FLAVOR STATES OF THE KNOT MODEL

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Abstract. We discuss flavor states of the knot model and their relation to the KM and the PMNS matrices. These states are eigenstates of absorption operators and are analogous to the coherent states of the Maxwell field. The underlying model has been proposed as a possible substructure of the standard model.

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1   Introduction

To describe the weak decays of the strange particles Cabibbo introduced a 2×2 mixing matrix that was later extended by Kobayashi and Maskawa to the 3×3 matrix labelled by the three-flavor states of the up and down quarks. Following the suggestion of Pontecorvo the neutrino oscillations may similarly be described by the PMNS matrix that expresses the three-flavor states as mixtures of the three mass states of the three leptonic neutrinos.[1]

Flavor states are eigenstates of the absorption operators that in turn depend on the dynamics of the model. Since we are here interested in the flavor states of the knot model, we shall first summarize the kinematical structure of this model as determined by the symmetry algebra, $SL_q(2)$, in preparation for introducing the dynamical assumptions, that are also subject to $SL_q(2)$, and that determine the flavor states.

2   Quantum Trefoils[2]

We require that a quantum knot be described by one member of an irreducible representation of the knot algebra $(SL_q(2))$, which is here denoted by $D^j_{mm'}$. It is also required that there be a correspondence between $D^j_{mm'}$, and a classical knot. Both requirements are met by allowing only those elements $D^j_{mm'}$, to represent quantum knots for which

$$(j, m, m') = \frac{1}{2}(N, w, r + 1) \tag{2.1}$$

where (N, w, r) are the number of crossings, the writhe, and the rotation that describe the projection of a 3-dimensional classical knot onto a 2-dimensional plane. The simplest classical knot is the trefoil having the 2d projection described by

$$(N, w, r) = (3, \pm 3, \pm 2) \tag{2.2}$$

By (2.1) the corresponding four quantum trefoils are represented by

$$D^{3/2}_{\frac{3}{2}\frac{3}{2}}, \ D^{3/2}_{-\frac{3}{2}\frac{3}{2}}, \ D^{3/2}_{\frac{3}{2}-\frac{1}{2}}, \ D^{3/2}_{-\frac{3}{2}-\frac{1}{2}} \tag{2.3}$$

There are four quantum trefoils but only two of the four corresponding classical trefoils can be topologically distinguished. Note also that $2m$ and $2m'$, belonging to the same

representation, are of same parity while w and r, describing the projection of a classical knot, are required to be of opposite parity.

3   Irreducible Representations of the Knot Algebra[2]

The $2j + 1$ dimensional representation of $SLq(2)$ may be written as follows:

$$D^j_{mm'}(a, b, c, d) = \sum_{\substack{0 \le s \le n_+ \\ 0 \le t \le n_-}} \mathcal{A}^j_{mm'}(q, s, t)\delta(s + t, n'_+)a^s b^{(n_+ - s)}c^t d^{(n_- - t)} \tag{3.1}$$

where

$$n_\pm = j \pm m \tag{3.2}$$

$$n'_\pm = j \pm m' \tag{3.3}$$

and the arguments (a, b, c, d) satisfy the knot algebra:[2]

$$\begin{aligned}
&ab = qba \quad bd = qdb \quad bc = cb \quad ad - qbc = 1 \\
&ac = qca \quad cd = qdc \qquad\qquad\quad da - q_1 cb = 1
\end{aligned} \tag{A}$$

where $q_1 = q^{-1}$.

The $\mathcal{A}^j_{mm'}$ are q-deformations of the Wigner coefficients that appear in irreducible representations of $SU(2)$.

The knot algebra (A) and hence $D^j_{mm'}(a, b, c, d)$ are defined only up to the gauge transformation

$$U_a(1) : \begin{aligned} a' &= e^{i\varphi_a}a \\ d' &= e^{-i\varphi_a}d \end{aligned} \qquad\qquad U_b(1) : \begin{aligned} b' &= e^{i\varphi_b}b \\ c' &= e^{-i\varphi_b}c \end{aligned} \tag{3.4}$$

Eqns. (3.4) leave the algebra (A) invariant and induce on the elements of every representation the following $U_a(1) \times U_b(1)$ gauge transformation

$$D^j_{mm'}(a', b', c', d') = e^{i(\varphi_a + \varphi_b)m} \, e^{i(\varphi_a - \varphi_b)m'} \, D^j_{mm'}(a, b, c, d) \tag{3.5}$$

4   Field Theory and Charges of Quantum Trefoils[2]

One may construct a field theory of the quantum knots by attaching $D^j_{mm'}$ to a standard field operator $\psi(x)$ as follows:

$$\Psi^j_{mm'} = \psi(x) \; D^j_{mm'} \tag{4.1}$$

By (3.5), the field operator $\Psi^j_{mm'}$ also transforms under the gauge transformations $U_a(1) \times U_b(1)$. If the attachment (4.1) is made consistently for both fermionic and bosonic fields one may construct a modified standard action that is invariant under $U_a \times U_b$, as is shown in Ref. 3. This invariance of the field action is a physical requirement since the relabelling of the algebra described by (3.4) cannot affect the physics. Then in view of this invariance there will be by Noether's theorem one conserved charge associated with $U_a(1)$ and a second conserved charge associated with $U_b(1)$. Then by (3.5) and (3.1) these charges may be defined by

$$Q(w) \equiv -k_w m \;\; = \;\; -k_w \; \frac{w}{2} \tag{4.2}$$

$$Q(r) \equiv -k_r m' \;\; = \;\; -k_r \; \frac{r+1}{2} \tag{4.3}$$

and may be referred to as the writhe and rotation charges. Here k_w and k_r are undetermined constants with the dimensions of an electric charge. In terms of $Q(w)$ and $Q(r)$, the $U_a(1) \times U_b(1)$ transformations on $\Psi^j_{mm'}$ become

$$\Psi^{j'}_{mm'} = e^{\frac{-i}{k_w}Q(w)\varphi(w)} e^{\frac{-i}{k_r}Q(r)\varphi(r)} \; \Psi^j_{mm'} \tag{4.4}$$

where $\varphi(w) = \varphi_a + \varphi_b$ and $\varphi(r) = \varphi_a - \varphi_b$ by (3.5).

