Voros product and the Pauli principle at low energies

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Abstract

Using the Voros star product, we investigate the status of the two particle correlation function to study the possible extent to which the previously proposed violation of the Pauli principle may impact at low energies. The results show interesting features which are not present in the computations made using the Moyal star product.

It is likely that at short distances spacetime has to be described by different geometrical structures and that the very concept of point and localizability may no longer be adequate. This is one of the main motivations for the introduction of noncommutative (NC) geometry [1]-[3] in physics. Attempts have been made to formulate quantum mechanics [4]-[11] and quantum field theories [12]-[15] on NC spacetime. However, for the former case, the issue of the lack of Lorentz invariance symmetry has remained a challenge since field theories defined on a NC spacetime with the commutation relation of the coordinate operators

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix, are obviously not Lorentz invariant.

The twist approach to NC quantum field theory has been developed to circumvent this problem [16]-[23]. It is triggered by the realization that it is possible to twist the coproduct of the universal enevelope $U(\mathcal{P})$ of the Poincaré algebra such that it is compatible with the star product.

An interesting consequence that follows from the twisted implementation of the Poincaré group is the violation of Pauli exclusion principle. In [24], it has been shown that the two particle correlation function for a free fermion gas in $2 + 1$ dimensions (with exclusively spatial noncommutativity, i.e. $\theta_{0i} = 0$, $i = 1, 2$) does not vanish in the limit $r \to 0$ which indicates that there is a finite probability that fermions may come very close to each other. The computation

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has been done by using twisted commutation relations among the creation and annihilation operators of the Schrödinger field which in turn has been obtained by taking the non-relativistic limit of the analogous relations for the Klein-Gordon field. A key ingredient in the entire analysis has been the Moyal twist element which reads

$$\mathcal{F}_M^{(\theta)} = e^{-\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} . \tag{2}$$

Recently, a comparison of NC field theories built using two different star products (twist elements): Moyal and Voros have been made [25]. It has been found that although the Green's functions are different for the two theories, the S-matrix is the same in both cases and is different from the commutative case. This motivates us to investigate the status of the two particle correlation function using deformed commutation relations obtained by incorporating the action of the Voros twist element [25]

$$\mathcal{F}_V^{(\theta)} = e^{-\theta\partial_+ \otimes \partial_-} . \tag{3}$$

on the usual pointwise product between two fields.

To begin the analysis, we first write down the mode expansion of a free nonrelativistic quantum field ψ of mass m (in units with $\hbar = 1$) in the NC plane as

$$\psi(\vec{x}, t) = \int d^2k \ a(k)\tilde{e}_k \tag{4}$$

where $\tilde{e}_k = e^{-i\frac{|\vec{k}|^2 t}{2m}}e^{i\vec{k}.\vec{x}}$ and $a(k)$ satisfy the usual (anti)commutation relation

$$a(k)a^\dagger(k') - \eta a^\dagger(k')a(k) = (2\pi)^2\delta^2(\vec{k} - \vec{k}') \tag{5}$$

where η is +1 for bosons and −1 for fermions. The deformation algebra involving $a(k)$ has already been derived in [26] for the relativistic case and can be readily shown to hold in the nonrelativistic case as well :

$$
\begin{aligned}
a(k) \star_M a(k') &= e^{-(1/2)\theta^{ij}k_i k'_j}a(k)a(k') \\
a(k) \star_M a^\dagger(k') &= e^{(1/2)\theta^{ij}k_i k'_j}a(k)a^\dagger(k') \\
a^\dagger(k) \star_M a(k') &= e^{-(1/2)\theta^{ij}k_i k'_j}a^\dagger(k)a(k')
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
a(k) \star_V a(k') &= e^{-\theta k_- k'_+}a(k)a(k') \\
a(k) \star_V a^\dagger(k') &= e^{\theta k_- k'_+}a(k)a^\dagger(k') \\
a^\dagger(k) \star_V a(k') &= e^{-\theta k_- k'_+}a^\dagger(k)a(k') .
\end{aligned}
\tag{7}
$$

