Nonrelativistic general covariant theory of gravity with a running constant λ

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In this paper, we investigate three important issues: ghost, stability and strong coupling, in the Horava-Melby-Thompson (HMT) setup of the Horava-Lifshitz (HL) theory with $\lambda \neq 1$. We first develop the general linear scalar perturbations of the Friedmann-Robertson-Walker universe with arbitrary spatial curvature, and then apply them to the case of the Minkowski background. We find that it is stable and the spin-0 graviton is eliminated. As a result, the strong coupling problem found in previous versions of the HL theory does not present here. We also study the ghost problem, and find explicitly the ghost-free conditions, which are identical to the ones obtained in the Sotiriou, Visser and Weinfurtner (SVW) setup with projectability condition. The vector and tensor perturbations are the same as presented previously in the SVW setup, in which the vector perturbations vanish identically. This implies that the gravitonal sector in the HMT setup is completely described by the spin-2 massless graviton.

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I. INTRODUCTION

Recently, Horava proposed a quantum gravity theory [1], motivated by the Lifshitz theory in solid state physics [2]. The Horava-Lifshitz (HL) theory is based on the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at high energies [3]. Along this kind of thoughts, Horava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

$$\mathbf{x} \to \ell \mathbf{x}, \quad t \to \ell^z t, \tag{1.1}$$

where $z \geq 3$, in order for the theory to be power-counting renormalizable in $(3+1)$-dimensional spacetimes [4]. At low energies, high-order curvature corrections become negligible, and the theory is expected to flow to $z = 1$, whereby the Lorentz invariance is "accidentally restored." Such an anisotropy between time and space can be realized nicely when one writes the metric in the Arnowitt-Deser-Misner (ADM) form [5],

$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \quad (i,\ j = 1,2,3). \tag{1.2}$$

Under the rescaling (1.1) with $z = 3$, a condition we shall assume in this paper, the lapse function N, the shift vector N^i, and the 3-metric g_{ij} scale as,

$$N \to N,\ N^i \to \ell^{-2}N^i,\ g_{ij} \to g_{ij}. \tag{1.3}$$

The gauge symmetry of the theory is the foliation-preserving diffeomorphism, Diff($M,\ \mathcal{F}$),

$$\tilde{t} = t - f(t), \quad \tilde{x}^i = x^i - \zeta^i(t, \mathbf{x}), \tag{1.4}$$

for which N, N^i and g_{ij} transform as

$$\begin{aligned} \delta N &= \zeta^k \nabla_k N + \dot{N} f + N\dot{f}, \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik}\dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\ \delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij}, \end{aligned} \tag{1.5}$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to the 3-metric g_{ij}, $N_i = g_{ik}N^k$, and $\delta g_{ij} \equiv \tilde{g}_{ij}\left(t, x^k\right) - g_{ij}\left(t, x^k\right)$, etc. From these expressions one can see that N and N_i play the role of gauge fields of the Diff($M,\ \mathcal{F}$) symmetry. Therefore, it is natural to assume that N and N_i inherit the same dependence on spacetime as the corresponding generators, in addition to the fact that the dynamical variables g_{ij} should in general depend on both time and space, that is,

$$N = N(t), \quad N_i = N_i(t,x), \quad g_{ij} = g_{ij}(t,x), \tag{1.6}$$

which is clearly preserved by Diff($M,\ \mathcal{F}$), and often referred to as the projectability condition.

Due to the restricted diffeomorphisms (1.4), one more degree of freedom appears in the gravitational sector - a spin-0 graviton. This is potentially dangerous, and needs to decouple in the IR regime, in order to be consistent with observations. Whether this is possible or not is still an open question [6]. In particular, it was shown that the spin-0 mode is not stable in the original version of the HL theory [1] as well as in the Sotiriou, Visser and Weinfurtner (SVW) generalization [7, 8], although these instabilities were all found in the Minkowski background. In the de Sitter spacetime, it was shown that it is stable [9]. So, one may take the latter as its legitimate background, similar to what happened in the massive gravity [10]. However, the strong coupling problem still exists [11, 12], although it might be circumvented by the Vainshtein mechanism [13], as recently showed in the spherical [6] and cosmological [11] spacetimes.

To cure the above problems, various versions of the theory were proposed [14, 15]. In particular, Horava and Melby-Thompson (HMT) showed that one can eliminate

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the spin-0 graviton by introducing two auxiliary fields, the $U(1)$ gauge field A and the Newtonian pre-potentail φ, by extending the Diff(M, $\mathcal{F}$) symmetry to include a local $U(1)$ symmetry [16],

$$U(1) \ltimes \text{Diff}(M,\ \mathcal{F}). \tag{1.7}$$

Under this extended symmetry, the special status of time still maintains, so that the anisotropic scaling (1.1) with $z > 1$ can be realized, whereby the UV behavior of the theory will be considerably improved. Under the local $U(1)$ symmetry, the relevant quantities transform as

$$\begin{aligned}\delta_\alpha A &= \dot{\alpha} - N^i\nabla_i\alpha, \quad \delta_\alpha\varphi = -\alpha,\\ \delta_\alpha N_i &= N\nabla_i\alpha, \quad \delta_\alpha g_{ij} = 0, \quad \delta N = 0,\end{aligned} \tag{1.8}$$

where α is the generator of the local $U(1)$ gauge symmetry. Under the Diff(M, $\mathcal{F}$), A and φ transform as,

$$\begin{aligned}\delta A &= \zeta^i\partial_i A + \dot{f}A + f\dot{A},\\ \delta\varphi &= f\dot{\varphi} + \zeta^i\partial_i\varphi.\end{aligned} \tag{1.9}$$

For the detail, we refer readers to [16, 17].

