Noncommutative Geometry and Supergravity.

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A spectral action associated with an Einstein-Cartan formulation of supergravity is proposed. To construct this action we make use of the Seeley-DeWitt coefficients in a Riemann-Cartan space. For consistency in its construction the Rarita-Schwinger action is added to the resulting spectral action.

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I. INTRODUCTION

The equivalence principle and gauge invariance are fundamental pillars of the two most successful theories in physics, general relativity and Yang-Mills theory. By means of them the basic interactions in our Universe can be understood, however these theories seem to be incompatible at the quantum level. This incompatibility might suggest that they are theories arising from some other more fundamental formulation. One of the most interesting proposals in the literature is the spectral action of noncommutative geometry. It involves new spectral geometry consistent with the physical measurements of distances. The usual emphasis on the points $x \in \mathcal{M}$ on a geometric space is replaced by the spectrum Σ of the Dirac operator D. It is assumed that the spectral action depends only on Σ. This is the spectral action principle. The spectrum is a geometric invariant that replaces diffeomorphism invariance. By applying this basic principle to the noncommutative geometry defined by the standard model it has been shown [1] that the dynamics of all interactions, including gravity are given by the spectral action. Its heat kernel expansion in terms of the Seeley-De Witt coefficients a_n results in an effective action up to the coefficient considered. For the gravitational sector of the spectral action, the first three terms on the expansion correspond to a constant, the usual Einstein-Hilbert action plus Weyl gravity and a Gauss-Bonnet topological invariant.

The Dirac operator considered in the spectral action principle proposal [1–4] is constructed with the Ricci rotation coefficients, which are assumed to be an explicit function of the tetrads and their derivatives, namely the ones resulting from solving the Riemannian torsion-free condition in standard general relativity. However, if the Ricci rotation coefficients were considered as independent variables, one would need to write the action up to the desired order of approximation in the Seeley-De Witt coefficients and then vary the action with respect to both the connection and the metric (or tetrads) independently, as in the Palatini first order formulation of general relativity. The Ricci connections would then be complicated functions of the tetrads and would depend on the parameters that appear at each order in the action. The Riemann tensor and tensors derived from it will depend on these generalized connections and not on the usual Riemannian connection of standard general relativity. It is also well known that theories with higher order terms in the Riemann, Ricci tensors or Ricci scalar do not provide the same equations of motion obtained from the second as opposed to those from the first order formalism [5]. The gravity action arising from the spectral action is usually given in terms of functions of the metric (or tetrads) and its derivatives obtained from using the Ricci rotation coefficients as explicit functions of the metric or tetrads and their derivatives as in the standard second order formulation of general relativity [5]. In [1], the authors derive this gravity spectral action. It is constructed as a function of the metric and its derivatives, by means of the Ricci rotation coefficients corresponding to the standard Riemannian connection.

Gravity theories with torsion have been of interest for many years [6]. Torsion is usually associated with some matter fields in the action. The Ricci rotation coefficients in the theory are modified by adding the contorsion to the standard connection. One associates this new connection with a Dirac operator. The presence of matter in the action yields, after solving the torsion constraint, a particular dependence of the torsion on the matter content of the theory. On the other hand, Seeley-De Witt coefficients have been calculated for connections in a Riemann-Cartan manifold [7], by means of them the expansion of

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the spectral action can be constructed. Actually, $\mathcal{N}=1$ supergravity can be understood as a gravity theory with a particular torsion. The action can be completely written as a function of the tetrads (metric), the torsion $S_{\mu\nu\rho}$ and the Rarita-Schwinger gravitino field ψ_μ. In this work we will, in section II, consider the Ricci rotation coefficients with the contorsion term $K_{\mu\nu\rho}$ and write the Seeley-De Witt coefficients up to a_4. On the other hand, if matter is present in the action one should consider the solution to the torsion constraint, for an action that consists of general relativity, the torsion content in the action and the matter action. This is the equivalent procedure as the one followed in the spectral action when considering the Ricci rotation coefficients as given by solving the standard torsion free condition in general relativity and taking this connection to get the whole gravity action. A similar procedure will be followed to obtain a spectral action associated with $\mathcal{N}=1$ supergravity. In section III, we will explicitly show how to construct supergravity as a gravity theory with torsion [8] and solve the torsion constraint to get the generalized Ricci rotation coefficients. These will be the appropriate coefficients to construct the corresponding spectral action. After calculating the torsion in terms of the matter fields, the Rarita-Schwinger field ψ_μ, the action is given in section IV in terms of the tetrads (metric) and the ψ_μ field. The Rarita-Schwinger action is the appropriate and consistent fermionic matter content to be added to the action.

