Causal Scattering Matrix and the Chronological Product

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Abstract. A causal scattering matrix is constructed by means of the mixed chronological and normal product of the free quantum fields of different variables $x \in \mathbf{R}^4$. This scattering matrix does not contain the diverging integrals.

1 Introduction

The scattering matrix connects the asymptotic Schrödinger equation solutions. Following Stueckelberg and Rivier [1] Bogoliubov [2] introduced the scattering matrix without making use of Schrödinger equation. Bogoliubov [2] defined the function $g(x)$ taking the values in the interval $[0,1]$ and representing the intensity of interaction switching. Then in the space-time domains where $g(x) = 0$ the interaction is absent, in the space-time domains where $g(x) = 1$ it is switched on absolutely and for $0 < g(x) < 1$ it is switched on partially. Now let $g(x)$ be not zero only in some finite space-time domain. In this case the fields are free in the sufficiently long ago past and in the sufficiently distant future. Bogoliubov [2] believed that the initial and final states should be connected by some operator $S(g)$. The operator $S(g)$ is naturally interpreted as the scattering operator for the case when the interaction is switched on with the intensity $g(x)$. Bogoliubov [2] believed also that the "physical" case when the interaction is switched on absolutely in the whole space-time must be considered in the given scheme by making use of the limit process when the space-time domain where $g(x) = 1$ spreads infinitely to the whole space-time. If for some matrix elements of the operator $S(g)$ the limit values exist, then these limit matrix elements ought be considered as the corresponding matrix elements of the scattering matrix S. The mathematical reason for the switching function $g(x)$ is very simple: the distributions should be integrated with the smooth functions rapidly decreasing at the infinity.

Let us formulate the main physical conditions the operator $S(g)$ should satisfy. In order to guarantee the theory covariance we need to demand

$$S(Lg) = U(L)S(g)U^*(L) \tag{1.1}$$

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where $Lg(x) = g(L^{-1}x)$ and $U(L)$ is a unitary operator by means of which the quantum wave functions transform under the transformations L from the Lorentz group and the group of translations. In order to conserve the amplitude state norm under the transformation from the initial state to the final state the operator $S(g)S^*(g)$ has to be the projector on the subspace of the asymptotic states. The operator $S(g)$ is defined on the subspace of the asymptotic states. Bogoliubov [2] required that the operator $S(g)$ satisfies the unitary condition:

$$S(g)S^*(g) = 1. \tag{1.2}$$

The identity operator is denoted by 1 and is often omitted.

Now let us take into account the causality condition according to which some event in the system can influence the evolution of the system in the future only and can not influence the behavior of the system in the past, in the time preceding the given event. Therefore we need to demand that a change in the interaction law in some space-time domain can change the motion in the succeeding moments only. Due to the book ([3], Section 17.5) we formulate the causality condition. We consider the case when the space-time domain G where the function $g(x)$ is not zero is divided into two separate domains G_1 and G_2 such that all time points of the domain G_1 lie in the past relative to some moment t and all time points of the domain G_2 lie in the future relative to t. Then the function $g(x)$ may be represented as a sum of two functions

$$g(x) = g_1(x) + g_2(x) \tag{1.3}$$

where the function g_1 is not zero in the domain G_1 only and the function g_2 is not zero in the domain G_2 only. The causality condition is called the relation

$$S(g_1 + g_2) = S(g_2)S(g_1). \tag{1.4}$$

Bogoliubov [2] defines the causal scattering matrix operator $S(g)$ by means of Lagrange function constructed from the normal products of the free quantum fields of one variable $x \in \mathbf{R}^4$. The causal scattering matrix operator $S(g)$ coefficients contain the diverging integrals.

In order to "quantize" the physicists change the powers of the classical variables in the classical Lagrange function for the normal products of the free field operators at the same space-time point. The "quantized" Lagrangian mechanics is not compatible with the causality condition (1.4).

This paper is the straightforward generalization of the paper [2]. In this paper Lagrange function is changed for a linear combination of the normal products of the free quantum fields of different variables $x \in \mathbf{R}^4$. The switching function $g(x)$ of one variable $x \in \mathbf{R}^4$ is changed for the set of the switching functions of different variables $x \in \mathbf{R}^4$. The causal scattering matrix operator satisfying the relation of the type (1.4) is constructed by means of the mixed chronological and normal product of the free quantum fields of different variables $x \in \mathbf{R}^4$. The causal scattering matrix coefficients do not contain the diverging integrals.

