Hamiltonian Dyson-Schwinger and FRG Flow Equations of Yang-Mills Theory in Coulomb Gauge

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Abstract. A new functional renormalization group equation for Hamiltonian Yang-Mills theory in Coulomb gauge is presented and solved for the static gluon and ghost propagators under the assumption of ghost dominance. The results are compared to those obtained in the variational approach.

Keywords: Yang-Mills, Hamiltonian, Coulomb gauge, functional renormalization group, Dyson-Schwinger equations
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INTRODUCTION

My talk is devoted to the application of functional renormalization group (FRG) flows to the Hamiltonian formulation of Yang-Mills theory in Coulomb gauge developed in our group [1].

The advantage of the Hamiltonian formulation is its close connection to physics. In the variational approach one makes an ansatz for the unknown vacuum wave functional which encodes all the physics [2, 1]. This ansatz can be systematically improved towards the full theory. The price to pay is the apparent loss of renormalization group invariance.

Renormalization group invariance is naturally built-in in the functional renormalization group approach to the Hamilton formulation of Yang-Mills theory put forward in [3]. Such an approach has the advantage of combining renormalization group invariance with the physical Hamiltonian picture.

HAMILTONIAN FLOW

In the FRG approach the quantum theory of a field φ is infrared regulated by adding the regulator term

$$\Delta S_k[\varphi] = \frac{1}{2}\varphi\cdot R_k\cdot\varphi \equiv \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\varphi(p)R_k(p)\varphi(-p) \tag{1}$$

to the classical action. The regulator function $R_k(p)$ is an effective momentum dependent mass with the properties

$$\lim_{p/k\to 0} R_k(p) > 0\ ,\ \lim_{k/p\to 0} R_k(p) = 0\ , \tag{2}$$

which ensures that $R_k(p)$ suppresses propagation of modes with $p \lesssim k$ while those with $p \gtrsim k$ are unaffected and the full theory at hand is recovered as the cut-off scale k is pushed to zero. Wetterich's flow equation for the effective action $\Gamma_k[\phi]$ of a field ϕ is given by

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)}[\phi]+R_k}\,\dot{R}_k, \tag{3}$$

where

$$\Gamma^{(n)}_{k,1\ldots n}[\phi] = \frac{\delta^n\Gamma_k[\phi]}{\delta\phi_1\ldots\delta\phi_n} \tag{4}$$

are the one-particle irreducible n-point functions (proper vertices), for reviews on gauge theories see [4]. The generic structure of the flow equation (3) is independent of the details of the underlying theory, but is a mere consequence of the form of the regulator term (1), i.e., that it is quadratic in the field. By taking functional derivatives of Eq. (3) one obtains the flow equations for the (inverse) propagators and proper vertices. For the two-point function this equation reads

$$\begin{aligned}\partial_t\Gamma^{(2)}_{k,12} = \frac{1}{2}\mathrm{Tr}\dot{R}_k\,\frac{1}{\Gamma_k^{(2)}+R_k}\Bigg\{-\Gamma^{(4)}_{k,12}\\ +\left[\Gamma^{(3)}_{k,1}\,\frac{1}{\Gamma_k^{(2)}+R_k}\,\Gamma^{(3)}_{k,2}+(1\leftrightarrow 2)\right]\Bigg\}\frac{1}{\Gamma_k^{(2)}+R_k},\end{aligned} \tag{5}$$

where all cyclic indices (summed over in the trace) have been suppressed.

In the Hamiltonian approach to Yang-Mills theory in Coulomb gauge the generating functional of static correlation functions reads

$$Z[J] = \int\mathscr{D}A\,\mathrm{Det}(-D\partial)|\psi[A]|^2\exp(J\cdot A)\ , \tag{6}$$

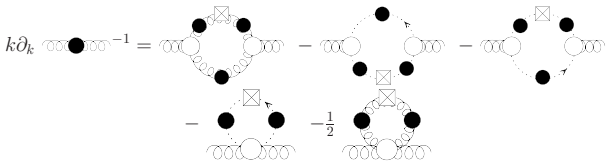

FIGURE 1. Flow equation of the gluon propagator. The spiral and dotted lines with black circles denote the regularized gluon and ghost propagators at cutoff momentum k, respectively. White circles stand for proper vertices at cutoff k, a regulator insertion $\dot{R}_k$ is represented by a square with a cross.

where the integration is over transversal gauge fields A and the Coulomb gauge condition has been implemented by the usual Faddeev-Popov method. Representing the Faddev-Popov determinant in the standard fashion by ghost fields, $c, \bar{c}$,

$$\mathrm{Det}(-D\partial) = \int \mathscr{D}\bar{c}\mathscr{D}c\, e^{-\int \bar{c}(-D\partial)c} \tag{7}$$

the underlying action reads

$$S[A,\bar{c},c] = -\ln|\psi[A]|^2 + \int \bar{c}(-D\partial)c\ . \tag{8}$$

The general flow equation (5) still holds provided that ϕ is interpreted as the superfield $\phi = (A,c,\bar{c})$. The FRG flow equations for the gluon and ghost propagators are diagrammatically given in Figs. 1, 2.