We now compute the two particle correlation function for a free gas in 2+1 dimensions using the canonical ensemble, i.e., we are interested in the matrix elements $\frac{1}{Z}\langle r_1, r_2|e^{-\beta H}|r_1, r_2\rangle$, ($\beta = 1/(k_B T)$) where Z is the canonical partition function and H is the non-relativistic Hamiltonian. The physical meaning of this function is quite simple; it tells us what the probability is to find particle two at position r_2, given that particle one is at r_1, i.e., it measures two particle correlations [27]. The relevant two particle state is given by

$$
\begin{aligned}
|r_1, r_2\rangle &= \hat{\psi}^\dagger(r_1) \star_V \hat{\psi}^\dagger(r_2)|0\rangle \\
&= \int \frac{d^2q_1}{(2\pi)^2}\frac{d^2q_2}{(2\pi)^2}e^*_{q_1}(r_1)e^*_{q_2}(r_2)a^\dagger(q_1) \star_V a^\dagger(q_2)|0\rangle .
\end{aligned}
\tag{8}
$$

The two particle correlation function can therefore be written as

$$C(r_1, r_2) = \frac{1}{Z}\langle r_1, r_2|e^{-\beta H}|r_1, r_2\rangle = \frac{1}{Z}\int d^2k_1 d^2k_2 e^{-\frac{\beta}{2m}(k_1^2 + k_2^2)}|\langle r_1, r_2|k_1, k_2\rangle|^2 \tag{9}$$

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where we have introduced a complete set of momentum eigenstates $|k_1, k_2\rangle$ and Z is the partition function of the system.

Using eq(s)(8, 7, 5) and noting that

$$|k_1, k_2\rangle = a^\dagger(k_1) \star_V a^\dagger(k_2)|0\rangle \tag{10}$$

we get

$$|\langle r_1, r_2|k_1, k_2\rangle|^2 = e^{-2\theta(k_{1x}k_{2x}+k_{1y}k_{2y})}\{2 + \eta e^{i(\vec{k_1}-\vec{k_2})\cdot\vec{r}_{12}}e^{i\theta k_1 \wedge k_2} + c.c\} \tag{11}$$

where $c.c$ implies complex conjugate of the second term in the above expression. Substituting this in eq(9), we obtain

$$C(r_1, r_2) = \frac{f(\theta)}{Z}\left(1 + \eta\frac{(1 - \frac{4m^2\theta^2}{\beta^2})}{(1 - \frac{3m^2\theta^2}{\beta^2})}e^{-\frac{m(1+\frac{2m\theta}{\beta})}{\beta(1-\frac{3m^2\theta^2}{\beta^2})}r^2}\right) \tag{12}$$

where $r = r_1 - r_2$. The partition function of the system can now be readily computed and reads

$$Z = \int d^2r_1 d^2r_2 \langle r_1, r_2|e^{-\beta H}|r_1, r_2\rangle$$

$$= f(\theta)A^2\left\{1 + \eta\frac{\pi\beta}{mA}\left(1 - \frac{2m\theta}{\beta}\right)\right\} \approx f(\theta)A^2 \tag{13}$$

in the limit of the area of the system $A \to \infty$. Substituting the above result in eq(12), we finally obtain

$$C(r_1, r_2) = \frac{1}{A^2}\left(1 + \eta\frac{(1 - \frac{16\pi^2\theta^2}{\lambda^4})}{(1 - \frac{12\pi^2\theta^2}{\lambda^4})}e^{-\frac{2\pi(1+\frac{4\pi\theta}{\lambda^2})}{\lambda^2(1-\frac{12\pi^2\theta^2}{\lambda^4})}r^2}\right) \tag{14}$$

where λ is the mean thermal wavelength given by

$$\lambda = \left(\frac{2\pi\beta}{m}\right)^{1/2} \quad ; \quad \beta = \frac{1}{k_B T} . \tag{15}$$

