Integral Octonions, Octonion XY-Product, and the Leech Lattice

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November 12, 2010

Abstract

The integral octonions arise from the octonion XY-product. A parallel is shown to exist with the quaternion Z-product. Connections to the laminated lattices, Λ_4, Λ_8, Λ_{16} and Λ_{24} (Leech), are developed.

1. Octonion Multiplication and XY-Product Variants.

Given any multiplication table for the 8-dimensional octonion algebra, **O**, one can construct an infinite number of variants, isomorphic to **O** itself, by replacing the original product with the XY-product,

$$A \circ_{XY} B \equiv (AX)(Y^\dagger B), \tag{1}$$

where $YX^\dagger$ must be a unit octonion, and is the identity of the new multiplication. The octonion identity is left unchanged in the case $Y^\dagger = X^{-1}$, giving rise to the X-product:

$$A \circ_X B \equiv (AX)(X^{-1}B). \tag{2}$$

in what follows, let e_a, a = 0,...,7, represent the 8 octonion units, with $e_0 = 1$ the identity (in **[1,2]** the index ∞ is used for the identity, but the index choice 0 makes it easier to program computers to do calculations). Our starting multiplication table is the commonly chosen cyclic multiplication:

$$e_a e_{a+1} = e_{a+3}, \quad a = 1, ..., 7, \tag{3}$$

where the indices are taken modulo 7, from 1 to 7. This particular table is invariant with respect to both index cycling and index doubling. That is,

$$e_a e_b = e_c \Longrightarrow e_{a+1} e_{b+1} = e_{c+1},$$
$$e_a e_b = e_c \Longrightarrow e_{2a} e_{2b} = e_{2c}.$$

2. Λ_8 Lattices.

Representations of the $E_8 = \Lambda_8$ lattice arise from the X-product. In particular, define

$$
\begin{aligned}
\Xi_0 &= & \{\pm e_a\}, \\
\Xi_2 &= & \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct}, \\
& & e_a(e_b(e_c e_d)) = \pm 1\}, \\
\Xi^{even} &= & \Xi_0 \cup \Xi_2, \\
\mathcal{E}_8^{even} &= & \text{span}\{\Xi^{even}\}, \\
\\
\Xi_1 &= & \{(\pm e_a \pm e_b)/\sqrt{2} : a, b \text{ distinct}\}, \\
\\
\Xi_3 &= & \{(\sum_{a=0}^{7} \pm e_a)/\sqrt{8} : \text{ even number of +'s}\}, \\
& & a, b, c, d \in \{0, ..., 7\}, \\
\Xi^{odd} &= & \Xi_1 \cup \Xi_3, \\
\mathcal{E}_8^{odd} &= & \text{span}\{\Xi^{odd}\}
\end{aligned}
\tag{4}
$$

(spans over the integers, Z). Ξ^{even} has $16 + 224 = 240$ elements, and Ξ^{odd} has $112 + 128 = 240$ elements, each a representation of the inner shell of an E_8 lattice. (One may think of these Λ_8 lattices as discrete versions of S^7, the algebra of unit octonions.)

These elements have an interesting relation to our chosen octonion multiplication: for all $X \in \Xi^{even} \cup \Xi^{odd}$, and for all pairs of octonion units e_a, e_b, there exists a unit e_c such that

$$e_a \circ_X e_b = \pm e_c. \tag{5}$$

3. Integral Octonions and the X-Product.

Define

$$\ell_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7). \tag{6}$$

Note that ℓ_0, like our multiplication table, is invariant with respect to index cycling and doubling. Therefore, the X-product

$$A \circ_{\ell_0} B = (A\ell_0)(\ell_0^{-1}B) \tag{7}$$

is also invariant with respect to index cycling and doubling. Its multiplication table is in some sense dual to that given above:

$$e_a \circ_{\ell_0} e_{a+2} = e_{a+3}, \quad a = 1, ..., 7.$$

What about the X-product

$$A \circ_{\ell_0^{-1}} B = (A\ell_0^{-1})(\ell_0 B)? \tag{8}$$

Since $\ell_0^{-1} = \frac{1}{4}(1 - e_1 - ... - e_7)$ has an odd number of plus signs (as coefficients of the units), we don't expect the product of any two units to be another unit using this X-product. For example,

$$e_1 \circ_{\ell_0^{-1}} e_2 = \frac{1}{2}(e_3 - e_5 + e_6 + e_7).$$

While the value on the righthand side of this equality is not an octonion unit, it is an element of $\mathcal{E}_8^{even}$, as are e_1 and e_2. This is a specific example of a more general result given below.

