilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{D}g_{\mu\nu}R$ is

$$I_{O(h)} = \frac{\left(1+a\right)^{\frac{D-4}{2}}}{\kappa\alpha}\int d^D x\sqrt{-\bar{g}}\left[\left(1+a\right)\left(\bar{g}^{\rho\mu}A^{(1)}_{\mu\rho}+h\right) - \left(1+a\right)^{\frac{4-D}{2}}\left(\alpha\Lambda_0+1\right)h\right], \tag{59}$$

where $A^{(1)}_{\mu\rho}$ is the first order term in the metric perturbation expansion of $A_{\mu\nu}$. Here, $a$ is defined as $\bar{A}_{\mu\nu} \equiv a\bar{g}_{\mu\nu}$ and for BINMG it becomes

$$\frac{\sigma}{m^2}\left(\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right) = -\sigma\lambda\bar{g}_{\mu\nu} \Rightarrow a = -\sigma\lambda, \tag{60}$$

which, when inserted to the action, yields the constraint $a > -1 \Rightarrow \sigma\lambda < 1$. For BINMG, $A_{\mu\nu}$ is $A_{\mu\nu} = \frac{\sigma}{m^2}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)$, then $A^{(1)}_{\mu\nu}$ and $\bar{g}^{\rho\mu}A^{(1)}_{\mu\rho}$ becomes

$$A^{(1)}_{\mu\nu} = \frac{\sigma}{m^2}\left(R^L_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R_L - 3\lambda m^2 h_{\mu\nu}\right), \qquad \bar{g}^{\rho\mu}A^{(1)}_{\mu\rho} = -\frac{\sigma}{2m^2}\left(R_L+2\lambda m^2 h\right), \tag{61}$$

where $R^L_{\mu\nu}$ and $R_L$ are the linearized Ricci tensor and the linearized curvature scalar with the definitions

$$R^L_{\mu\nu} \equiv \frac{1}{2}\left(\bar{\nabla}_\sigma\bar{\nabla}_\mu h^\sigma_\nu + \bar{\nabla}_\sigma\bar{\nabla}_\nu h^\sigma_\mu - \Box h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\right), \qquad R_L \equiv \left(g^{\mu\nu}R_{\mu\nu}\right)_L. \tag{62}$$

Then, for BINMG with $\alpha = -\frac{1}{2m^2}$ and $\kappa \to \kappa^2$, the $O\left(h\right)$ action becomes

$$\begin{aligned} I_{O(h)} &= -\frac{2m^2}{\kappa^2\sqrt{1+a}}\int d^3x\sqrt{-\bar{g}}\left\{\left(1+a\right)\left[-\frac{\sigma}{2m^2}\left(R_L+2\lambda m^2 h\right)\right] + \left(1+a\right)h - \sqrt{1+a}\left(1-\frac{\lambda_0}{2}\right)h\right\} \\ &= -\frac{2m^2}{\kappa^2\sqrt{1+a}}\int d^3x\sqrt{-\bar{g}}\left[\left(1+a\right)h - \sqrt{1+a}\left(1-\frac{\lambda}{2}\right)h\right], \end{aligned} \tag{63}$$

then the constant curvature background equation of motion can be found as in (43) from the coefficient of $h^{\mu\nu}$.

Now, let us turn to the explicit calculation of $O\left(h^2\right)$ action for BINMG. In [12], the second order action in metric perturbation for (58) in three dimensions was calculated as

$$\begin{aligned} I_{O(h^2)} = &-\frac{1}{\kappa\alpha\sqrt{1+a}}\int d^3x\sqrt{-\bar{g}}\left\{\frac{1}{2}A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)} - \frac{1}{4}\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2 - \left(1+a\right)\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}\right. \\ &\left. + h^{\mu\nu}\left(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}A^{(1)}_{\rho\sigma}\right) - \frac{1}{4}\left[1-\sqrt{1+a}\left(\alpha\Lambda_0+1\right)\right]\left(h^2-2h^2_{\mu\nu}\right)\right\}. \end{aligned} \tag{64}$$