The above result show many interesting features. Note that there exists two thermal wavelengths $\lambda_1 = 2\sqrt{\pi\theta}$ and $\lambda_2 = \sqrt{2\sqrt{3}\pi\theta}$ where the second term in the above expression vanishes. These wavelengths correspond to two temperatures $T_1 = 1/(2k_B m\theta)$ and $T_2 = 1/(\sqrt{3}k_B m\theta)$ at which the correlation function is completely independent of whether the particles are bosons or fermions. For temperatures much less than T_1 the correlations exhibit mild deviations away from that of commutative bosons or fermions. However, for temperatures T lying between T_1 and T_2 (i.e. $T_1 < T < T_2$), the bosons start behaving as fermions and vice versa. These temperatures are of course extremely high. Indeed, assuming θ to be of the order of the Planck length squared and restoring $\hbar$ one finds them to be of the order of 10^{48}K. For temperatures $T > T_2$, the exponential in eq(14) becomes positive and correlation grow exponentially with distance. This signals the onset of instability, which can best be seen if one considers the exchange potential defined by $V(r) = -k_B T \log C(r_1, r_2)$. This becomes unstable or complex above this temperature. This suggests the existence of a high energy cut off $E = k_B T_2$. These features are completely absent when the computations are made using the Moyal star product [24] and are summarized in Figures 1a and b, which shows the exchange potential for bosons and fermions respectively for values of the dimensionless variable $\alpha = \frac{\theta}{\lambda^2}$ below T_1 ($\alpha = 0.0796$) and between T_1 and T_2 ($\alpha = 0.919$). Here r is measured in units of λ. For reference the commutative results ($\alpha = 0$) are also shown. The transgression of statistics in the temperature range between T_1 and T_2 can clearly be seen.
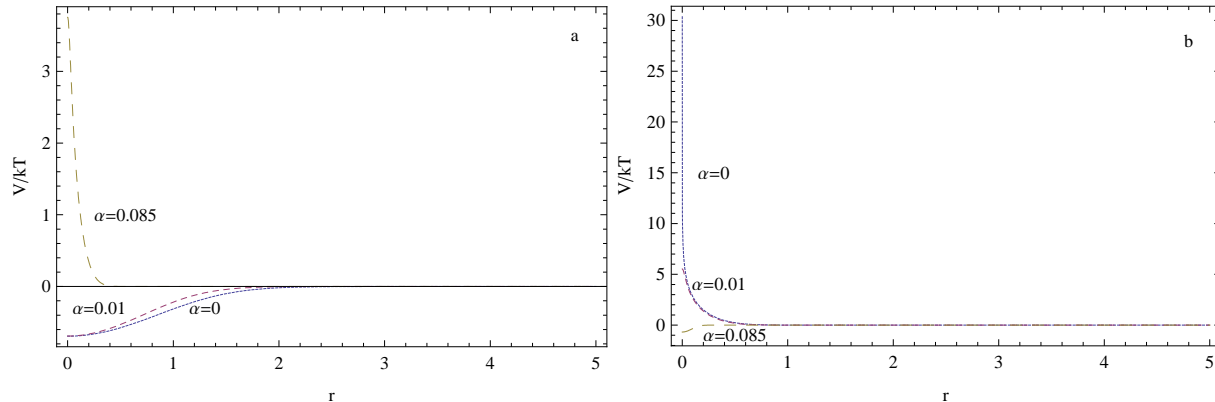
Figure 1: The exchange potential in units of $k_B T$ for bosons (a) and fermions (b). The solid curve represents the commutative result ($\alpha = 0$), the short dashed curve ($\alpha = 0.01$) is the result below T_1 and the long dashed curve ($\alpha = 0.085$) is the result for a temperature between T_1 and T_2. The latter exhibits for (a) fermionic rather than bosonic and (b) bosonic rather than fermionic behaviour.

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