2 Chronological Product

Consider a free real scalar field and a free spin field given by the distributions $\varphi(x)$ and $\psi_\alpha(x)$ taking the values in the set of Hilbert space operators with the commutation relations (11.3) and (13.4) from the book [3]

$$[\varphi(x), \varphi(y)] = \frac{1}{i} D_{m^2}(x - y), \tag{2.1}$$

$$[\psi_\alpha(x), \bar{\psi}_\beta(y)]_+ \equiv \psi_\alpha(x)\bar{\psi}_\beta(y) + \bar{\psi}_\beta(y)\psi_\alpha(x) = \frac{1}{i}\left(i\sum_{\mu=0}^{3}\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} + m\right) D_{m^2}(x-y)$$

where the Pauli - Jordan distribution ([3], relation (10.18))

$$D_{m^2}(x) = \frac{i}{(2\pi)^3}\int d^4k(\theta(k^0)\delta((k,k)-m^2) - \theta(-k^0)\delta((k,k)-m^2))e^{-i(k,x)}, \tag{2.2}$$

$$(k,x) = \sum_{\mu,\nu=0}^{3}\eta_{\mu\nu}k^\mu x^\nu.$$

Here $\eta_{\mu\nu}$ is the diagonal 4×4 - matrix with the diagonal elements $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$ and $\gamma^\mu_{\alpha\beta}$ is the Dirac matrices ([3], relations (6.18)). The commuting relations of the free vector field $U_\mu(x)$ and the free electromagnetic field $A_\mu(x)$ are given by the relations (11.27) and (12.4) from the book [3]. These relations are similar to the relations (2.1). The operator valued distributions $\varphi(x)$, $U_\mu(x)$, $A_\mu(x)$, $\psi_\alpha(x)$ and all its possible derivatives are called the free quantum fields and denote $u_\alpha(x)$.

The vacuum expectation of the product of two free fields is given by the relations (10.17), (16.12) and (16.14) from the book [3]

$$< \varphi(x)\varphi(y) >_0 = \frac{1}{i}D^-_{m^2}(x-y) \equiv \frac{1}{(2\pi)^3}\int d^4k\theta(k^0)\delta((k,k)-m^2)e^{-i(k,x-y)}, \tag{2.3}$$

$$< \psi_\alpha(x)\bar{\psi}_\beta(y) >_0 = \frac{1}{i}\left(i\sum_{\mu=0}^{3}\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu} + m\right) D^-_{m^2}(x-y).$$

The vacuum expectations $< U^*_\lambda(x)U_\nu(y) >_0$ and $< A_\lambda(x)A_\nu(y) >_0$ are similar to the vacuum expectations (2.3). The vacuum expectations of another free fields products are either the derivatives of the distributions (2.3) or are equal to zero.

The free fields normal product is given by the relations (16.17) from the book [3]

$$:1: = 1, \ :u_\alpha(x): = u_\alpha(x),$$

$$u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n) =: u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n): + \sum_{1\le k<l\le n} < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0 \times$$
$$: u_{\alpha(1)}(x_1)\cdots \widehat{u_{\alpha(k)}(x_k)}\cdots \widehat{u_{\alpha(l)}(x_l)}\cdots u_{\alpha(n)}(x_n): + \cdots, \ n = 2,3,... \tag{2.4}$$

The subsequent summings in the equality (2.4) run over two pairs of numbers from $1,...,n$, over three pairs of numbers from $1,...,n$, etc. In the book (([3]), Section 16.2) the definition (2.4) is called the Wick theorem for the normal products. The normal product may be also defined in the following way

$$: u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n) := u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n) - \sum_{1\le k<l\le n} < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0 \times$$
$$u_{\alpha(1)}(x_1)\cdots \widehat{u_{\alpha(k)}(x_k)}\cdots \widehat{u_{\alpha(l)}(x_l)}\cdots u_{\alpha(n)}(x_n) + \cdots, \ n = 2,3,... \tag{2.5}$$

In the equality (2.5) the subsequent summing run over two pairs of numbers from $1,..,n$, over three pairs of numbers from $1,..,n$, etc. The summing over an even (odd) number of pairs has the sign plus (minus). The relation (2.5) for $n = 2$ coincides with the relation (2.4) for $n = 2$.

