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A New Entropic Force Scenario and Holographic Thermodynamics

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Abstract

We propose a new holographic program of gravity in which we introduce a surface stress tensor. Our proposal differs from Verlinde's in several aspects. First, we use an open or a closed screen, a temperature is not necessary but a surface energy density and pressure are introduced. The surface stress tensor is proportional to the extrinsic curvature. The energy we use is Brown-York energy and the equipartiton theorem is violated by a non-vanishing surface pressure. We discuss holographic thermodynamics of a gas of weak gravity and find a chemical potential, and show that Verlinde's program does not lead to a reasonable thermodynamics. The holographic entropy is similar to the Bekenstein entropy bound.

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I. INTRODUCTION

That Gravity may not be a fundamental force has long been suspected, and gained more credit and interest since Verlinde's work [1] (see also [2]). Verlinde proposed that gravity is not only an emergent force, but specifically an entropic force. His work builds upon an early work of Jacobson [3], and there have appeared many following up papers afterwards [4].

The fact that gravity is emergent is not surprising, given the tremendous progress in AdS/CFT in the last decade. This correspondence tells that at least for an anti-de Sitter space or a related space-time, quantum gravity can be fully described by a quantum field theory without gravity in a lower dimension. It is certainly true that gravity remains a fundamental force in the bulk description, but this theory may be less fundamental than the corresponding holographic field theory, since the latter has a well-developed understanding including its definition and quantization. Some would like to say that the holographic field theory offers a nonperturbative defintion of quantum gravity in the bulk.

The program of Verlinde is more general than gauge/gravity correspondence, however it is also rather primitive, since we have little information about the theory on a holographic screen. The main ingredients in Verlinde's scenario including a temperature, the equipartition theorem of energy, and an entropy increase when a test particle approaches the holographic screen. However, the temperature depends on the local bulk geometry of the full space-time, and we do not know how this relation arises. Moreover, the equipartition sounds very peculiar.

In this paper, we shall challenge Verlinde's definition of energy, and the assumption of the equipartition theorem. The main reason is that Verlinde's energy does not give us a well-behaved thermodynamics, specifically, the holographic entropy of a bulk gas computed using Verlinde's definition of energy has an density independent area term. We propose to use Brown-York semi-local energy to replace Verlinde energy. In doing so, we introduce a surface stress tensor on a holographic screen, this tensor give us both the surface energy density as well as the surface pressure-this is missing in Verlinde's program.

The surface stress tensor is proportional to the extrinsic curvature of the screen in space-time. This may hint that we need more geometric data than Verlinde does. However, in deriving the Einstein equations, only two physical quantities are involved, namely the surface energy density and the surface pressure, compared with Verlinde temperature, we need one more scalar. This may not be a disadvantage, rather it is an advantage since we know more about what is living on the screen. We do not have to assume the equipartition, and we show that the equipartition rule is violated by a non-vanishing surface pressure.

Another difference between our proposal and Verlinde's is that we use an open or closed screen, while Verlinde uses a closed one. We do not need a temperature, though in discussing holographic thermodynamics we adopt Verlinde temperature. The use of a closed screen in Verlinde's program introduces the problem of possible negative temperature, thus his derivation of the Eintein equations is incomplete. In his derivation, a total energy on the closed screen is needed, while in our derivation we need to study the energy flow through an open/closed surface. Thus our derivation is similar to Jacobson's original derivation [3], the difference is that our holographic screen is time-like and his is null, so we have more data on the screen since the surface pressure always vanishes on a null screen.

One of the present authors and Yi Pang proved a no-go theorem some time ago [5], stating that it is impossible to accommodate a high derivative theory of gravity in Verlinde's program. It appears that there is no such a no-go theorem in our program, we may replace the extrinsic curvature as the surface stress tensor by one involving higher derivatives too to derive equations of motion in a high derivative theory.

We also study holographic thermodynamics of a spherically symmetric and static system using our program. One derives one more important quantity, the chemical potential, for this system. We obtain a very interesting result: The holographic entropy of a gas of weak gravity has a form parametrically similar to the Bekenstein entropy bound. We take this as an indication that our program is correct. We also show that there is always a density independent area term in the holographic entropy of the gas if we use Verlinde's program. Note that, the holographic entropy is usually much larger than the statistical entropy of the gas, we explain this fact by the tremendous contribution to entropy from gravity. It has been known for a long time that the usual statistical entropy of a collapsing system violates the second law of thermodynamics, our explanation is that we need to use the holographic entropy rather than the usual statistical entropy. We expect that the holographic entropy smoothly crosses-over to the black hole entropy in a black hole formation process.

It remains to derive a general, geometric formula for the chemical potential for a general system, we leave this to a future work.

The plan of this paper is the following. We explain our holographic derivation of the

Einstein equations in sect.2, and compare our proposal to those of Verlinde and Jacobson in sect.3. We discuss holographic thermodynamics of a gas of weak gravity in sect.4 and show that Verlinde's program does not give us a reasonable answer to this problem in sect.5. We concldue in sect.6. The appendix gives the solution of a gas of weak gravity to the second order in G.

II. THE EINSTEIN EQUATIONS AND HOLOGRAPHIC SCREENS

In this section, we will derive the Einstein equations from the holographic principle in a different way from Verlinde's.

Let us first briefly review Verlinde's derivation of the Einstein equations. There are a few important ingredients in Verlinde's scenario. The first is a time-like holographic screen and a temperature on it. This temperature is motivated by Unruh temperature, and plays an imperative role in the derivation of the Newton's law of gravitation. We will follow Verlinde to consider a time-like holographic screen and his definition of temperature on it, however, we shall not assume that the screen is closed, not to mention an equipotential one. A time-like holographic screen in our proposal can be either closed or open, and does not have to be an equipotenttial surface so long if the temperature on the surface is positive everywhere. In fact, in the following derivation of the Einstein equations, we do not even need a temperature, the temperature is introduced only when we discuss holographic thermodynamics in Sect.4. For Verlinde, a closed surface can not be any closed surface since most of time the temperature may become negative.