APPROXIMATION SCHEMES AND NUMERICAL SOLUTION

The FRG flow equations embody an infinite tower of coupled equations for the flow of the propagators and the proper vertices. These equations have to be truncated to get a closed system. We shall use the following truncation: we only keep the gluon and ghost propagators, to wit

$$\Gamma^{(2)}_{k,AA} = 2\omega_k(p)\,, \qquad \Gamma^{(2)}_{k,\bar{c}c} = \frac{p^2}{d_k(p)}\ , \tag{9}$$

In addition, we keep the ghost-gluon vertex $\Gamma^{(3)}_{k,A\bar{c}c}$, which we approximate by the bare vertex, i.e., we do not solve its FRG flow equation. The latter approximation is justified by Taylor's non-renormalization theorem extended to Coulomb gauge. The above truncation removes the tadpole diagrams from Figs. 1, 2. Moreover, we shall assume infrared ghost dominance and discard gluon loops. Then the flow equations of the ghost and gluon propagator reduce to the ones shown in Figs. 3, 4.

FIGURE 2. Flow equation of the ghost propagator.

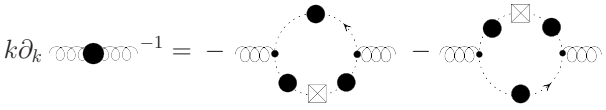

FIGURE 3. Truncated flow equation of the gluon propagator. The bare vertices at $k=\Lambda$ are symbolized by small dots.

These flow equations are solved numerically using the regulators

$$\begin{aligned} &R_{A,k}(p) = 2pr_k(p)\ ,\ R_{c,k}(p) = p^2 r_k(p)\ , \\ &r_k(p) = \exp\left[\frac{k^2}{p^2} - \frac{p^2}{k^2}\right] \end{aligned} \tag{10}$$

and the perturbative initial conditions at the large momentum scale $k=\Lambda$,

$$d_\Lambda(p) = d_\Lambda = const.\,, \quad \omega_\Lambda(p) = p + a\ . \tag{11}$$

With these initial conditions, the flow equations for the ghost and gluon propagators are solved under the constraint of infrared scaling for the ghost form factor. The resulting full flow of the ghost dressing function is shown in Fig. 5. As the IR cut-off momentum k is decreased, the ghost form factor $d_k(p)$ (constant at $k=\Lambda$) builds up infrared strength and the final solution at $k=k_{min}$ is shown in Fig. 7 together with the one for the gluon energy $\omega_{k_{min}}(p)$ in Fig. 6. It is seen that the IR exponents, i.e., the slopes of the curves $d_{k_{min}}(p), \omega_{k_{min}}(p)$ do not change as the minimal cut-off k_{min} is lowered. Let us stress that we have assumed infrared scaling of the ghost form factor but not the horizon condition $d^{-1}_{k=0}(p=0)=0$. The latter was obtained from the integration of the flow equation but not put in by hand (the same is also true for the infrared analysis of the Dyson-Schwinger equations (DSEs) following from the variational Hamiltonian approach, i.e., assuming scaling the DSEs yield the horizon condition).

In Coulomb gauge the inverse ghost form factor $d^{-1}(p)$ has been shown to represent the dielectric function of the Yang-Mills vacuum [5], $\varepsilon(p) = d^{-1}(p)$. Then the so-called horizon condition $d^{-1}(0)=0$ implies that the Yang-Mills vacuum is a perfect dual color superconductor. In the variational approach one can show that the infrared exponents of the ghost and gluon propagators,

$$\omega(p\to 0) \sim 1/p^\alpha\ , \qquad d(p\to 0) \sim 1/p^\beta\ , \tag{12}$$

are related by a sum rule under the assumption of a trivial scaling of the ghost-gluon vertex [6],

$$\alpha = 2\beta - 1\ . \tag{13}$$

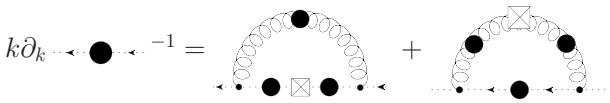

FIGURE 4. Truncated flow equation of the ghost propagator.

