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6D supergravity without tensor multiplets

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Abstract: We systematically investigate the finite set of possible gauge groups and matter content for $\mathcal{N} = 1$ supergravity theories in six dimensions with no tensor multiplets, focusing on nonabelian gauge groups which are a product of SU(N) factors. We identify a number of models which obey all known low-energy consistency conditions, but which have no known string theory realization. Many of these models contain novel matter representations, suggesting possible new string theory constructions. Many of the most exotic matter structures arise in models which precisely saturate the gravitational anomaly bound on the number of hypermultiplets. Such models have a rigid symmetry structure, in the sense that there are no moduli which leave the full gauge group unbroken.

Contents

1. Introduction

Six-dimensional theories of gravity with minimal supersymmetry provide a rich domain in which to study fundamental questions about the space of supersymmetric string theory vacua and consistency constraints on low-energy supergravity theories. Such theories can contain gauge groups and matter in various representations, providing structure analogous to the symmetries and particles seen in our observed four-dimensional region of the universe. At the same time, we have sufficient analytic control over supergravity in six dimensions to begin to systematically address global questions about the space of possible theories.

In a recent series of papers [1, 2, 3, 4] (summarized in [5]), it was shown that when the number of tensor multiplets is less than 9, there are a finite number of possible distinct gauge groups and matter content compatible with known low-energy consistency conditions for 6D $\mathcal{N} = 1$ supergravity. It was furthermore shown that the anomaly cancellation conditions for such theories give rise to an integral lattice which provides a direct connection to topological F-theory data for constructing such theories. In some cases, the resulting topological data cannot be consistent with F-theory geometry in any known fashion, so that a class of theories can be identified which satisfy all known low-energy consistency conditions and yet are not realized in conventional F-theory or any other known string construction.

In this paper we initiate a systematic analysis of the class of theories with $T = 0$ tensor multiplets. In this case the anomaly cancellation conditions are particularly simple, so that a complete solution and classification of possible low-energy models is tractable. We focus here on models with gauge group of the form $G = SU(N_1) \times \cdots \times SU(N_k)$, and consider all possible structures of matter representations. We find that as the rank of the gauge group factors decreases, exotic matter representations arise in the low-energy theory. Some of these exotic matter types are not currently reproducible from F-theory or any other approach to string compactification. We give a systematic classification of the new kinds of matter representations which can arise in these apparently consistent low-energy theories. The new representations may give hints for new codimension 2 singularity structures in F-theory. Or they may be associated with novel pathologies obstructing a UV completion, which may be identifiable from the low-energy theory.

A number of specific 6D theories without tensor multiplets have previously been constructed using particular string compactifications. While perturbative heterotic compactifications on a K3 surface have a single tensor multiplet arising from the reduction of the anti-self-dual part of the 10D B field, by passing through a tensionless string transition the tensor multiplet can be removed from the spectrum [6, 7, 8]. In F-theory this leads to the geometrically simplest class of 6D vacuum construction, based on an elliptic fibration over $\mathbb{P}^2$. Other $T = 0$ models have been identified using type I constructions and Gepner models [10, 11]. The gauge groups and matter representations in these models can also be realized through F-theory constructions of the type considered in this paper. While we focus on gauge groups built from $SU(N)$ factors, the results can be easily extended to all semi-simple Lie groups.

In Section 2, we review the basic structure of low-energy 6D supergravities, specializing to the case without tensor multiplets. We also review the F-theory construction of such models. In Section 3, we summarize the results of our analysis and give explicit examples of apparently-consistent models containing exotic matter representations. Some comments and conclusions are given in Section 4. Appendices contain proofs of several technical results used in the main text.

Note that although in this paper we refer to certain features of low-energy theories in relation to possible F-theory constructions, the classification of apparently consistent low-energy models is independent of any assumptions about the UV completion of the models. More explicit analysis of F-theory realizations of some of the models presented here will appear elsewhere [9].

2. 6D supergravity without tensor multiplets

In this section we briefly review the structure of 6D supergravity theories, specializing to the case without tensor multiplets. Theories with an arbitrary number of tensor multiplets were developed in [12], and anomaly cancellation in such models was analyzed in [13, 14, 4], generalizing the Green-Schwarz anomaly cancellation mechanism for theories with one tensor multiplet described in [15, 16]. We follow here the notation and conventions of [4], to which the reader is referred for further background.

2.1 6D supergravity and anomaly cancellation

Classical $\mathcal{N} = 1$ supergravity in six dimensions contains fields in four distinct representations of the supersymmetry algebra. Each theory contains a single gravity multiplet, whose bosonic components describe the metric tensor $g_{\mu\nu}$ and a self-dual 2-form field $B^+_{\mu\nu}$. The theory can contain any number T of tensor multiplets, each of which contains an anti-self-dual 2-form field. In this paper we specialize to the case $T = 0$. The gravity theory can also be coupled classically to an arbitrary gauge group G described by V vector fields ($V =$ dim G), and H hypermultiplets containing scalars transforming in an arbitrary representation of G. In this paper we restrict attention to theories having gauge groups with nonabelian structure

$$G = G_1 \times \cdots \times G_k = SU(N_1) \times \cdots \times SU(N_k)\,. \tag{2.1}$$

A similar analysis can be done for other nonabelian gauge group structures. Abelian gauge group factors do not significantly modify the story for the nonabelian part of the theory, and will be discussed elsewhere.

Quantum (semi-classical) consistency of supergravity theories in six dimensions requires that anomalies cancel through the Green-Schwarz mechanism [15, 13, 14]. As described in [4], in the case $T = 0$ the anomaly cancellation conditions can be written in terms of a set of integers b_i associated with the simple factors G_i of the gauge group

$$H_{\rm total} - V = H_{\rm neutral} + H - V = 273 \tag{2.2}$$

$$3b_i = \frac{1}{6}\left[\sum_R x^i_R A^i_R - A^i_{\rm adj}\right] \tag{2.3}$$

$$0 = \sum_R x^i_R B^i_R - B^i_{\rm adj} \tag{2.4}$$

$$b_i^2 = \frac{1}{3}\left[\sum_R x^i_R C^i_R - C^i_{\rm adj}\right] \tag{2.5}$$

$$b_i b_j = \sum_{RS} x^{ij}_{RS} A^i_R A^j_S. \tag{2.6}$$

In these anomaly cancellation conditions the quantities x^i_R denote the number of matter fields which transform in the irreducible representation R of gauge group factor G_i. Similarly, x^{ij}_{RS} denotes the number of matter fields transforming under representation $R \times S$ of

$G_i \times G_j$. The constants A_R, B_R, C_R are group theory coefficients defined through

$$\mathrm{tr}_R F^2 = A_R \mathrm{tr} F^2 \tag{2.7}$$

$$\mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R (\mathrm{tr} F^2)^2 \,. \tag{2.8}$$

We denote by H the number of matter hypermultiplets carrying nonabelian charges and H_{neutral} the number of neutral hypermultiplets. Note that because we have specialized to models with simple gauge group factors $SU(N_i)$, the normalization factors λ_i appearing in the anomaly cancellation conditions as presented in [4] are all unity ($\lambda_i = 1$) and do not appear in our equations. Note that conjugate representations R and $\bar{R}$ contribute in the same way to the anomaly conditions. We will not distinguish between representations and their conjugates in our analysis here; the information about whether each matter field is in a particular representation or its conjugate (or a linear combination) represents an additional discrete degree of freedom which parameterizes the full set of possible models.

In addition to local anomalies, quantum consistency requires the absence of global anomalies [20]. For $T = 0$ models with $SU(N)$ gauge group factors, the absence of global anomalies is guaranteed for any model without local anomalies. This result is proven in Appendix A.

Another low-energy condition, which was used in [2, 4] to prove that the number of gauge groups and matter representations associated with consistent theories is finite for $T < 9$, is the constraint that all gauge kinetic terms have the proper sign [13]. In theories with $T = 0$ this is simply the constraint that all b_i have the same sign. As we demonstrate below, with the sign conventions chosen here, there are no models consistent with anomaly cancellation which have $b_i \leq 0$, so the gauge kinetic term sign condition is automatically satisfied.

In a general 6D supergravity theory, the tensor multiplet moduli define the coupling constants, or the strength of the gauge interactions relative to gravity. Theories with $T = 0$ are, therefore, intrinsically gravitational with all interaction strengths set by the Planck scale.

2.2 F-theory realizations of $T = 0$ 6D models

F-theory [17, 18, 8] is a very general approach to constructing string vacua in even dimensions. F-theory is particularly useful in describing models without tensor multiplets. We briefly summarize the basic aspects of F-theory realizations of 6D theories here, specializing to the case $T = 0$. The F-theory picture will not be used in deriving the results in the remainder of the paper, so readers without background in this area can skip this section if they like. The main result we take from this summary is the condition (2.13) which places a bound on the range of possible low-energy models with F-theory realizations. A more detailed discussion of F-theory constructions and the correspondence with low-energy 6D supergravity models is given in [4].

F-theory models in six dimensions are constructed using a Calabi-Yau threefold which admits an elliptic fibration with section. For $T = 0$, the base B of the elliptic fibration is just complex projective space $\mathbb{P}^2$. This is thus the simplest class of F-theory compactifications below 8 dimensions. The gauge group of an F-theory compactification is determined by the codimension one singularities in the fibration. The Kodaira type of each singularity determines an associated nonabelian gauge group factor in a way which is now well understood [18, 8]. When compactifying to 6 space-time dimensions, the elliptic fibration can develop further codimension two singularities. These correspond to matter transforming under various representations of the gauge group. Some types of codimension two singularities have been analyzed and associated to specific types of matter representations [21, 14, 22, 23, 24]. A complete analysis of codimension two singularities is still lacking, however, and the complete range of possible matter representations which can be realized in F-theory is not yet systematically understood.

