

Gravity and Yang-Mills theory

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Abstract

Three of the four forces of Nature are described by quantum Yang-Mills theories with remarkable precision. The fourth force, gravity, is described classically by the Einstein-Hilbert theory. There appears to be an inherent incompatibility between quantum mechanics and the Einstein-Hilbert theory which prevents us from developing a consistent quantum theory of gravity. The Einstein-Hilbert theory is therefore believed to differ greatly from Yang-Mills theory (which does have a sensible quantum mechanical description). It is therefore very surprising that these two theories actually share close perturbative ties. This article focuses on these ties between Yang-Mills theory and the Einstein-Hilbert theory. We discuss the origin of these ties and their implications for a quantum theory of gravity.

Gravity, although the oldest force known to man, is the force we understand the least. The weakness of the gravitational force precludes the possibility of performing simple experiments that will teach us more about this omnipresent force. In addition, the fact that the gravitational constant κ is dimensionful seems to produce a plethora of problems when attempting to unite quantum mechanics and general relativity.

Apart from gravity there are three other *fundamental* forces in Nature: the electromagnetic force, the weak force and the strong force. These three forces are all described by quantum Yang-Mills theories and theoretical predictions have, with stunning accuracy, matched experimental observations. This leads us naturally back to gravity: to wonder why, unlike the other three forces, it refuses to play well in the quantum mechanical playground. Experimental checks on the general theory of relativity have proved very successful and this suggests that the correct quantum theory of gravity will prove to be an “extension” of Einstein’s theory rather than a replacement.

There are many reasons to believe that gravity is indeed very different from the other three forces. The Einstein-Hilbert Lagrangian, which describes gravity, differs greatly from the Yang-Mills Lagrangian suggesting that the theories they describe should also differ drastically.

In this article we paint a picture that conveys quite the opposite impression - a picture where Yang-Mills and gravity seem to have far more in common than previously believed. We explore close ties between the theories that go beyond superficial on-shell relations.

We begin with a quick review of why the two theories are expected to behave very differently. The Yang-Mills action reads

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x \text{Tr} (F^{\mu\nu} F_{\mu\nu}) , \quad (1)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu] . \quad (2)$$

A_μ represents the gauge field and g the dimensionless Yang-Mills coupling constant. Gravity, on the other hand, is governed by the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} R , \quad (3)$$

where R is the Ricci scalar, g the determinant of the metric and κ the dimensionful gravitational coupling constant. Significant differences between the two theories include

1. Yang-Mills theory has only cubic and quartic interaction vertices while gravity involves infinitely many interaction vertices.
2. The coupling constant g is dimensionless while κ has the dimensions of length.
3. The trace in Yang-Mills, due to the gauge group, is absent in gravity.

Each of these is a fairly substantial difference in its own right. Given these manifest differences, close ties between the two theories are all the more surprising.

The first clear indications of a concrete connection between gravity and Yang-Mills theory arose from the Kawai-Lewellyn-Tye (KLT) relations [1]. These relations tells us that tree-level scattering amplitudes in pure gravity are the “square” of tree-level scattering amplitudes in pure Yang-Mills theory. Thus, rather unexpectedly, graviton scattering is expressible as a sum of products of pieces of non-abelian gauge theory scattering amplitudes [2]. Since scattering amplitudes contain information about the physics of a process, this points to a much stronger relationship between gravity and Yang-Mills theory than one might anticipate from a cursory glance at the respective Lagrangians.