In section 5 we look at the laminated lattice Λ_4 over the quaternion algebra $\mathbf{H}$. This algebra is known to be closed with respect to quaternion multiplication, giving rise to the algebra of integral quaternions. However, surprisingly, the Λ_8 lattice $\mathcal{E}_8^{even}$ does not close with respect to our given octonion multiplication. Weirdly, the set $\Xi^{even}[0 - a]$, derived from Ξ^{even} by replacing each occurrence of e_0 in elements of Ξ^{even} with e_a, and vice versa, is multiplicatively closed. These representations of unit integral octonions are also derivable from the promised general result.

X-Product Integral Octonion Result

$$X \in \Xi^{odd}, \text{ and } A, B \in \Xi^{even} \implies A \circ_{X^\dagger} B \in \Xi^{even}. \tag{9}$$

Proof:
To prove (9) we will set up a partial multiplication table for the Ξ_m (since these sets are finite, proving these results even with a computer is not difficult). In general,

$$A \in \Xi^{even}, \text{ and } Y \in \Xi^{odd} \implies YA \in \Xi^{odd}, AY^\dagger \in \Xi^{odd\dagger},$$

and

$$X, Y \in \Xi^{odd} \implies Y^\dagger X \in \Xi^{even}.$$

(Interestingly these results are not commutative. For example, $X \in \Xi_1$, and $Y \in \Xi_3 \implies YX, XY^\dagger \in \Xi^{even}[0 - a], (a \neq 0)$.) Therefore,

$$X \in \Xi^{odd}, \text{ and } A, B \in \Xi^{even} \implies AX^\dagger \in \Xi^{odd\dagger} \text{ and } XB \in \Xi^{odd} \implies A \circ_{X^\dagger} B = (AX^\dagger)(XB) \in \Xi^{even}.$$

That is, Ξ^{even}, which is not multiplicatively closed with respect to the given octonion product, is closed with respect to these X-products. $\square$

As a corollary, given the identity

$$A(BX) = (A \circ_{X\dagger} B)X.$$

we see that Ξ^{even} closes as a set of actions on each $X \in \Xi^{odd}$, and more generally on Ξ^{odd} itself [10]. Note that of all the Ξ_k, only Ξ_3 is not invariant with respect to octonion conjugation.

4. Integral Octonions, the XY-Product and Λ_{16} Lattices.

The set of $X, Y \in \mathbf{O}$ satisfying the property that for all units e_a and e_b there exists some unit e_c such that,

$$e_a \circ_{XY} e_b = (e_a X)(Y^{-1} e_b) = \pm e_c, \tag{10}$$

gives rise to two copies of Λ_{16} [6]. In particular, (10) is satisfied if $X \in \Xi^{even} \cup \Xi^{odd}$ and there exists some unit e_d such that $Y = \pm e_d X$. In the case $Y^{-1} = X^{-1} e_d^\dagger$ one can show using X-product identities that for all $A, B \in \mathbf{O}$,

$$(AX)((X^{-1} e_d^\dagger)B) = A \circ_X (e_d^\dagger \circ_X B). \tag{11}$$

If $X^{-1} \in \Xi^{odd}$, then Ξ^{even} is closed with respect to X-product multiplication. If $A, B \in \Xi^{even}$, then $C = (e_d^\dagger \circ_X B) \in \Xi^{even}$, so $A \circ_X C \in \Xi^{even}$. That is,

$$A, B \in \Xi^{even} \text{ and } X^{-1} \in \Xi^{odd} \implies (AX)((X^{-1} e_d^\dagger)B) \in \Xi^{even}.$$

So $\mathcal{E}_8^{even}$ are octavian integers with respect to this XY-product. The identity of this set of integers is just e_d.