The discriminant locus of an elliptic fibration is given by a divisor class Δ in the base B. This can be decomposed into a sum of components

$$\Delta = \sum_i N_i \xi_i + Y \tag{2.9}$$

where ξ_i are irreducible effective divisors giving rise to nonabelian gauge factors and Y is a residual effective divisor. For the gauge groups we are considering here, with all simple factors of the form $SU(N_i)$, for example, ξ_i corresponds to an A_{N_i-1} singularity for $N_i > 3$ (there are several possible ways of realizing the groups $SU(2)$ and $SU(3)$). The structure of the group of divisors on $\mathbb{P}^2$ is very simple. All divisors are integer multiples of the hyperplane divisor H, so $\xi_i = \beta_i H$ with $\beta_i > 0$ for irreducible effective divisors. The canonical class of $\mathbb{P}^2$ is $K = -3H$. The Kodaira condition stating that the total space of the elliptic fibration is a Calabi-Yau manifold is

$$-12K = 36H = \Delta = \sum_i N_i \xi_i + Y\,. \tag{2.10}$$

As shown in [3, 4], the correspondence between components of the discriminant locus and the anomaly structure [14, 23] can be used to construct a map from any given low-energy theory to topological data for an F-theory fibration. For $\mathbb{P}^2$, this map is uniquely defined and quite simple. Associated with the coefficient b_i for each simple factor in the gauge group there is a divisor $\xi_i = b_i H$, so the map is

$$b_i \to \beta_i\,. \tag{2.11}$$

From this we see that low-energy theories arising from F-theory must satisfy several constraints. Because ξ_i must be effective, we have

$$b_i > 0\,. \tag{2.12}$$

In [4] a more general version of this condition is described, corresponding to the constraint that the divisors ξ_i must lie in the Mori cone. [Note that in writing the anomaly condition (2.3), from the point of view of the more general formalism of [4], we have chosen a sign for the gravitational anomaly coefficient a (corresponding to the low-energy manifestation of the canonical class $K = -3H$) which leads to the sign of the constraint (2.12)].

The Kodaira constraint gives the upper bound

$$36 \geq \sum_i N_i b_i \,, \tag{2.13}$$

since the residual divisor locus Y must also be effective. As we will see, the positivity condition (2.12) is automatically satisfied for any low-energy theory, but the Kodaira condition (2.13) provides a criterion for showing that some apparently consistent low-energy theories cannot be realized in F-theory as it is currently understood. In [4] some other F-theory constraints on low-energy theories are described, such as the condition that the anomaly coefficients live in a unimodular lattice; these other constraints are automatically satisfied for models with $T = 0$.

3. Possible models and exotic matter representations

In this section we begin with some general observations regarding the structure of $T = 0$ 6D models and then present the results of a systematic analysis of those models with gauge group of the form (2.1).

3.1 Block decomposition of models

In systematically determining what kinds of models are possible, we can use the fact that the anomaly equations depend primarily on the integers b_i associated with each gauge group factor separately. Only the cross-term component (2.6) of the anomaly factorization condition depends upon more than one distinct b_i. By using the other anomaly conditions we can constrain the gauge group factors $SU(N_i)$ and matter transforming under each factor independently. We can then treat these factors and associated matter as "blocks" which can be combined to build models with multiple gauge group factors. This general approach is discussed in [3] and used there to construct $T = 1$ models with gauge groups which are products of $SU(N)$ factors with a restricted class of representations.

For models with $T = 0$ the classification of blocks is particularly simple; the integer b_i associated with each factor $SU(N_i)$ in the gauge group places strong constraints on allowed representations. To see how the possible representation content is structured it is helpful to go into slightly more detail regarding the properties of the group theory coefficients A_R, B_R, C_R. As discussed for example in [2] (see also [25]), these group theory coefficients can be computed for any particular representation using two diagonal generators T_{12}, T_{34} which, in the fundamental representation, take the form

$$(T_{12})_{ab} = \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2} \tag{3.1}$$

$$(T_{34})_{ab} = \delta_{a3}\delta_{b3} - \delta_{a4}\delta_{b4} \tag{3.2}$$

The group theory factors A_R, B_R, C_R can be computed in terms of traces of these generators. For $SU(N), N > 3$, we have

$$A_R = \frac{1}{2}\mathrm{tr}_R T_{12}^2 \tag{3.3}$$

$$B_R + 2C_R = \frac{1}{2}\mathrm{tr}_R T_{12}^4 \tag{3.4}$$

$$C_R = \frac{3}{4}\mathrm{tr}_R T_{12}^2 T_{34}^2 \tag{3.5}$$

In these traces, we sum over all basis states in the representation R, which can be represented in terms of the Young tableaux with various labelings of the associated Young diagram D_R. For $SU(2)$ and $SU(3)$ there is no fourth order Casimir, or generator T_{34}, so we can take $B_R = 0$ and use (3.4) to compute C_R. We will find it useful to work with the linear combination

$$g_R := \frac{1}{12}\left(2C_R + B_R - A_R\right) = \frac{1}{24}\left(\mathrm{tr}_R T_{12}^4 - \mathrm{tr}_R T_{12}^2\right) . \tag{3.6}$$

Since in any given state in the representation $T_{12}^2 \leq T_{12}^4$, we see that

$$g_R \geq 0, \qquad \forall R\,. \tag{3.7}$$

For representations given by Young diagrams with a single column there are no states with $|\langle T_{12}\rangle| > 1$ and therefore $g_R = 0$; all other representations have $g_R > 0$.

For a gauge group factor $SU(N)$ with corresponding anomaly integer b, we can take a linear combination of the anomaly conditions (2.3), (2.4), (2.5) to get

$$\sum_R x_R\, g_R = \frac{1}{2}\left(2g_{\mathrm{adj}} + b^2 - 3b\right) = \frac{(b-1)(b-2)}{2} \tag{3.8}$$

where we have used $g_{\mathrm{adj}} = C_{\mathrm{adj}}/6 = 1$. For models with an F-theory construction, the anomaly integer b is the degree of the curve realizing the corresponding gauge group. The quantity $g := \sum_R x_R g_R = (b-1)(b-2)/2$ is then the (arithmetic) genus of this curve. We thus refer to g_R as the "genus" of representation R, anticipating that for situations with an F-theory realization it will have this geometric interpretation. In F-theory, the number of adjoint hypermultiplets in the low-energy theory is given by the geometric genus g_g of the curve. The genus-degree formula for a general, possibly singular, curve relates the arithmetic and geometric genera

$$g = (b-1)(b-2)/2 = g_g + \sum_P \frac{m_P(m_P-1)}{2}, \tag{3.9}$$

where the sum is over all singular points P of the curve, and m_P is the multiplicity at point P [26]. This relationship provides a clue towards realizing general matter representations in F-theory through codimension-2 singularities. This point will be explored further in [9].

Some examples of group theory coefficients, dimensions, and genera are shown in Table 1.

Rep.	Dimension	A_R	B_R	C_R	g_R
$\yng(1)$	N	1	1	0	0
Adjoint	N^2-1	$2N$	$2N$	6	1
$\yng(1,1)$	$\frac{N(N-1)}{2}$	$N-2$	$N-8$	3	0
$\yng(2)$	$\frac{N(N+1)}{2}$	$N+2$	$N+8$	3	1
$\yng(1,1,1)$	$\frac{N(N-1)(N-2)}{6}$	$\frac{N^2-5N+6}{2}$	$\frac{N^2-17N+54}{2}$	$3N-12$	0
$\yng(2,1)$	$\frac{N(N^2-1)}{3}$	N^2-3	N^2-27	$6N$	$N-2$
$\yng(3)$	$\frac{N(N+1)(N+2)}{6}$	$\frac{N^2+5N+6}{2}$	$\frac{N^2+17N+54}{2}$	$3N+12$	$N+4$
$\yng(1,1,1,1)$	$\frac{N(N-1)(N-2)(N-3)}{24}$	$\frac{(N-2)(N-3)(N-4)}{6}$	$\frac{(N-4)(N^2-23N+96)}{6}$	$\frac{3(N^2-9N+20)}{2}$	0
$\yng(2,2)$	$\frac{N^2(N+1)(N-1)}{12}$	$\frac{N(N-2)(N+2)}{3}$	$\frac{N(N^2-58)}{3}$	$3(N^2+2)$	$\frac{(N-1)(N-2)}{2}$

Table 1: Values of the group-theoretic coefficients A_R, B_R, C_R, dimension and genus for some representations of $SU(N)$.

We can now easily show that there are no $SU(N)$ factors in the gauge group with $b < 0$. For such a factor, from (3.8) we have

$$\sum_R x_R g_R = \frac{1}{2}\left(2 + b^2 - 3b\right) \geq 3\,, \tag{3.10}$$

so some representations with nonzero genus must be included. From (2.3) we have

$$A_{\text{adj}} + 18b = 2N - 18|b| = \sum_R x_R A_R\,. \tag{3.11}$$

Since all A_R are positive, this implies $N > 9$. But for $N > 9$ a matter hypermultiplet in any representation R with $g_R > 0$ satisfies $x_R A_R > 2N - 18$, with the exception of a single hypermultiplet in the 2-index symmetric representation $\yng(2)$. In that case, however, we would have $\sum_R x_R g_R = 1 < 3$, and so there are no gauge group factors with $b < 0$.

3.2 Single factors

We can now address the question of what group factors $SU(N)$ and associated representations can appear in a low-energy supergravity theory satisfying the anomaly constraints. Unlike the situation for theories with $T > 0$ tensor fields, for $T = 0$ theories the fact that there are a finite number of possible "blocks" associated with gauge group factors $SU(N)$ and particular matter representations follows without using the gravitational anomaly bound (2.2). This result is proven in Appendix B. The consequence of the finiteness of the set of blocks, independent of any bound on the number of hypermultiplets in the matter representations, is that we can simply enumerate all possible gauge group blocks. We can then in principle figure out all ways of combining these blocks to form low-energy models consistent with anomaly cancellation.