The second ingredient of Verlinde's proposal is the equipartition theorem, namely, assuming the number of degrees of freedom N be proportional to the area, the energy of this area is $TN/2$. This is a peculiar assumption, since in statistical mechanics it is valid only for a gas with high temperature without interaction. We will not assume this. The third ingredient is that the holographic energy on the screen is equal to the Tolman-Komar mass, we will also differ from Verlinde for this point. The reason for us to abandon the Tolman-Komar mass is that this definition does not offer us a consistent thermodynamics on the holographic screen.

To define Verlinde temperature, we need the definition of a generalization of Newton's potential [6]

$$\phi = \frac{1}{2}\log(-\xi^a \xi_a), \tag{1}$$

where ξ^a is a local time-like Killing vector. The definition of Verlinde temperature is

$$T = \frac{\hbar}{2\pi} e^{\phi} N^a \nabla_a \phi, \tag{2}$$

where N^a is a vector normal to the time-like surface Σ (this is a 2+1 dimensional surface whose constant time slice is B), whose precise definition will be given shortly. Next, the number of degrees of freedom on an area dA of B is $dN = dA/(G\hbar)$, so according to the equipartition assumption the energy on B is

$$E = \frac{1}{2}\int_B T dN, \tag{3}$$

after some manipulations, this is equal to $1/(4\pi G)\int_V R_{ab} n^a \xi^b dV$, where V is the volume enclosed by B and n^a is normal to V(different from N^a). If one takes this to be equal to the Tolman-Komar mass $2\int_V (T_{ab} - \frac{1}{2}T g_{ab}) n^a \xi^b dV$, one may see that the Einstein equations appear in an integral form.

Of course the above is not a complete derivation of the Einstein equations. Let us leave aside the issue whether ξ^a and n^a can be arbitrary and independent of each other. To derive the differential equations from the integral equation, V must be arbitrary and arbitrarily small, this can not be the case since to have the temperature T always positive on B, the normal vector N^a can not reverse its direction.

We now turn to our derivation of the Einstein equations. Our proposal is mainly motivated by the work of Brown and York [7] and that of Jacobson [3]. We shall use the semi-local energy of Brown and York rather than the Tolman-Komar mass, this is a good definition with a nice thermodynamics on holographic screens. We shall use an open screen (we will not introduce a temperature in this section, so it can be closed), this is motivated by Jacobson's work which was the source of Verlinde's work after all. In [3], Jacobson used a null open screen, here we will use a time-like open screen. A time-like holographic screen encodes much more information than a null one, as we shall see later.

To begin, let us start with some definitions. Our holographic screen is a 2+1 dimensional time-like hypersurface Σ, which can be open or closed, and we will work on a patch of it, M is the 3+1 dimensional space-time. We use x^a, g_{ab}, ∇_a, y_i, γ_{ij} and D_i (here a,b run from

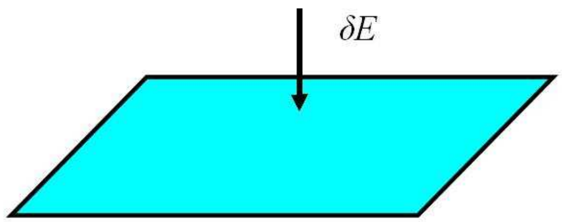

FIG. 1: An energy flux δE passing through an open patch on the holographic screen.

0 to 3, and i,j run from 0 to 2) to denote the coordinates, metric and covariant derivatives on M and Σ, respectively.

Similar to Jacobson's idea [3], we consider an energy flux δE passing through an open patch on the holographic screen $d\Sigma = dAdt$ (see FIG. 1)

$$\delta E = \int_{\Sigma} T_{ab}\xi^a N^b dAdt, \tag{4}$$

where T_{ab} is the stress tensor of matter in the bulk M, ξ^a is the Killing vector, and N^a is the unit vector normal to Σ. To define the normal N^a we may assume Σ be specified by a function $f_\Sigma(x^a)$ with x^a subject to

$$f_\Sigma(x^a) = c. \tag{5}$$

Obviously, a vector normal to Σ is proportional to $g^{ab}\nabla_b f$. Thus, after normalization it follows that

$$N^a = \frac{g^{ab}\nabla_b f_\Sigma}{\sqrt{\nabla_b f_\Sigma \nabla^b f_\Sigma}}. \tag{6}$$

According to the holographic principle, every physical process happening in the bulk corresponds to a process on the holographic screen. The above energy flux Eq.(4) represents change of energy on one side of the holographic screen, it is then natural to assume that it is equal to the change of energy on the patch of screen through which the bulk energy flows.

To calculate the change of energy on the screen, we need to introduce the surface energy density σ on the screen or more generally the surface stress tensor τ_{ij}. The relations between the surface energy density, the surface energy flux j and the surface stress tensor are

$$\sigma = u_i\tau^{ij}\xi_j \ , \ j = -m_i\tau^{ij}\xi_j, \tag{7}$$

where u_i, m_i are the unit vectors normal to the screen's boundary $\partial\Sigma$ (we will give the expressions of u_i and m_i under Eq.(11)), ξ_i is the Killing vector on Σ. In quasi-static spacetime u^i is related to ξ^i by $u^i = e^{-\phi}\xi^i$, and ϕ is the Newton's potential defined by $\phi = \frac{1}{2}\log(-\xi^i\xi_i)$ on the screen. Note that σ and j are the energy density and energy flux on the screen measured by the observer at infinity. Apparently, central to our discussion is the choice of τ_{ij}, we will ultimately use the Brown-York expression, but for now let us be more general. Naturally, τ^{ij} should depend only on geometry of the boundary, thus we assume the following

$$\tau^{ij} = n(K^{ij} - K\gamma^{ij}), \tag{8}$$

where n is a constant to be determined, and K^{ij} is the extrinsic curvature on Σ defined by

$$K_{ij} = -e_i^a e_j^b \nabla_a N_b, \tag{9}$$

where $e_j^b = \frac{\partial x^a}{\partial y^i}$ is the projection operator satisfying $N_a e_i^a = 0$. It should be stressed that Eq.(8) is the key ansatz in our scheme to derive the Einstein equations. We also note that one may replace the coefficient of the term K in Eq.(8) by any other number, this does not affect the derivation of the Einstein equations.