With the explicit form of $A_{\mu\nu}$ for BINMG, let us calculate each term separately. First, the second line of the above equation takes the following form by use of the definition of the linearized Einstein tensor $\mathcal{G}^L_{\mu\nu} \equiv R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - 2\Lambda h_{\mu\nu}$ in three dimensions and by use of the equation of motion;

$$\int d^3x\,\sqrt{-\bar{g}}\left\{h^{\mu\nu}\left(A^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}A^{\alpha(1)}_{\alpha}\right) - \frac{1}{4}\left[1-\sqrt{1+a}\left(1-\frac{\lambda_0}{2}\right)\right]h^{\mu\nu}\left(\bar{g}_{\mu\nu}h - 2h_{\mu\nu}\right)\right\}$$
$$= \frac{\sigma}{m^2}\int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu}\left[\mathcal{G}^L_{\mu\nu} + \frac{1}{4}\bar{g}_{\mu\nu}R_L + \frac{\lambda m^2}{4}\left(\bar{g}_{\mu\nu}h - 2h_{\mu\nu}\right)\right]. \quad (65)$$

Secondly, let us calculate the terms quadratic in $A_{\mu\nu}$. There are two such terms $A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)}$ and $\left(\bar{g}^{\mu\nu}A^{(1)}_{\mu\nu}\right)^2$, and the first one becomes

$$\int d^3x\,\sqrt{-\bar{g}}\frac{1}{2}A^{(1)}_{\mu\nu}A^{\mu\nu}_{(1)} = \frac{1}{2m^4}\int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu}\Big[-\frac{1}{4}\left(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + 2\lambda m^2\bar{g}_{\mu\nu}\right)R_L$$
$$-\frac{1}{2}\left(\Box\mathcal{G}^L_{\mu\nu} - \lambda m^2\bar{g}_{\mu\nu}R_L\right) - \lambda m^2\mathcal{G}^L_{\mu\nu} + \lambda^2m^4h_{\mu\nu}\Big], \quad (66)$$

by using $\int d^3x\,\sqrt{-\bar{g}}R_L^2$ and $\int d^3x\,\sqrt{-\bar{g}}R^{\mu\nu}_L R^L_{\mu\nu}$ which can be found as

$$\int d^3x\,\sqrt{-\bar{g}}R_L^2 = \int d^3x\,\sqrt{-\bar{g}}\left[-h^{\mu\nu}\left(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + 2\lambda m^2\bar{g}_{\mu\nu}\right)R_L\right], \quad (67)$$

$$\int d^3x\,\sqrt{-\bar{g}}R^{\mu\nu}_L R^L_{\mu\nu} = -\frac{1}{2}\int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu}\Big[\left(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + 2\lambda m^2\bar{g}_{\mu\nu}\right)R_L + \left(\Box\mathcal{G}^L_{\mu\nu} - \lambda m^2\bar{g}_{\mu\nu}R_L\right)$$
$$-10\lambda m^2R^L_{\mu\nu} + \lambda m^2\bar{g}_{\mu\nu}R_L + 12\lambda^2m^4h_{\mu\nu}\Big], \quad (68)$$

where the background Bianchi identity and integration by parts have been used. The other term reads

$$\int d^3x\,\sqrt{-\bar{g}}\left[-\frac{1}{4}\left(A^{\alpha(1)}_{\alpha}\right)^2\right] = \frac{1}{16m^4}\int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu}\Big[\left(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu + 2\lambda m^2\bar{g}_{\mu\nu}\right)R_L$$
$$-4\lambda m^2\bar{g}_{\mu\nu}R_L - 4\lambda^2m^4\bar{g}_{\mu\nu}h\Big]. \quad (69)$$