One conceptually straightforward way to see how to perform an enumeration of individual blocks is as follows. Fix N and b. Then (3.8) gives a bound on the sum of the non-negative values g_R associated with the matter representations transforming under $SU(N)$. This gives a finite partition problem, to which all solutions can be found. Each solution of the partition problem corresponds to a set of values for the x_R associated with representations with nonzero genus g_R. As noted above, the representations with $g_R > 0$ are all associated with Young diagrams having more than one column. We can then fix the x_R for all R with $g_R > 0$ and treat (2.5) as a second partition problem. Since all C_R are positive except for the fundamental representation, this gives a set of possible combinations of coefficients x_R for all representations besides the fundamental. We can then use (2.4) to determine the number of fundamental representations, which must be nonnegative.

As an example of how this analysis works we begin by considering the set of blocks with $b = 1$. From (3.8) we have

$$b = 1: \qquad 2\sum_R x_R\, g_R = (b-1)(b-2) = 0\,. \tag{3.12}$$

Thus, $x_R = 0$ for any representation with $g_R > 0$, and we cannot include any representations other than those with a single column. The anomaly condition (2.5) becomes

$$\sum_R x_R\, C_R = 9\,. \tag{3.13}$$

For $N > 7$, the coefficients C_R satisfy $C_R > 9$ for all one-column representations other than the two-index antisymmetric (A2) and fundamental (F) representations. So in these cases the only solution is $x_{A2} = 3$. The anomaly condition (2.4) then becomes

$$\sum_R x_R\, B_R = x_F + 3(N-8) = B_{\rm adj} = 2N\,, \tag{3.14}$$

so

$$x_F = 24 - N\,. \tag{3.15}$$

Thus, for $b = 1$ there are no possible blocks with $N > 24$, and the only possible blocks with $N > 7$ are $SU(N)$ factors with matter content

$$(24-N)\times\Box + 3\times\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}, \qquad (b=1, N \leq 24, H-V = (2+45N-N^2)/2 \leq 273)\,. \tag{3.16}$$

(Recall that when describing the hypermultiplet matter content of any block or model we denote by H the number of matter hypermultiplets which carry nonabelian charges; as long as this quantity satisfies $H - V \leq 273$, uncharged hypermultiplets can be added to saturate the gravitational anomaly condition (2.2).)

For $N \leq 7$, other $b = 1$ blocks are possible. It is easy to verify that including the 3-antisymmetric (A3) representation at the second step of the above analysis for $SU(7)$ gives a block satisfying the anomaly cancellation conditions with

$$SU(7): \qquad 22\times\Box + 1\times\begin{smallmatrix}\Box\\ \Box\\ \Box\end{smallmatrix}, \qquad (b=1, H-V = 141)\,. \tag{3.17}$$

A similar block can be constructed for $SU(6)$ with 20 fundamental, one A2, and one A3 representation. Since for $SU(5)$ the A3 and A2 representations are conjugate (and therefore treated as equivalent in this analysis), this exhausts the range of possibilities for $b = 1$. Note that all these blocks automatically satisfy the gravitational anomaly bound $H - V \leq 273$.

A similar analysis for $b = 2$ again allows only single-column representations, which now restrict $N \leq 12$ and includes $SU(N)$ blocks of the form

$$(48 - 4N) \times \square + 6 \times \begin{smallmatrix}\square\\\square\end{smallmatrix}, \qquad (b = 2, N \leq 12, H - V = 1 + 45N - 2N^2 \leq 273) \tag{3.18}$$

for all $N \leq 12$. Other $b = 2$ blocks are possible for $6 \leq N \leq 10$: blocks with single 3-antisymmetric (A3) representations are possible at $N = 10, 9$ with $H - V > 273$ and at $N = 8, 7, 6$ with $H - V \leq 273$. For $SU(6)$ there are also blocks with two and three A3 representations, and for $SU(7)$ there is a block with two A3 representations; all these blocks satisfy the gravitational anomaly bound $H - V \leq 273$. There is also a single $b = 2$ block with gauge group $SU(8)$ and a 4-antisymmetric (A4) representation

$$SU(8): \qquad 32 \times \square + 1 \times \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}, \qquad (b = 2, H - V = 263)\,. \tag{3.19}$$

This exhausts the range of possibilities for $b = 2$ blocks.

Continuing to $b = 3$, there is now a nonzero contribution to the genus,

$$b = 3: \qquad 2\sum_R x_R\, g_R = (b - 1)(b - 2) = 2\,. \tag{3.20}$$

There is, therefore, necessarily a matter representation with more than one column, which has $g_R = 1$. The only possibilities are the adjoint and two-index symmetric representations for general N (note that the representation $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ in Table 1 for $SU(3)$ has $g_R = 1$, but is also the adjoint of $SU(3)$). For each choice of representation saturating the genus $g = 1$, there are various possible combinations of n-antisymmetric single-column representations which can solve the partition problem for the C's. The largest N for which a one-block model appears with $b = 3$ which satisfies the gravitational anomaly bound on the number of hypermultiplets is $N = 9$; the matter content of this model is

$$SU(9): \qquad 5 \times \square + 4 \times \begin{smallmatrix}\square\\\square\end{smallmatrix} + 1 \times \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} + 1 \times \text{Adj}, \qquad (b = 3, H - V = 273)\,. \tag{3.21}$$

Note that the blocks listed explicitly above (3.16, 3.17, 3.18, 3.19, 3.21) have $H - V \leq 273$, and therefore, by adding neutral hypermultiplets, can be completed to anomaly-free low-energy supergravity theories with single factor gauge groups $G = SU(N)$. The model in (3.21) precisely saturates the gravitational anomaly bound with $H - V = 273$. This model therefore has no neutral hypermultiplets and is "rigid" in the sense that deformation along any scalar modulus will break the symmetry of the model. As we will see, many of the most exotic matter representations arise in such rigid models.

All the models described above furthermore satisfy the Kodaira bound from F-theory $\sum_i b_i N_i = bN \leq 36$. We might therefore expect that these models have F-theory realizations. While the fundamental and antisymmetric matter representations have standard F-theory realizations, however, the 3-index and 4-index representations are more exotic. These representations were also encountered in $T = 1$ models in [3]. In the case of the

N	max b	(total blocks)	# $SU(N)$ models	# satisfy Kodaira
13-24	1 (1)	(1)	1	1
12	2 (2)	(2)	2	2
11	2 (3)	(4)	2	2
10	2 (4)	(6)	2	2
9	3 (4)	(8)	3	3
8	8 (8)	(22)	15	14
7	4 (7)	(28)	16	16
6	6 (8)	(147)	48	47
5	8 (14)	(186)	23	16
4	16 (34)	(3893)	207	154
3	597 (597)		10100	262
2	$24297 \leq b_{\max} < 36647$		$\sim 5 \times 10^7$	176

Table 2: Summary of possible distinct matter representations for gauge group factors $SU(N)$. Numbers in parentheses refer to possible blocks without constraint on number of hypermultiplets, numbers without parentheses refer to possible anomaly-free models with single nonabelian factor in total gauge group $SU(N)$. Last column gives number of single factor models which satisfy Kodaira constraint $bN \leq 36$ needed for F-theory realization. Number of blocks not individually satisfying gravitational anomaly bound becomes very large at $N = 3$, as does number of blocks for $N = 2$ even with gravitational anomaly constraint. We have not precisely computed the number of blocks in these categories.

3-index representations, a codimension two singularity structure has been identified in F-theory which realizes this matter representation for $N = 6, 7, 8$ [24] through local enhancement of the singularity type to E_6, E_7 and E_8 respectively. We are not aware, however, of any known F-theory realization of the 4-index antisymmetric representation, or of the 3-index antisymmetric representation for $N = 9$.

We have systematically analyzed the set of all possible $SU(N)$ blocks with arbitrary matter representations for $T = 0$ and any N. A summary of the results of this analysis appears in Table 2. We carried out this analysis by finding all of the finite number of solutions for the partition problem for each combination of N and b, within the bounded range of b's for which a solution can be found for each N. For $N \geq 4$ we have explicitly computed all blocks, dividing the set into those which do or do not individually satisfy the gravitational anomaly bound $H - V \leq 273$. For $SU(2)$ and $SU(3)$, the total number of blocks becomes quite large. For $SU(3)$ we have only explicitly computed the number of blocks which individually satisfy the gravitational anomaly bound, and for $SU(2)$ we have only estimated the number of blocks and their range and computed some specific examples, as described below. The detailed analysis of upper bounds on b for each fixed N is given in the Appendix.

We now describe briefly a few interesting aspects of the results summarized in Table 2 and highlight a few specific blocks of interest.

$N > 8$:

For $N > 9$, there are known F-theory realizations of all matter representations appearing in all single-block models. Furthermore, the Kodaira constraint is satisfied for all single blocks with $N \geq 8$. Thus, it seems likely that all the single-block $SU(N)$ models with $N > 9$ which are anomaly-free can be realized in F-theory. The only unusual representation which arises at $N = 9$ is the 3-index antisymmetric representation mentioned above in the model (3.21).

$N = 8$:

At $N = 8$ we find several novel features. As mentioned above, there is an $SU(8)$ model with a 4-index antisymmetric representation. There is also a somewhat exotic model with

$$SU(8): \qquad 1 \times \yng(2,2) \qquad (b = 8, H - V = 273)\,. \tag{3.22}$$

This is the only $SU(8)$ model containing a block with $b > 4$ and is another example of a model with rigid symmetry. There is no known F-theory realization of the "box" matter representation appearing in this model. Furthermore, this model violates the Kodaira condition ($bN = 64 > 32$). Nonetheless, the numerology seems to work out rather nicely for this model, suggesting that there may possibly be some new class of string compactification which could realize this model.