Since we expect that the most elementary particles, the elementary fermions, are quantum trefoils in any natural knot model, we next make a direct comparison between the $Q(w)$ and $Q(r)$ charges of the four quantum trefoils and the charge and hypercharge of the four fermion families of the standard theory where the members of each family are denoted by (f_1, f_2, f_3) in Table 4.1.[2] The knot entries in the table are determined by (2.1), (4.2), and (4.3).

In Table (4.1) we have assumed a single value of k:

$$k_r = k_w = k \tag{4.5}$$

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<div align="center">**Table 4.1**</div>

Standard Representation					Trefoil Representation				
(f_1, f_2, f_3)	t	t_3	t_0	Q_e	(w,r)	$D^{N/2}_{\frac{w}{2}\frac{r+1}{2}}$	Q_w	Q_r	$Q_w + Q_r$
$(e,\mu,\tau)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	$(3,2)$	$D^{3/2}_{\frac{3}{2}\frac{3}{2}}$	$-k\left(\frac{3}{2}\right)$	$-k\left(\frac{3}{2}\right)$	$-3k$
$(\nu_e,\nu_\mu,\nu_\tau)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$(-3,2)$	$D^{3/2}_{-\frac{3}{2}\frac{3}{2}}$	$-k\left(-\frac{3}{2}\right)$	$-k\left(\frac{3}{2}\right)$	0
$(d,s,b)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	$(3,-2)$	$D^{3/2}_{\frac{3}{2}-\frac{1}{2}}$	$-k\left(\frac{3}{2}\right)$	$-k\left(-\frac{1}{2}\right)$	$-k$
$(u,c,t)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	$(-3,-2)$	$D^{3/2}_{-\frac{3}{2}-\frac{1}{2}}$	$-k\left(-\frac{3}{2}\right)$	$-k\left(-\frac{1}{2}\right)$	$2k$

which is also the same for all trefoils. If we set $k = e/3$, we find that the four fermion families are related to the four quantum trefoils as follows:

$$Q_w = et_3 \tag{4.6}$$

$$Q_r = et_0 \tag{4.7}$$

$$Q_w + Q_r = Q_e \tag{4.8}$$

in agreement with the standard model where there is the independent relation for the electric charge

$$Q_e = e(t_3 + t_0) \tag{4.9}$$

If one aligns the trefoils and the fermion families in any order different from that in Table 4.1, one needs more than a single value of k to relate (t_3, t_0) to (Q_w, Q_r). It is important that we choose $k_r = k_w$ and that we also choose a single value of k for the four quantum trefoils. Note that it is also not possible to exchange t_3 and t_0 in (4.6) and (4.7). Therefore the correspondence between the four fermion families and the four trefoils, is empirically fixed and unique. The value of k as $e/3$ follows from the identification of the total charge of the trefoil, $Q_w + Q_r$, with Q_e. One may also read directly from the table

$$(j, m, m') = 3(t, -t_3, -t_0) \tag{4.10}$$

and enter the information conveyed by (4.10) into (4.1) as follows:

$$\Psi^{3/2}(t_3, t_0, n) = \psi(t_3, t_0, n)D^{3/2}_{-3t_0-3t_0}|n\rangle \tag{4.11}$$

where $\psi(t_3, t_0, n)$ is the quantum field of the standard model that represents the fermion with electroweak $SU(2) \times U(1)$ quantum numbers (t_3, t_0). Here $|n\rangle$ lies in the state space defined by the knot algebra where $n = 0, 1, 2$ labels the generation, e.g. (e, μ, τ). Then $D^{3/2}_{-3t_3-3t_0}|n\rangle$ may be regarded as an "internal state function" reminiscent of a classical knot and providing substructure to the elementary quantum fields of the standard model.

We shall now propose that the non-trivial correspondence embodied in Table (4.1) and expressed by (4.15) for the elementary fermions holds more generally in the following form

$$\Psi^t_{t_3 t_0}(n) = \psi(t, t_3, t_0, n) D^{3t}_{-3t_3-3t_0}|n> \tag{4.12}$$

i.e., we assume that (t, t_3, t_0) are related to (j, m, m') just as in the special case $t = \frac{1}{2}$:

$$3t = j \tag{4.13j}$$

$$3t_3 = -m \tag{4.13m}$$

$$3t_0 = -m' \tag{4.13m'}$$

In other words we assume that there is an underlying $SLq(2)$ symmetry of the elementary particles that may be expressed through the internal state functions $D^j_{mm'} |n\rangle$. For $j \geq 1$ not all states (m, m') of $D^j_{mm'}$ are filled. The occupied states are labelled by $D^{3t}_{-3t_3-3t_0}$ according to (4.13) and are determined by the intersection of the electroweak $SU(2) \times U(1)$ and the $SU_q(2)$ symmetries. For example, the analogue of Table 4.1 for the elementary fermions is Table 4.2 for the elementary bosons of the Weinberg-Salam model. The $|n\rangle$ in (4.12) are intended to represent the possible states of excitation of the quantum knot.

Table 4.2

	t	t_3	t_0	$D^{3t}_{-3t_3-3k_0}$
W^+	1	1	0	D^3_{-30}
W^-	1	-1	0	D^3_{30}
W^3	1	0	0	D^3_{00}
W^0	0	0	0	D^0_{00}

We adopt the following rule: If a particle is labelled in the standard model by electroweak quantum numbers (t, t_3, t_0) then attach to the quantum field operator of that particle the

factor $D^{3t}_{-3t_3-3t_0}(a,b,c,d)$. This factor is to be understood as an element of the $j = 3t$ representation of the $SL_q(2)$ algebra and may be interpreted as the replacement of the point particle of the standard model by a solitonic structure described solely by this factor. The extension of (4.11) to (4.12) expresses the conservation of t_3 and t_0 everywhere in the modified model as a joint consequence of the $U_a \times U_b$ and the $SU(2) \times U(1)$ invariance.