On the screen, the change of energy has two sources, one due to variation of the energy density σ, another is due to energy flowing out the patch to other parts of the screen, given by the energy flux j. The energy variation on the patch is then given by

$$\delta E = \int (u_i\tau^{ij}\xi_j)dA|_t^{t+dt} - \int m_i\tau^{ij}\xi_j dydt = -\int_{\partial\Sigma}(M_i)\tau^{ij}\xi_j\sqrt{h}dz^2 = -\int_{\Sigma} D_i\tau^{ij}\xi_j dAdt, \tag{10}$$

where the first term in the first equality is due to change of the density and the second term is the energy flow through the patch boundary parameterized by y (see Fig. 2). These two terms can be naturally written in a uniform form, since the boundary of the patch consists of two space-like surfaces ($d\Sigma$ at t and $t+dt$), and a time-like boundary. h is the determinant of the reduced metric on $\partial\Sigma$, D_i is the covariant derivative on Σ. M_i is a unit vector in Σ and is normal to $\partial\Sigma$. Let us choose a suitable function $f_{\partial\Sigma}(y^i) = c$ on Σ to denote the boundary $\partial\Sigma$, then we have

$$M^i = \frac{\gamma^{ij}D_j f_{\partial\Sigma}}{\sqrt{D_j f_{\partial\Sigma} D^j f_{\partial\Sigma}}}. \tag{11}$$

Notice that when M^i is along the direction of $dy^0(dt)$ it becomes u^i, and when along the direction of $dy^i(dy^1, dy^2)$ it becomes m^i.

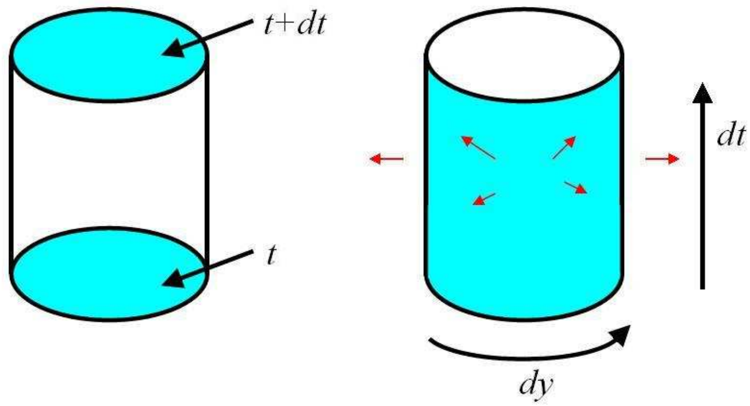

FIG. 2: Left panel: The first term $\int (u_i \tau^{ij} \xi_j) dA|_t^{t+dt}$ in Eq.(10) is due to change of the density. Right panel: The second term $-\int m_i \tau^{ij} \xi_j dy dt$ is the energy flow (denoted by the red arrows) through the patch boundary parameterized by y.

From Eq.(8) and the Gauss-Codazzi equation $(R_{ab} - R g_{ab}/2) N^a e_i^b = -D_j (K^j{}_i - K \gamma^j{}_i)$, one can rewrite Eq.(10) as

$$\delta E = \int_\Sigma n (R_{ab} - \frac{R}{2} g_{ab}) \xi^i e_i^a N^b dA dt. \tag{12}$$

According to the holographic principle if we equate Eq.(4), the energy flow through the patch of the holographic screen Σ, and Eq.(12), the energy change on this patch, we will obtain the Einstein equations.

In the following derivations, for simplicity we will focus only on a quasi-static process. Recalling that in Eq.(5) we use $f_\Sigma(x^a) = c$ to denote the holographic screen Σ. In the quasi-static limit, $f_\Sigma(x^a)$ is independent of time, so $N_a \sim (0, \partial_1 f_\Sigma, \partial_2 f_\Sigma, \partial_3 f_\Sigma)$. Note that $\xi^a \simeq (1, 0, 0, 0)$ in the quasi-static limit, thus we have $N_a \xi^a \to 0$. The Killing vector ξ_i on Σ is induced from the Killing vector ξ_a in the bulk M:

$$\xi_i = \xi_a e_i^a, \quad D_{(i} \xi_{j)} = \nabla_a (\xi_b - N_c \xi^c N_b) e_{(i}^a e_{j)}^b = K_{ij} N_a \xi^a \to 0. \tag{13}$$

Thus, in the quasi-static limit, we can rewrite Eq.(4) as

$$\delta E = \int_\Sigma T_{ab} \xi^i e_i^a N^b dA dt. \tag{14}$$

Equating Eq.(14) and Eq.(12) results in

$$n(R_{ab}-\frac{R}{2}g_{ab})e_i^a N^b = T_{ab}e_i^a N^b. \tag{15}$$

Note that $g_{ab}e_i^a N^b = 0$, thus we can add a cosmological constant term Λg_{ab} to the left of the above equation. Defining the Newton's constant as $G = \frac{1}{8\pi n}$, we then get the Einstein equations

$$R_{ab}-\frac{R}{2}g_{ab}+\Lambda g_{ab} = 8\pi G T_{ab}. \tag{16}$$

Substituting $n = \frac{1}{8\pi G}$ into Eq.(8), we obtain

$$\tau^{ij} = \frac{1}{8\pi G}(K^{ij}-K\gamma^{ij}), \tag{17}$$

this is just the quasi-local stress tensor of Brown-York defined in [7]. As emphasized in [7], τ^{ij} characterizes the entire system, including contributions from both the gravitational field and the matter fields. In addition, generally τ^{ij} need to be subtracted in order to derive the correct energy. We will discuss this issue in the next section.

Note that, if we replace the extrinsic curvature in τ_{ij} in Eq.(8))by the combination of the extrinsic curvature and higher derivative terms, we may derive the equations of motion in a high derivative theory, thus avoiding the no-go theorem of [5].