$N \leq 6$

At $N = 6$ and below, the range of possible representations expands significantly, and models which violate the Kodaira condition begin to proliferate. There is one model at $N = 6$ which has another exotic representation

$$SU(6): \qquad 2 \times \yng(1,1) + 2 \times \yng(1,1,1) + 2 \times \yng(2,1,1,1) + 2 \times \text{Adj}, \qquad (b = 6, H - V = 273)\,. \tag{3.23}$$

This is another example of a model with rigid symmetry, although this model is (just barely) within the Kodaira bound.

At $N = 5$ and below an increasing range of exotic representations becomes possible. At the end of this section we summarize the set of representations which can be realized in models satisfying the Kodaira condition for any N. One particularly simple and interesting block with $N = 4$ is

$$SU(4): \qquad 1 \times \yng(2,2) + 64 \times \yng(1), \qquad (b = 4, H - V = 261)\,. \tag{3.24}$$

For models not satisfying the Kodaira bound, an even wider range of representations can be realized; for example, for $N = 4$ there are single block models violating the Kodaira bound which have the representations $\yng(4)$ and $\yng(3,2,1)$. Most of these exotic representations appear in models which precisely saturate or almost saturate the gravitational anomaly bound. For example, one $SU(4)$ model at $b = 16$ has

$$SU(4): \qquad 3 \times \yng(3,2,1) + 3 \times \yng(2,2) + 1 \times \yng(4), \qquad (b = 16, H - V = 272)\,. \tag{3.25}$$

At $N = 3$ the range of possibilities increases still further. The distribution of blocks across values of b is rather non-uniform. There are an enormous range of blocks not satisfying the gravitational anomaly bound and having $b < 500$ which we have not attempted to completely enumerate. Among those blocks individually satisfying the gravitational anomaly bound, most are distributed across values of $b < 70$, with more blocks at values of b divisible by 3. The most blocks satisfying the gravitational anomaly bound occur at $b = 24$ (910 blocks). There are only a few values of $b > 70$ with allowed such blocks, including 44 blocks at $b = 93$, followed by 3 blocks at $b = 105$ and single blocks each at $b = 153, 168, 408$ and 597. The matter content for $b = 597$ is given by

$$SU(3): \qquad 1 \times \square\!\square\!\square\!\square\!\square\!\square\,(\text{S6}) + 1 \times \ (\text{S21}) \qquad (b = 597, H - V = 273)\,. \tag{3.26}$$

Even without imposing the gravitational anomaly bound, there are only blocks possible for three distinct values of $b > 500$. At $b = 521$ there are 79,151 different blocks possible with $H - V \leq 1000$; at $b = 522$ there are 40 such blocks. The only block possible with $b > 522$ is (3.26). It is striking that the largest possible $SU(3)$ block precisely saturates the gravitational anomaly bound.

For $SU(2)$ we have not computed all blocks explicitly, even restricting to blocks satisfying the gravitational anomaly bound, as the number of possibilities is very large. The best upper bound we have found for b for $SU(2)$ is 36,647 (see Appendix B). We have sampled the distribution by computing the number of blocks satisfying the gravitational anomaly bound for multiples of 20, $b = 20k$, up to $b = 1000$, and for multiples of 250 up to $b = 20,000$. The number of blocks at fixed b seems to peak around $b = 420$, where there are 65,459 distinct $SU(2)$ blocks. The number of blocks starts to drop significantly after $b = 1000$, with for example 11,121 blocks at $b = 1000$, 835 blocks at $b = 2000$, and 12 blocks at $b = 4000$. As for $N = 3$, however, there are individual blocks out to much larger values of b. We have found blocks satisfying $H - V \leq 273$ for b up to 24,297, though there are probably sporadic blocks appearing for larger b up to close to the bound of 36,647 (though these must be rare; for example 24,297 is the only value of b between 24,000 and 25,500 which admits a block). Based on the partial data we have computed, we estimate the number of blocks satisfying the gravitational anomaly bound to be on the order of 5×10^7. The total number of blocks without imposing the gravitational anomaly constraint is much larger, but still finite. An example of an $SU(2)$ block with a very large value of b satisfying the gravitational anomaly bound is the following block with $b = 10,750$

$$\begin{aligned} SU(2): \qquad & 1 \times \square\!\square(\text{S2}) + 1 \times (\text{S3}) + 1 \times (\text{S4}) + 1 \times (\text{S5}) + 1 \times (\text{S6}) \\ & +1 \times (\text{S17}) + 1 \times (\text{S55}) + 1 \times (\text{S69}) + 1 \times (\text{S85}), \\ & (b = 10750, H - V = 252)\,. \end{aligned} \tag{3.27}$$

An example of a block with larger b which violates the gravitational anomaly bound is

$$\begin{aligned} SU(2): \qquad & 24530 \times \square + 8380 \times \square\!\square + 1 \times (\text{S12}) + 1 \times (\text{S29}) + 1 \times (\text{S43}) \\ & +1 \times (\text{S113}), \qquad (b = 18000, H - V = 74398)\,. \end{aligned} \tag{3.28}$$

This block, in fact, wildly violates the gravitational anomaly bound, and it can be shown fairly easily that no model satisfying the gravitational anomaly bound can contain this block. For $SU(2)$ there are many such single blocks at large b that satisfy the single block anomaly equations but violate the gravitational anomaly bound. Thus, as the rank decreases the gravitational anomaly bound becomes a more important constraint in restraining the class of allowed models, even though the gravitational anomaly bound alone is sufficient to prove that the number of blocks is finite.

We conclude this description of single $SU(N)$ factor matter blocks in $T = 0$ models with a brief summary of all novel representations which can appear in single block models satisfying the F-theory Kodaira constraint, but for which no F-theory realization is known. There is no argument we are aware of which rules out these representations in F-theory; indeed it seems likely that some of these representations can be realized by new codimension 2 singular structures. A more detailed F-theory analysis of such singularity types will be considered elsewhere. Note that further representations can appear when multiple blocks are considered, so this list is not a complete list of all possible matter types for $T = 0$ models.

Matter representations with standard F-theory constructions are the fundamental ($\ydiagram{1}$), 2-antisymmetric (A2 $= \ydiagram{1,1}$), and adjoint representations [8]. The 2-symmetric (S2 $= \ydiagram{2}$) was identified in terms of a double point singularity in F-theory in [14] and the local singularity structure associated with 3-antisymmetric representations ($\ydiagram{1,1,1}$) have also been identified in F-theory for $SU(6)$ [21, 22, 24], $SU(7)$ [22, 24], and $SU(8)$ [24].

The novel matter representations which can appear in a model satisfying the Kodaira constraint, where the gauge group has a single nonabelian factor $SU(N)$ are as follows

$\ydiagram{1,1,1}$: Appears for $SU(N)$, $N = 9, 8, 7, 6$.

$\ydiagram{1,1,1,1}$: Appears for $SU(8)$ as in the single block model (3.19)

$\ydiagram{2,1}$: Appears for $SU(N)$, $N = 5, 4$ (Adjoint for $SU(3)$).

$\ydiagram{2,1,1}$: Appears for $SU(5)$ (Adjoint for $SU(4)$).

$\ydiagram{2,2}$: Appears for $SU(4)$.

$\ydiagram{3}$: Appears for $SU(N)$, $N = 4, 3, 2$.

$\ydiagram{3,1,1}$: Appears for $SU(4)$.

$\ydiagram{3,1}$: Appears for $SU(3)$.

$\ydiagram{4}$: Appears for $SU(2)$.

$\ydiagram{5}$: Appears for $SU(2)$.

3.3 Two-factor combinations

In principle, given the complete list of all possible single blocks one can construct all multi-block models satisfying the gravitational anomaly bound by simply considering all possible ways in which matter can be multiply charged between blocks in a fashion compatible with (2.6). Since the number of jointly charged hypermultiplets grows quickly as the number of blocks increases, the ways of combining multiple blocks are actually quite constrained.

We have used the complete analysis of single blocks to construct in this fashion all possible two-block models with gauge group $SU(N) \times SU(M)$ for $4 \leq N \leq M$. We present here some examples of the features which can appear in such two-block models.

From the cross-term anomaly constraint (2.6), it follows that any pair of blocks must share matter which transforms under each gauge group factor, satisfying the summation relation

$$b_i b_j = \sum_{RS} x^{ij}_{RS} A^i_R A^j_S \,. \tag{3.29}$$

The simplest type of matter charged under two gauge group factors is bifundamental matter, familiar from various string constructions. In this case $A^i_R = A^j_S = 1$. There is a simple family of two-block models with matter content of the form

$$\begin{aligned} G &= SU(N) \times SU(24-N) \\ b_1 = b_2 &= 1 \\ \text{matter} &= 3(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \cdot) + 3(\cdot \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 1(\Box \times \Box)\,. \end{aligned} \tag{3.30}$$

Another family of models takes the form

$$\begin{aligned} G &= SU(N) \times SU(12-N) \\ b_1 = b_2 &= 2 \\ \text{matter} &= 6(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \cdot) + 6(\cdot \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 4(\Box \times \Box) \end{aligned} \tag{3.31}$$

for $N \leq 12$. The family of models (3.31), including the single block model with $b = 2, N = 12$ were previously constructed by Schellekens using Gepner models [11].