The KLT relations are easy to write down but before doing so, we need to address the differences, between the two theories, listed above in points 2 and 3. To this end we define, for Yang-Mills, color-stripped partial amplitudes

$$A_n^{\text{tree}} = g^{(n-2)} \text{Tr}(\dots) \times A_n^{\text{tree}} . \quad (4)$$

A_n^{tree} is a tree-level scattering process involving n gluons in pure Yang-Mills theory. All information regarding the color structure is contained within the trace and A_n^{tree} now represents the color-stripped partial amplitude. Similarly, in the case of gravity, we define

$$M_n^{\text{tree}} = \left(\frac{\kappa}{2}\right)^{(n-2)} \times M_n^{\text{tree}} , \quad (5)$$

where M_n^{tree} represents a tree-level gravity scattering process and $\kappa^2 = 32\pi G_N$ is the coupling in terms of the Newton constant. M_n^{tree} is now κ -independent and represents the coupling-stripped gravity amplitude. We are now in a position to write down the KLT relations

$$M_n^{\text{tree}} \propto (A_n^{\text{tree}})^2 , \quad (6)$$

suggesting the possibility of a much broader and intriguing relation of the form

$$\text{Gravity} \sim (\text{Yang-Mills}) \times (\text{Yang-Mills}) . \quad (7)$$

Such a broad relationship however would require far more than simple tree-level connections. If the relationship is indeed deeper it ought to stem from similarities at the off-shell level: in particular, *can we relate the two theories directly at the level of the Lagrangians?* It turns out that the answer to this question is at least partially a yes and the rest of this paper will expand on this point.

We start by observing that *both* the gauge field and the graviton field, in four dimensions, involve exactly two physical degrees of freedom¹. We make this degrees-of-freedom equality manifest in the Lagrangians by working in light-cone gauge where only physical helicity states are present.

The structure of the light-cone gauge Yang-Mills Lagrangian, in momentum space, is

$$L_{\text{YM}} \sim \bar{A} p_\mu^2 A + g f_1 \bar{A} A A + g f_2 \bar{A} \bar{A} A + g^2 f_3 \bar{A} \bar{A} A A , \quad (8)$$

¹These degrees of freedom correspond to the positive and negative helicity states under the $SO(2)$ little group in four dimensions.

where \bar{A} , A are $-$, $+$ helicity states of the gauge field, p_μ^2 stems from the d'Alembertian in the kinetic term, g is the Yang-Mills coupling constant and f_1, f_2, f_3 represent momentum coefficients².

It turns out, from explicit calculations, that tree-level scattering amplitudes constructed using the $\bar{A}AA$ (helicity $-++$) vertex in (8) vanish [3]. So, when focusing on amplitude calculations it makes sense to try and eliminate this vertex by means of a field redefinition. Such a canonical change-of-variables, for Yang-Mills theory, was found in [4,5]. The change-of-variables maps the first two terms of the interacting Yang-Mills Lagrangian in (8) into a free Lagrangian in new variables

$$\bar{A} p_\mu^2 A + g f_1 \bar{A}AA \rightarrow \bar{B} p_\mu^2 B .$$

This is achieved by the following canonical redefinition of the gauge field [4,5]

$$\begin{aligned} A &\rightarrow B + b_1 B^2 + b_2 B^3 + \dots + b_{n-1} B^n , \\ \bar{A} &\rightarrow \bar{B} + c_1 B \bar{B} + c_2 B^2 \bar{B} + \dots + c_{n-1} B^{n-1} \bar{B} , \end{aligned}$$

where b_n, c_n are functions of the momenta and \bar{B}, B represent the “shifted” gauge field. This canonical change-of-variables results in a new form for the Yang-Mills Lagrangian

$$L'_{\text{YM}} \sim \bar{B} p_\mu^2 B + g F_1 \bar{B} \bar{B} B + g^2 F_2 \bar{B} \bar{B} B B + \dots + g^{n-1} F_{n-1} \bar{B} \bar{B} B^n , \quad (9)$$

where the F_n are the momentum coefficients for the new interaction vertices. The price paid for eliminating the offending term from (8) is the appearance of infinitely many interaction vertices *exactly like in gravity*. Note that all interaction vertices in (9) involve exactly two negative helicity fields³.