Let us consider $\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}$ which is ,

$$\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu} = \frac{\sigma}{m^2}\left(\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} - \frac{3}{2}R^{(2)} - \frac{1}{2}hR_L\right), \quad (70)$$

and using

$$\int d^3x\,\sqrt{-\bar{g}}R_{(2)} = \int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu}\left(-\frac{1}{2}\mathcal{G}^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L + \lambda m^2h_{\mu\nu} - \frac{\lambda m^2}{2}\bar{g}_{\mu\nu}h\right), \quad (71)$$

and

$$\int d^3x\,\sqrt{-\bar{g}}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} = h^{\mu\nu}\left(\frac{1}{2}\mathcal{G}^L_{\mu\nu} + \lambda m^2h_{\mu\nu} - \frac{\lambda m^2}{2}\bar{g}_{\mu\nu}h\right), \quad (72)$$

one gets

$$\int d^3x\,\sqrt{-\bar{g}}\bar{g}^{\mu\nu}A^{(2)}_{\mu\nu}=\int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu}\left[\frac{\sigma}{4m^2}\left(5\mathcal{G}^L_{\mu\nu}+\bar{g}_{\mu\nu}R_L+\lambda m^2\bar{g}_{\mu\nu}h-2\lambda m^2h_{\mu\nu}\right)\right]. \tag{73}$$

This computation is somewhat lengthy, and one needs

$$\bar{g}^{\nu\sigma}h^{\mu}_{\beta}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L=h^{\mu\nu}\left(-R^L_{\mu\nu}+3\lambda m^2h_{\mu\nu}-\lambda m^2\bar{g}_{\mu\nu}h\right), \tag{74}$$

and the two expressions involving linearized Christoffel connection whose definition is $\left(\Gamma^{\rho}_{\mu\nu}\right)_L\equiv\frac{1}{2}\bar{g}^{\rho\lambda}\left(\bar{\nabla}_\mu h_{\nu\lambda}+\bar{\nabla}_\nu h_{\mu\lambda}-\bar{\nabla}_\lambda h_{\mu\nu}\right)$,

$$\int d^3x\,\sqrt{-\bar{g}}\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L=\int d^3x\,\sqrt{-\bar{g}}\left[-\frac{1}{2}h^{\mu\nu}\left(\bar{\nabla}^{\sigma}\bar{\nabla}_\mu h_{\nu\sigma}+\bar{\nabla}^{\sigma}\bar{\nabla}_\nu h_{\mu\sigma}-\frac{3}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h\right)\right.$$
$$\left.+h^{\mu\nu}\left(3\lambda m^2h_{\mu\nu}-\frac{\lambda m^2}{2}\bar{g}_{\mu\nu}h\right)+\frac{1}{4}h^{\mu\nu}\bar{g}_{\mu\nu}R_L\right], \tag{75}$$

$$\int d^3x\,\sqrt{-\bar{g}}\bar{g}^{\nu\sigma}\bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L=\int d^3x\,\sqrt{-\bar{g}}\left[-\frac{3}{4}h^{\mu\nu}\Box h_{\mu\nu}+\frac{1}{4}h^{\mu\nu}\left(\bar{\nabla}^{\sigma}\bar{\nabla}_\mu h_{\nu\sigma}+\bar{\nabla}^{\sigma}\bar{\nabla}_\nu h_{\mu\sigma}\right)\right]. \tag{76}$$

Collecting all the terms and making use of the equations of motion, one obtains

$$I_{O(h^2)}=-\frac{1}{2\kappa^2\sqrt{1-\sigma\lambda}}\int d^3x\,\sqrt{-\bar{g}}h^{\mu\nu} \tag{77}$$
$$\times\left\{\left(\sigma-3\lambda\right)\mathcal{G}^L_{\mu\nu}+\frac{1}{m^2}\left[\frac{1}{4}\left(\bar{g}_{\mu\nu}\Box-\bar{\nabla}_\mu\bar{\nabla}_\nu+2\lambda m^2\bar{g}_{\mu\nu}\right)R_L+\left(\Box\mathcal{G}^L_{\mu\nu}-\lambda m^2\bar{g}_{\mu\nu}R_L\right)\right]\right\},$$

which can be compared to (25) of [21]. Then, one can observe that this is the $O\left(h^2\right)$ of NMG with the redefined parameters given in (52).

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[36] In fact, pure $f(R)$ theory with the Lagrangian density $R + \alpha R^n$ is also unitary, but for the discussion in this paper $n = 2$ theory is relevant.