But (9) can clearly be generalized even further:

XY-Product Integral Octonion Result

$$A, B \in \Xi^{even} \text{ and } X, Y \in \Xi^{odd\dagger} \text{ and } |XY^{-1}| = 1 \implies A \circ_{XY} B \in \Xi^{even}. \tag{12}$$

Proof:
In general, and on the assumption my computer code was without error:

$$\Xi^{even} \Xi^{odd\dagger} = \Xi^{odd\dagger},$$
$$\Xi^{odd} \Xi^{even} = \Xi^{odd},$$
$$\Xi^{odd\dagger} \Xi^{odd} = \Xi^{even}.$$

Therefore, generalizing the results above:

$$(\Xi^{even} \Xi^{odd\dagger})(\Xi^{odd} \Xi^{even}) = \Xi^{odd\dagger} \Xi^{odd} = \Xi^{even},$$
$$(\Xi^{odd} \Xi^{even})(\Xi^{odd\dagger} \Xi^{odd}) = \Xi^{odd} \Xi^{even} = \Xi^{odd},$$
$$(\Xi^{odd\dagger} \Xi^{odd})(\Xi^{even} \Xi^{odd\dagger}) = \Xi^{even} \Xi^{odd\dagger} = \Xi^{odd\dagger}.$$

So there exist XY-products under which each of these lattice inner shells, Ξ^{even}, Ξ^{odd}, and $\Xi^{odd\dagger}$, is multiplicatively closed. $\square$

In [1] there are seven copies of the octavian integers defined starting from Ξ^{even}, but needing that strange switching of indices in each case to make the set multiplicatively closed ($\Xi^{even}[0 - a]$). We see here that the XY-product can be used to unravel the octavian integers so that we need only use Ξ^{even}, not a rotated copy. This is reminiscent of the way the XY-product unravels triality. In particular, for all $g \in SO_8$ acting on $\mathbf{O}$ there exist unit elements $X, Y \in \mathbf{O}$ (not unique) such that for all $A, B \in \mathbf{O}$,

$$g[A \circ_{XY} B] = g[A]g[B]. \tag{13}$$

That is, replacing our starting product with the XY-product on the left hand side above means we needn't perform triality rotations on g to achieve equality. If $g \in G_2$, the automorphism group of **O**, then $X = Y = \pm 1$, so $g[AB] = g[A]g[B]$.

As to $\Xi^{even}[0 - a]$ being multiplicatively closed, note that if we define $d_a = (1 + e_a)/\sqrt{2}$, then

$$\Xi^{even}[0 - a] = d_a \Xi^{even} d_a.$$

Therefore, exploiting a Moufang identity and the result (12) above, noting that $d_a \in \Xi^{odd}$ and $d_a \in \Xi^{odd\dagger}$,

$$\begin{aligned}
\Xi^{even}[0 - a]\Xi^{even}[0 - a] &= (d_a \Xi^{even} d_a)(d_a \Xi^{even} d_a) \\
&= d_a((\Xi^{even} d_a)(d_a \Xi^{even}))d_a \\
&= d_a(\Xi^{even})d_a = \Xi^{even}[0 - a].
\end{aligned}$$

That is, $\Xi^{even}[0 - a]$ are octavian integers as a consequence of a special application of (12).

5. Integral Quaternions, the Z-Product and Λ_4 Lattices.

The 4-dimensional quaternion algebra, **H**, is associative, so for all $A, B, X \in$ **H**, $X \neq 0$, $(AX)(X^{-1}B) = AB$. **H** hasn't got an X-product like that defined for **O**. But the generalization of the XY-product leads to the following definition of the quaternion Z-product:

$$(AX)(Y^\dagger B) = AXY^\dagger B = A \bullet_Z B = AZB, \tag{14}$$

where $Z = XY^\dagger$ must be s unit quaternion. The automorphism group of **H** is SO_3. A general element of the full SO_4 group of actions on **H** takes the form

$$g[A] = UAV^{-1}, \tag{15}$$

with $V^{-1}U$ a unit quaternion. Let $Z = V^{-1}U$, then

$$g[A \bullet_Z B] = g[A]g[B]. \tag{16}$$

If $g \in SO_3$, then $Z = 1$, and $g[AB] = g[A]g[B]$. So the quaternion Z-product bears the same relationship to SO_3 and SO_4 as the octonion XY-product bears to G_2 and SO_8.