There are a variety of other two-block combinations possible with bifundamental matter and higher values of b's. When we consider larger values of b_i, b_j, more interesting combinations can also arise. There are some models which contain representations of the form $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \Box$. For example, the two-block model with largest $N \leq M$ with such a representation has gauge group and matter content

$$\begin{aligned} G &= SU(5) \times SU(7) \\ (b_1, b_2) &= (4, 2) \\ \text{matter} &= 2(\Box \times \cdot) + 1(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \cdot) + 3(\Box\!\Box \times \cdot) + 2(\cdot \times \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}) + 2(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \Box) + 2(\Box \times \Box) \\ H - V &= 273\,. \end{aligned} \tag{3.32}$$

There is a similar model with gauge group $SU(5) \times SU(6)$, but with $SU(5)$ adjoints instead of symmetric representations.

$$\begin{aligned} G &= SU(5) \times SU(6) \\ (b_1, b_2) &= (4, 2) \\ \text{matter} &= 4(\Box \times \cdot) + 3(\text{Adj} \times \cdot) + 3(\cdot \times \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}) + 2(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \Box) + 2(\Box \times \Box) \\ H - V &= 273\,. \end{aligned} \tag{3.33}$$

These models both saturate the gravitational anomaly, have similar representation content, and satisfy the Kodaira constraint.

As the rank of the gauge group factors drops, more exotic matter multiplets charged under two factors appear. For example, for $SU(4) \times SU(4)$ there are models containing matter which transforms in a non-trivial non-fundamental representation of two gauge groups. One example is given by the model

$$\begin{aligned} G &= SU(4) \times SU(4) \\ (b_1, b_2) &= (2, 2) \\ \text{matter} &= 32(\yng(1) \times \cdot) + 32(\cdot \times \yng(1)) + 1(\yng(1,1) \times \yng(1,1)) \\ H - V &= 262\,. \end{aligned} \tag{3.34}$$

Another interesting class of models are those which contain two blocks $SU(N) \times SU(M)$ for large M and small N. For example we find the following three models

$$\begin{aligned} G &= SU(2) \times SU(24) \\ (b_1, b_2) &= (88, 1) \\ \text{matter} &= 1(\yng(8) \times \cdot) + 1(\yng(12) \times \cdot) + 1(\yng(2) \times \yng(1,1)) \\ H - V &= 272\,. \end{aligned} \tag{3.35}$$

$$\begin{aligned} G &= SU(3) \times SU(24) \\ (b_1, b_2) &= (22, 1) \\ \text{matter} &= 1(\yng(6) \times \cdot) + 1(\yng(1) \times \yng(1,1)) \\ H - V &= 273\,. \end{aligned} \tag{3.36}$$

$$\begin{aligned} G &= SU(2) \times SU(19) \\ (b_1, b_2) &= (27, 1) \\ \text{matter} &= 1(\yng(1) \times \cdot) + 2(\yng(2) \times \cdot) + 1(\yng(3) \times \cdot) + 1(\yng(4) \times \cdot) \\ &\quad +1(\yng(5) \times \cdot) + 1(\yng(6) \times \cdot) \\ &\quad +1(\yng(1) \times \yng(1,1)) + 1(\yng(3) \times \yng(1)) \\ H - V &= 273\,. \end{aligned} \tag{3.37}$$

These are the only multiblock models with a gauge group larger than $SU(18)$ that has non-bifundamental jointly charged matter. These models all severely violate the Kodaira bound. It is perhaps interesting to note that models containing $SU(N)$ factors with $N = 20, 21, 22, 23$ cannot have jointly charged matter other than bifundamental matter as in the family of models (3.30)

3.4 Matter transforming under more than two factors

We have also considered models containing more than two blocks which when taken together satisfy the gravitational anomaly bound, and which contain matter charged under more than two gauge group factors. A limited class of such multiply-charged matter representations are known to appear in F-theory constructions. In particular tri-fundamental representations of $SU(2) \times SU(2) \times SU(N)$ can arise at a point where the singularity structure is enhanced to D_{N+2} [22], and tri-fundamentals of $SU(2) \times SU(3) \times SU(N)$ can be realized from E_{N+3} singularities for $N \leq 5$. In [3] we identified apparently consistent low-energy models with $T = 1$ containing tri-fundamental matter charged under the three gauge group factors $SU(2) \times SU(3) \times SU(6)$. While we have not done a completely systematic search, we have identified a number of the interesting matter structures of this type which can arise in $T = 0$ models. We list here some of the possibilities. While this list is not necessarily comprehensive, it should serve to demonstrate the kinds of multiply-charged matter representations which may be possible.

3-charged matter

As for $T = 1$, at $T = 0$ we find tri-fundamental matter charged under $SU(2) \times SU(3) \times SU(6)$. Such matter appears in the following 3-block model

$$\begin{aligned}
G &= SU(2) \times SU(3) \times SU(6) \\
(b_1, b_2, b_3) &= (3, 2, 1) \\
\text{matter} &= 1(\Box \times \Box \times \Box) + 36(\Box \times \cdot \times \cdot) + 30(\cdot \times \Box \times \cdot) + 12(\cdot \times \cdot \times \Box) \\
&\quad +1(\Box\!\Box \times \cdot \times \cdot) + 3(\cdot \times \cdot \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) \\
H - V &= 272\,.
\end{aligned} \tag{3.38}$$

Matter charged under $SU(2) \times SU(4) \times SU(4)$ appears in the model*

$$\begin{aligned}
G &= SU(2) \times SU(4) \times SU(4) \\
(b_1, b_2, b_3) &= (2, 4, 4) \\
\text{matter} &= 2(\Box \times \Box \times \Box) + 4(\cdot \times \Box \times \Box) + 2(\cdot \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) \\
&\quad +8(\Box \times \cdot \times \cdot) + 3(\cdot \times \text{Adj} \times \cdot) + 3(\cdot \times \cdot \times \text{Adj}) \\
H - V &= 273\,.
\end{aligned} \tag{3.39}$$

There is also matter charged under $SU(3) \times SU(3) \times SU(3)$, appearing in the model

$$\begin{aligned}
G &= SU(3) \times SU(3) \times SU(3) \\
(b_1, b_2, b_3) &= (2, 2, 2) \\
\text{matter} &= 1(\Box \times \Box \times \Box) + 1\,[(\Box \times \Box \times \cdot) + \text{cyclic}] + 27\,[(\Box \times \cdot \times \cdot) + \text{cyclic}] \\
H - V &= 273\,.
\end{aligned} \tag{3.40}$$

Both these models containing tri-fundamental matter satisfy the Kodaira constraint. There is also an interesting combination of 3 blocks of the form (3.24) which contains matter charged under $SU(4) \times SU(4) \times SU(4)$.

*Thanks to D. Morrison for suggesting this possibility.

$$
\begin{aligned}
G &= SU(4) \times SU(4) \times SU(4) \qquad (3.41)\\
(b_1, b_2, b_3) &= (4, 4, 4)\\
\text{matter} &= 4(\square \times \square \times \square) + 1\left[\left(\begin{smallmatrix}\square\!\square\\ \square\!\square\end{smallmatrix} \times \cdot \times \cdot\right) + \text{cyclic}\right]\\
H - V &= 271\,.
\end{aligned}
$$

It is possible to combine four SU(3) blocks to have multiple tri-fundamentals between groups of 3 of the SU(3)'s

$$
\begin{aligned}
G &= SU(3) \times SU(3) \times SU(3) \times SU(3) \qquad (3.42)\\
(b_1, b_2, b_3, b_4) &= (3, 3, 3, 3)\\
\text{matter} &= 1\left[(\square \times \square \times \square \times \cdot) + \text{cyclic}\right] + 3\left[(\square \times \square \times \cdot \times \cdot) + 5 \text{ permutations}\right]\\
&\quad + 1\left[\left(\begin{smallmatrix}\square\!\square\\ \square\end{smallmatrix} \times \cdot \times \cdot \times \cdot\right) + \text{cyclic}\right]\\
H - V &= 270\,.
\end{aligned}
$$

Matter charged under more than three factors

We have found a few exotic models in which matter can be charged under more than 3 gauge group factors.

There is a combination of 4 SU(2) factors carrying a 4-fundamental in a model which satisfies the Kodaira constraint

$$
\begin{aligned}
G &= SU(2)^4 \qquad (3.43)\\
b_i &= 4\\
\text{matter} &= 2(\square \times \square \times \square \times \square) + 8\left[(\square \times \square \times \cdot \times \cdot) + 5 \text{ permutations}\right]\\
&\quad + 3\left[(\square\!\square \times \cdot \times \cdot \times \cdot) + \text{cyclic}\right]\\
H - V &= 248\,.
\end{aligned}
$$

And there is a more exotic combination of 8 SU(2) factors at $b = 8$ where each block has 128 fundamental representations and one S4 (5-dimensional) representation

$$
\begin{aligned}
G &= SU(2)^8 \qquad (3.44)\\
b_i &= 8\\
\text{matter} &= 1(\square \times \square \times \square \times \square \times \square \times \square \times \square \times \square)\\
&\quad + 1\left[(\square\!\square\!\square\!\square \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot) + \text{cyclic}\right]\\
H - V &= 272\,.
\end{aligned}
$$

4. Conclusions and outlook

4.1 Summary of results

We have systematically investigated the possible gauge groups and matter structure for 6D $\mathcal{N} = 1$ supergravity models without tensor multiplets. We have restricted attention to gauge groups which are a product of $SU(N)$ factors, although the methods used here would apply equally well to general semi-simple gauge groups. Anomaly cancellation conditions provide strong constraints which limit the range of possibilities for such models. Potentially interesting results of our analysis include the following:

New matter representations:

We have identified a number of $SU(N)$ matter representations which are not ruled out by low-energy consistency conditions, but whose realization in string theory is not yet known. In F-theory, $T = 0$ models constitute the simplest class of compactifications on an elliptically-fibered Calabi-Yau threefold, with a base manifold $B = \mathbb{P}^2$. Some of the novel matter representations we have found are compatible with topological F-theory constraints, and may be realized by new codimension two singularities in F-theory. Explicitly identifying and classifying such codimension two singularities and associated matter representations in F-theory will be addressed elsewhere. It may also be interesting to look for realizations of these matter representations in other string vacuum constructions.