As shown explicitly in [18], the $U(1)$ symmetry pertains specifically to the case $\lambda = 1$, where λ is a coupling constant that characterizes the deviation of the kinetic part of action from the corresponding one given in general relativity. It is exactly because of this deviation that causes all the problems, including ghost, instability and strong coupling. Therefore, it was considered as a remarkable feature of this nonrelativistic general covariant theory, in which λ is forced to be one. However, this claim was challenged recently [19], and argued that the introduction of the Newtonian pre-potential is so powerful that action with $\lambda \neq 1$ also has the $U(1)$ symmetry. It should be noted that even $\lambda = 1$ in the tree level, it is still subjected to quantum corrections. This is in contrast to the relativistic case, where $\lambda = 1$ is protected by the Lorentz symmetry, Diff(M),

$$\tilde{x}^\mu = x^\mu - \zeta^\mu(t, \mathbf{x}),\ (\mu = 0, 1, 2, 3) \tag{1.10}$$

even in the quantum level.

In this paper we investigate the HMT setup with any given coupling constant λ. In [17] we studied the HMT setup with $\lambda = 1$. It is found not difficult to generalize those studies to the general case of $\lambda \neq 1$. Specifically, in Sec. II we briefly review the theory by presenting all the field equations and conservation laws when matter is present for $\lambda \neq 1$. In Sec. III we study the Friedmann-Robertson-Walker (FRW) universe with any given spatial curvature in such a setup, and derive the generalized Friedmann equation and conservation law of energy, while in Sec. IV we develop the general formulas for the linear scalar perturbations of the FRW universe. Applying these formulas to the Minkowski background, in Sec.V we study the stability problem, and show explicitly that it is stable and the spin-0 graviton is eliminated even for $\lambda \neq 1$. This conclusion is the same as that obtained by da Silva for the maximal symmetric spacetimes with detailed balance condition, in which the Minkowski spacetime is not a solution of the theory [19]. In Sec. VI, we study the ghost and strong coupling problems, and derive the ghost-free conditions in terms of λ, which are the same as those obtained in the SVW setup [8]. The strong coupling problem is solved automatically since now the spin-0 graviton vanishes identically. Finally, in Sec. VII, we present our main conclusions and discussing remarks.

II. NON-RELATIVISITC GENERAL COVARIANT HL THEORY WITH ANY λ

When $\lambda \neq 1$, the total action can be written as [19],

$$\begin{aligned}S &= \zeta^2\int dtd^3xN\sqrt{g}\Big(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_\lambda\\ &\qquad + \frac{1}{\zeta^2}\mathcal{L}_M\Big),\end{aligned} \tag{2.1}$$

where $g = \det g_{ij}$, and

$$\begin{aligned}\mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2,\\ \mathcal{L}_\varphi &= \varphi\mathcal{G}^{ij}\Big(2K_{ij} + \nabla_i\nabla_j\varphi\Big),\\ \mathcal{L}_A &= \frac{A}{N}\Big(2\Lambda_g - R\Big),\\ \mathcal{L}_\lambda &= \big(1-\lambda\big)\Big[\big(\nabla^2\varphi\big)^2 + 2K\nabla^2\varphi\Big].\end{aligned} \tag{2.2}$$

Here Λ_g is a coupling constant, and the Ricci and Riemann terms all refer to the three-metric g_{ij}, and

$$\begin{aligned}K_{ij} &= \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right),\\ \mathcal{G}_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R + \Lambda_g g_{ij}.\end{aligned} \tag{2.3}$$

$\mathcal{L}_M$ is the matter Lagrangian density, which in general is a function of all the dynamical variables, $U(1)$ gauge field, and the Newtonian prepotential, $\mathcal{L}_M = \mathcal{L}_M\big(N,\ N_i,\ g_{ij},\ \varphi,\ A;\ \chi\big)$, where χ denotes collectively the matter fields. $\mathcal{L}_V$ is an arbitrary Diff(Σ)-invariant local scalar functional built out of the spatial metric, its Riemann tensor and spatial covariant derivatives, without the use of time derivatives.

Note the difference between the notations used here and the ones used in [16, 19]. In this paper, we shall use directly the notations and conventions defined in [8] and [17], which will be referred, respectively, to as Paper I and Paper II, without further explanations. However, in order to have the present paper as independent as possible, it is difficult to avoid repeating the same materials, although we shall try to limit it to its minimum.

In [7], by assuming that the highest order derivatives are six and that the theory respects the parity, SVW constructed the most general form of $\mathcal{L}_V$,

$$\mathcal{L}_V = \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2}\left(g_2R^2 + g_3R_{ij}R^{ij}\right)$$

$$+\frac{1}{\zeta^4}\left(g_4R^3+g_5R\,R_{ij}R^{ij}+g_6R^i_jR^j_kR^k_i\right)$$
$$+\frac{1}{\zeta^4}\left[g_7R\nabla^2R+g_8\left(\nabla_iR_{jk}\right)\left(\nabla^iR^{jk}\right)\right], \quad (2.4)$$

where the coupling constants $g_s\,(s=0,1,2,\dots 8)$ are all dimensionless. The relativistic limit in the IR requires $g_1=-1$ and $\zeta^2=(16\pi G)^{-2}$ [7].