5   The Electroweak Interactions[4]

In the $SLq(2)$ model the solitonic fermions interact by the emission and absorption of solitonic bosons. Denote the generic fermion-boson interaction by

$$\bar{\mathcal{F}}'' \, \mathcal{B}' \, \mathcal{F} \tag{5.1}$$

where

$$\mathcal{F} = \mathbf{F}(p, s, t_3, t_0) \left(D^{3/2}_{-3t_3-3t_0} \right) |n> \tag{5.2}$$

$$\bar{\mathcal{F}}'' = <n''| \left(\bar{D}^{3/2}_{-3t_3-3t_0} \right)'' \, \bar{\mathbf{F}}''(p, s, t_3, t_0) \tag{5.3}$$

$$\mathcal{B}' = \mathbf{B}'(p, s, t_3, t_0) \left(D^{3t}_{-3t_3-3t_0} \right)' \tag{5.4}$$

and the pair (p, s) refer to momentum and spin. Then (5.1) becomes

$$(\bar{\mathbf{F}}''\mathbf{B}'\mathbf{F}) <n''| \bar{D}^{3/2}_{-3t_3''-3t_0''} \, D^{3t}_{-3t_3'-3t_0'} \, D^{3/2}_{-3t_3-3t_0} |n> \tag{5.5}$$

The matrix elements of the standard model will then be modified by the following form factors:

$$<n''| \bar{D}^{3/2}_{-3t_3''-3t_0''} \, D^{3t}_{-3t_3'-3t_0'} \, D^{3/2}_{-3t_3-3t_0} |n> \tag{5.6}$$

Here n and n'' take on the values $0, 1, 2$ corresponding to the 3 generations in each family of fermions. These form factors are 2 parameter numerical functions that are in principle observable. To calculate them one needs the solitonic factors $D^j_{mm'}(a, b, c, d)$ shown in Tables (5.1) and (5.2). In previous work we have taken the $|n\rangle$ to be eigenstates of b and c and they have been assumed to be eigenstates of mass.

The solitonic factors have been computed according to (3.1) and are all monomials except for the neutral W^0 and W^3. The numerical factors $\mathcal{A}^j_{mm'}$ have been dropped but may be

Table 5.1

(f_1, f_2, f_3)	t	t_3	t_0	Q	$D^{3t}_{-3t_3 -3t_0}$
(e, μ, τ)	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	$D^{3/2}_{\frac{3}{2}\frac{3}{2}} \sim a^3$
$(\nu_e, \nu_\mu, \nu_\tau)$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$D^{3/2}_{-\frac{3}{2}\frac{3}{2}} \sim c^3$
(d, s, b)	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}e$	$D^{3/2}_{\frac{3}{2}-\frac{1}{2}} \sim ab^2$
(u, c, t)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}e$	$D^{3/2}_{-\frac{3}{2}-\frac{1}{2}} \sim cd^2$

Table 5.2

	t	t_3	t_0	Q	$D^{3t}_{-3t_3 -3t_0}$
W^+	1	1	0	e	$D^3_{-30} \sim c^3 d^3$
W^-	1	-1	0	$-e$	$D^3_{30} \sim a^3 b^3$
W^3	1	0	0	0	$D^3_{00} \sim f_3(b, c)$
W^0	0	0	0	0	$D^0_{00} \sim f_0(b, c)$

computed according to

$$\mathcal{A}^j_{mm'} = \left[\frac{\left\langle n'_+ \right\rangle_{q_1}! \left\langle n'_- \right\rangle_{q_1}!}{\left\langle n_+ \right\rangle_{q_1}! \left\langle n_- \right\rangle_{q_1}!} \right]^{\frac{1}{2}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \tag{5.7}$$

where

$$\left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_q = \frac{\langle n \rangle_q!}{\langle n-s \rangle_q! \langle s \rangle_q!} \quad \text{with} \quad \langle n \rangle_q = \frac{q^n - 1}{q - 1}; \quad q_1 = q^{-1} \tag{5.8}$$

Since we require that the fermion-boson interaction be expressed by (5.1), and that the total action be both $SU(2) \times U(1)$ and $U_a(1) \times U_b(1)$ invariant, (5.1) and (5.5) must share this invariance. Then in view of these invariances and since $(4.13m)$ and $(4.13m')$ hold for $\mathcal{F}$, they must also hold for $\mathcal{B}$. Hence these relations are not simply conjectured extensions but they are an essential requirement of the electroweak model. Eq. $(4.13j)$ is not required but is allowed.

6 The Dynamics of the Quantum Knot

$D^j_{mm'}(q(a, b, c, d))$ is a kinematic factor describing a generic quantum knot. The corresponding classical knot (N, w, r) has arbitrary size and shape. To describe the oscillations of this quantum knot in a field theory one replaces the classical Fourier modes by quantum oscillators determined by a quantum Hamiltonian. The kinematics of the model is determined by $SL_q(2)$ and we shall restrict the Hamiltonian by the same symmetry. Then the normal modes of the field operators that describe the physical particles are determined both dynamically and kinematically by $SL_q(2)$. (It is similarly possible to restrict both the dynamic and kinematic dependence of states of the quantized hydrogen atom by a single (rotation) group (without explicitly introducing the Coulomb potential.)[5] There is, however, no uniquely defined way of dynamically implementing this symmetry.