Let us prove the relation (2.5) by making use of the relation (2.4). Let us change every distribution $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0$ in the right-hand side of the relation (2.4) for the distribution

$$< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0 - < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0 \tag{2.6}$$

equal to zero. Then we get the relation

$$\begin{aligned} &: u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n) :=: u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n) : + \\ \sum_{1 \leq k < l \leq n} &(< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0 - < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0) \times \\ &: u_{\alpha(1)}(x_1) \cdots \widehat{u_{\alpha(k)}(x_k)} \cdots \widehat{u_{\alpha(l)}(x_l)} \cdots u_{\alpha(n)}(x_n) : + \cdots \end{aligned} \tag{2.7}$$

Choose the first term $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0$ in every sum (2.6) of the equality (2.7). Adding the first term $: u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n) :$ we get due to the relation (2.4) the first term of the right-hand side of the relation (2.5). Let us choose the second term $- < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0$ in one sum (2.6) of the relation (2.7) and the first term $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0$ in all other sums (2.6). Due to the relation (2.4) we get the second term of the right-hand side of the relation (2.5). If we continue this process, we transform the relation (2.7) into the relation (2.5).

Let the vacuum expectation of normal product of arbitrary number $n > 0$ of free fields vanish. Let also $< 1 >_0 = 1$. Hence the relation (2.4) implies that the vacuum expectation of any odd number of free fields vanishes and the vacuum expectation of any even number of free fields is equal to

$$\begin{aligned} &< u_{\alpha(1)}(x_1) \cdots u_{\alpha(2n)}(x_{2n}) >_0 = \\ \sum_{\sigma} &< u_{\alpha(\sigma(1))}(x_{\sigma(1)})u_{\alpha(\sigma(2))}(x_{\sigma(2)}) >_0 \cdots < u_{\alpha(\sigma(2n-1))}(x_{\sigma(2n-1)})u_{\alpha(\sigma(2n))}(x_{\sigma(2n)}) >_0 \end{aligned} \tag{2.8}$$

where the summing runs over all permutations σ of the numbers $1, ..., 2n$ not changing the order in any pair of the numbers $2k - 1$, $2k$: $\sigma(2k - 1) < \sigma(2k)$ for any $k = 1, ..., n$.

By making use of the relations (2.5), (2.8) we get the rule for calculation of the vacuum expectation

$$< u_{\beta(1)}(y_1) \cdots u_{\beta(m)}(y_m) : u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n) : u_{\beta'(1)}(y'_1) \cdots u_{\beta'(m')}(y'_{m'}) >_0 .$$

In the sum (2.8) for the vacuum expectation

$$< u_{\beta(1)}(y_1) \cdots u_{\beta(m)}(y_m)u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n)u_{\beta'(1)}(y'_1) \cdots u_{\beta'(m')}(y'_{m'}) >_0 \tag{2.9}$$

it is needed to cancel all terms containing at least one multiplier $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0$, $1 \leq k < l \leq n$. Hence it is possible to let $x_1 = \cdots = x_n$ in the vacuum expectation

$$< u_{\beta(1)}(y_1) \cdots u_{\beta(m)}(y_m) : u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n) : u_{\beta'(1)}(y'_1) \cdots u_{\beta'(m')}(y'_{m'}) >_0 .$$

Therefore there exists the integral

$$\int d^4x_1 \cdots d^4x_n : u_{\alpha(1)}(x_1) \cdots u_{\alpha(n)}(x_n) : h(x_1, .., x_n) \tag{2.10}$$

for the distribution

$$h(x_1,..,x_n)=g(x_1)\delta(x_2-x_1)\cdots\delta(x_n-x_1),\ g(x_1)\in D(\mathbf{R}^4). \tag{2.11}$$

The integral (2.10), (2.11) exists for free fields only. The normal product of interacting fields is not defined. It is impossible to let $x_1=\cdots=x_n$ in the expectation (2.9) of the free fields product $u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n)$. The integral (2.10), (2.11) we denote as

$$\int d^4x : u_{\alpha(1)}(x)\cdots u_{\alpha(n)}(x) : g(x). \tag{2.12}$$

If Fermi fields (the operators $\psi_\alpha(x)$, $\bar{\psi}_\alpha(x)$ and their derivatives) are included in the normal product (2.10) in the even combinations, the operator (2.10) is called polylocal ([3], Section 16.8). The polylocal operator (2.12) is called local ([3], Section 16.8). Due to the book ([3], Section 18.3) "the interaction Lagrangian should be the local, Hermitian and Lorentz covariant combination of field operator functions." We consider the polylocal combination of the field operator functions (2.10). If we take the distribution (2.11), we get the local interaction Lagrangian (2.12).