Then, it can be shown that the Hamiltonian and momentum constraints are given, respectively, by,

$$\int d^3x\sqrt{g}\left[\mathcal{L}_K+\mathcal{L}_V-\varphi\mathcal{G}^{ij}\nabla_i\nabla_j\varphi-\big(1-\lambda\big)\big(\nabla^2\varphi\big)^2\right] = 8\pi G\int d^3x\sqrt{g}\,J^t, \quad (2.5)$$

$$\nabla_j\Big[\pi^{ij}-\varphi\mathcal{G}^{ij}-\big(1-\lambda\big)g^{ij}\nabla^2\varphi\Big]=8\pi GJ^i, \quad (2.6)$$

where

$$\begin{aligned} J^t &\equiv 2\frac{\delta\left(N\mathcal{L}_M\right)}{\delta N},\\ \pi^{ij} &\equiv -K^{ij}+\lambda Kg^{ij},\\ J^i &\equiv -N\frac{\delta\mathcal{L}_M}{\delta N_i}. \end{aligned} \quad (2.7)$$

Variation of the action (2.1) with respect to φ and A yield,

$$\mathcal{G}^{ij}\Big(K_{ij}+\nabla_i\nabla_j\varphi\Big)+\big(1-\lambda\big)\nabla^2\Big(K+\nabla^2\varphi\Big) = 8\pi GJ_\varphi, \quad (2.8)$$

$$R-2\Lambda_g=8\pi GJ_A, \quad (2.9)$$

where

$$J_\varphi\equiv-\frac{\delta\mathcal{L}_M}{\delta\varphi},\quad J_A\equiv2\frac{\delta\left(N\mathcal{L}_M\right)}{\delta A}. \quad (2.10)$$

On the other hand, the dynamical equations now read,

$$\begin{aligned} &\frac{1}{N\sqrt{g}}\left\{\sqrt{g}\Big[\pi^{ij}-\varphi\mathcal{G}^{ij}-\big(1-\lambda\big)g^{ij}\nabla^2\varphi\Big]\right\}_{,t}\\ &=-2\left(K^2\right)^{ij}+2\lambda KK^{ij}\\ &\quad+\frac{1}{N}\nabla_k\left[N^k\pi^{ij}-2\pi^{k(i}N^{j)}\right]\\ &\quad-2\big(1-\lambda\big)\Big[\big(K+\nabla^2\varphi\big)\nabla^i\nabla^j\varphi+K^{ij}\nabla^2\varphi\Big]\\ &\quad+\big(1-\lambda\big)\Big[2\nabla^{(i}F^{j)}_\varphi-g^{ij}\nabla_kF^k_\varphi\Big]\\ &\quad+\frac{1}{2}\Big(\mathcal{L}_K+\mathcal{L}_\varphi+\mathcal{L}_A+\mathcal{L}_\lambda\Big)g^{ij}\\ &\quad+F^{ij}+F^{ij}_\varphi+F^{ij}_A+8\pi G\tau^{ij}, \end{aligned} \quad (2.11)$$

where $\left(K^2\right)^{ij}\equiv K^{il}K_l^j,\ f_{(ij)}\equiv\left(f_{ij}+f_{ji}\right)/2$, and

$$\begin{aligned} F^{ij} &\equiv \frac{1}{\sqrt{g}}\frac{\delta\left(-\sqrt{g}\mathcal{L}_V\right)}{\delta g_{ij}}=\sum_{s=0}^{8}g_s\zeta^{n_s}\left(F_s\right)^{ij},\\ F^{ij}_\varphi &= \sum_{n=1}^{3}F^{ij}_{(\varphi,n)},\\ F^i_\varphi &= \Big(K+\nabla^2\varphi\Big)\nabla^i\varphi+\frac{N^i}{N}\nabla^2\varphi,\\ F^{ij}_A &= \frac{1}{N}\left[AR^{ij}-\Big(\nabla^i\nabla^j-g^{ij}\nabla^2\Big)A\right], \end{aligned} \quad (2.12)$$

with $n_s=(2,0,-2,-2,-4,-4,-4,-4,-4)$. The stress 3-tensor τ^{ij} is defined as

$$\tau^{ij}=\frac{2}{\sqrt{g}}\frac{\delta\left(\sqrt{g}\mathcal{L}_M\right)}{\delta g_{ij}}, \quad (2.13)$$

and the geometric 3-tensors $(F_s)_{ij}$ and $F^{ij}_{(\varphi,n)}$ are defined as,