If the knot oscillates like the standard quantum mechanical harmonic oscillator, the Hamiltonian is of the following form:

$$H = \frac{\hbar\omega}{2}(a\bar{a} + \bar{a}a) \tag{6.1}$$

where $\bar{a}$ and a are raising and lowering operators and

$$[a, \bar{a}] = 1 \tag{6.2}$$

Since the raising and lowering operators of the $SL_q(2)$ algebra, that correspond to $\bar{a}$ and a, are d and a respectively, the analogue of (6.1) in the knot algebra is

$$H = \frac{\hbar\omega}{2}(ad + da) \tag{6.3}$$

but by the algebra (A)

$$[a, d] = (q - q_1)bc \tag{6.4}$$

and

$$\frac{1}{2}(ad + da) = 1 + \frac{1}{2}(q + q_1)bc \tag{6.5}$$

We may generalize the $SL_q(2)$ analogue of the Hamiltonian of the harmonic oscillator if we replace (6.3) by a more general function of $ad + da$, or of bc by (6.5), or with a still different H by

$$H = H(b, c)\frac{\hbar\omega}{2} \tag{6.6}$$

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Since b and c commute, they have common eigenstates. Let $|0\rangle$ be the ground state and let

$$b|0\rangle = \beta|0\rangle \tag{6.7}$$

$$c|0\rangle = \gamma|0\rangle \tag{6.8}$$

$$|n\rangle \sim d^n|0\rangle \tag{6.9}$$

Then by the algebra (A)

$$b|n\rangle = q^n\beta|n\rangle \quad \text{and} \quad c|n\rangle = q^n\gamma|n\rangle \tag{6.10}$$

Let the Hamiltonian of the quantum knot be $H(b,c)$. Let us consider the states of this knot defined by $D^j_{mm'}|n\rangle$. We may then compute

$$H(b,c)D^j_{mm'}|n\rangle = H(b,c)\left[\sum_{s,t}\mathcal{A}^j_{mm'}\delta(s+t,n'_+)a^s b^{n+-s}c^t d^{n_--t}\right]|n\rangle \tag{6.11}$$

$$= D^j_{mm'}H(q_1^{n_a-n_d}b, q_1^{n_a-n_d}c)|n\rangle \tag{6.12}$$

where n_a and n_d are the exponents of a and d respectively, and $n_a - n_d$ is the same for every term of $D^j_{mm'}$. Then one has

$$\begin{aligned}H(b,c)D^j_{mm'}|n\rangle &= D^j_{mm'}H(q_1^{n_a-n_d}q^n\beta, q_1^{n_a-n_d}q^n\gamma)|n\rangle\\ &= E^j_{mm'}(n)\mathcal{D}^j_{mm'}|n\rangle\end{aligned} \tag{6.13}$$

where the eigenvalues of H are

$$E^j_{mm'}(n) = H(\lambda\beta, \lambda\gamma) \tag{6.14}$$

and

$$\lambda = q^{n-(m+m')} \tag{6.15}$$

The eigenstates of H are the $D^j_{mm'}|n\rangle$ and the indices on $D^j_{mm'}$ are the eigenvalues of the integrals of motion. Since $(m,m') = 3(-t_3, -t_0)$ by (4.10), we have by (6.15)

$$\lambda = \lambda(n, Q_e) \tag{6.16}$$

where

$$\lambda(n, Q_e) = q^{n+\frac{3}{e}Q_e} \tag{6.17}$$

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and where Q_e is the electric charge of the knot. Here $\lambda(n, Q_e)$ is different for each value of Q_e, which labels the fermion family, and each value of n, which labels the generation. For example, $Q_e = -e$ for leptons and $n = 0, 1, 2$, where the $(0, 1, 2)$ states are electron, muon, and tauon states. The index n labels states of different mass, while the operators a and d have matrix elements connecting adjacent states of different mass and the same charge.

Finally, in order that the H introduced in (6.6) qualify as the Hamiltonian of an elementary fermionic knot we shall require that it be compatible with the fermion mass term in the standard theory, namely

$$\mathcal{M} = \bar{L}\varphi R + \bar{R}\varphi L \tag{6.18}$$

where L and R are left- and right-chiral Lorentz spinors and φ is the Higgs field, a Lorentz scalar, so that product $\bar{L}\varphi R$ is Lorentz invariant. In the Lagrangian of the standard model L and φ are isotopic doublets. $(\bar{L}\varphi)$ and R are separately isotopic singlets and $\mathcal{M}$ is invariant under the gauged $SU(2) \times U(1)$ group.

In the knot model L is aditionally a fermionic knot with the charge structure $D^{3/2}_{-3t_3-3t_0}$. If a knot singlet is assigned to φ, then φ is neutral (unitary gauge) while the right chiral spinor must have the same knot state as the left chiral spinor, namely, $D^{N/2}_{-3t_3-3t_0}$, in order to preserve the $U_a(1) \times U_b(1)$ invariance. Then the standard Higgs mechanism is still possible with $\varphi \sim D^0_{00}$.

If L and R are now assigned the same internal state, and we treat the mass term in the same way as the other terms of the Lagrangian, then we have

$$L \quad \to \quad \chi_L(t_3, t_0, n) D^{3/2}_{-3t_3-3t_0}|n\rangle \tag{6.19}$$

$$R \quad \to \quad \chi_R(t_3, t_0, n) D^{3/2}_{-3t_3-3t_0}|n\rangle \tag{6.20}$$

where $\chi_L(t_3, t_0, n)$ and $\chi_R(t_3, t_0, n)$ are the standard fermionic chiral fields for the particle labelled (t_3, t_0, n).

Then

$$\mathcal{M}(t_3, t_0, n) = \langle n|\bar{D}^{3/2}_{-3t_3-3t_0} D^{3/2}_{-3t_3-3t_0}|n\rangle(\bar{\chi}_L\varphi\bar{\chi}_R + \bar{\chi}_R\bar{\varphi}\chi_L) \tag{6.21}$$

By the argument of the standard theory

$$\bar{\chi}_L\varphi\chi_R + \bar{\chi}_R\bar{\varphi}\chi_L \tag{6.22}$$

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may be reduced to

$$\rho(\bar{\chi}_L \chi_R + \bar{\chi}_R \chi_L) = \rho \bar{\chi} \chi \tag{6.23}$$

where ρ is the vacuum expectation value of φ, the Higgs field. Then by (6.18)

$$\mathcal{M}(t_3, t_0, n) = m(t_3, t_0, n)\bar{\chi}\chi \tag{6.24}$$

and by (6.21)

$$m(t_3, t_0, n) = \rho(t_3, t_0)\langle n|\bar{D}^{3/2}_{-3t_3-3t_0} D^{3/2}_{-3t_3-3t_0}|n\rangle \tag{6.25}$$

which is compatible with (6.6) and (6.17) as shown, for example, in Ref. 4, where the energy levels given by (6.25) are fixed as polynomial functions of q^n.