II. SPECTRAL ACTION WITH TORSION

Instead of the well known geometry of space-time, the basic data of noncommutative geometry consists of an involutive algebra $\mathcal{A}$ of operators in a Hilbert space $\mathcal{H}$, which plays the role of the algebra of coordinates and a self-adjoint operator D in $\mathcal{H}$, which plays the role of the inverse line element. A fundamental principle in the noncommutative approach is that the usual emphasis on points in space-time is replaced by the spectrum of the operator D. The spectral action principle states that the physical action depends only on the spectrum of the Dirac operator. These ideas were the origin of the spectral action given in [1].

The bosonic part of the spectral action is

$$S = Tr[f(\frac{D}{\Lambda})] \ , \tag{1}$$

where D is a Dirac type operator acting on the Hilbert space $\mathcal{H} = L^2(M,S)$ of L^2-spinors. The action is uniquely defined and the coefficients in the spectral expansion are calculated from the heat kernel expansion.

For the gravitational Dirac operator we have

$$D = e^{\mu}{}_{a}\gamma^a(\partial_\mu + \omega_\mu) \ , \tag{2}$$

where ω_μ is the spin connection on $\mathcal{M}$, $\omega_\mu = \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab}$ with ω_μ related to $e_\mu{}^a$ by the vanishing of the covariant derivative which allows us to express $\omega_\mu{}^{ab}$ as functions of the tetrads as in standard Einstein tetradic gravity, the Riemannian torsion free condition. The first Seeley-De Witt coefficients for the pure gravitational Dirac operator from the heat kernel expansion are

$$\begin{aligned} a_0 &= \frac{1}{4\pi^2}\int d^4x\sqrt{g} \ , \\ a_2 &= -\frac{1}{48\pi^2}\int d^4x\sqrt{g}R \ , \\ a_4 &= \frac{1}{4\pi^2}\frac{1}{360}\int d^4x\sqrt{g}\,(-18C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 11R^*R^*) \ , \end{aligned} \tag{3}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor of conformal gravity $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ and the Euler characteristic χ_E is given by, $\chi_E = \frac{1}{32\pi}\int d^4x\sqrt{g}R^*R^*$ with $R^*R^* = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. With these results the action derived from the spectral triple of the Dirac operator (2) is

$$\begin{aligned} S = \\ \int d^4\sqrt{g}\,\{\alpha + \beta R + \gamma(-18C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 11R^*R^*)\} \ , \end{aligned} \tag{4}$$

where α, β and γ are constants. The spectral action gives the Hilbert-Einstein action with corrections, this approach is also valid for the standard model [9].

As already mentioned in the previous section our interest is in a connection to supergravity that will be treated as a torsion theory of gravity. For this reason we consider a connection with torsion and show the corresponding Seeley-DeWitt coefficients up to a_4.

Lets start by adding an antisymmetric field $K_{\mu ab}$ to the usual Ricci rotation coefficients

$$\omega_{\mu ab} = \tilde{\omega}_{\mu ab} + K_{\mu ab} \ , \tag{5}$$

here $\tilde{\omega}_{\mu ab} = \frac{1}{2}e^{\nu}{}_{a}\nabla_\mu e_{\nu b}$ is the usual spin connection [1] and $K_{\mu ab} = K_{\mu\nu\rho}e^{\nu}{}_{a}e^{\rho}{}_{b}$, $K_{\mu\nu\rho}$ the contorsion tensor, which in terms of the torsion $S_{\mu\nu\rho}$ is

$$K_{\mu\nu\rho} = -S_{\mu\nu\rho} + S_{\nu\rho\mu} - S_{\rho\mu\nu} \ . \tag{6}$$

The Dirac operator has the form

$$D_\mu\psi_\rho = \partial_\mu\psi_\rho + \frac{1}{2}(\tilde{\omega}_{\mu ab} + K_{\mu ab})\sigma^{ab}\psi_\rho \ , \tag{7}$$

the Riemann and Ricci tensors and the Ricci scalar in the spectral action should now be constructed with this new Dirac operator that depends on the modified connection. The generalized Ricci scalar, Ricci tensor and Riemann tensor are

$$R = \tilde{R} + 2\nabla_\mu K_\nu{}^{\mu\nu} + K_\mu{}^\mu{}_d K_\nu{}^{d\nu} - K_\nu{}^\mu{}_d K_\mu{}^{d\nu},$$
$$R_{\mu a} = \tilde{R}_{\mu a} + \nabla_\mu K_{\nu a}{}^\nu - \nabla_\nu K_{\mu a}{}^\nu + K_{\mu a d} K_\nu{}^{d\nu} - K_{\nu a d} K_\mu{}^{d\nu}, \tag{8}$$
$$R_{\mu\nu ab} = \tilde{R}_{\mu\nu ab} + \nabla_\mu K_{\nu ab} - \nabla_\nu K_{\mu ab} + K_{\mu a d} K_\nu{}^d{}_b - K_{\nu a d} K_\mu{}^d{}_b \ .$$

We can now write the corresponding action by directly substituting the expressions (8) in (4). $\tilde{R}$, $\tilde{R}_{\mu\nu}$ and $\tilde{R}_{\mu\nu ab}$ are the standard terms that depend on the usual spin connection without torsion. Furthermore, we have not made any assumptions about the dependence of the torsion on a specific kind of matter. In the next section we will show that $\mathcal{N}=1$ supergravity can be constructed as a gravity theory with torsion, where the resulting torsion will be given in terms of the Rarita-Schwinger field.