Let q_m, m = 0,1,2,3, be a quaternion basis, with q_0 the identity. Define

$$\begin{aligned}
\Upsilon_0 &= & \{\pm q_m\}, \\
\Upsilon_2 &= & \{(\pm q_m \pm q_n \pm q_r \pm q_s)/2 : m, n, r, s \text{ distinct}, \\
\Upsilon^{even} &= & \Upsilon_0 \cup \Upsilon_2, \\
\mathcal{D}_4^{even} &= & \text{span}\{\Upsilon^{even}\}, \\
\\
\Upsilon_1 &= & \{(\pm q_m \pm q_n)/\sqrt{2} : m, n \text{ distinct}\}, \\
\Upsilon^{odd} &= & \Upsilon_1, \\
\mathcal{D}_4^{odd} &= & \text{span}\{\Upsilon^{odd}\}
\end{aligned} \tag{17}$$

Both Υ^{even} and Υ^{odd} have 24 elements, and constitute the inner shells of D_4 lattices. The elements Υ^{even} are the Hurwitz units of the set of Hurwitz integers (see [1]). They are multiplicatively closed. That being the case, Υ^{even} is multiplicatively closed using the Z-product for all $Z \in \Upsilon^{even}$. If $X, Y \in \Upsilon^{odd}$ and the Z-product with $Z = XY^\dagger$ preserves quaternion units ($q_m \bullet_Z q_n = \pm q_k$), then there exists some index j such that $Y = \pm q_j X$. Above we saw that in expanding this unit preserving property from the X-product to the XY-product led to an expansion of associated $E_8 = \Lambda_8$ lattices to Λ_{16}. Here we get an expansion of $D_4 = \Lambda_4$ to $E_8 = \Lambda_8$. I will not work out the details here. They are similar to the more complicated case developed in [6]. I conclude this section by noting that Υ^{even} is closed under the Z-product for $Z = XY^\dagger \in \Upsilon^{even}$.

6. Octonion Triples and the Leech Lattice.

Before proceeding we'll define variations on the Ξ_k:

$$
\begin{aligned}
\mathcal{A}_0 &= \quad \{\pm e_a\}, \\
\mathcal{A}_2 &= \quad \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct}, \\
&\qquad e_a(e_b(e_c e_d)) = \pm 1\}, \\
\mathcal{A}^{even} &= \quad \mathcal{A}_0 \cup \mathcal{A}_2, \\[4pt]
\mathcal{A}_1 &= \quad \{(\pm e_a \pm e_b) : a, b \text{ distinct}\}, \\[4pt]
\mathcal{A}_3 &= \quad \{(\textstyle\sum_{a=0}^{7} \pm e_a)/2 : \text{ even number of +'s}\}, \\
&\qquad a, b, c, d \in \{0, ..., 7\}, \\
\mathcal{A}^{odd} &= \quad \mathcal{A}_1 \cup \mathcal{A}_3.
\end{aligned}
$$

The only change is to the odd elements, which now have rational coefficients 1 and $\frac{1}{2}$. So these elements are no longer unit octonions, and $\ell_0 \in \mathcal{A}^{odd}$. Still, in general, if $U \in \mathcal{A}_k$, any k, then for all basis elements e_a and e_b there exists an e_c such that

$$
(e_a U)(U^{-1} e_b) = \pm e_c = (e_a(e_b U)) U^{-1}. \tag{18}
$$

The last equality follows from Moufang identities, and it implies by induction that for an arbitrary set of octonion units e_a, e_b, ... e_d there exists a unit e_c such that

$$
e_a(e_b(...(e_d U)...)) = \pm e_c U. \tag{19}
$$

That is, nested products of units from the left on any $U \in \mathcal{A}^{even} \cup \mathcal{A}^{odd}$ collapse to a product of a single unit, to within a sign. In each case, the resulting unit $\pm e_c$ depends upon U.

In **[9]** I introduced a representation of the Leech lattice over $\mathbf{O}^3$. Quite frankly, I no longer have much of an idea how I arrived at this representation, but more recently Robert Wilson, applying more rigorous and reproducible mathematical methods, independently derived a representation of Λ_{24} over $\mathbf{O}^3$ that I will show verifies my initial guess.