7 Flavor States

The states $|n\rangle$ appearing in the form factor (5.6) are to be interpreted as mass states since they are states of the general Hamiltonian (6.6). However, instead of taking the operator in (5.6) between the states $|n\rangle$, eigenstates of b and c, one may take the same operator between eigenstates of d and a as follows:

$$\langle d'|M|a'\rangle = \sum \langle d'|b''\rangle\langle b''|M|b'\rangle\langle b'|a'\rangle \tag{7.1}$$

where a' and d' are eigenvalues of a and d and

$$|a'\rangle = \sum |b'\rangle\langle b'|a'\rangle \tag{7.2}$$

$$|d'\rangle = \sum |b'\rangle\langle b'|d'\rangle \tag{7.3}$$

In Ref. (3) with M given by (5.6) the matrix $\langle n''|M|n'\rangle = \langle b''|M|b'\rangle$ describing quark-quark interactions was proposed as a representation of the KM matrix. We now describe this matrix in the (a, d) instead of the (b, c) representation. Since the (d, a) operators are emission and absorption operators while the (b, c) operators define mass states, we shall describe the $|d'\rangle$ and $|a'\rangle$ states as flavor states. By (7.2) and (7.3) they are superpositions of mass states. Since the eigenstates of the emission and absorption operators correspond to the coherent states of the Maxwell field, we shall also refer to the corresponding $SL_q(2)$ states as coherent states.

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In the photon case the raising operator increases the number of photons by one. Here the raising operator increases the number of $|n+1\rangle$ particles and simultaneously decreases the number of $|n\rangle$ particles, in each case also by one. The lowering operator, as usual, acts oppositely to the raising operator. In the photon case the index n is the number of photons in one mode, while here n refers to the generation.

8 The Coherent States of $SU_q(2)$

We consider the unitary algebra $SU_q(2)$ obtained from $SL_q(2)$ by setting

$$d = \bar{a} \tag{8.1}$$

$$c = -q_1\bar{b} \tag{8.2}$$

Then the $SU_q(2)$ algebra is

$$
\begin{aligned}
ab = qba \quad & a\bar{a} + b\bar{b} = 1 \\
a\bar{b} = q\bar{b}a \quad & \bar{a}a + q_1^2\bar{b}b = 1
\end{aligned}
\tag{8.3}
$$

From the algebra one has

$$b\bar{a}^n = q^n \bar{a}^n b \tag{8.4}$$

Let $|0\rangle$ be the lowest eigenstate of b and let β be the lowest eigenvalue. Then

$$b \cdot \bar{a}^n|0\rangle = \beta q^n \cdot \bar{a}^n|0\rangle \tag{8.5}$$

Then the $\bar{a}^n|0\rangle$ are eigenstates of b with eigenvalues βq^n. Let

$$|n\rangle = \bar{a}^n|0\rangle \tag{8.6}$$

up to a normalization. Then

$$\bar{a}|n\rangle = \bar{\lambda}_n|n+1\rangle \tag{8.7}$$

$$\langle n|a = \langle n+1|\lambda_n \tag{8.7}'$$

and

$$\langle n|a\bar{a}|n\rangle = |\lambda_n|^2\langle n+1|n+1\rangle \tag{8.8}$$

Normalizing all states by setting $\langle n|n\rangle = 1$, we have

$$|\lambda_n|^2 = \langle n|a\bar{a}|n\rangle \tag{8.9}$$

By (8.3)

$$\begin{aligned} |\lambda_n|^2 &= \langle n|1 - b\bar{b}|n\rangle \\ &= 1 - q^{2n}|\beta|^2 \end{aligned} \tag{8.10}$$

We have assumed that the mass states $|n\rangle$ are eigenstates of the Hamiltonian:

$$H|n\rangle = E_n|n\rangle \tag{8.11}$$

where H is a function of

$$\frac{1}{2}(a\bar{a} + \bar{a}a) = 1 - \frac{1}{2}(1 + q_1^2)\bar{b}b \tag{8.12}$$

The eigenstates of the Hamiltonian are then also eigenstates of $\bar{b}b$. They are orthogonal since H and $\bar{b}b$ are hermitian.

We define the coherent states $|\alpha\rangle$ as eigenstates of the absorption and emission operators:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{8.13}$$

$$\langle\alpha|\bar{a} = \langle\alpha|\alpha^* \tag{8.14}$$

To express the coherent states as a superposition of the mass states, i.e. as

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle \tag{8.15}$$

we need the coefficients $\langle n|\alpha\rangle$.

By (8.7)$'$

$$\langle n|a|\alpha\rangle = \lambda_n\langle n+1|\alpha\rangle \tag{8.16}$$

and by (8.13)

$$\langle n|a|\alpha\rangle = \alpha\langle n|\alpha\rangle \tag{8.17}$$

Then by (8.16) and (8.17)

$$\begin{aligned} \langle n+1|\alpha\rangle &= \frac{\alpha}{\lambda_n}\langle n|\alpha\rangle \tag{8.18} \\ &= \frac{\alpha}{\lambda_n}\frac{\alpha}{\lambda_{n-1}}\ldots\langle 0|\alpha\rangle \tag{8.19} \end{aligned}$$

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where by (8.10)

$$\lambda_n = |1 - q^{2n}|\beta|^2|^{1/2}e^{i\varphi_n} \tag{8.20}$$

Then

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\prod\limits_0^{n-1}\lambda_s}\langle 0|\alpha\rangle \qquad n \geq 1 \tag{8.21}$$

and

$$\sum_{n\geq 0}\langle \alpha|n\rangle\langle n|\alpha\rangle = \sum_{n\geq 1}\frac{|\alpha|^{2n}}{|\prod\limits_0^{n-1}\lambda_s|^2}|\langle 0|\alpha\rangle|^2 + |\langle 0|\alpha\rangle|^2 \tag{8.22}$$

or if the $|n\rangle$ are complete

$$\langle \alpha|\alpha\rangle = \sum_{n\geq 1}\frac{|\alpha|^{2n}}{\prod\limits_0^{n-1}|\lambda_s|^2}|\langle 0|\alpha\rangle|^2 + |\langle 0|\alpha\rangle|^2 \tag{8.23}$$

Then, normalizing $\langle \alpha|\alpha\rangle = 1$, one has

$$|\langle 0|\alpha\rangle|^{-2} = \sum_{n\geq 1}\frac{|\alpha|^{2n}}{|\prod\limits_0^{n-1}\lambda_s|^2} + 1 \tag{8.24}$$

As usual $\langle \alpha|n\rangle$ is the adjoint of $\langle n|\alpha\rangle$, but the bracket $\langle n|\alpha\rangle$ between the mass and flavor states is not unitary.