III. SUPERGRAVITY AS A TORSION GRAVITY THEORY

Let us begin by briefly reviewing the second and the first order formalisms of $\mathcal{N}=1$ supergravity. In the second order formalism [10], the action of $\mathcal{N}=1$ supergravity is written as a function of the tetrads $e_\mu{}^a$ and the gravitino field ψ^a_μ, in the following manner

$$I = \frac{1}{2}\int [eR(\{\}) - \epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu\mathcal{D}_\nu\psi_\rho - e\ (\frac{1}{4}\bar{\psi}_\alpha\gamma^\alpha\psi_\beta\ \bar{\psi}^\nu\gamma_\nu\psi^\beta + \frac{1}{8}\bar{\psi}^\nu\gamma^\alpha\psi_\beta\ \bar{\psi}_\alpha\gamma^\beta\psi_\nu - \frac{1}{16}\bar{\psi}_\nu\gamma_\alpha\psi_\beta\ \bar{\psi}^\nu\gamma^\alpha\psi^\beta) - \frac{i}{8}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu\sigma^{\kappa\sigma} \times [\ \bar{\psi}_\nu\gamma_\sigma\psi_\kappa - \bar{\psi}_\kappa\gamma_\nu\psi_\sigma + \bar{\psi}_\sigma\gamma_\kappa\psi_\nu]\psi_\rho]d^4x\ , \tag{9}$$

where $R(\{\})$ is the Ricci scalar calculated using the Christoffel symbols, ψ_α is a vector spinor ($\bar{\psi}_\alpha = \psi^T_\alpha C^{-1}$, C the charge conjugation matrix), $\gamma^\alpha = e^\alpha{}_b\gamma^b$, where γ^b are the flat space $\gamma-$ matrices in the Majorana representation

$$\sigma^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right], \quad \gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3\ , \tag{10}$$

we also define

$$\mathcal{D}_\mu\psi_\rho = \partial_\mu\psi_\rho + \frac{1}{4}e^\nu{}_a\nabla_\mu e_{\nu b}\sigma^{ab}\psi_\rho\ . \tag{11}$$

The variation of (9) with respect to $e_\mu{}^a$ gives Einstein equations of motion and variation respect to ψ_α yields the Rarita-Schwinger equation

$$\epsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\nu\mathcal{D}_\rho\psi_\lambda = 0\ , \tag{12}$$

the third order terms that might be expected can be shown to be zero by means of Fierz transformations.
One can also write $\mathcal{N}=1$ supergravity in the first order formalism

$$I = \frac{1}{2}\int [eR - i\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho]d^4x\ , \tag{13}$$

the curvature is calculated as a function of the Ricci rotation coefficients $\omega_{\mu ab}$

$$R_{\mu\nu ab} = \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b\ , \tag{14}$$

and the covariant derivative is

$$D_\mu\psi_\rho = \partial_\mu\psi_\rho + \frac{1}{2}\omega_{\mu ab}\sigma^{ab}\psi_\rho\ , \tag{15}$$

with

$$\omega_{\mu ab} = \tilde{\omega}_{\mu ab} + K_{\mu ab}\ , \tag{16}$$

where $K_{\mu ab}$ is given in terms of the torsion (6) which after variation gives $S_{\mu\nu\rho}$ as

$$S_{\mu\nu\rho} = \frac{1}{4}\bar{\psi}_\mu\gamma_\rho\psi_\nu\ . \tag{17}$$

This expression has the proper symmetry properties for $S_{\mu\nu\rho}$ ($S_{\mu\nu\rho} = -S_{\nu\mu\rho}$) because of the antisymmetry of products of ψ_ρ. The action now can be varied with respect to $e_\mu{}^a$ to give a much simpler expression for the gravitational field equations

$$R^\mu_{\ a} - \frac{1}{2}e^\mu{}_a R = T^\mu_{\ a}\ , \tag{18}$$

where $T^\mu_{\ a}$ is the stress tensor for the spin-$\frac{3}{2}$ field given by

$$T^\mu_{\ a} = -\frac{i}{2e}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_a D_\nu\psi_\rho\ . \tag{19}$$