To find the relationships between the bulk energy and the energy on the holographic screen, let us consider the case of a small closed screen. Using Eq.(4) and Eq.(10), together with the conversation of energy in the bulk

$$\int \sqrt{-g}\nabla_a(T^{ab}\xi_b)dx^4 = -(U_a T^{ab}\xi_b)dV\mid_t^{t+dt} + \int_\Sigma T_{ab}\xi^a N^b dAdt = 0, \tag{18}$$

we obtain

$$(u_i\tau^{ij}\xi_j)dA|_t^{t+dt} = (U_a T^{ab}\xi_b)dV\mid_t^{t+dt}, \tag{19}$$

where we have used the fact that a closed screen has no time-like boundary, thus there is no the second term in Eq.(10). The above relation describes the relationship between the bulk energy and the energy on the holographic screen. Clearly, they are different from each other at most by a time independent constant. This constant is related to the subtraction scheme of the energy on the screen, and generally it depends on the position of the screen. Since the constant is independent of time, the subtraction scheme does not affect our derivation of the Einstein equations. One may view Eq.(19) as a consequence of the Einstein equations.

In all derivations of the Einstein equations through the holographic principle, two data sets are involved, one set is the bulk data, another set is the screen data. In our derivation, the important screen data are the surface stress tensor, while in Jacobson's derivation, they are the entropy and Unruh temperature. In Verlinde's version, they are Verlinde temperature and the equipartition assumption. In the next section, we compare our proposal to Jacobson's as well as Verlinde's.

III. COMPARISON WITH JACOBSON AND VERLINDE

The energy on a holographic screen is completely determined by the surface energy density σ, as in Eq.(7) and Eq.(8). We will show that this is different from Verlinde's simple postulate $\frac{1}{2}TA/G\hbar$, thus the quasi-local energy of Brown-York is different from that of Verlinde. Nevertheless, a part of the quasi-local energy surface energy is similar to that of Verlinde.

If we ignore the subtraction of the energy on the screen, we find in the static space

$$E = \int (u_i \tau^{ij} \xi_j) dA = \int \frac{1}{8\pi G} u_i (K^{ij} - K\gamma^{ij}) \xi_j dA = \int \frac{T}{4G\hbar} dA + \int \frac{1}{8\pi G} K \exp(\phi) dA. \quad (20)$$

In the derivation of the above equation the following formulas have been used

$$\frac{2\pi T}{\hbar} = N^b U^a \nabla_a \xi_b = N^b u^i e_i^a \nabla_a \xi_b = N_a u^i (e_j^a D^j \xi_i + N^a K_{ij} \xi^j) = u^i K_{ij} \xi^j, \quad (21)$$

$$u_i \gamma^{ij} \xi_j = -\exp[\phi], \quad (22)$$

where ϕ is the Newton's potential, and $U^a = e^{-\phi} \xi^a$ [1]. We see that the first term in Eq.(20) is precisely half of Verlinde's energy, but the second term is not.

Let the surface stress tensor of Brown-York assume the perfect fluid form

$$\tau_{ij} = e^{-\phi} (\sigma + p) u_i u_j + e^{-\phi} p \gamma_{ij}. \quad (23)$$

σ and p are the the energy density and pressure on the screen measured by the observer at infinity which are related with the local energy density and pressure on the screen by a red-shift $e^{-\phi}$. One can check that $T \sim \sigma + 2p$ and $K \sim \sigma - 2p$, thus our quasi-local energy agrees with Verlinde's energy only when the surface pressure $p = 0$, this is not generally valid, and we shall see that p will play an important role in holographic thermodynamics.

More precisely $TA/(2G\hbar) = (\sigma + 2p)A$, we rewrite the energy on the screen as

$$E = \int \frac{T}{2G\hbar} dA - 2\int p\, dA. \tag{24}$$

If we are to insist T be the physical temperature on the holographic screen, then the above formula tells us that the first term is the result of the equipartition theorem, if the second term vanishes. A non-vanishing pressure represents a deviation from the equipartition theorem. Thus, only a system of surface dust satisfies the equipartition theorem, or more practically, a system of motionless spin degrees of freedom.

To repeat what we briefly mentioned in the previous section, Verlinde derives the Einstein equations from assumptions of the equipartition rule and the Tollman-Collman Mass

$$M = \frac{TN}{2} = \frac{TA}{2G\hbar} = \frac{1}{4\pi G}\int_V R_{ab} n^a \xi^b dV, \tag{25}$$

$$M = 2\int_V (T_{ab} - \frac{T}{2} g_{ab}) n^a \xi^b dV, \tag{26}$$

where V is a spacelike hypersurface denoting the space volume, n^a is the unit vector normal to V. What should be mentioned is that, the Stokes theorem must be applied in order to get a volume integral from a surface integral $\frac{1}{2G\hbar}\int TdA$ in the first line of Eq.(25). This requires that the screen should be closed. However, as we said already, a negative temperatiure problem will arise since for an arbitrary closed screen Verlinde temperature can not keep positive in general. Imagine an arbitrary small surface B, if T is positive, then it becomes negative when one goes around to the opposite side, the reason is that the sign of N^a will change on the opposite side of a closed screen, leading to the change of the sign of the temperature, which is related to N^a by $T = \frac{\hbar}{2\pi} N^a \nabla_a e^{\phi}$.

While in our proposal, we do not need a temperature. When we wish to keep the virtue of Verlinde's derivation of the entropic force, we can consider an open screen on which T is always positive. We also abandon the unnatural assumption of equipartition rule for a thermodynamics system on the holographic screen we know little about.