Violation of Kodaira bound:

All F-theory realizations of $T = 0$ 6D theories satisfy the Kodaira constraint, which follows from the condition that the elliptic fibration in the F-theory model must have a total space which is Calabi-Yau. The Kodaira constraint can be expressed in terms of a condition on the field content and anomaly structure of the low-energy 6D theory. There are a finite number of models, of which we have described several explicitly here, which satisfy the anomaly cancellation condition on the low-energy theory but which seem to violate the Kodaira constraint. It would be very interesting to know whether these represent models which can be built using an as-yet unknown string construction, or whether they suffer from some UV inconsistency which may be identifiable from the structure of the low-energy theory. It is also possible that the map from [4] which we have used to pull back the Kodaira constraint to the low-energy theory may need to be modified in the presence of exotic singularities; this might make it possible to reconcile some of the models found here with F-theory constraints.

Models with rigid symmetry:

Many of the most unusual matter representations we have found live in models which either completely or almost completely saturate the gravitational anomaly bound $H - V \leq 273$. When this bound is saturated, there are no uncharged hypermultiplets, and any deformation of the model will break the symmetry and reduce the matter content. Thus, these models are delicately balanced configurations which exist only at specific points in the moduli space of 6D supergravity theories. Many of the explicit models we have found which go outside the domain of established F-theory constructions turn out to precisely saturate the gravitational anomaly bound and exhibit remarkable numerological/group theory structure, suggesting that some novel stringy mechanism may enable the existence of these theories as quantum-complete theories of supergravity.

Diversity at low rank:

When the rank of the gauge group factors is large, in general we find that the associated models contain matter associated with well-known F-theory singularity types and clearly satisfy the Kodaira constraint. As the rank of the factors decreases, however, more exotic types of matter appear and more models arise which violate the Kodaira constraint. Models containing only $SU(2)$ factors become difficult to classify, and admit a wide range of representations. This observation matches with recent work on $U(1)$ factors in 6D supergravity models, to appear elsewhere, which shows that the range of matter charged under abelian factors is less constrained than for matter charged under nonabelian factors.

It would be interesting to understand whether there is some systematic reason for the increase in diversity of models at low rank.

4.2 Outlook

There are a number of further directions in which one can hope to gain further insight into the theories described here. As mentioned above, further analysis of codimension 2 singularities in F-theory will be helpful in identifying which of the models found here have acceptable F-theory constructions. It has also recently been suggested that a more general class of brane constructions in F-theory may give rise to additional matter content from nonabelian structure in the Higgs field of the 7-brane world-volume theory [19]. It would be interesting to see if such constructions could give rise to some of the exotic matter representations encountered here.

Another point of view which may be helpful in understanding the models described here is to consider connectivity of the space of theories. By Higgsing some of the theories with higher symmetry and more complicated matter content, the models reduce in complexity to models with better understood stringy descriptions. For example, by successively Higgsing pairs of fundamental representations in the $SU(8)$ model (3.19) with the exotic 4-antisymmetric representation, this model reduces to a standard $SU(5)$ model with only fundamental and antisymmetric matter content. Finding a string realization of simpler models and moving backwards up such a Higgsing chain may help us to understand how the more complicated matter structures found here can be realized in string theory. Other continuous transitions should be possible which connect the various branches of the 6D $T = 0$ moduli space, for example by tuning parameters which change the "degree" b_i of the various $SU(N_i)$ factors. Finally, there are points in the moduli space where tensionless strings arise and 29 hypermultiplets are traded for a single tensor multiplet. Such transitions connect the space of models described here to the $T = 1$ models described in [3]. Note, however, that such transitions cannot occur for many of the most exotic models found here since when the gravitational anomaly is almost saturated there are not enough neutral hypermultiplets to effect a transition of this kind. Thus, some of the most exotic phenomena we have identified here are probably unique to models without tensor multiplets.

We hope that the variety of new apparently consistent supergravity models identified here will stimulate some further understanding of new string constructions or will help to generate new constraints on quantum theories of gravity. Lessons of this type for the moduli space of 6D supergravity theories may have implications for our understanding of 4D gravity coupled to gauge groups and matter.

Appendices

A. Global anomalies

In this section we prove that locally non-anomalous blocks with gauge group $SU(2)$ and $SU(3)$ in $T = 0$ theories are free of global anomalies of the kind addressed in [20]. The problem with $SU(2)$ and $SU(3)$ charged chiral fermions is that the fermion measure might obtain a phase under global gauge transformations, which are gauge transformations that are not homotopic to the identity. This happens for only for the gauge groups $SU(2)$ and $SU(3)$ among the SU groups in six dimensions because $\pi_6(SU(2)) = \mathbb{Z}_{12}$, $\pi_6(SU(3)) = \mathbb{Z}_6$ while π_6 is trivial for the other $SU(N)$ gauge groups.

Let us first consider $SU(3)$ in six dimensions. Defining the global gauge transformation that generates $\pi_6(SU(3)) = \mathbb{Z}_6$ as g, we need to determine the phase $2\pi\alpha_r$ a chiral fermion measure in representation r acquires when acted on by g. Note that α_r is defined up to integers.

This problem was essentially solved in [27], but let us phrase it in a language convenient for our purposes. The result of [27], [28] is that if an $SU(4)$ representation R is broken into $\sum_i r_i$ of $SU(3)$ representations r_i in a canonical embedding then

$$\sum \alpha_{r_i} = \frac{B_R}{3!} \tag{A.1}$$

Without loss of generality, let us embed the $SU(3)$ in the upper left corner of $SU(4)$. Recall from section 3,

$$B_R + 2C_R = \frac{1}{2}\mathrm{tr}_R T_{12}^4 \tag{A.2}$$

$$C_R = \frac{3}{4}\mathrm{tr}_R T_{12}^4 T_{34}^4 \tag{A.3}$$

Note that C_R is a multiple of 3 as $\mathrm{tr}_R T_{12}^4 T_{34}^4$ is integral, and is a multiple of 4. This can be seen from looking at which Young tableaux contribute to the trace for a given representation (for an explicit proof, see [4]). If R is broken into $\sum_i r_i$, it is clear that

$$B_R + 2C_R = \frac{1}{2}\mathrm{tr}_R T_{12}^4 = \sum_i \frac{1}{2}\mathrm{tr}_{r_i} T_{12}^4 = 2\sum_i C_{r_i} \tag{A.4}$$

and therefore

$$2\sum_i C_{r_i} \equiv B_R \pmod 6 \equiv \sum_i 6\alpha_{r_i} \pmod 6 \tag{A.5}$$

Since the measure of a chiral fermion in the trivial representation does not acquire a phase under global transformations, and by taking $R = \mathbf{4}$ we find that $\alpha_{\mathbf{3}} = C_{\mathbf{3}}/3$. It is straightforward to carry this through by going through the representations of $SU(4)$, and showing by induction that actually

$$\alpha_r = \frac{C_r}{3} \tag{A.6}$$

For $SU(2)$ the situation is similar. Denoting the generator of $\pi_6(SU(2)) = \mathbb{Z}_{12}$ by g' we need to find the phase $2\pi\alpha_{r'}$ the chiral fermion measure in representation r' of $SU(2)$ acquires when acted on by g'.

Let us embed $SU(2)$ into $SU(3)$. It is known(for example as stated in [27]), that g' maps to g when we do the embedding. Therefore we see that if an $SU(3)$ representation r is broken into $\sum_i r'_i$ of $SU(2)$ representations r'_i in a canonical embedding

$$\sum_i \alpha_{r'_i} = \alpha_r = \frac{C_r}{3} = \sum_i \frac{C_{r'_i}}{3} \tag{A.7}$$

By an almost word-by-word duplication of the argument for $SU(3)$, we obtain

$$\alpha_{r'} = \frac{C_{r'}}{3} \tag{A.8}$$

This is a satisfying result, because we see from the anomaly equations on the C factors

$$\sum_i x_r \alpha_r - \alpha_{\text{adj}} = \sum_i x_r \frac{C_r}{3} - \frac{C_{\text{adj}}}{3} = b^2 \tag{A.9}$$

which is an integer when b is an integer. Note that the far left hand side is the phase (divided by 2π) the fermion measure obtains under the global transformation given by the generator of π_6 of the gauge group. Therefore if the far right hand side is integral our theory would not have any global gauge anomalies. But as proven in [4], for $T = 0$, b is always integral for non-anomalous blocks. So an $SU(2)$ or $SU(3)$ block of a $T = 0$ theory free of perturbative anomalies does not have global anomalies!

B. Proof of bounds on b

Using group theory identities, one can find constraints on the matter content of individual blocks. In this Appendix, we show that even without the $H - V$ constraint we can bound the degree b of an $SU(N)$ block just from group theory constraints. The only equations we use are the anomaly cancellation conditions

$$18b_i + A^i_{\text{adj}} = \sum_R x^i_R A^i_R \tag{B.1}$$

$$B^i_{\text{adj}} = \sum_R x^i_R B^i_R \tag{B.2}$$

$$3b_i^2 + C^i_{\text{adj}} = \sum_R x^i_R C^i_R \tag{B.3}$$

In section B.1 we make some useful statements based on the Weyl character formula. In section B.2 we see how this bounds b for gauge groups larger than $SU(3)$. In section B.3 we discuss the process of bounding b's for the gauge groups $SU(2)$ and $SU(3)$. In section B.4 we summarize the results. A useful reference for this section is [29], chapters 12 and 13.