However, the action of supergravity can also be written as function of the tetrads (metric), the gravitino field and the torsion $S_{\mu\nu\rho}$, all these formalisms give the same equations of motion. This last formalism is not well known and we present it here [8]. We begin with the Lagrangian density (13) which by using $\omega_{\mu ab}$, $K_{\mu ab}$ and $D_\mu\psi_\rho$ can be expressed as

$$L = \frac{1}{2}eR(\{\}) + \frac{1}{2}e(-4S_{\alpha\beta}{}^\alpha S^{\nu\beta}{}_\nu - 2S^\nu{}_\beta{}^\alpha S_{\alpha\nu}{}^\beta + S_{\nu\beta\alpha}S^{\nu\beta\alpha}) - \frac{1}{2}i\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu\mathcal{D}_\nu\psi_\rho - \frac{1}{4}i\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu e^\kappa{}_a e^\sigma{}_b\sigma^{ab} \times [\ S_{\nu\kappa\sigma} - S_{\kappa\sigma\nu} + S_{\sigma\nu\kappa}]\psi_\rho\ . \tag{20}$$

If we were to vary (20) with respect to $e_\mu^{\ a}$ we would obtain

$$R^\mu_{\ a} - \frac{1}{2}e^\mu_{\ a}R = T^\mu_{\ a}\ , \tag{21}$$

with $R_\mu^{\ a}$ as a function of the affine connection $\Gamma^\sigma_{\ \mu\nu}$ which contains derivatives of the torsion tensor $S_{\mu\nu\rho}$. However this expression does not possesses derivatives of $S_{\mu\nu\rho}$ and up to this point it is not clear how to handle (20) in order to get (21). To clear up this point and explain the procedure let us begin by varying (20) with respect to the torsion $S_{\mu\nu\rho}$. Defining

$$V^{\nu cd} = -\frac{1}{4}i\epsilon^{\sigma\beta\nu\rho}\bar{\psi}_\sigma\gamma_5\gamma_\beta\sigma^{cd}\psi_\rho\ , \tag{22}$$

we get

$$\begin{aligned}\delta L = \delta S_{\lambda\mu\nu}\{e\ [4e^\nu_{\ a}e^{[\lambda a}S^{\mu]\rho}_{\ \ \ \ \rho} - 2S^{\nu[\lambda\mu]} + S^{\lambda\mu\nu}] \\ + V^{\lambda\mu\nu} - V^{\nu\lambda\mu} + V^{\mu\nu\lambda}\}\ ,\end{aligned} \tag{23}$$

now using the definition of $\sigma^{\gamma\delta}$ and the well known algebra of the $\gamma's$ we can write

$$V^{\nu\gamma\delta} = \frac{1}{2}\epsilon^{\sigma\beta\nu\rho}\epsilon^{abcd}e_{\beta a}e^\gamma_{\ b}e^\delta_{\ c}e^\mu_{\ d}S_{\sigma\rho\mu}\ , \tag{24}$$

with $S_{\sigma\rho\mu} = \frac{1}{4}\bar{\psi}_\sigma\gamma_\mu\psi_\rho$, (23) gives zero if and only if

$$S_{\sigma\rho\mu} = -S_{\rho\sigma\mu}. \tag{25}$$

This result is the same as in the first order formalism and in the first order explicit torsion procedure. The second step is to vary with respect to tetrads and to show how the derivatives of the torsion $S_{\sigma\rho\mu}$ appear in the Lagrangian. Defining

$$R(\{\}) = \tilde{R}, \qquad R_{\mu\nu}(\{\}) = \tilde{R}_{\mu\nu}\ , \tag{26}$$