Finally, we compare our proposal with that of Jacobson [3]. We were actually motivated by mimicking his discussion for an open time-like screen, rather than a null one. In Jacobson's discussion, he starts with an equation similar to Eq.(4)

$$\delta Q = \int_H T_{ab} \xi^a k^b dA d\lambda, \tag{27}$$

(H denotes the null surface, λ is the affine parameter and $k^a = \frac{dx^a}{d\lambda}$ is the tangent vector to H.) and then assumes that the energy flow is given by the variation of energy on the null surface which in turn is given by $T\delta S$ according to the first law of thermodynamics. Assuming $dS = 1/4\delta A$, he arrives at the Einstein equations with a undetermined cosmological constant. The role of $T\delta S$ is played by the first term in Eq.(10). On null surface we replace N^a and u^i by k^a and l^a respectively, where l^a is a vector on H satisfying $l^a k_a = -1, l^a l_a = 0$. Assume $\tau_{ab} = nK_{ab} = n\nabla_a k_b$, we can derive

$$\begin{aligned}\delta Q &= \int (l_a \tau^{ab}\xi_b dA)|_0^{d\lambda} = n\int (l^a K_{ab}\xi^b dA)|_0^{d\lambda} \\ &= n\int (l^a \nabla_a k_b \xi^b dA)|_0^{d\lambda} = n\int (-l^a k^b \nabla_a \xi_b dA)|_0^{d\lambda} \\ &= \frac{2\pi n}{\hbar}\int (TdA)|_0^{d\lambda} = \frac{(8\pi n)}{\hbar} T\delta(\frac{dA}{4}) = T\delta S. \end{aligned} \tag{28}$$

In the above derivations, we have used the formulas $T = \frac{\hbar}{2\pi} l^a k^b \nabla_b \xi_a$, $k_a \xi^a = 0$ and $n = \frac{1}{8\pi G}$. On the null surface, the second term in Eq.(10) always vanishes, since there is no energy flow along the null surface, this also means $p = 0$ on the null surface. $p = 0$ explains why the first law can be written in a simple form $\delta E = T\delta S$.

To summarize, our proposal differs from Verlinde's proposal in that it works also for an open screen, thus it is more appropriate for derivation of the Einstein equations since a patch of open screen can be arbitrarily small. We do not have to assume the equipartition theorem of energy. More importantly, as we shall see, the knowledge about p is useful for discussing holographic thermodynamics (See Table.I for details). Finally, Jacobson's derivation is similar to ours but there is also no information about p (which vanishes on a null screen), thus his program contains little information about details of thermodynamics (see Table.II for details).

IV. HOLOGRAPHIC THERMODYNAMICS AND THE ENTROPY BOUND

We have seen in the previous two sections that not only we can derive the Einstein equations of gravity, we also have an opportunity to discuss holographic thermodynamics, since we have almost the ingredients, energy, temperature, pressure. We shall see, the final ingredient will be the chemical potential μ.

Let us for simplicity consider the situation in which Verlinde temperature on the holographic screen is the same everywhere. Since the number of degrees of freedom plays no

TABLE I: A comparison between Verlinde's and our proposals

Verlinde's proposal	Our proposal
Closed holographic screen	Open or closed time-like screen
Temperature T	Without or with T
Tolman-Komar mass	Brown-York energy
Equipartition	Surface stress tensor

TABLE II: A comparison with Jacobson's and our proposals

Jacobson's proposal	Our proposal
Open null screen	Open or closed time-like screen
T only	T, p, μ
First law: $dE = TdS$	First law: $dE = TdS - pdA + \mu dN_f$

role in our proposal, we reserve N_f for the number of particles on the screen, (whatever it is, there may be no particles but spin degrees of freedom etc.) if there is such a thing (N will denote the lapse function in the ADM formalism). The first law of thermodynamics on the holographic screen reads

$$dE = TdS - pdA + \mu dN_f, \tag{29}$$

where the symbol dA is not same as in the previous sections, but denotes the change of area of the holographic screen. The above law can be understood in two different but related ways. The first is when the holographic screen evolves with time in a time-dependent background. In such a background, for a given time there is a local Killing vector so Verlinde temperature is well-defined, the above equation is then about evolution of energy, entropy, area and the number of screen particles. The second interpretation is the following. We may imagine that we move the holographic screen in space-time in a way we like, this can be a quasi-static process (we move it very slowly), then the first law is a statement about change of physical quantities on different screens.

The first interpretation of Eq.(29) is useful for a situation in cosmology, we will not discuss this in the present paper. We will employ to second interpretation to discuss the holographic entropy of a gas, as well as black holes.

For a static mass distribution with rotational symmetry, we consider holographic screens be spheres with constant r, thus, verlinde temperature is constant everywhere on a given screen. Due to rotational symmetry, σ, p and μ are also constant on a given screen. The reason for us to consider these special cases is that in our proposal so far, there is suggestion about the form of the chemical potential μ, but it is nevertheless non-vanishing in general.

We start with the general static, spherically symmetric metric

$$ds^2 = -N^2 dt^2 + h^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{30}$$

where N and h are functions of r, and Σ is a hypersurface described by $r =$constant.

In [7], Brown and York suggest a subtraction procedure for the stress tensor on Σ, since without subtraction we will get divergent results. The subtraction does not change our derivation of the Einstein equations, since the subtraction is against a fixed background. For an asymptotically flat space, Brown and York propose to deduct the contributions of energy from the flat background spacet-ime. This yields the same result at infinity as the ADM energy for an asymptotically flat space. We will use this procedure for a black hole. We will also use this procedure inside a gas, since N and h approaches 1 at the origin of the gas.

Thus, we focus on spherically symmetric asymptotically flat space-time in this paper. For more details of the subtraction scheme, please refer to Sec. VI of [7]. In the following, we will use the results of [7] without explaining their derivation.

Following the subtraction scheme used in [7], the surface energy density on screen becomes

$$\sigma = u_i \xi_j \tau^{ij} = \frac{N}{4\pi G}\left(\frac{1}{r} - \frac{1}{rh}\right), \tag{31}$$

while the surface momentum density j_a vanishes (due to spherical symmetry). And the the pressure p is

$$p = \frac{N}{8\pi G}\left(\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r}\right). \tag{32}$$

where σ and p are energy density and pressure measured by the observer at infinity which are related with those defined in [7] by a redshift. One can easily find that σ, p vanish in flat space-time. Thus, the quasi-local energy is

$$E = \int_B d^2x\sqrt{\sigma}\sigma = \frac{N}{G}\left(r - \frac{r}{h}\right). \tag{33}$$

The Verlinde temperature on the screen is

$$T = \frac{\hbar}{2\pi} e^{\phi} N^b \nabla_b \phi = \frac{\hbar}{2\pi}\frac{N'}{h}. \tag{34}$$

We are ready to discuss the first law of thermodynamics on screen

$$dE = TdS - pdA + \mu dN_f, \tag{35}$$

where S is entropy on the sphere etc, the area is $A = 4\pi r^2$, the above law is a statement for adiabatically changing the radius r of the sphere.