9 The KM and the PMNS Matrices

Since there are only three generations in each family of fermions, there are only three mass states, which we label $n = 0, 1, 2$; and there are only three flavor states which we label by $\alpha_0, \alpha_1, \alpha_2$. Because we identify the flavor states as the coherent states, we have by (8.20) and (8.21)

$$\langle 1|\alpha_i\rangle = \frac{\alpha_i}{[1 - |\beta|^2]^{1/2}}\langle 0|\alpha_i\rangle \qquad i = 0, 1, 2 \tag{9.1}$$

$$\langle 2|\alpha_i\rangle = \frac{\alpha_i^2}{[(1 - |\beta|^2)(1 - q^2|\beta|^2)]^{1/2}}\langle 0|\alpha_i\rangle \tag{9.2}$$

By (8.15)

$$|\alpha_i\rangle = |0\rangle\langle 0|\alpha_i\rangle + |1\rangle\langle 1|\alpha_i\rangle + |2\rangle\langle 2|\alpha_i\rangle \qquad i = 0, 1, 2 \tag{9.3}$$

We shall take the mass states $|n\rangle$ orthonormal. Then we have

$$\langle\alpha_i|\alpha_i\rangle = |\langle 0|\alpha_i\rangle|^2 + |\langle 1|\alpha_i\rangle|^2 + |\langle 2|\alpha_i\rangle|^2 \tag{9.4}$$

Normalizing the coherent states, $\langle\alpha_i|\alpha_i\rangle = 1$, one has by (9.1), (9.2) and (9.4)

$$|\langle 0|\alpha_i\rangle| = \left[1 + \frac{|\alpha_i^2|}{1 - |\beta|^2} + \frac{|\alpha_i^4|}{(1 - |\beta|^2)(1 - q^2|\beta|^2)} \right]^{-1/2} \tag{9.5}$$

In our earlier work the elements of the "KM" matrix were expressed in the $|n\rangle$ or mass basis where the $|n\rangle$ are eigenstates of b and c. We now express the same operators in the $|\alpha\rangle$ or coherent basis, i.e. eigenfunctions of the absorption (a) and creation operators ($\bar{a}$). The elements of the KM matrix in this basis are

$$\langle u(i)|W^+|d(j)\rangle = \sum_{nn'}\langle u(i)|u(n)\rangle\langle u(n)|W^+|d(n')\rangle\langle d(n')|d(j)\rangle \tag{9.6}$$

$$\langle d(j)|W^-|u(i)\rangle = \sum_{nn'}\langle d(j)|d(n)\rangle\langle d(n)|W^-|u\langle n'|\rangle\langle u(n')|u(i)\rangle \tag{9.7}$$

where $u(i)$ and $d(j)$ are the up (u, c, t) and down (d, s, b) quark triplets. Here i and j describe flavor states while n and n' describe mass states. In these equations W^+ and W^- refer to charged W fields and are represented by D^3_{-30} and D^3_{30} respectively as in Table 5.2 and Eq. (5.6).

The associated form factors are special cases of (5.6). Corresponding to (9.6) and (9.7) we have

$$\langle u(n)|W^+|d(n')\rangle \sim \langle n|\bar{D}^{3/2}_{-\frac{3}{2}-\frac{1}{2}}D^3_{-30}D^{3/2}_{\frac{3}{2}-\frac{1}{2}}|n'\rangle \quad \text{and} \tag{9.8}$$

$$\langle d(n)|W^-|u(n')\rangle \sim \langle n|\bar{D}^{3/2}_{\frac{3}{2}-\frac{1}{2}}D^3_{30}D^{3/2}_{-\frac{3}{2}-\frac{1}{2}}|n'\rangle \tag{9.9}$$

With the same model for the PMNS matrix, the form factor is

$$\langle n|\bar{D}^{3/2}_{-\frac{3}{2}\frac{3}{2}}D^j_{00}D^{3/2}_{-\frac{3}{2}\frac{3}{2}}|n'\rangle \tag{9.10}$$

where $n = 0, 1, 2$ label the three generations, e.g. the e, μ and τ neutrino states. Here the $|n\rangle$ represent mass states. Since charge and hypercharge are conserved, the product $\bar{D}^{3/2}D^jD^{3/2}$ is neutral. It therefore lies in the $(b, \bar{b})$ subalgebra and has no off-diagonal elements. Then

$$\langle n'|\bar{D}^{3/2}_{m''p''}D^j_{m'p'}D^{3/2}_{mp}|n\rangle = \langle n'|F(b, \bar{b})|n\rangle \tag{9.11}$$

$$= \delta(n', n)F_n \tag{9.12}$$

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In terms of flavor states one then has

$$\langle i|F(b,\bar{b})|j\rangle = \sum_n \langle i|n\rangle F_n \langle n|j\rangle \tag{9.13}$$

The "KM" and "PMNS" matrices, which are diagonal in mass states $|n\rangle$, are not diagonal in the flavor states $|i\rangle$.

If one requires $\langle i|F(b,\bar{b})|j\rangle$ to be a $SU(3)$ matrix, then it is natural to parametrize by q, $|\beta|^2$, and the complex eigenvalues of α.