and using $|$ for the covariant derivative not including the torsion, we find

$$\begin{aligned}
&\delta\ L = \frac{\delta L}{\delta e_\mu^{\ a}}\delta e_\mu^{\ a} = \\
&\delta\ e_\mu^{\ a}[\frac{1}{2}e(-\tilde{R}^{\nu\beta} + \frac{1}{2}e^\nu_{\ a}e^{\beta a}\tilde{R})\frac{\delta(e_{\nu b}e_\beta^{\ b})}{\delta e_\mu^{\ a}} - \frac{1}{2}i\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_a D_\nu\psi_\rho - V^{\nu\mu\sigma}[S_{\nu\kappa\sigma} - S_{\kappa\sigma\nu} + S_{\sigma\nu\kappa}]e^\kappa_{\ a} - V^{\nu\kappa\mu}[S_{\nu\kappa\sigma} - S_{\kappa\sigma\nu} + S_{\sigma\nu\kappa}]e^\sigma_{\ a} \\
&+ V^{\nu cd}\frac{\delta(e^\sigma_{\ c}e_{\sigma d|\nu})}{\delta e_\mu^{\ a}} + \frac{1}{2}ee^\mu_{\ a}(-4S_{\alpha\beta}^{\ \ \ \alpha}S^{\nu\beta}_{\ \ \ \nu} - 2S^\nu_{\ \beta}{}^\alpha S_{\alpha\nu}^{\ \ \ \beta} + S_{\nu\beta\alpha}S^{\nu\beta\alpha}) + \frac{1}{2}e(4S^{\mu\delta}_{\ \ \ \lambda}\ S_{\beta\delta}^{\ \ \ \beta}\ e^\lambda_{\ a} + 4S_\tau^{\ \delta\mu}\ S_{\beta\delta}^{\ \ \ \beta}\ e^\tau_{\ a} + 4S^{\rho\mu}_{\ \ \ \rho}\ S_{\beta\delta}^{\ \ \ \beta}\ e^\delta_{\ a} \\
&+ 4S^\rho_{\ \kappa\rho}\ S_\beta^{\ \mu\beta}\ e^\kappa_{\ a} + 4S^{\rho\delta}_{\ \ \ \rho}\ S^\mu_{\ \delta\sigma}\ e^\sigma_{\ a} + 4S^{\rho\delta}_{\ \ \ \rho}\ S_{\beta\delta}^{\ \ \ \mu}\ e^\beta_{\ a} + 2S^{\mu\sigma\rho}\ S_{\rho\delta\sigma}\ e^\delta_{\ a} + 2S_\rho^{\ \sigma\beta}\ S_{\beta\ \sigma}^{\ \mu}\ e^\rho_{\ a} + 2S^{\delta\sigma\mu}\ S_{\beta\delta\sigma}\ e^\beta_{\ a} + 2S^{\sigma\delta}_{\ \ \ \lambda}\ S^\mu_{\ \delta\sigma}\ e^\lambda_{\ a} \\
&+ 2S^{\delta\mu\beta}\ S_{\beta\delta\sigma}\ e^\sigma_{\ a} + 2S^\delta_{\ \kappa}{}^\beta\ S_{\beta\delta}^{\ \ \ \mu}\ e^\kappa_{\ a} - S^{\mu\delta\sigma}\ S_{\beta\delta\sigma}\ e^\beta_{\ a} - S_\rho^{\ \delta\sigma}\ S^\mu_{\ \delta\sigma}\ e^\rho_{\ a} - S^{\beta\mu\sigma}\ S_{\beta\delta\sigma}\ e^\delta_{\ a} - S^\beta_{\ \kappa}{}^\sigma\ S_{\beta\ \sigma}^{\ \mu}\ e^\kappa_{\ a} - S^{\beta\delta\mu}\ S_{\beta\delta\sigma}\ e^\sigma_{\ a} \\
&- S^{\beta\delta}_{\ \ \ \lambda}\ S_{\beta\delta}^{\ \ \ \mu}\ e^\lambda_{\ a})]\ ,
\end{aligned} \tag{27}$$

to be able to perform this calculation it is necessary to know how to handle the fifth term in the last expression. For this purpose it is useful to notice that (22) can be transformed by means of the substitution

$$e\epsilon_{\beta\omega\phi\sigma}g^{\omega\gamma}g^{\phi\delta}g^{\sigma\mu} = -\epsilon^{abcd}e_{\beta a}e^\gamma_{\ b}e^\delta_{\ c}e^\mu_{\ d}\ , \tag{28}$$

then one rewrites the product $\epsilon_{\beta\omega\phi\theta}\epsilon^{\sigma\beta\nu\rho}$ in terns of products of $\delta_\mu^{\ \nu}$ which can be expressed as products of the metric $g^{\mu\nu}$ to give

$$V^{\nu\gamma\delta} = -e\ T^{\nu\gamma\delta}\ , \tag{29}$$

with

$$T^{\nu\gamma\delta} = e^\nu_{\ a}e^{\delta a}S^{\gamma\mu}_{\ \ \ \mu} + S^{\delta\gamma\nu} - e^\nu_{\ a}e^{\gamma a}S^{\delta\mu}_{\ \ \ \mu}\ , \tag{30}$$

to vary

$$V^{\nu cd}\frac{\delta(e^\sigma_{\ c}e_{\sigma d|\nu})}{\delta e_\mu^{\ a}}\ , \tag{31}$$

one expands the covariant derivative, integrates by parts and interchanges derivatives of $V^{\nu\gamma\sigma}$ in the following form