To compute the change of energy, we use Eq.(33), we have

$$\begin{aligned} dE &= \frac{1}{G}[N'(r - \frac{r}{h}) + N(1 - \frac{1}{h}) + Nr\frac{h'}{h^2}]dr \\ &= -pdA + \frac{1}{8\pi G}(\frac{Nh'}{h^2} + N')dA, \end{aligned} \tag{36}$$

where we used Eq.(32) for the expression of p. The second term contains both TdS and μdN_f.

To solve these two terms, let us assume that when the sphere sweeps vacuum, $dS = 0$, namely there is no heat flowing into the sphere. Physically, this is natural since one does not expect any change of number of states on the screen, for example a Schwarzschild black hole has a fixed entropy. Note that in the vacuum, we have the relation $Nh = 1$, this is valid certainly for a Schwarzschild black hole. We will assume that $N_f \sim \frac{A}{G\hbar}$, or without loss of generality, take $N_f = \frac{A}{G\hbar}$, and sweep all coefficients into the definition of μ.

Take a Schwarzschild black hole as an example, with the metric $ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, we have

$$\frac{p}{\rho} = \frac{\frac{N'r}{(1-N)} - 1}{2} = \frac{(1 - \frac{GM}{r}) - \sqrt{1 - \frac{2GM}{r}}}{2(\sqrt{1 - \frac{2GM}{r}} - (1 - \frac{2GM}{r}))}. \tag{37}$$

we read μ directly from Eq.(35) Eq.(36)

$$\begin{aligned} \mu &= \frac{\hbar N'}{8\pi h}(h + \frac{Nh'}{N'h}) \\ &= \frac{T}{4}(h + \frac{Nh'}{N'h}) \\ &= \frac{T}{4}[(h-1) + (1 + \frac{Nh'}{N'h}) - C(Nh - 1)] \\ &= \frac{\hbar}{8\pi}(\frac{xN'}{h} - N' - \frac{(1-x)Nh'}{h^2} - \frac{CN'(Nh-1)}{h}), \end{aligned} \tag{38}$$

where in the third line we added a term $C(Nh-1)$, this term vanishes in the region of vacuum, we also used the fact that for a vaccum region $(1+\frac{Nh'}{N'h}=0)$, so we obtain a general expression

$$\mu=\frac{T}{4}(h+\frac{Nh'}{N'h})=\frac{T}{4}[x(h-1)+(1-x)(h+\frac{Nh'}{N'h})-C(Nh-1)], \tag{39}$$

the above expression is certainly valid for a vacuum region, and in particular for region outside a Schwarzschild black hole. Now, let us take a step further, assuming that the above is also valid for a region with any kind of matter, with parameters x and C to be determined. This is certainly a stretch of logic. We think this assumption is rather reasonable for the following reason. We have learned that according to Brown and York, both surface energy density σ and surface pressure p are local functions of the local geometry (here only N and h and their derivatives are involved), it is natural to assume that μ is also a function of local geometry.

In fact, beside the term $C(Nh-1)$, other terms such as $C_n((Nh)^n-1))$ are also possible. However, in our following discussion about a gas with weak gravity, this term amounts to redefining C as $n\ C_n$. From Eqs.(35, 36, 39), one get the entropy

$$dS=\frac{1}{4G\hbar}(x(1+\frac{Nh'}{N'h})+C(Nh-1))dA. \tag{40}$$

In the following, we are mainly interested in the holographic thermodynamics of a gas with weak gravity. We can use this case to determine paramter x. This metric of this case is solved in the appendix , and we have

$$ds^2=-(1+ar^2+br^4)dt^2+(1+cr^2+dr^4)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{41}$$

where a,b,c,d can be found in the appendix, and we solved the metric up to the second order in G, this is needed for determining the holographic entropy of the gas. To this order, we find that two parameters of the gas need to be introduced, namely $w_1=P(0)/\rho(0)$ and $w_2=dP/d\rho(0)$. in order not to violate the dominant energy condition $w_1\le 1$, and there is no constraint on w_2.

Substituting the solution into Eq.(40), we find that in general there is an area term $\sim xr^2/G\hbar$ (Eq.(70) in the Appendix), this is certainly too large for a gas. To make this term absent, we take $x = 0$, thus

$$\begin{aligned}\mu &= \frac{T}{4}[(h+\frac{Nh'}{N'h})+C(Nh-1)] \\ &= \frac{\hbar}{8\pi}(N'+\frac{Nh'}{h^2}+C(NN'-\frac{N'}{h})),\end{aligned} \tag{42}$$

and

$$S = \frac{C\pi^2\rho(1+w_1)r^4}{\hbar}. \tag{43}$$

This is our main result in this section. This holographic entropy is much larger than the usual entropy of a hot gas, even for the case when $w_1 = w_2 = 1/3$, namely radiation. The entropy increases with w_1 and reaches maximum for $w_1 = 1$, the largest value allowed by the dominant energy condition. This holographic entropy is quite similar to the Bekenstein bound $2\pi Mr$. To our approximation, $M = (4\pi/3)\rho r^3$, thus the Bekenstein bound is $(8\pi^2/3)\rho r^4/\hbar$. Now, if we assume that the Bekenstein bound is saturated by our holographic entropy for radiation $w_1 = 1/3$, then we need to take $C = 2$. If the Bekentstein bound is saturated for $w_1 = 1$, then $C = 4/3$. We leave determination of C to a future work.