$$\begin{aligned} (F_0)_{ij} &= -\frac{1}{2}g_{ij},\\ (F_1)_{ij} &= R_{ij}-\frac{1}{2}Rg_{ij},\\ (F_2)_{ij} &= 2\left(R_{ij}-\nabla_i\nabla_j\right)R-\frac{1}{2}g_{ij}\left(R-4\nabla^2\right)R,\\ (F_3)_{ij} &= \nabla^2R_{ij}-\left(\nabla_i\nabla_j-3R_{ij}\right)R-4\left(R^2\right)_{ij}\\ &\quad+\frac{1}{2}g_{ij}\left(3R_{kl}R^{kl}+\nabla^2R-2R^2\right),\\ (F_4)_{ij} &= 3\left(R_{ij}-\nabla_i\nabla_j\right)R^2-\frac{1}{2}g_{ij}\left(R-6\nabla^2\right)R^2,\\ (F_5)_{ij} &= \left(R_{ij}+\nabla_i\nabla_j\right)\left(R_{kl}R^{kl}\right)+2R\left(R^2\right)_{ij}\\ &\quad+\nabla^2\left(RR_{ij}\right)-\nabla^k\left[\nabla_i\left(RR_{jk}\right)+\nabla_j\left(RR_{ik}\right)\right]\\ &\quad-\frac{1}{2}g_{ij}\left[\left(R-2\nabla^2\right)\left(R_{kl}R^{kl}\right)\right.\\ &\quad\left.-2\nabla_k\nabla_l\left(RR^{kl}\right)\right],\\ (F_6)_{ij} &= 3\left(R^3\right)_{ij}+\frac{3}{2}\left[\nabla^2\left(R^2\right)_{ij}\right.\\ &\quad\left.-\nabla^k\Big(\nabla_i\left(R^2\right)_{jk}+\nabla_j\left(R^2\right)_{ik}\Big)\right]\\ &\quad-\frac{1}{2}g_{ij}\left[R^k_lR^l_mR^m_k-3\nabla_k\nabla_l\left(R^2\right)^{kl}\right],\\ (F_7)_{ij} &= 2\nabla_i\nabla_j\left(\nabla^2R\right)-2\left(\nabla^2R\right)R_{ij}\\ &\quad+\left(\nabla_iR\right)\left(\nabla_jR\right)-\frac{1}{2}g_{ij}\left[\left(\nabla R\right)^2+4\nabla^4R\right],\\ (F_8)_{ij} &= \nabla^4R_{ij}-\nabla_k\left(\nabla_i\nabla^2R^k_j+\nabla_j\nabla^2R^k_i\right)\\ &\quad-\left(\nabla_iR^k_l\right)\left(\nabla_jR^l_k\right)-2\left(\nabla^kR^l_i\right)\left(\nabla_kR_{jl}\right)\\ &\quad-\frac{1}{2}g_{ij}\left[\left(\nabla_kR_{lm}\right)^2-2\left(\nabla_k\nabla_l\nabla^2R^{kl}\right)\right], \end{aligned} \quad (2.14)$$

$$\begin{aligned} F^{ij}_{(\varphi,1)} &= \frac{1}{2}\varphi\left\{\Big(2K+\nabla^2\varphi\Big)R^{ij}-2\Big(2K^j_k+\nabla^j\nabla_k\varphi\Big)R^{ik}\right.\\ &\quad-2\Big(2K^i_k+\nabla^i\nabla_k\varphi\Big)R^{jk}\\ &\quad\left.-\Big(2\Lambda_g-R\Big)\Big(2K^{ij}+\nabla^i\nabla^j\varphi\Big)\right\},\\ F^{ij}_{(\varphi,2)} &= \frac{1}{2}\nabla_k\left\{\varphi\mathcal{G}^{ik}\Big(\frac{2N^j}{N}+\nabla^j\varphi\Big)\right. \end{aligned}$$

$$+\varphi\mathcal{G}^{jk}\Big(\frac{2N^i}{N}+\nabla^i\varphi\Big)-\varphi\mathcal{G}^{ij}\Big(\frac{2N^k}{N}+\nabla^k\varphi\Big)\bigg\},$$

$$F^{ij}_{(\varphi,3)} \;=\; \frac{1}{2}\Big\{2\nabla_k\nabla^{(i}f^{j)k}_{\varphi}-\nabla^2 f^{ij}_{\varphi}-\big(\nabla_k\nabla_l f^{kl}_{\varphi}\big)\,g^{ij}\Big\}, \tag{2.15}$$

where

$$f^{ij}_{\varphi} \;=\; \varphi\bigg\{\Big(2K^{ij}+\nabla^i\nabla^j\varphi\Big)-\frac{1}{2}\Big(2K+\nabla^2\varphi\Big)g^{ij}\bigg\}. \tag{2.16}$$

The matter components $(J^t,\ J^i,\ J_\varphi,\ J_A,\ \tau^{ij})$ satisfy the conservation laws,

$$\int d^3x\sqrt{g}\bigg[\dot{g}_{kl}\tau^{kl}-\frac{1}{\sqrt{g}}\left(\sqrt{g}J^t\right)_{,t}+\frac{2N_k}{N\sqrt{g}}\left(\sqrt{g}J^k\right)_{,t} -2\dot{\varphi}J_\varphi-\frac{A}{N\sqrt{g}}\left(\sqrt{g}J_A\right)_{,t}\bigg]=0, \tag{2.17}$$

$$\nabla^k\tau_{ik}-\frac{1}{N\sqrt{g}}\left(\sqrt{g}J_i\right)_{,t}-\frac{J^k}{N}\left(\nabla_k N_i-\nabla_i N_k\right) -\frac{N_i}{N}\nabla_k J^k+J_\varphi\nabla_i\varphi-\frac{J_A}{2N}\nabla_i A=0. \tag{2.18}$$