$$\begin{aligned}V^{\nu cd}\frac{\delta(e^\sigma_{\ c}e_{\sigma d|\nu})}{\delta e_\mu^{\ a}} = V^{\nu\mu\sigma}e_{\sigma a|\nu} - V^{\nu\mu\sigma}_{\ \ \ \ \ ,\nu}e_{\sigma a} \\ - V^{\nu\mu\sigma}e_{\sigma a,\nu} - V^{\nu\mu\sigma}e_{\delta a}\{^{\ \delta}_{\sigma\ \nu}\} + V^{\nu\gamma\mu}_{\ \ \ \ \ ,\gamma}e_{\nu a} \\ + V^{\mu\gamma\tau}_{\ \ \ \ \ ,\gamma}e_{\tau a}\ ,\end{aligned} \tag{32}$$

the use of (30) in (32) gives

$$V^{\ \nu cd}\frac{\delta(e^{\sigma}{}_{c}e_{\sigma d|\nu})}{\delta e_{\mu}{}^{a}} = -e\ e_{\rho a}[S^{\mu\rho\sigma}{}_{|\sigma} + S^{\mu\sigma\rho}{}_{|\sigma} + S^{\rho\sigma\mu}{}_{|\sigma}$$
$$+ 2e^{\rho}{}_{b}e^{\mu b}S^{\sigma\gamma}{}_{\gamma|\sigma} - 2e^{\rho}{}_{b}e^{\sigma b}S^{\mu\gamma}{}_{\gamma|\sigma}]\ . \tag{33}$$

In this form the derivatives of the torsion that are necessary to obtain the Einstein equations appear. Substituting (25) and (33) in (27), one obtains

$$\begin{aligned}
&\tilde{R}^{\ \mu}{}_{a} - \frac{1}{2}e^{\mu}{}_{a}\tilde{R} - e^{\kappa}{}_{a}[-S^{\mu}{}_{\kappa}{}^{\rho}{}_{|\rho} - S^{\mu\rho}{}_{\kappa|\rho} - S_{\kappa}{}^{\rho\mu}{}_{|\rho} + 2S^{\mu\rho}{}_{\rho|\kappa}] \\
&+ \frac{1}{2}e^{\mu}{}_{a}[4S^{\sigma\rho}{}_{\rho|\sigma} + S_{\rho\beta}{}^{\rho}S^{\nu\beta}{}_{\nu} - S_{\nu\beta\alpha}S^{\nu\beta\alpha} + 2S^{\nu}{}_{\beta}{}^{\rho}S_{\rho\nu}{}^{\beta}] \\
&- e^{\kappa}{}_{a}[-2S^{\sigma\rho}{}_{\rho}S^{\mu}{}_{\kappa\sigma} + 2S^{\sigma\rho}{}_{\rho}S_{\sigma}{}^{\mu}{}_{\kappa} - 2S^{\sigma\rho}{}_{\rho}S_{\kappa\sigma}{}^{\mu} \\
&+ 2S^{\delta\sigma\mu}S_{\kappa\delta\sigma} - S^{\beta\delta\mu}S_{\beta\delta\kappa}] = -\frac{i}{2e}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}_{\lambda}\gamma_{5}\gamma_{a}D_{\nu}\psi_{\rho}\ ,
\end{aligned} \tag{34}$$

with $S_{\mu\nu\rho}$ given by (17), expression (34) can be identified with the usual equations of motion. As is well known, and was mentioned above, variation with respect ψ_μ gives the Rarita-Schwinger equation. If we want to relate our procedure to standard Einstein-Cartan torsion theory we would need to know how to vary the action (20) with respect to the metric. It is then necessary to chose a gauge; one possibility [11] is the so called Brill-Wheeler (Pauli) condition $\delta e_{\mu}{}^{b} = \frac{1}{4}[e^{\tau b}\delta_{\mu}{}^{\sigma} + e^{\sigma b}\delta_{\mu}{}^{\tau}]\delta g_{\sigma\tau}$.

In the next section we will use the torsion (17) and substitute it in (8) to get the appropriate terms that will build the spectral action associated to $\mathcal{N}=1$ supergravity.

IV. SPECTRAL ACTION ASSOCIATED TO SUPERGRAVITY

In the following we will then consider supergravity as an Einstein-Cartan theory. The Dirac operator is given by

$$D_{SG} = \gamma^{a}e^{\mu}_{a}[\partial_{\mu} + \frac{1}{2}(\tilde{\omega}_{\mu bc} + K_{\mu bc})\sigma^{bc}]\ , \tag{35}$$

where the contorsion term $K_{\mu\nu\rho}$ is constructed from the torsion that arises in the supergravity expression (17). The expansion of action (1) will be given by the action (4) with the Ricci scalar, the Ricci and Riemann tensors (8) and $K_{\mu\nu\rho}$ (6) constructed from $S_{\mu\nu\rho} = \frac{1}{4}\bar{\psi}_{\mu}\gamma_{\rho}\psi_{\nu}$. Notice that, as we mentioned in the Introduction, we could calculate the spectral action leaving the contorsion as simply the form given in (6), and then vary the resulting action with respect to $S_{\mu\nu\rho}$ as in [8]. This would give different equations of motion than those obtained by substituting $S_{\mu\nu\rho}$ by $\frac{1}{4}\bar{\psi}_{\mu}\gamma_{\rho}\psi_{\nu}$. We have decided that this would not make contact with the concepts of [1] where the action depends only on torsion-free gravity and the Dirac field, so we will not attempt it here. Furthermore, the appropriate matter action consistent with the torsion (17) to be added to the spectral action is the Rarita-Schwinger action and the final action will then be