Interestingly, our result is consistent with Verlinde's entropic force. He assumes

$$Fdr = TdS, \tag{44}$$

where Fdr is the work done by gravitational force on the gas when the gas is squeezed into the holographic screen, or the change of gravitational energy when we move the holographic screen adiabatically outward. Under weak gravity approximation, we have $F = C'\rho\phi 4\pi r^2$ so

$$C'\rho\phi 4\pi r^2 dr = \frac{\hbar}{2\pi}ardS, \tag{45}$$

this leads to $S = \frac{C'\pi^2\rho r^4}{\hbar}$.

In retrospect, our result is quite natural. We study the situation with weak gravity expansion, the expansion parameter is $G\rho r^2$, if the first term in the expansion of S is $O(r^2/G\hbar)$, the second must be $O(\rho r^4/\hbar)$. The interesting aspect of our discussion is that we can make the area term dsiappear, thus the leading term is $O(\rho r^4/\hbar)$, and we predict a w_1 dependent factor $1 + w_1$, increasing when w_1 increases. It is also interesting that the form is very close to the Bekenstein bound, thanks to the factor π^2. The undetermined numerical factor C should be a rational number, as we believe that μ is to be determined by a geometrical method. As we shall see, the trouble with Verlinde's definition of energy is that the area term is always there.

Finally, we make a note that the holographic entropy, though much larger than the usual statistical entropy of a gas, may help to resolve some longstanding puzzles. It has been known for a long time that the usual statistical entropy of a collapsing system violates the second law of thermodynamics, our explanation is that we need to use the holographic entropy rather than the usual statistical entropy. We expect that the holographic entropy smoothly crosses-over to the black hole entropy in a black hole formation process.

V. A TROUBLE WITH VERLINDE ENERGY

In this section, we work with Verlinde's definition of energy to see whether we can have a reasonable thermodynamics of a gas, we shall see that there will be always an area term, this indicates that Verlinde energy is not a good choice.

Verlinde's definition of energy M is given by the equipartition theorem, which in turn upon using the Einstein equations is equal to the Komar mass or ADM mass

$$M = \frac{N_f T}{2} = \frac{AT}{2G\hbar}, \tag{46}$$

where A is the screen area, and N_f is the number of used bits on screen which is supposed to be equal to the area [1], this is the same as our choice of number of "particles" in the previous section. The general form of the first law of thermodynamics on the screen is

$$dM = TdS - pdA + \mu dN_f. \tag{47}$$

For Schwarzschild black hole $dM = 0$, $dS = 0$, so we can easily derive

$$\mu = G\hbar p = \frac{N\hbar}{8\pi}(\frac{h'}{Nh} + \frac{1}{rh} - \frac{1}{r}). \tag{48}$$

We choose as in the previous section

$$\mu = G\hbar p - CT(Nh - 1), \tag{49}$$

where the second term vanishes for a region of vacuum. One may use another function with the energy dimension instead of T, we will discussion this more general choice shortly. For a gas with weak gravity discussed in the previous section and the appendix, using the above chemical potential, one gets

$$dS = \frac{dM}{T} + \frac{(p - \frac{\mu}{G\hbar})dA}{T} = \frac{(CT(Nh-1) + \frac{T}{2})dA}{G\hbar T} + \frac{A}{2G\hbar} d\ln T, \tag{50}$$

where $T = \frac{2G\hbar\rho r(1+3w_1)}{3}$ in leading order, thus $\frac{Ad\ln T}{2G\hbar} \approx \frac{Ad\ln r}{2G\hbar} = \frac{2\pi r^2}{G\hbar}\frac{dr}{r} = d\frac{A}{4G\hbar}$.
The first term in the second equality of the above formula is of higher order in r than the second term, whihle the second term gives $S = \frac{3A}{4G\hbar}$ in the leading order, we certainly do not expect an area law independent of the gas energy density, this term is also way too larger than the Bekenstein bound. We conclude that there is no reasonable holographic thermodynamics with Verlinde's choice of energy, with the choice of the chemical potential in (49).

To disapprove Verlinde's energy, let us consider a more general choice of chemical potential

$$\mu = G\hbar p + Tf(N, h, N^{(n)}, h^{(n)}), \tag{51}$$

where f is a function of N and h and their derivatives. Thus, we derive from the first law that

$$dS = \frac{dM}{T} + f(N, h, N^{(n)}, h^{(n)})\frac{dA}{G\hbar}, \tag{52}$$

the first term will always give rise to an area contribution to S, we want to know the behavior of f for a small r, in this case $N, h \to 1$, $N^{(1)}, h^{(1)} \to 0$, this is the same behavior as the vacuum, For a region of large r outside of a black hole, $dS = f(1,1,0,0)dA/G\hbar$, if f does not contain higher derivatives of N, h than the first order, so $f(1,1,0,0) = 0$, and this implies as for a gas when $r \to 0$, $f \to 0$ thus it is at least proportional to r, so the second term in (52) does not contain an area term, and we can not use it to cancel the area term coming from the frist term in (52). If f contains, for instance, the second derivatives of N and h, these quantities depend on ρ, however, we can not use an ρ dependent area term to cancel an ρ independent area term. This argument is equally applicable if even higher order derivatives are present in f.

We conclude that Verlinde's choice of energy is not suitable for a reasonable holographic thermodynamics.

VI. CONCLUSION

In the present work, we suggest a new program, in which a holographic screen is either open or closed. The surface stress tensor is proportional to the extrinsic curvature, thus the energy we obtain is actually Brown-York semi-local energy. This energy includes gravitational contribution.

By using the new energy definition, we are able to avoid assuming the equipartition theorem of energy. With an open screen we do not have to face the negative temperature problem, thus the derivation of the Einstein equations is more complete. In view of this, our proposal is more appropriate for studying a time-dependent background, in particular cosmology.

We also discuss holographic thermodynamics of a spherically symmetric and static system, in particular a gas of weak gravity, and obtain an interesting result for the holographic entropy. This entropy is similar to the Bekenstein bound. This suggests that there is a tremendous contribution to entropy from gravity. This may resolve the entropy jump puzzle in a black hole formation process, and also may resolve violation of the second law of thermodynamics in a collapsing gravitational system.

There remain many interesting problems. The most important one is to derive a general and geometric formula for the chemical potential, as it is as important as the surface energy density and surface pressure, and contains useful information for holography.