III. COSMOLOGICAL MODELS

The homogeneous and isotropic universe is described by,

$$\bar{N}=1,\ \ \bar{N}_i=0,\ \ \bar{g}_{ij}=a^2(t)\gamma_{ij}, \tag{3.1}$$

where $\gamma_{ij}=\delta_{ij}\big(1+\frac{1}{4}kr^2\big)^{-2}$, with $r^2\equiv x^2+y^2+z^2,\ k=0,\pm1$. As in Paper I, we use symbols with bars to denote the quantities of background. Using the $U(1)$ gauge freedom of Eq.(1.8), on the other hand, we can always set

$$\bar{\varphi}=0. \tag{3.2}$$

Then, we find

$$\begin{aligned}
\bar{K}_{ij} &= -a^2H\gamma_{ij},\quad \bar{R}_{ij}=2k\gamma_{ij},\\
\bar{F}^{ij}_{A} &= \frac{2k\bar{A}}{a^4}\gamma^{ij},\quad \bar{F}^{ij}_{\varphi}=0,\quad \bar{F}^{i}_{\varphi}=0,\\
\bar{F}^{ij} &= \frac{\gamma^{ij}}{a^2}\left(-\Lambda+\frac{k}{a^2}+\frac{2\beta_1k^2}{a^4}+\frac{12\beta_2k^3}{a^6}\right),
\end{aligned} \tag{3.3}$$

where $H=\dot{a}/a,\ \Lambda\equiv\zeta^2g_0/2$, and

$$\beta_1\equiv\frac{3g_2+g_3}{\zeta^2},\quad \beta_2\equiv\frac{9g_4+3g_5+g_6}{\zeta^4}. \tag{3.4}$$

Hence, we obtain

$$\begin{aligned}
\bar{\mathcal{L}}_K &= 3\big(1-3\lambda\big)H^2,\ \ \bar{\mathcal{L}}_\varphi=0=\bar{\mathcal{L}}_\lambda,\\
\bar{\mathcal{L}}_A &= 2\bar{A}\Big(\Lambda_g-\frac{3k}{a^2}\Big),\\
\bar{\mathcal{L}}_V &= 2\Lambda-\frac{6k}{a^2}+\frac{12\beta_1k^2}{a^4}+\frac{24\beta_2k^3}{a^6}.
\end{aligned} \tag{3.5}$$

It can be shown that the super-momentum constraint (2.6) is satisfied identically for $\bar{J}^i=0$, while the Hamiltonian constraint (2.5) yields,

$$\frac{1}{2}\big(3\lambda-1\big)H^2+\frac{k}{a^2}=\frac{8\pi G}{3}\bar{\rho}+\frac{\Lambda}{3}+\frac{2\beta_1k^2}{a^4}+\frac{4\beta_2k^3}{a^6}, \tag{3.6}$$

where $\bar{J}^t\equiv-2\bar{\rho}$. On the other hand, Eqs.(2.8) and (2.9) give, respectively,

$$H\left(\Lambda_g-\frac{k}{a^2}\right)=-\frac{8\pi G}{3}\bar{J}_\varphi, \tag{3.7}$$

$$\frac{3k}{a^2}-\Lambda_g=4\pi G\bar{J}_A, \tag{3.8}$$

while the dynamical equation (2.11) reduces to

$$\frac{1}{2}\big(3\lambda-1\big)\frac{\ddot{a}}{a} \;=\; -\frac{4\pi G}{3}(\bar{\rho}+3\bar{p})+\frac{1}{3}\Lambda-\frac{2\beta_1k^2}{a^4} -\frac{8\beta_2k^3}{a^6}+\frac{1}{2}\bar{A}\left(\frac{k}{a^2}-\Lambda_g\right), \tag{3.9}$$

where $\bar{\tau}_{ij}=\bar{p}\,\bar{g}_{ij}$.

The conservation law of momentum (2.18) is satisfied identically, while the one of energy (2.17) reads,

$$\dot{\bar{\rho}}+3H\left(\bar{\rho}+\bar{p}\right)=\bar{A}\bar{J}_\varphi. \tag{3.10}$$

It is interesting to note that the energy of matter is not conserved in general, due to its interaction with the gauge field $\bar{A}$ and the Newtonian pre-potential $\bar{\varphi}$.

IV. COSMOLOGICAL SCALAR PERTURBATIONS

The linear scalar perturbations of the metric in terms of the conformal time η are given by,

$$\begin{aligned}
\delta N &= a\phi,\quad \delta N_i=a^2B_{|i},\\
\delta g_{ij} &= -2a^2\big(\psi\gamma_{ij}-E_{|ij}\big),\\
A &= \bar{A}+\delta A,\quad \varphi=\bar{\varphi}+\delta\varphi.
\end{aligned} \tag{4.1}$$