B.1 The Weyl Character Formula

We will use the Weyl character formula (equation (XIII.37) of [29])

$$\text{tr}_\lambda e^\rho = \frac{\sum_{w\in W} \text{sign}(w) e^{\langle \lambda+\delta, w\rho\rangle}}{\sum_{w\in W} \text{sign}(w) e^{\langle \delta, w\rho\rangle}} \tag{B.4}$$

In this formula tr_λ denotes the trace of the representation with highest weight vector λ. ρ is an element of the Lie algebra $\rho = \rho_a T^a$ where T^a is the Cartan basis, and $(\rho_1, \cdots, \rho_r)$ are coordinates on the weight space. Brackets denote inner products in the weight space. R^+ is the set of positive roots of the Lie algebra, and W is the Weyl group. The vector δ is defined to be half the sum of the positive roots

$$\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \tag{B.5}$$

For $\rho = s\delta$ the equation simplifies to

$$\begin{aligned} &\mathrm{tr}_\lambda e^{s\delta} \\ &= \prod_{\alpha \in R^+} \left(\frac{e^{\frac{s}{2}\langle \alpha, \lambda+\delta \rangle} - e^{-\frac{s}{2}\langle \alpha, \lambda+\delta \rangle}}{e^{\frac{s}{2}\langle \alpha, \delta \rangle} - e^{-\frac{s}{2}\langle \alpha, \delta \rangle}} \right) \\ &= \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda+\delta \rangle}{\langle \alpha, \delta \rangle} \prod_{\alpha \in R^+} \left(\frac{1 + \frac{1}{6}\langle \alpha, \lambda+\delta \rangle^2 (\frac{s}{2})^2 + \frac{1}{120}\langle \alpha, \lambda+\delta \rangle^4 (\frac{s}{2})^4 + \cdots}{1 + \frac{1}{6}\langle \alpha, \delta \rangle^2 (\frac{s}{2})^2 + \frac{1}{120}\langle \alpha, \delta \rangle^4 (\frac{s}{2})^4 + \cdots} \right) \end{aligned} \tag{B.6}$$

This is due to the relation (equation (XIII.13) of [29])

$$\sum_{w \in W} \mathrm{sign}(w) e^{\langle \delta, w\rho \rangle} = \prod_{\alpha \in R^+} (e^{\frac{1}{2}\langle \alpha, \rho \rangle} - e^{-\frac{1}{2}\langle \alpha, \rho \rangle}) \tag{B.7}$$

which holds for an arbitrary vector ρ.

Expanding in s, and looking at the terms of order 0, 2, and 4, we find that

$$\begin{aligned} D_\lambda = \mathrm{tr}_\lambda 1 &= \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda+\delta \rangle}{\langle \alpha, \delta \rangle} \\ A_\lambda (\mathrm{tr}_f \delta^2) = \mathrm{tr}_\lambda \delta^2 &= \frac{D_\lambda}{12} \sum_{\alpha \in R^+} (\langle \alpha, \lambda+\delta \rangle^2 - \langle \alpha, \delta \rangle^2) \\ B_\lambda (\mathrm{tr}_f \delta^4) + C_\lambda (\mathrm{tr}_f \delta^2)^2 = \mathrm{tr}_\lambda \delta^4 &= \frac{D_\lambda}{120} \sum_{\alpha \in R^+} (-\langle \alpha, \lambda+\delta \rangle^4 + \langle \alpha, \delta \rangle^4) \\ &\quad + \frac{D_\lambda}{48} (\sum_{\alpha \in R^+} (\langle \alpha, \lambda+\delta \rangle^2 - \langle \alpha, \delta \rangle^2))^2 \end{aligned} \tag{B.8}$$

Now $\mathrm{tr}_f \delta^2$ and $\mathrm{tr}_f \delta^4$ are explicitly calculable. We consider $SU(N)$ groups with the normalization $\mathrm{tr}_f T^a T^b = 2\delta_{ab}$. We use the fact that for $SU(N)$ the positive roots are given by

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + \alpha_j \tag{B.9}$$

for $i \leq j$ where α_i for $i = 1, 2, \cdots, N-1$ are the simple roots of $SU(N)$ whose Cartan matrix is given by

$$(\alpha_i \cdot \alpha_j) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \tag{B.10}$$

Then,

$$\delta = \frac{1}{2}\sum_{i\leq j}\alpha_{ij} = \sum_{i=1}^{N-1}\frac{i(N-i)}{2}\alpha_i \tag{B.11}$$

By explicit calculation one may show that

$$\langle\alpha_i,\delta\rangle = 1 \tag{B.12}$$

and therefore taking the dual basis of $\{\alpha_i\}$ to be $\{\beta_i\}$,

$$\delta = \sum_i \beta_i \tag{B.13}$$

Now we use the fact that the highest weight vector for the fundamental representation is β_1. Then using (B.8) we find that

$$\begin{aligned}
\mathrm{tr}_f\delta^2 &= \frac{N}{12}\sum_{i\leq j}(\langle\alpha_{ij},\beta_1+\delta\rangle^2 - \langle\alpha_{ij},\delta\rangle^2) \\
&= \frac{N}{12}\sum_{i\leq j}((\delta_{i1}+(j-i+1))^2-(j-i+1)^2) \\
&= \frac{N}{12}\sum_{j=1}^{N-1}((j+1)^2-j^2) = \frac{N(N-1)(N+1)}{12}
\end{aligned} \tag{B.14}$$

and likewise

$$\begin{aligned}
\mathrm{tr}_f\delta^4 &= \frac{N}{120}\sum_{i\leq j}(-\langle\alpha_{ij},\beta_1+\delta\rangle^4+\langle\alpha_{ij},\delta\rangle^4)+\frac{N}{48}\sum_{i\leq j}(\langle\alpha_{ij},\beta_1+\delta\rangle^2-\langle\alpha_{ij},\delta\rangle^2))^2 \\
&= \frac{N}{120}\sum_{i\leq j}(-(\delta_{i1}+(j-i+1))^4+(j-i+1)^4)+\frac{N}{48}(N^2-1)^2 \\
&= \frac{N}{120}\sum_{j=1}^{N-1}(-(j+1)^4+j^4)+\frac{N}{48}(N^2-1)^2 \\
&= -\frac{N}{120}(N^4-1)+\frac{N}{48}(N^2-1)^2 \\
&= \frac{1}{240}N(N-1)(N+1)(3N^2-7)
\end{aligned} \tag{B.15}$$

Furthermore, plugging in the equation for A_λ to the equation for the fourth order invariants and dividing both sides by $(\mathrm{tr}_f\delta^2)^2$ we find that

$$y_N B_\lambda + C_\lambda \leq \frac{3A_\lambda^2}{D_\lambda} \tag{B.16}$$

where we have defined

$$y_N \equiv \frac{\mathrm{tr}_f\delta^4}{(\mathrm{tr}_f\delta^2)^2} = \frac{3(3N^2-7)}{5N(N^2-1)} \tag{B.17}$$

All the results of the current section hold for $SU(2)$ and $SU(3)$ also if we set $B_\lambda = 0$ by hand, which we can certainly do for these groups.

B.2 Restriction on b

For a single $SU(N)$ block with $N \geq 4$ define

$$\frac{\sum_R x_R(y_N B_R + C_R)}{\sum_R x_R A_R} = \frac{3b^2 + 6 + 2Ny_N}{18b + 2N} \equiv \eta \tag{B.18}$$

This means that there must exist a representation R_0 with

$$\frac{y_N B_{R_0} + C_{R_0}}{A_{R_0}} \geq \eta \tag{B.19}$$

since by definition, $y_N B_{R_0} + C_{R_0}$ and A_{R_0} are positive. Then by inequality (B.16),

$$(\frac{D_{R_0}}{3})\eta \leq A_{R_0} \leq \sum_R x_R A_R = 18b + 2N \tag{B.20}$$

Plugging in the definition of η, we find that

$$D_{R_0} \leq \frac{324(b + N/9)^2}{b^2 + (2 + 2y_N N/3)} \tag{B.21}$$

The maximum value for the right hand side of the above equation is obtained for $b = (18/N + 6y_N)$ and plugging in this value of b we obtain the inequality

$$D_{R_0} \leq 324 + \frac{4N^2}{2 + 2y_N N/3} \equiv D_N \tag{B.22}$$

Hence we obtain

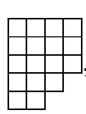
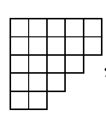
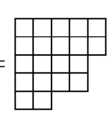
$$\frac{b^2 + 2 + 2Ny_N/3}{18b + 2N} = \frac{\eta}{3} \leq \max_{R:D_R<D_N}(\frac{A_R}{D_R}) \tag{B.23}$$

This procedure gives an upper bound b_u on b, for all $N \geq 4$. For $N = 2, 3$ we are able to obtain slightly improved bounds as we have explicit expressions for A_R, C_R, D_R for these groups. This is helpful since the enumeration of $SU(2)$ and $SU(3)$ blocks takes much more time numerically compared to blocks with $N \geq 4$. We will elaborate on this in section B.3.

Implementing (B.23) to obtain a upper bound of b is a rather tedious, but straightforward task. In practice we must find all representations with $D_R < D_N$ and find the maximum value of A_R/D_R among those representations. This can be done by using the following useful

Fact : Given two representations R_1 and R_2 represented by young-diagrams Y_1, Y_2, if Y_2 can be obtained by attaching columns of blocks to the right of Y_1, then necessarily $D_{R_1} < D_{R_2}$.