$$Tr[f(\frac{D_{SG}}{\Lambda})] + (\psi, D_{SG}\psi)_{RS}\ , \tag{36}$$

and the Dirac operator D_{SG} is then

$$\begin{aligned}
&D_{\mu} = \partial_{\mu} \\
&+ \frac{1}{2}[\tilde{\omega}_{\mu ab} - \frac{1}{4}(\bar{\psi}_{\mu}\gamma_{\rho}\psi_{\nu} - \bar{\psi}_{\nu}\gamma_{\mu}\psi_{\rho} + \bar{\psi}_{\rho}\gamma_{\nu}\psi_{\mu})e^{\nu}{}_{a}e^{\rho}{}_{b}]\sigma^{ab},
\end{aligned} \tag{37}$$

therefore we will get an action of the form

$$\begin{aligned}
&S = \int d^{4}xe(\alpha + \beta R) + (\psi, D_{SG}\psi) \\
&+ \int d^{4}xe\gamma\{\ -18(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^{2}) \\
&+ 11R^{*}R^{*}\}
\end{aligned} \tag{38}$$

where the first terms correspond to the standard $\mathcal{N}$=1 supergravity action and

$$\begin{aligned}
R =&\tilde{R} + \nabla_{\mu}(\bar{\psi}^{\mu}\gamma_{\nu}\psi^{\nu}) - \frac{1}{4}\bar{\psi}_{\alpha}\gamma^{\alpha}\psi_{\beta}\bar{\psi}^{\nu}\gamma_{\nu}\psi^{\beta} \\
&- \frac{1}{8}\bar{\psi}^{\nu}\gamma^{\alpha}\psi_{\beta}\bar{\psi}_{\alpha}\gamma^{\beta}\psi_{\nu} + \frac{1}{16}\bar{\psi}_{\nu}\gamma_{\alpha}\psi_{\beta}\bar{\psi}^{\nu}\gamma^{\alpha}\psi^{\beta}\ ,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&R_{\ \mu\nu} = \tilde{R}_{\mu\nu} + \frac{1}{2}\nabla_{\mu}(\bar{\psi}_{\nu}\gamma^{\alpha}\psi_{\alpha}) \\
&- \frac{1}{4}\nabla_{\alpha}(\bar{\psi}_{\nu}\gamma^{\alpha}\psi_{\mu} + \bar{\psi}_{\nu}\gamma_{\mu}\psi^{\alpha} + \bar{\psi}_{\mu}\gamma_{\nu}\psi^{\alpha}) \\
&+ \frac{1}{8}(\bar{\psi}_{\nu}\gamma_{\beta}\psi_{\mu}\bar{\psi}^{\beta}\gamma^{\alpha}\psi_{\alpha} + \bar{\psi}_{\nu}\gamma_{\mu}\psi_{\beta}\bar{\psi}^{\beta}\gamma^{\alpha}\psi_{\alpha} + \bar{\psi}_{\mu}\gamma_{\nu}\psi_{\beta}\bar{\psi}^{\beta}\gamma^{\alpha}\psi_{\alpha}) \\
&- \frac{1}{16}(\bar{\psi}_{\nu}\gamma_{\beta}\psi_{\alpha}\bar{\psi}^{\beta}\gamma^{\alpha}\psi_{\mu} + \bar{\psi}_{\nu}\gamma_{\beta}\psi_{\alpha}\bar{\psi}^{\beta}\gamma_{\mu}\psi^{\alpha} + \bar{\psi}_{\nu}\gamma_{\beta}\psi_{\alpha}\bar{\psi}_{\mu}\gamma^{\beta}\psi^{\alpha} \\
&+ \bar{\psi}_{\nu}\gamma_{\alpha}\psi_{\beta}\bar{\psi}^{\beta}\gamma^{\alpha}\psi_{\mu} + \bar{\psi}_{\nu}\gamma_{\alpha}\psi_{\beta}\bar{\psi}^{\beta}\gamma_{\mu}\psi^{\alpha} + \bar{\psi}_{\nu}\gamma_{\alpha}\psi_{\beta}\bar{\psi}_{\mu}\gamma^{\beta}\psi^{\alpha} \\
&+ \bar{\psi}_{\alpha}\gamma_{\nu}\psi_{\beta}\bar{\psi}^{\beta}\gamma^{\alpha}\psi_{\mu} + \bar{\psi}_{\alpha}\gamma_{\nu}\psi_{\beta}\bar{\psi}^{\beta}\gamma_{\mu}\psi^{\alpha} + \bar{\psi}_{\alpha}\gamma_{\nu}\psi_{\beta}\bar{\psi}_{\mu}\gamma^{\beta}\psi^{\alpha})\ ,
\end{aligned} \tag{40}$$