In both the KM and PMNS cases one makes use of the matrix $\langle n|\alpha\rangle$ that describes the mixing of mass states in the flavor states. The observational consequences are, however, very different in the two cases: the KM matrix describes transitions between quarks of different charge that are mediated by the $W^{\pm}$ field and in this case the $\langle n|\alpha\rangle$ matrix simply changes the basis from mass to flavor states; the PMNS matrix, on the other hand, describes neutral transitions between neutrinos of different mass. In both the quark and neutrino cases the different mass states travel at different velocities and oscillate at different frequencies but only in the neutrino case does the particle move far enough for the interference to be observed.

The probability of a neutrino being detected after the time t in the flavor states ν_j if it is emitted in the flavor state ν_i is

$$P_{i\to j} = |\langle \nu_j|\nu_i(t)\rangle|^2 \tag{9.14}$$

where the flavor states are superpositions of mass states ν_n:

$$\nu_i = \sum U_{in}\nu_n \tag{9.15}$$

The mass states are orthonormal

$$\langle \nu_n, \nu_{n'}\rangle = \delta_{nn'} \tag{9.16}$$

and propagate according to

$$\nu_n(t) = e^{i(E_n t - \vec{p}_n \vec{x})}\nu_n(0) \tag{9.17}$$

If $p \gg m$

$$\nu_n(t) \sim e^{-im_n^2 t/2E_n}\nu_n(0) \tag{9.18}$$

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Then

$$P_{i \to j}(t) = |U_{jn}^* U_{in} e^{-im_n t/2E_b}|^2 \qquad (9.19)$$

Constructive interference makes it possible for a neutrino created with a given flavor to change its flavor during propagation.

10 An Alternative Implementation of the $SL_q(2)$ Symmetry[6]

The previous sections of this paper have been based on discussions of the $SL_q(2)$ algebra where the mass states are identified with eigenstates of b and c and an associated Hamiltonian. The eigenvalues of this Hamiltonian are functions of q^n rather than n; the latter measures the eigenvalues of the Hamiltonian of the standard harmonic oscillator, and the q^n form a geometric rather than an arithmetical progression. We shall now describe an implementation of the $SL_q(2)$ symmetry that is closer to the standard procedures which are dependent on the familiar quantum oscillator. In this way one arrives at a different presentation of the flavor states, but the energy levels still turn out to depend on q^n.

The invariant matrix of $SL_q(2)$ is a 2-dimensional square root of -1, namely

$$\epsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \qquad (10.1)$$

Any 2-dimensional representation of $SL_q(2)$:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (10.2)$$

satisfies

$$T \epsilon_q T^t = T^t \epsilon_q T = \epsilon_q \qquad (10.3)$$

and the elements of T satisfy the knot algebra (A).

One may define a 2-dimensional vector basis Λ of $SL_q(2)$ by

$$\Lambda^t \epsilon_q \Lambda = q^{-1/2} \qquad (10.4)$$

Then (10.4) is invariant under $\Lambda' = T\Lambda$. Let

$$\Lambda = \begin{pmatrix} D_x \\ x \end{pmatrix} \qquad (10.5)$$

Then

$$D_x x - q x D_x = 1 \qquad (10.6)$$

This equation will be satisfied if D_x is chosen as a q difference operator:

$$D_x \psi(x) = \frac{\psi(qx) - \psi(x)}{qx - x} \qquad (10.7)$$

Then $x D_x$ is a "basic dilatation operator":

$$x D_x = \left\langle x \frac{d}{dx} \right\rangle \qquad (10.8)$$

$$= \frac{q^{x\frac{d}{dx}} - 1}{q - 1} \qquad (10.9)$$

If we introduce

$$P_x = \frac{\hbar}{i} D_x \qquad (10.10)$$

Then

$$(P_x x - q x P_x)\psi(x) = \frac{\hbar}{i}\psi(x) \qquad (10.11)$$

If $q = 1$, (10.11) is the Heisenberg commutator.

We also introduce the q-commutator and rewrite (10.6) as

$$[P_x, x]_q = -i\hbar \qquad (10.12)$$

We may quantize by (10.4) with Λ given by (10.5) and (10.10).We may also quantize by any Λ' related to Λ by

$$\Lambda' = T\Lambda \qquad T \epsilon S L_q(2) \qquad (10.13)$$

If q is near unity (as it must be insofar as the standard theory $(q = 1)$ is approximately correct) then $q = 1 + \epsilon$ and by 10.7)

$$D_x \psi(x) = \frac{\psi(x + \epsilon x) - \psi(x)}{\epsilon x} \qquad (10.14)$$

and D_x resembles the difference operator on a lattice space, and q may play the role of a dimensionless regulator.

Let us next apply this method of quantization to a harmonic oscillator.

11   The q-Quantized Oscillator[6]

Let us quantize according to (10.4) and (10.13) by taking

$$\Lambda' = Z = \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \tag{11.1}$$

Then

$$Z^t \epsilon_q Z = q^{-1/2} \tag{11.2}$$

or

$$z\bar{z} - q\bar{z}z = 1 \tag{11.3}$$

or

$$[z, \bar{z}]_q = 1 \tag{11.4}$$

Let us interpret z and $\bar{z}$ as absorption and emission operators and retain the usual harmonic oscillator Hamiltonian written in terms of these operators:

$$H = \frac{\hbar\omega}{2}(z\bar{z} + \bar{z}z) \tag{11.5}$$

Then if we modify the Heisenberg equation of motion in accordance with (11.4) we have

$$i\hbar\dot{z} = [z, H]_q \tag{11.6}$$

Assuming (11.5) and (11.6) we find the usual harmonic dependence

$$z \sim e^{i\omega t} \tag{11.7}$$

Denote the eigenstates of H by $|n\rangle$. Then z and $\bar{z}$ will satisfy (11.3) if

$$z|n\rangle = \langle n\rangle_q^{1/2}|n-1\rangle \qquad \langle n|\bar{z} = \langle n-1|\langle n\rangle_q^{1/2} \tag{11.8}$$

$$\bar{z}|n\rangle = \langle n+1\rangle_q^{1/2}|n+1\rangle \qquad \langle n|z = \langle n+1\rangle_q^{1/2}\langle n+1| \tag{11.9}$$

where $\langle n\rangle_q$ is the "basic number"

$$\langle n\rangle_q = \frac{q^n - 1}{q - 1} \tag{11.10}$$

Then

$$z\bar{z}|n\rangle = \langle n+1\rangle_q|n\rangle \tag{11.11}$$

$$\bar{z}z|n\rangle = \langle n\rangle_q|n\rangle \tag{11.12}$$

and

$$(z\bar{z} - q\bar{z}z)|n\rangle = (\langle n+1\rangle_q - q\langle n\rangle_q)|n\rangle = |n\rangle \tag{11.13}$$