In both our papers the final result breaks up the inner shell of Λ_{24}, which is of order 196560, into three subsets with orders $3 \times 240 = 720$, $3 \times 240 \times 16 = 11520$, and $3 \times 240 \times 16 \times 16 = 184320$, the sum of all three orders being 196560. The biggest of these subsets is the one I want to focus on. Translating Wilson's notation into my own, the elements of this subset take the form

$$
(\ (P\ell_0^{\dagger})e_a, \ \pm P e_c, \ \pm (P e_a) e_c \ ),
$$

where $P \in \mathcal{A}^{odd\dagger}$, the two $\pm$ signs are independent, the indices $a, c \in \{0, ..., 7\}$ are independent, and to achieve the full subset we include all permutations of these three octonion components. As a first step in connecting to my representation we take the conjugate of each of the three components,

$$
(\ e_a(\ell_0 P), \ \pm e_c P, \ \pm e_c(e_a P) \ ),
$$

where now it is understood that $P \in \mathcal{A}^{odd}$, and although it may seem like I am playing fast and loose with the signs, as long as we have two independent $\pm$ signs on two of these three terms all is ok. Next we cyclicly permute these three terms to the left, keep the $\pm$ signs on the second two, and replace P with $e_c P$ (keep in mind, this result takes advantage of sign flexibility):

$$
(\ P, \ \pm e_c(e_a(e_c P)), \ \pm e_a(\ell_0(e_c P)) \ ).
$$

The two e_c units appearing in the second component cancel (the resulting sign change we absorb into the $\pm$), leaving us with

$$(\; P, \;\; \pm e_a P, \;\; \pm e_a(\ell_0(e_c P)) \;) = (\; P, \;\; \pm e_a P, \;\; \pm(e_a \ell_0 e_a)(e_b P) \;), \tag{20}$$

where e_b depends on e_a, e_c, and P. To within multiplication by a scalar, this is now in the form of the representation presented in **[9]** (and I owe a big debt to Robert Wilson for putting that representation on a firm mathematical footing).

The remaining elements of this representation of the Leech lattice, also conforming to **[9]** and **[11]**, take the forms,

$$(\; 2Q, \;\; \pm 2 e_a Q, \; 0 \;) = (\; \ell_0^\dagger P, \;\; \pm e_a(\ell_0^\dagger P), \; 0 \;), \tag{21}$$

where $Q \in \mathcal{A}^{even}$, or $P = \ell_0 Q \in \mathcal{A}^{odd}$, and we include all permutations ($3 \times 240 \times 16 = 11520$ elements, and note that $(\ell_0^\dagger)^{-1} = \frac{1}{2}\ell_0$)), and

$$(\; 2P, \;\; 0, \; 0 \;), \tag{22}$$

where $P \in \mathcal{A}^{odd}$, and we include all permutations ($3 \times 240 = 720$ elements).

7. Motivation.

This paper has ostensibly nothing to do with physics, but my purpose in exploring these ideas does. For years I pursued applications of the four division algebras, **R**, **C**, **H** and **O**, to physics, and these efforts met with considerable success. The dimensions of these algebras, 1, 2, 4, 8 ($= 2^k$, $k = 0, 1, 2, 3$), are mathematically resonant. This finite sequence of integers is associated with myriad generative mathematical notions. But this sequence does not include the integer 24, nor does it make sense that it should. There is another finite sequence I suggest that is resonant in a different way: 1, 2, 8, 24. This tidbit from Wikipedia in speaking about the Leech lattice:

> "This arrangement of 196560 unit balls centred about another unit ball is so efficient that there is no room to move any of the balls; this configuration, together with its mirror-image, is the only 24-dimensional arrangement where 196560 unit balls simultaneously touch another. This property is also true in 1, 2 and 8 dimensions, with 2, 6 and 240 unit balls, respectively, based on the integer lattice, hexagonal tiling and E_8 lattice, respectively."

As we have seen, Λ_{24} can be nicely represented in the 24-dimensional $\mathbf{O}^3$. Likewise, $\Lambda_8 = E_8$ has a nice representation in $\mathbf{H}^2$ (as well as in simply $\mathbf{O}$), and Λ_2 and Λ_1 find their most natural expressions in $\mathbf{C}^1$ and $\mathbf{R}^1$, respectively. In **[12]**, inspired by all of this, I took the algebra $\mathbf{T} = \mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$, which in part found a roll of my application of the division algebras to physics in my hyper-spinor space $\mathbf{T}^2$, and expanded this to

$$\mathbf{T}^6 = \mathbf{R}^1 \otimes \mathbf{C}^1 \otimes \mathbf{H}^2 \otimes \mathbf{O}^3.$$