Under the gauge transformations (1.4), they transform as

$$\begin{aligned}
\tilde{\phi} &= \phi-\mathcal{H}\xi^0-\xi^{0'},\quad \tilde{\psi}=\psi+\mathcal{H}\xi^0,\\
\tilde{B} &= B+\xi^0-\xi',\quad \tilde{E}=E-\xi,\\
\tilde{\delta\varphi} &= \delta\varphi-\xi^0\bar{\varphi}',\quad \tilde{\delta A}=\delta A-\xi^0\bar{A}'-\xi^{0'}\bar{A},
\end{aligned} \tag{4.2}$$

where $f=-\xi^0,\ \zeta^i=-\xi^{|i},\ \mathcal{H}\equiv a'/a$, and a prime denotes the ordinary derivative with respect to η. Under the $U(1)$ gauge transformations, on the other hand, we find that

$$\begin{aligned}
\tilde{\phi} &= \phi,\quad \tilde{E}=E,\quad \tilde{\psi}=\psi,\quad \tilde{B}=B-\frac{\epsilon}{a},\\
\tilde{\delta\varphi} &= \delta\varphi+\epsilon,\quad \tilde{\delta A}=\delta A-\epsilon',
\end{aligned} \tag{4.3}$$

where $\epsilon = -\alpha$. Then, the gauge transformations of the whole group $U(1) \ltimes \text{Diff}(M,\ \mathcal{F})$ will be the linear combination of the above two. Since we have six unknown and three arbitrary functions, the total number of the gauge-invariants of $U(1) \ltimes \text{Diff}(M,\ \mathcal{F})$ is $N = 6 - 3 = 3$. These gauge-invariants can be constructed as,

$$\begin{aligned}
\Phi &= \phi + \frac{a}{a-\bar{\varphi}'}\left[\frac{\delta\varphi'}{a} + \mathcal{H}\big(B-E'\big) + \big(B-E'\big)'\right] \\
&\quad + \frac{1}{\big(a-\bar{\varphi}'\big)^2}\left(\bar{\varphi}'' - \mathcal{H}\bar{\varphi}'\right)\Big[\delta\varphi + a\big(B-E'\big)\Big], \\
\Psi &= \psi - \frac{\mathcal{H}}{a-\bar{\varphi}'}\Big[\delta\varphi + a\big(B-E'\big)\Big], \\
\Gamma &= \delta A + \left[\frac{a+\bar{A}}{a-\bar{\varphi}'}\delta\varphi + \frac{a(\bar{A}+\bar{\varphi}')}{a-\bar{\varphi}'}\big(B-E'\big)\right]'. \quad (4.4)
\end{aligned}$$

Using the $U(1)$ gauge freedom (4.3), we shall set

$$\delta\varphi = 0. \quad (4.5)$$

This gauge choice is completely fixed the $U(1)$ gauge. Then, considering Eq.(3.2), we find that the above expressions reduce to

$$\begin{aligned}
\Phi &= \phi + \mathcal{H}\big(B-E'\big) + \big(B-E'\big)', \\
\Psi &= \psi - \mathcal{H}\big(B-E'\big), \\
\Gamma &= \delta A + \Big[\bar{A}\big(B-E'\big)\Big]',\ (\bar{\varphi} = \delta\varphi = 0). \quad (4.6)
\end{aligned}$$

The expressions for Φ and Ψ now take precisely the same forms as those defined in Paper I, which are also identical to those given in general relativity [20]. In Papers I and II, the quasi-longitudinal gauge,

$$\phi = 0 = E, \quad (4.7)$$

was imposed. In this paper, we shall adopt this gauge for the metric perturbations, and the gauge of Eq.(4.5) for the Newtonian pre-potential. We shall refer them as the "generalized" quasi-longitudinal gauge, or simply the quasi-longitudinal gauge.

Then, to first-order the Hamiltonian and momentum constraints become, respectively,

$$\begin{aligned}
&\int \sqrt{\gamma}d^3x\Bigg[\left(\vec{\nabla}^2 + 3k\right)\psi - \frac{(3\lambda-1)\mathcal{H}}{2}\left(\vec{\nabla}^2 B + 3\psi'\right) \\
&\quad - 2k\Big(\frac{2\beta_1}{a^2} + \frac{6\beta_2 k}{a^4} + \frac{3g_7}{\zeta^4 a^4}\vec{\nabla}^2\Big)\left(\vec{\nabla}^2 + 3k\right)\psi \\
&\quad - 4\pi G a^2 \delta\mu\Bigg] = 0, \quad (4.8)
\end{aligned}$$

$$(3\lambda-1)\psi' - 2kB - (1-\lambda)\vec{\nabla}^2 B = 8\pi G a q, \quad (4.9)$$

where $\delta\mu \equiv -\delta J^t/2$ and $\delta J^i \equiv a^{-2}q^{|i}$. These are the same as given by Eqs.(4.19) and (4.21) in Paper I. On the other hand, the linearized equations (2.8) and (2.9) reduce, respectively, to

$$\begin{aligned}
&\Big(\Lambda_g - \frac{k}{a^2}\Big)\Big[\vec{\nabla}^2 B + 3\big(\psi' + 2\mathcal{H}\psi\big)\Big] \\
&\quad + \frac{2\mathcal{H}}{a^2}\Big[\vec{\nabla}^2\psi + 3\big(2k - a^2\Lambda_g\big)\psi\Big] \\
&\quad + \frac{1-\lambda}{a^2}\vec{\nabla}^2\Big(\vec{\nabla}^2 B + 3\psi'\Big) = 8\pi G a\delta J_\varphi, \quad (4.10) \\
&\vec{\nabla}^2\psi + 3k\psi = 2\pi G a^2 \delta J_A. \quad (4.11)
\end{aligned}$$