The advantage of the new form of the Yang-Mills Lagrangian is that certain classes of scattering amplitudes (referred to as MHV) become trivial to compute. For example, the amplitude for $A_{(-++)}$ is obtained trivially by taking the coefficient F_2 from (9) and putting it on-shell. To compute the same process starting from (8), one would have to deal with not just the quartic vertex but also contact diagrams arising from combinations of the two cubic vertices.

We now have a “close to on-shell physics” form for the Yang-Mills Lagrangian where amplitude structures are manifest. Since amplitudes in Yang-Mills are related to those in gravity, it seems natural to look for a similar amplitude-friendly form for the gravity Lagrangian - such a gravity Lagrangian ought to have manifest similarities to (9).

We start with the momentum-space Einstein-Hilbert Lagrangian in light-cone gauge [6]

$$L_{\text{EH}} \sim \bar{h} p_\mu^2 h + \kappa l_1 \bar{h} h h + \kappa l_2 \bar{h} \bar{h} h + \kappa^2 l_3 \bar{h} \bar{h} h h + \dots \quad (10)$$

where \bar{h} , h are $-$, $+$ helicity states of the graviton and l_1, l_2, l_3, \dots represent momentum coefficients. Notice that this structure, which is similar in some ways to (8), reveals the key difference highlighted earlier: that Yang-Mills theory involves only a finite number of interaction vertices (making it renormalizable) while gravity involves infinitely many

²Integrals over momenta and δ -functions are not shown here explicitly.

³The apparent disparity in the Lagrangian between $+$ and $-$ is an artifact of the convention-choice [3].

interaction vertices. As with Yang-Mills, amplitudes arising from the first cubic vertex in (10) vanish motivating the elimination of this vertex.

A canonical field redefinition, analagous to the one in Yang-Mills, was found in [7]

$$\begin{aligned} h &\rightarrow C + q_1 C^2 + q_2 C^3 + \dots + q_{n-1} C^n , \\ \bar{h} &\rightarrow \bar{C} + r_1 C \bar{C} + r_2 C^2 \bar{C} + \dots + r_{n-1} C^{n-1} \bar{C} , \end{aligned}$$

where q_n, r_n represent functions of the momenta and \bar{C}, C the “shifted” graviton. This redefinition successfully eliminates the “bad” vertex resulting in the new form

$$L'_{\text{EH}} \sim \bar{C} p_\mu^2 C + \kappa L_1 \bar{C} \bar{C} C + \kappa^2 L_2 \bar{C} \bar{C} C C + \dots + \kappa^{n-1} L_{n-1} \bar{C} \dots \bar{C} C \dots C + \dots \quad (11)$$

where the L_n are the momentum coefficients for the new interaction vertices.

We are now in a position to compare the two structures in (9) and (11). Up to second order in the coupling constants we find that gravity does indeed behave like the “square” of Yang-Mills. For instance

$$L_2 \sim (F_2) \times (F_2) . \quad (12)$$

This is a much stronger statement than a KLT relation since it relates *off-shell* coefficients in a Lagrangian as opposed to on-shell amplitudes. Unfortunately, beyond second order in the coupling new problems crop up. This happens because gravity vertices in (11) involve varying numbers of fields of both helicities, unlike Yang-Mills where all vertices in (9) involve exactly two negative helicity fields. A complete understanding of the relationship in (7) could prove invaluable for a detailed finiteness-analysis [8] of gravity or supergravity.

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In this article, we have reviewed two equivalent forms of the Yang-Mills Lagrangian. The first has a finite number of interaction vertices while the second involves infinitely many such vertices. We have also described two equivalent light-cone Lagrangians for gravity, both involving infinitely many vertices. We conclude by asking whether there exists a third equivalent form of the gravity Lagrangian that involves only a finite number of interaction vertices⁴. If yes, what implications will such a form of the Lagrangian have for the ultra-violet properties of gravity?

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⁴The answer to this question is most likely a No. However, where the search for a quantum theory of gravity is concerned, concrete avenues of progress are rare and this approach, even if unsuccessful, is certain to teach us more about the UV structure of gravity - this makes it worth pursuing to its logical conclusion.

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