$$\begin{aligned}
&R_{\ \mu\nu\rho\sigma} = \tilde{R}_{\mu\nu\rho\sigma} + \frac{1}{4}\nabla_{\mu}(\bar{\psi}_{\rho}\gamma_{\sigma}\psi_{\nu} + \bar{\psi}_{\rho}\gamma_{\nu}\psi_{\sigma} + \bar{\psi}_{\nu}\gamma_{\rho}\psi_{\sigma}) \\
&- \frac{1}{4}\nabla_{\nu}(\bar{\psi}_{\rho}\gamma_{\sigma}\psi_{\mu} + \bar{\psi}_{\rho}\gamma_{\mu}\psi_{\sigma} + \bar{\psi}_{\mu}\gamma_{\rho}\psi_{\sigma}) \\
&+ \frac{1}{16}(\bar{\psi}_{\rho}\gamma_{\alpha}\psi_{\mu}\bar{\psi}^{\alpha}\gamma_{\sigma}\psi_{\nu} + \bar{\psi}_{\rho}\gamma_{\alpha}\psi_{\mu}\bar{\psi}^{\alpha}\gamma_{\nu}\psi_{\sigma} + \bar{\psi}_{\rho}\gamma_{\alpha}\psi_{\mu}\bar{\psi}_{\nu}\gamma^{\alpha}\psi_{\sigma} \\
&+ \bar{\psi}_{\rho}\gamma_{\mu}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\sigma}\psi_{\nu} + \bar{\psi}_{\rho}\gamma_{\mu}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\nu}\psi_{\sigma} + \bar{\psi}_{\rho}\gamma_{\mu}\psi_{\alpha}\bar{\psi}_{\nu}\gamma^{\alpha}\psi_{\sigma} \\
&+ \bar{\psi}_{\mu}\gamma_{\rho}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\sigma}\psi_{\nu} + \bar{\psi}_{\mu}\gamma_{\rho}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\nu}\psi_{\sigma} + \bar{\psi}_{\mu}\gamma_{\rho}\psi_{\alpha}\bar{\psi}_{\nu}\gamma^{\alpha}\psi_{\sigma}) \\
&- \frac{1}{16}(\bar{\psi}_{\rho}\gamma_{\alpha}\psi_{\nu}\bar{\psi}^{\alpha}\gamma_{\sigma}\psi_{\mu} + \bar{\psi}_{\rho}\gamma_{\alpha}\psi_{\nu}\bar{\psi}^{\alpha}\gamma_{\mu}\psi_{\sigma} + \bar{\psi}_{\rho}\gamma_{\alpha}\psi_{\nu}\bar{\psi}_{\mu}\gamma^{\alpha}\psi_{\sigma} \\
&+ \bar{\psi}_{\rho}\gamma_{\nu}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\sigma}\psi_{\mu} + \bar{\psi}_{\rho}\gamma_{\nu}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\mu}\psi_{\sigma} + \bar{\psi}_{\rho}\gamma_{\nu}\psi_{\alpha}\bar{\psi}_{\mu}\gamma^{\alpha}\psi_{\sigma} \\
&+ \bar{\psi}_{\nu}\gamma_{\rho}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\sigma}\psi_{\mu} + \bar{\psi}_{\nu}\gamma_{\rho}\psi_{\alpha}\bar{\psi}^{\alpha}\gamma_{\mu}\psi_{\sigma} + \bar{\psi}_{\nu}\gamma_{\rho}\psi_{\alpha}\bar{\psi}_{\mu}\gamma^{\alpha}\psi_{\sigma})\ .
\end{aligned} \tag{41}$$

The last term in (38) is given by the Euler characteristic and $\tilde{R}$, $\tilde{R}_{\mu a}$ and $\tilde{R}_{\mu\nu ab}$ correspond to the standard

Riemannian connection $\tilde{\omega}_{\mu ab} = \frac{1}{2} e^{\nu}{}_{a} \nabla_{\mu} e_{\nu b}$. It has been shown that the Dirac operators can be considered as dynamical variables of Euclidean supergravity restricted by certain conditions [12]. The physical meaning of the interaction terms involved in (38-41), subject to those restrictions is beyond the scope of this work and is left for future study. A similar possible construction of spectral actions corresponding to extended supergravities is also a matter of further research.

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