Therefore the commutator (11.3) is satisfied. By (11.5)

$$H|n\rangle = \frac{\hbar\omega}{2}(\langle n+1\rangle_q + \langle n\rangle_q)|n\rangle \tag{11.14}$$

The eigenvalues of H in the limit $q = 1$ are

$$\frac{1}{2}\hbar\omega(2n+1)$$

in agreement with the standard harmonic oscillator. For general values of q, however, one has

$$\frac{1}{2}\left[\langle n+1\rangle_q + \langle n\rangle_q\right] = \frac{1}{2}\frac{q^{n+1} + q^n - 2}{q-1}$$

in agreement with (6.17) in its dependence on q^n rather than n. In fact, one may define on the algebra (8.3) as a special case of (6.6), a linear function of b, namely

$$H_b = \left(\frac{1}{2}\frac{q+1}{q-1}\frac{b}{\beta} - \frac{1}{q-1}\right)\hbar\omega$$

such that

$$H_b|n\rangle = E_n|n\rangle$$

where the $|n\rangle$ are the eigenstates of b, with eigenvalues βq^n by (8.5), and

$$\begin{aligned}
E_n &= \left(\frac{1}{2}\frac{q+1}{q-1}q^n - \frac{1}{q-1}\right)\hbar\omega \\
&= \frac{1}{2}\left(\langle n+1\rangle_q + \langle n\rangle_q\right)\hbar\omega
\end{aligned}$$

with the spectra and eigenstates of the Hamiltonian of the z-oscillator, namely

$$H_z = \frac{1}{2}(\bar{z}z + z\bar{z})\hbar\omega$$

with

$$z\bar{z} - q\bar{z}z = 1$$

according to (11.3), (11.5) and (11.14).

The coherent states are eigenstates of the absorption operator. Denote the coherent states by $|\zeta\rangle$. Then

$$z|\zeta\rangle = \zeta|\zeta\rangle \tag{11.15}$$

We express $|\zeta\rangle$ as a superposition of the eigenstates $|n\rangle$ of H i.e.,

$$|\zeta\rangle = \sum |n\rangle\langle n|\zeta\rangle \tag{11.16}$$

The $|n\rangle$ and $|\zeta\rangle$ states are again states of mass and flavor respectively.

To compute $\langle n|\zeta\rangle$ note

$$\langle n|z|\zeta\rangle = \langle n+1\rangle_q^{1/2}\langle n+1|\zeta\rangle \tag{11.17}$$

and

$$\langle n|z|\zeta\rangle = \zeta\langle n|\zeta\rangle \tag{11.18}$$

Then

$$\langle n+1|\zeta\rangle = \frac{\zeta}{\langle n+1\rangle_q^{1/2}}\langle n|\zeta\rangle \tag{11.19}$$

and

$$\langle n|\zeta\rangle = \frac{\zeta^n}{\langle n\rangle_q^{1/2}!}\langle 0|\zeta\rangle \tag{11.20}$$

One may normalize by setting

$$\langle 0|\zeta\rangle = e_q^{-|\zeta|^2} \tag{11.21}$$

where

$$e_q^{|\zeta|^2} = \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{\langle n_q\rangle!} \tag{11.22}$$

is the twisted exponential. Then

$$\langle \zeta|\zeta\rangle = \sum\langle \zeta|n\rangle\langle n|\zeta\rangle = 1 \tag{11.23}$$

If $q = 1$ then the basic numbers $\langle n\rangle_q$ are replaced by n and one recovers the familiar results for the coherent states of the harmonic oscillator. Since there are again only three ococupied states, one has

$$|\zeta_i\rangle = |0\rangle\langle 0|\zeta_i\rangle + |1\rangle\langle 1|\zeta_i\rangle + |2\rangle\langle 2|\zeta_i\rangle \tag{11.24}$$

Since the mass states are again orthonormal, one has

$$\langle \zeta_i | \zeta_i \rangle = |\langle 0 | \zeta_i \rangle|^2 + |\langle 1 | \zeta_i \rangle|^2 + |\langle 2 | \zeta_i \rangle|^2 \tag{11.25}$$

where by (11.20)

$$\langle 1 | \zeta_i \rangle = \zeta_i \langle 0 | \zeta_i \rangle \tag{11.26}$$

$$\langle 2 | \zeta_i \rangle = \frac{\zeta_i^2}{(1+q)^{1/2}} \langle 0 | \zeta_i \rangle \tag{11.27}$$

Normalize $\langle \zeta | \zeta \rangle = 1$. Then by (11.26)-(11.28)

$$\langle 0 | \zeta_i \rangle = \left[1 + |\zeta_i|^2 + \frac{|\zeta_i|^4}{(1+q)} \right]^{-1/2} \tag{11.28}$$

We now have two representations of the mass-flavor mixing matrix: either $\langle n | \alpha \rangle$ in Section 9 or $\langle n | \zeta \rangle$ in Section 11. Both representations are allowed by $SU_q(2)$.

12   References

1. N. Cabbibo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and K. Maskawa, Prog. Theor. Phys. **49**, 282 (1972); B. Pontecorvo, JETP **6**, 429 (1957); Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).

2. R. J. Finkelstein, Int. J. Mod. Phys. A**24**, 2307 (2009).

3. R. J. Finkelstein, ArXiv: hep.th/0701124 v1.

4. A. C. Cadavid and R. J. Finkelstein, *ibid.* A**21**, 4269 (2006).

5. R. J. Finkelstein, J. Math. Phys. **8**, 443 (1967).

6. R. J. Finkelstein and Eran Marcus, J. Math. Phys. **36**, 2652 (1995).