This follows simply from the fact that the dimension of a representation is associated with the number of distinct labelings of the boxes which are horizontally non-decreasing and vertically increasing. Adding columns to the right, there is always at least one labeling of Y_2 for each labeling of Y_1 by simply repeating entries on each row in the added columns. For example, if we define

$Y_1 =$ , $\quad Y_2 =$ , $\quad Y_3 =$  (B.24)

N	3	4	5	6	7	8	9	10	11	12	13	14	15	$N \geq 16$
Bound on b	617	126	40	24	14	12	11	6	6	5	3	3	3	2

Table 3: Upper bound on b for individual block of group $SU(N)$.

the dimension of representation Y_1 is smaller than that of Y_2. Meanwhile, it is not guaranteed that the dimension of Y_1 is smaller than the dimension of Y_3.

Starting from single column representations one may span a tree of young diagrams by attaching columns of varying length to the right until one runs into a diagram with $D_R > D_N$. Since the dimension strictly increases at each step, all the branches of the tree will eventually terminate and one will be able to obtain all the representations with bounded dimension.

Although we have thus found an upper bound on b for each group $SU(N)$, which in principle makes the problem of enumerating blocks into a finite algorithm, in practice it is helpful to reduce the bound somewhat to make the enumeration of blocks more tractable. It turns out that we can further restrict b by utilizing the condition

$$A_{R_0} \leq 18b_u + 2N \tag{B.25}$$

That is, it can be the case that

$$\frac{b^2 + 2 + 2Ny_N/3}{18b + 2N} = \frac{\eta}{3} \leq \max_{\substack{R: D_R < D_N \\ \text{and } A_R \leq 18b_u + 2N}} (\frac{A_R}{D_R}) \tag{B.26}$$

further restricts b below the bound coming from (B.23).

For example, in the case of $SU(7)$ one finds that

$$\max_{R: D_R < 386} (\frac{A_R}{D_R}) = \frac{495}{492} = 1.07\cdots \tag{B.27}$$

and hence

$$b \leq 19 \tag{B.28}$$

But one finds that for $b \leq 19$, $18b + 2N \leq 356$, so

$$\max_{\substack{R: D_R < 386 \\ \text{and } A_R \leq 356}} (\frac{A_R}{D_R}) = \frac{165}{210} = 0.78\cdots \tag{B.29}$$

and hence b is further restricted to the value

$$b \leq 14 \tag{B.30}$$

The bound on b obtained this way for $4 \leq N \leq 17$ is given in table 3. For $N \geq 18$,

$$324 + \frac{4N^2}{2 + 2y_N N/3} < \frac{N(N-1)(N-2)}{6} \tag{B.31}$$

This means that the representation with maximum A/D in an $SU(N)$, $N \geq 18$ block is either the adjoint, symmetric, antisymmetric or fundamental. The maximum value of A/D of these is given by the symmetric and hence it must be the case that

$$\frac{b^2 + 2 + 2Ny_N/3}{18b + 2N} \leq \frac{2(N+2)}{N(N+1)} \tag{B.32}$$

A little bit of algebra shows that this implies $b \leq 2$ for $N \geq 18$. This completes the data needed for Table 3. Given the upper bound on b for each N it is therefore a finite problem to enumerate all possible gauge factor + matter "blocks". As noted in section 3, in most cases the actual maximum b for each N is smaller than that given in Table 3. In particular, above $N = 12$ only blocks with $b = 1$ are possible.

B.3 Comments on $SU(2)$ and $SU(3)$ Blocks

The b values of the $SU(2)$ blocks can be restricted in an equivalent fashion. The reason we are addressing them separately is because the bound on b obtained for $SU(2)$ using the equations in the previous section is very high. In particular, the most naive bounds on b for $SU(2)$ is of order 10^5.

We will first provide the most naive bounds on b we can get for $SU(2)$ and $SU(3)$. As mentioned in the previous section we can get slightly better bounds because the equations for A, C, D are simple enough to manipulate directly.

For $SU(2)$ all representations are m-symmetric representations. The dimension and group theory coefficients of the representation are

$$A_m = m(m+1)(m+2)/6 \tag{B.33}$$

$$C_m = A_m(3m^2 + 6m - 4)/10 \tag{B.34}$$

$$D_m = m + 1\,. \tag{B.35}$$

Also recall that $B_m = 0$ for all representations. The anomaly equations (B.1), (B.3) can be written as,

$$\sum_m m(m+1)(m+2)x_m = 108b + 24 \tag{B.36}$$

$$\sum_m m^2(m+2)^2(m+1)x_m = 60b^2 + 36b + 192\,. \tag{B.37}$$

Taking the largest m with $x_m \neq 0$ to be M, we find that,

$$\frac{10b^2 + 6b + 32}{18b + 4} \leq M(M+2) \tag{B.38}$$

and hence,

$$\left(\frac{10b^2 + 6b + 32}{18b + 4}\right)^{3/2} \leq M(M+2)\sqrt{M(M+2)} < M(M+2)(M+1) \leq 108b + 24 \tag{B.39}$$

This gives the bound $b \leq 68018$.

The representations of $SU(3)$ can be represented by a pair of numbers x and y which denote the number of two block/one block columns of its young diagram. Then the dimension and group theory coefficients of representation (x, y) are given by,

$$A_{x,y} = \frac{1}{24}XY(X+Y)(X^2+Y^2+XY-3) \tag{B.40}$$

$$C_{x,y} = \frac{1}{120}XY(X+Y)(X^2+Y^2+XY-3)(X^2+Y^2+XY-\frac{9}{2}) \tag{B.41}$$

$$D_{x,y} = \frac{1}{2}XY(X+Y)\,. \tag{B.42}$$

where we have defined $X = (x+1)$, $Y = (y+1)$. Writing out the anomaly equations and going through a similar process as in the $SU(2)$ case one obtains the bound $b \leq 617$.

Up to now we have been using that fact that in order for a block to satisfy the anomaly equations, we must have some representation R with a large

$$\frac{C_R}{A_R} \geq \frac{3b^2 + C_{\text{adj}}}{18b + A_{\text{adj}}} \sim \mathcal{O}(b) \tag{B.43}$$

We have been ruling out b values for which all such large representations R have $A_R > 18b + A_{\text{adj}}$. Finding all solutions to the anomaly equations for $SU(2)$ and $SU(3)$ for the range of b values constrained only by this condition turns out to be a demanding task numerically, and it is useful to further restrict the allowed values of b. To do this, we generalize the strategy employed up to now.

Suppose b_0 is a value not ruled out by the previous arguments. This means that there exists a representation R satisfying

$$\frac{C_R}{A_R} \geq \frac{3b_0^2 + C_{\text{adj}}}{18b_0 + A_{\text{adj}}} \tag{B.44}$$

and

$$A_R \leq 18b_0 + A_{\text{adj}} \tag{B.45}$$

within the block. Let $S(b_0) = \{R_1, \cdots, R_k\}$ be all the representations that satisfy these two equations. The fact that A, C, D, A/D and C/A are all strictly increasing functions with respect to m in the case of $SU(2)$ and of x and y in the case of $SU(3)$ is helpful in constructing this list.

Now assume that we have the representation R_1 in a block with given b_0. Now if R_1 satisfied,

$$3b_0^2 + C_{\text{adj}} = C_{R_1}, \qquad \text{and} \qquad 18b_0 + A_{\text{adj}} = A_{R_1} \tag{B.46}$$

we would have a solution for a single block whose matter content is given by just one R_1. Suppose this were not the case. † By the same line of argument as before we must have a representation R satisfying,

$$\frac{C_R}{A_R} > \frac{3b_0^2 + C_{\text{adj}} - C_{R_1}}{18b_0 + A_{\text{adj}} - A_{R_1}} \tag{B.47}$$

†In fact, we can show that for $SU(2)$ and $SU(3)$ there cannot be a block with a single matter representation, i.e. a block with $\sum_R x_R = 1$.

and

$$A_R < 18b_0 + A_{\text{adj}} - A_{R_1} \tag{B.48}$$

If no such representation R exists, the representation R_1 cannot show up in a block with $b = b_0$. If the situation were same for all the representations in $S(b_0) = \{R_1, R_2, \cdots, R_k\}$ we could rule out $b = b_0$. We can iterate this process to further rule out b values.

We have employed this procedure for $SU(2)$ and the initial bound $68,018$ has been pulled down to $36,647$. For $SU(3)$ we have iterated this process 5 times and were able to rule out 288 values of b in the range $b \leq 617$.

We can describe the problem of constructing single block models in a very concise way as a partition problem for $SU(2)$ and $SU(3)$ as the coefficients of the anomaly equations are all positive for these cases. While the situation is similar for $SU(3)$ we will for simplicity only depict the process for $SU(2)$. The problem is to find a combination of representations where the A_R, C_R, D_R values add up to $(4 + 18b, 8 + 3b^2, D)$ with $D \leq 276$. This is a straightforward partition problem, whose algorithmic solution is simplified by the fact that A_m/D_m and C_m/A_m are monotonically increasing functions of m. We have implemented an algorithm which computes all such partitions for fixed b and checked a representative sample of b's in the allowed range as described in the main text.

One last note is that the maximum b value possible for single block models turns out to be within an order of magnitude of the upper bounds for small gauge groups. For $SU(3)$ the maximum b possible for an $SU(3)$ is $b = 597$ while the bound is 617, and for $SU(2)$ we were able to find a block with $b = 24,297$ while the bound is given by $36,647$.

B.4 Summary

To summarize, just from the group theory we find that each individual block cannot have a gauge group larger than 24. The b values are bounded as in table 3 $SU(N)$ with $N \geq 3$. The best bound we have for $SU(2)$ is $36,647$. As discussed in the main text, given an upper bound on b, for each fixed $N \leq 24$ we can solve the finite partition problem for each b to enumerate all possible blocks.

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