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Appendix

In this appendix we discuss an approximate solution for a gas with weak gravity. We assume the gas is spherically symmetric and static, so the metric is

$$ds^2 = -(1 + ar^2 + br^4)dt^2 + (1 + cr^2 + dr^4)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{53}$$

where a b c d are coefficients to be determined. The expansion in r^2 is controled by $G\rho$. By weak gravity we mean $G\rho r^2 \ll 1$. Since both the energy density and pressure are smooth at the origin $r = 0$, so there are no linear terms of r in the metric.

For expansion up to the order $O(r^4)$, the most general gas is characterized by two parameters, we will use the following two parameters w_1 and w_2

$$w_1 = \frac{P(0)}{\rho(0)}, \quad w_2 = \frac{dP}{d\rho}(0), \tag{54}$$

where ρ and P are the energy density and pressure in spacetime M, respectively. From the Einstein equations

$$G^0_0 = -3c + (5c^2 - 5d)r^2 = -8\pi G\rho(r), \tag{55}$$

$$G^r_r = (2a - c) + (-2a^2 + 4b - 2ac + c^2 - d)r^2 = 8\pi GP(r), \tag{56}$$

$$G^\theta_\theta = (2a - c) + (-3a^2 + 8b - 3ac + 2c^2 - 2d)r^2 = 8\pi GP(r), \tag{57}$$

we obtain

$$a = \frac{4G\pi(1 + 3w_1)\rho}{3}, \qquad c = \frac{8G\pi\rho}{3}, \tag{58}$$

$$b = \frac{(5 - \frac{1}{w_2})}{20}(a^2 + ac) = \frac{4G^2\pi^2\rho^2(1 + w_1)(1 + 3w_1)(5w_2 - 1)}{15w_2}, \tag{59}$$

$$d = c^2 - \frac{1}{5w_2}(a^2 + ac) = -\frac{16G^2\pi^2\rho^2(3(1 + w_1)(1 + 3w_1) - 20w_2)}{45w_2}, \tag{60}$$

where $\rho = \rho(0)$. Note that when $w_2 \to 0$, b and d will reach infinity which implies that there is a lower limit for w_2. We also note by the way that the post-Newtonian formalism is not appropriate for the solutions we are interested (that requires both w parameters to be very small.)

We then have

$$\begin{aligned} N &= \sqrt{1 + ar^2 + br^4} \\ &\approx 1 + \frac{2G\pi\rho(1 + 3w_1)r^2}{3} - \frac{2G^2\pi^2\rho^2r^4(3(1 + w_1)(1 + 3w_1) - 10w_2 - 30w_1w_2)}{45w_2}, \end{aligned} \tag{61}$$

and

$$h = \sqrt{1 + cr^2 + dr^4} \approx 1 + \frac{4G\pi\rho r^2}{3} + \frac{8G^2\pi^2\rho^2 r^4(5w_2 - 3(1 + w_1)(1 + 3w_1))}{15w_2}. \tag{62}$$

We obtain Verlinde temperature T, surface pressure p, quqsi-local energy E on the screen Σ and pressure P in the bulk M as below

$$\begin{aligned} T &= \frac{\hbar N'}{2\pi h} \\ &\approx \frac{2G\hbar\rho(1 + 3w_1)r}{3} - \frac{4G^2\hbar\pi\rho^2 r^3(1 + 3w_1)(1 + w_1)}{15w_2}, \end{aligned} \tag{63}$$

$$p = \frac{N}{8\pi G}(\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r}) \approx \frac{\rho w_1 r}{2} - \frac{G\pi r^3\rho^2(2 + 3w_1)}{9}, \tag{64}$$

$$\begin{aligned} E &= \frac{N}{G}(r - \frac{r}{h}) \\ &\approx \frac{4\pi\rho r^3}{3} - \frac{8\pi^2 r^5 G\rho^2(3 + 12w_1 + 9w_1^2 - 10w_2 - 15w_1w_2)}{45w_2}, \end{aligned} \tag{65}$$

$$P = \rho w_1 - \frac{2\pi G\rho^2}{3}(1 + 3w_1)(1 + w_1)r^2. \tag{66}$$

$\rho(r)$ can be obtained from P by the definition of two w parameters.

The positivity of temperature (63) requires $w_1 > -1/3$, namely the gas generate attractive force only, requiring and bulk pressure (66) be positive, we have $w_1 \geq 0$, but this is not a physics condition. The Dominant Energy Condition ($\rho \geq \mid P \mid$) implies $w_1 \leq 1$, so the physical range of w_1 is $-1/3 < w_1 \leq 1$.

Now, we investigate the first law of thermodynamics on the screen to derive the holographic entropy of the gas:

$$dE = TdS - pdA + \mu dN_f. \tag{67}$$

As discussed in Sec. IV, the general expression of μ is

$$\mu = \frac{T}{4}(h + \frac{Nh'}{N'h}) = \frac{T}{4}[x(h - 1) + (1 - x)(h + \frac{Nh'}{N'h}) - C(Nh - 1)], \tag{68}$$

from which we have

$$dS = \frac{1}{4G\hbar}(x(1+\frac{Nh'}{N'h}) + C(Nh-1))dA. \tag{69}$$

Using Eq.(61) and Eq.(62), we obtain

$$\begin{aligned} S &= x(\frac{3\pi r^2}{G\hbar}\frac{1+w_1}{1+3w_1} + \frac{2\pi^2\rho r^4}{15\hbar}\frac{(3w_1-1)(3(1+w_1)(1+3w_1)-5w_2(5+3w_1))}{(1+3w_1)^2 w_2}) \\ &+ \frac{C\pi^2\rho(1+w_1)r^4}{\hbar}. \end{aligned} \tag{70}$$

The first term $x\frac{3\pi r^2}{G\hbar}\frac{1+w_1}{1+3w_1}$ is too large for gas, so to make this term absent, we take $x=0$, thus

$$S = \frac{C\pi^2\rho(1+w_1)r^4}{\hbar}. \tag{71}$$

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