The spinor space $\mathbf{T}^2$ elegantly accounts for one generation (family) of quarks and leptons (and their anti-particles), but theoretical consensus puts the total number of families at three. Clearly, if $\mathbf{T}^2$ accounts for one family, then $\mathbf{T}^6$ would account for three, but the dimensionality of $\mathbf{T}^6$ is wrong for a conventional spinor space. I do not view this as a deterrence, but as indication that the mathematical tools needed to fully exploit $\mathbf{T}^6$ may not yet be available to us. My suspicion is that ternary algebras may be involved (**[13]**). Λ_8, as we have seen, can be given an algebraic structure, turning this lattice into the octavian integers. The question is: can Λ_{24} be given its own algebraic structure, perhaps involving a ternary multiplication? No idea yet.

Finally, and to cement the notion that I may be barking mad, I present some possibly coincidental numerology thoughts. The two finite sequences of resonant dimensions given above are

n	1	2	3	4
$\frac{1}{2}2^n$	1	2	4	8
F_n	1	1	2	3
$\frac{1}{2}F_n 2^n$	1	2	8	24

and F_n are the first 4 numbers in the Fibonacci sequence, starting from n = 1.

Why should the Fibonacci sequence have anything to do with this finite sequence of dimensions associated with special lattices? No idea, and I wouldn't mention it at all except for the following coincidences relating to the sequence of corresponding kissing numbers, 2, 6, 240, 196560, for the associated lattices Λ_1, Λ_2, Λ_8, Λ_{24}:

$$\prod_{k=1}^{3} F_k = 2, \quad \prod_{k=1}^{4} F_k = 6, \quad \prod_{k=1}^{6} F_k = 240, \quad \prod_{k=1}^{8} F_k = \frac{196560}{3}.$$

Why 3,4,6,8? Not sure, although each is 1 plus a prime for the first 4 primes, 2,3,5,7. So? Not sure. Haven't a clue. I'd like a clue, but - sadly - I don't have one.

8. Some thoughts from the original version of this paper.

The sets Ξ_k defined in (4) are associated with our chosen multiplication table defined in (3). Any change in the multiplication table will result in a change in these sets. Each element X in some Ξ_k gives rise to an X-product variant of our original multiplication table that takes octonion units to units (see (5)). Associated with this new multiplication table will be an altered collection of sets like our original Ξ_k. We will denote these altered sets

$$^X\Xi_k.$$

In particular, the sets $^X\Xi_0$ and $^X\Xi_1$ will not be altered, as they are independent of the multiplication table. However,

$$^X\Xi_2 = \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct}, e_a \circ_X (e_b \circ_X (e_c \circ_X e_d)) = \pm 1\},$$

and $^X\Xi_3$ will be similar to Ξ_3, with the number of minus signs in the sum being odd or even, depending on the product.

One of the remarkable properties of the octonions is that any sum of nested products from the left (or right) can be expressed as a sum of nested products from the right (or left). In particular, for all X in some Ξ_k, and all indices a = 1,...,7,

$$X e_a = \frac{1}{2}(-e_a X + e_p(e_q X) + e_r(e_s X) + e_u(e_v X)), \tag{23}$$

where the indices a,p,q,r,s,u,v are distinct, accounting for all the indices from 1 to 7, and $e_p e_q = e_r e_s = e_u e_v = e_a$. Using the identity (20) we can rewrite (27):

$$X e_a = \frac{1}{2}(-e_a + e_p \circ_X e_q + e_r \circ_X e_s + e_u \circ_X e_v)X.$$

By (5) this reduces to,

$$X e_a = \frac{1}{2}(-e_a \pm e_b \pm e_c \pm e_d)X = PX.$$

P must have a norm 1, and the only way for such a linear combination of octonion units to have norm 1 is if either $P = \pm e_m$ (eg., this certainly occurs if $X = \pm 1$), or if all the indices a,b,c,d are distinct. Because the original set of 7 indices accounted for all the indices from 1 to 7, in this latter case the remaining 4 indices above must satisfy

$$e_a \circ_X (e_b \circ_X (e_c \circ_X e_d)) = \pm 1.$$

So in general,

$$X e_a = P X, \text{ with } P \in {}^X \Xi^{even}.$$

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