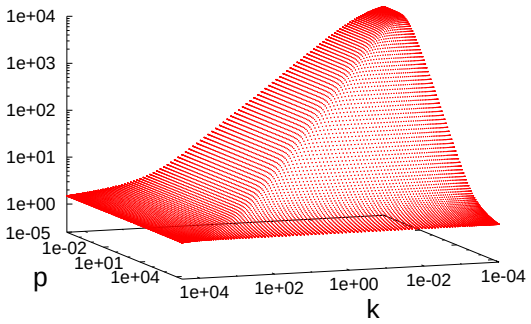

FIGURE 5. Flow $d_k(p)$ of the ghost form factor.

The infrared exponents extracted from the numerical solutions of the flow equations are

$$\alpha = 0.28\,, \quad \beta = 0.64\,. \tag{14}$$

They satisfy the sum rule found in [6] but are smaller than the ones of the DSE. Moreover, the present approach allows to prove the uniqueness of the sum rule (13) [3], analogously to the proof in Landau gauge [7].

Replacing the propagators with running cut-off momentum scale k under the loop integrals of the flow equation by the propagators of the full theory,

$$d_k(p) \to d_{k=0}(p)\ ,\ \omega_k(p) \to \omega_{k=0}(p)\ , \tag{15}$$

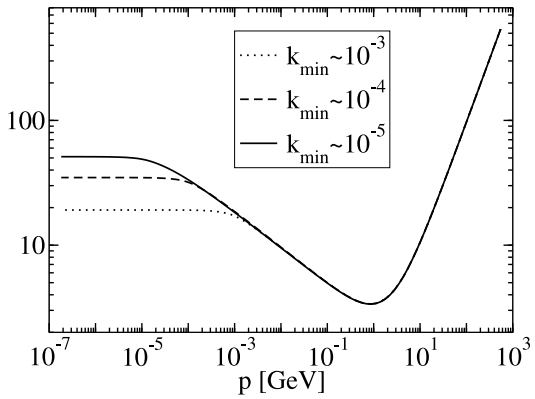

FIGURE 6. Inverse gluon propagator ω at three minimal cutoff values k_{min}.

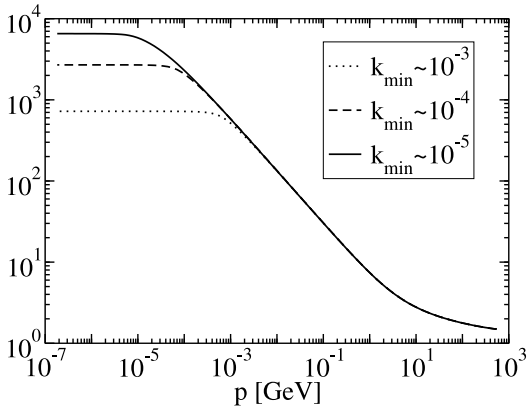

FIGURE 7. Inverse ghost form factor d at three minimal cutoff values k_{min}.

amounts to taking into account the tadpole diagrams [3]. Then the flow equations can be analytically integrated and turn precisely into the DSEs obtained in the variational approach to the Hamiltonian formulation of Yang-Mills theory [1], with explicit UV regularization by subtraction. This establishes the connection between these two approaches and highlights the inclusion of a consistent UV renormalization procedure in the present approach.

The above results encourage further studies, which include the flow of the potential between static color sources as well as dynamic quarks.

REFERENCES

1. C. Feuchter and H. Reinhardt, Phys. Rev. D **70**, 105021 (2004) [arXiv:hep-th/0408236].
   H. Reinhardt and C. Feuchter, Phys. Rev. D **71**, 105002 (2005) [arXiv:hep-th/0408237].
2. A. P. Szczepaniak and E. S. Swanson, Phys. Rev. D **65**, 025012 (2002) [arXiv:hep-ph/0107078].
3. M. Leder, J. M. Pawlowski, H. Reinhardt and A. Weber, arXiv:1006.5710 [hep-th].
4. D. F. Litim and J. M. Pawlowski, hep-th/9901063; J. M. Pawlowski, Annals Phys. **322** (2007) 2831 [arXiv:hep-th/0512261]. H. Gies, hep-ph/0611146.
5. H. Reinhardt, Phys. Rev. Lett. **101** (2008) 061602 [arXiv:0803.0504 [hep-th]].
6. W. Schleifenbaum, M. Leder and H. Reinhardt, Phys. Rev. D **73**, 125019 (2006) [arXiv:hep-th/0605115].
7. C. S. Fischer and J. M. Pawlowski, Phys. Rev. D **80** (2009) 025023 [arXiv:0903.2193 [hep-th]]; Phys. Rev. D **75** (2007) 025012 [arXiv:hep-th/0609009].