The linearly perturbed dynamical equations can be divided into the trace and traceless parts. The trace part reads,

$$\begin{aligned}
\psi'' &+ 2\mathcal{H}\psi' - \mathcal{F}\psi - \frac{1}{3(3\lambda-1)}\gamma^{ij}\delta F_{ij} \\
&- \frac{1}{3a(3\lambda-1)}\Big(2\vec{\nabla}^2 - 3k + 3\Lambda_g a^2\Big)\delta A \\
&+ \frac{2\bar{A}}{3a(3\lambda-1)}\Big(\vec{\nabla}^2 + 6k - 3\Lambda_g a^2\Big)\psi \\
&+ \frac{1}{3}\vec{\nabla}^2\left(B' + 2\mathcal{H}B\right) = \frac{8\pi G a^2}{(3\lambda-1)}\delta\mathcal{P}, \quad (4.12)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F} &= \frac{2a^2}{3\lambda-1}\left(-\Lambda + \frac{k}{a^2} + \frac{2\beta_1 k^2}{a^4} + \frac{12\beta_2 k^3}{a^6}\right), \\
\delta\tau^{ij} &= \frac{1}{a^2}\left[\left(\delta\mathcal{P} + 2\bar{p}\psi\right)\gamma^{ij} + \Pi^{|\langle ij\rangle}\right], \\
f_{|\langle ij\rangle} &\equiv f_{|ij} - \frac{1}{3}\gamma_{ij}\vec{\nabla}^2 f, \quad (4.13)
\end{aligned}$$

and $\delta F_{ij} = \sum g_s\zeta^{n_s}\delta(F_s)_{ij}$, with $\delta(F_s)_{ij}$ given by Eq. (A1) in Paper I. To avoid repeating, we shall not write them down here, and refer readers directly to Paper I. The traceless part is given by

$$\begin{aligned}
&\Big(B' + 2\mathcal{H}B\Big)_{|\langle ij\rangle} + \delta F_{\langle ij\rangle} - \frac{1}{a}\left(\delta A - \bar{A}\psi\right)_{|\langle ij\rangle} \\
&\quad = -8\pi G a^2 \Pi_{|\langle ij\rangle}. \quad (4.14)
\end{aligned}$$

Since $\mathcal{L}_M$ does not depends on λ, the perturbed parts of the conservation laws (2.17) and (2.18) are the same as those given in Paper II with $\bar{\varphi} = 0 = \delta\varphi$, which can be written as [17],

$$\begin{aligned}
&\int \sqrt{\gamma}d^3x\Bigg\{\delta\mu' + 3\mathcal{H}\left(\delta\mathcal{P} + \delta\mu\right) - 3\left(\bar{\rho} + \bar{p}\right)\psi' \\
&\quad + \frac{1}{2a^4}\bigg[\Big(a^3\bar{J}_A\Big)_{,\eta}\delta A + \bar{A}\Big(a^3\big(\delta A - 3\bar{A}\psi\big)\Big)_{,\eta}\bigg]\Bigg\} = 0, \\
&\quad (4.15)
\end{aligned}$$

$$\begin{aligned}
&q' + 3\mathcal{H}q - a\delta\mathcal{P} - \frac{2a}{3}\Big(\vec{\nabla}^2 + 3k\Big)\Pi \\
&\quad + \frac{1}{2}\bar{J}_A\delta A = 0, \quad (4.16)
\end{aligned}$$

where $\bar{J}_A$ is given by Eq.(3.8).

This completes the general description of linear scalar perturbations in the FRW background with any spatial curvature in the framework of the HMT setup with any given λ [19].

V. STABILITY OF THE MINKOWSKI SPACETIME

The stability of the maximal symmetric spacetimes in the HMT setup with $\lambda \neq 1$ was considered in [19] with detailed balance condition. Since the Minkowski is not a solution of the theory when detailed balance condition is imposed, so the analysis given in [19] does not include the Minkowski spacetime as the background. However, for the potential given by Eq.(2.4), the detailed balance condition is broken, and the Minkowski spacetime now is a solution of the theory. Therefore, in this section we study the stability of the Minkowski spacetime with any given λ. The case with $\lambda = 1$ was considered in Paper II, so in this section we consider only the case with $\lambda \neq 1$.

It is easy to show that the Minkowski spacetime,

$$a = 1, \quad \bar{A} = \bar{\varphi} = k = 0, \tag{5.1}$$

is a solution of the HMT theory even with $\lambda \neq 1$, provided that

$$\Lambda_g = \Lambda = \bar{J}_A = \bar{J}_\varphi = \bar{\rho} = \bar{p} = 0. \tag{5.2}$$

Then, the linearized Hamiltonian constraint (4.8) is satisfied identically, while the super-momentum constraint (4.9) yields,

$$\partial^2 B = \frac{3\lambda - 1}{1-\lambda}\dot{\psi}, \tag{5.3}$$

where $\partial^2 = \delta^{ij}\partial_i\partial_j$. Eqs.(4.10) and (4.11) reduce to,

$$\partial^2\left(\partial^2 B + 3\dot{\psi}\right) = 0, \tag{5.4}$$

$$\partial^2\psi = 0. \tag{5.5}$$

Then, we have $\delta F_{ij} = -\psi_{,ij}$, and the trace and traceless parts of the dynamical equations reduce, respectively, to

$$\ddot{\psi} - \frac{2}{3(3\lambda-1)}\partial^2\delta A + \frac{1}{3}\partial^2\dot{B} = 0, \tag{5.6}$$

$$\dot{B} = \delta A - \psi. \tag{5.7}$$

It can be shown that Eqs.(5.4) and (5.6) are not independent, and can be obtained from Eqs.(5.3), (5.5) and (5.7). Eq.(5.5) shows that ψ is not propagating, and with proper boundary conditions, we can set $\psi = 0$. Then, Eqs.(5.3) and (5.7) show that B and δA are also not propagating, and shall vanish with proper boundary conditions. Therefore, we finally have

$$\psi = B = \delta A = 0. \tag{5.8}$$

Thus, the scalar perturbations even with $\lambda \neq 1$ vanish identically in the Minkowski background. Hence, the spin-0 graviton is eliminated in the HMT setup even for any given coupling constant λ.

VI. GHOST AND STRONG COUPLING

To consider the ghost and strong coupling problems, we first note that they are closely related to the fact that $\lambda \neq 1$. The parts that depend on λ are the kinetic part, $\mathcal{L}_K$, and the interaction part between the kinetic extrinsic curvature and the Newtonian pre-potenital, $\mathcal{L}_\lambda$. With the gauge choice $\varphi = 0$, we can see that the latter vanishes identically. Then, the only part that is related to these two problems is the kinetic action S_K. In the flat FRW background, its quadratic part is given by [9],

$$\begin{aligned} S^{(2)}_K &= \zeta^2\int d\eta d^3x a^2\Big\{\left(1-3\lambda\right)\Big[3\psi'^2 + 6\mathcal{H}\psi\psi' \\ &\quad + 2\psi'\partial^2 B + \frac{9}{2}\mathcal{H}^2\psi^2\Big] + (1-\lambda)B\partial^4 B\Big\}, \end{aligned} \tag{6.1}$$

From the super-momentum constraint (4.9), we find that

$$\partial^2 B = \frac{3\lambda-1}{1-\lambda}\psi' - \frac{8\pi G a q}{1-\lambda}. \tag{6.2}$$

Substituting it into Eq.(6.1), we find that the term proportional to ψ'^2 is given by

$$S^{(2)}_K = \int d\eta d^3x a^2\left[\frac{2(3\lambda-1)}{\lambda-1}\psi'^2 + ...\right]. \tag{6.3}$$

Thus, the ghost-free conditon requires $(3\lambda-1)/(\lambda-1)$, or equivalently,

$$i)\ \lambda > 1, \quad \text{or} \quad ii)\quad \lambda < \frac{1}{3}, \tag{6.4}$$

which are precisely the conditions obtained in Paper I in the SVW setup [8].

On the other hand, the spin-0 gravity vanishes identically in the maximal symmetric backgeropunds [19], including the one of the Minkowski studied in the last section. Then from the analysis given in [11], we can see that the strong problem does not exist in these spacetimes.

VII. CONCLUSIONS

Recently, Horava and Melby-Thompson [16] proposed a new version of the HL theory of gravity, in which the spin-0 graviton, appearing in all the previous versions of the HL theory, is eliminated by introducing a Newtonian pre-potential φ and a local $U(1)$ gauge field A. Such a setup was orginally believed valid only for $\lambda = 1$. However, da Silva recently argued that the HMT setup can be easily generalized to the case with $\lambda \neq 1$. With such a generalization, the three challenging questions, ghost, stability and strong coupling, all related with $\lambda \neq 1$ and plagued in most of the previous versions of the HL theory [6, 14], raise again.

In this paper, we addressed these issues, by first developing the linear scalar perturbations of the FRW spacetimes for any given λ. In particular, in Sec. II we derived all the field equations and the corresponding conservation laws, while in Sec. III we studied the cosmological models of the FRW universe with any given spatial curvature k. When $\bar{J}_A = 0$, from Eq.(3.8) we can see that the spatial curvature must vanish, $k = 0$. This is the case when the matter of the universe is described by a scalar field [19], which is true in most of the inflationary models. Therefore, the HMT setup naturally gives rise to a flat FRW universe.

Then, in Sec. IV we presented the general formulas for the linear scalar perturbations. By studying the general gauge transformations of $U(1) \ltimes Diff(M, \mathcal{F})$, we found that there are only three gauge-invaraint quantities, and constructed them explicitly, given by Eq.(4.6). Applying these formulas to the Minkowski spacetime, in Sec. V we showed explicitly that the Minkowski spacetime is stable, and the correwsponding spin-0 graviton is eliminated by the gauge field even for $\lambda \neq 1$.

In Sec. VI, we considered the ghost and strong coupling problems, and found that the ghost-free conditions are the same as these found in Paper I in the SVW setup, given explicitly by Eq.(6.4). Since the spin-0 graviton does not appear in the maximal symmetric spacetimes at all, including the Minkowski one, the strong coupling problem does not exist, as that considered in the case with $\lambda = 1$ in [21].

The gauge field A and the Newtonian pre-potentail φ have no contributions to the vector and tensor perturbations, so the results presented in [22] in the SVW setup can be equally applied to the HMT setup even with $\lambda \neq 1$, where $\xi = 1 - \lambda$. In particular, it was shown that the vector perturbations vanish. Combining it with the result obtained in this paper, one can see that the only non-vanishing part is the tensor one. As a result, the gravitional section in the HMT setup is described purely by the spin-2 massless graviton even with $\lambda \neq 1$.

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