The chronological product of the field operators is defined by the relation (19.1) from the book [3]

$$T(u_{\alpha(1)}(x_1);\cdots;u_{\alpha(n)}(x_n))=(-1)^p u_{\alpha(j_1)}(x_{j_1})\cdots u_{\alpha(j_n)}(x_{j_n}),\ x^0_{j_1}\geq x^0_{j_2}\geq\cdots\geq x^0_{j_n} \tag{2.13}$$

where p is the parity of the Fermi fields permutation corresponding to the permutation j transforming the numbers $1,2,...,n$ into the numbers $j_1,j_2,...,j_n$. Due to the paper [2]: "Let us note as Stueckelberg did that the usual definition of T - product by means of introduction the chronological order for the operators is effective only without the coincidence of the arguments $x_1,...,x_n$. In view of the corresponding coefficient functions singularity their "redefinition" in the domains of the arguments coincidence is not done explicitly and presents a special problem."
It is necessary to change the closed set $x^0_{j_1}\geq x^0_{j_2}\geq\cdots\geq x^0_{j_n}$ for the open set $x^0_{j_1}>x^0_{j_2}>\cdots>x^0_{j_n}$ in the definition (2.13). The distribution may be restricted only to the open set. The correct relation (2.13) does not define the chronological product in the domains of the time arguments coincidence.

In the book ([3], Section 19.2) another definition of the field operators chronological product for $n=2,3,...$ is obtained

$$T(u_{\alpha(1)}(x_1);\cdots;u_{\alpha(n)}(x_n))=:u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n):+\sum_{1\leq k<l\leq n}$$
$$<T(u_{\alpha(k)}(x_k);u_{\alpha(l)}(x_l))>_0 : u_{\alpha(1)}(x_1)\cdots\widehat{u_{\alpha(k)}(x_k)}\cdots\widehat{u_{\alpha(l)}(x_l)}\cdots u_{\alpha(n)}(x_n):+\cdots \tag{2.14}$$

In the following terms in the equality (2.14) the summings run over two pairs of the numbers from $1,..,n$, over three pairs, etc. Due to the relations (19.6) and (19.9) from the book [3]

$$<T(\varphi(x);\varphi(y))>_0=\frac{1}{i}D^c_{m^2}(x-y)\equiv\lim_{\epsilon\to+0}\frac{1}{(2\pi)^4 i}\int d^4k\frac{e^{i(k,x-y)}}{m^2-(k,k)-i\epsilon}, \tag{2.15}$$

$$<T(\psi_\alpha(x);\bar{\psi}_\beta(y))>_0=\frac{1}{i}\left(i\sum_{\mu=0}^{3}\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial x^\mu}+m\right)D^c_{m^2}(x-y).$$

The vacuum expectations $< T(A_\lambda(x); A_\nu(y)) >_0$ and $< T(U_\lambda^*(x); U_\nu(y)) >_0$ are similar to the vacuum expectations (2.15). The distributions $< T(u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2)) >_0$ for other free fields are the derivatives of the distributions (2.15) or are equal to zero. In the book ([3], Section 19.2) the relation (2.14) is called the Wick theorem for chronological products. If we replace the distributions $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0$ with the distributions $< T(u_{\alpha(k)}(x_k); u_{\alpha(l)}(x_l)) >_0$ in the relation (2.4), then we get the relation (2.14).

Due to the relation (14.12) from the book [3]

$$D^c_{m^2}(x) = \begin{cases} D^-_{m^2}(x), & x^0 > 0, \\ D^-_{m^2}(-x), & x^0 < 0. \end{cases} \tag{2.16}$$

Hence the distribution $D^c_{m^2}(x) - D^-_{m^2}(x) = 0$ for $x^0 > 0$. The definitions (2.3), (2.15) imply Lorentz invariance of the distribution $D^c_{m^2}(x) - D^-_{m^2}(x)$. Hence the support of this distribution lies in the closed lower light cone. The distribution $D^-_{m^2}(x)$ satisfies the Klein - Gordon equation

$$((\partial_x, \partial_x) + m^2)D^-_{m^2}(x) = 0, \ (\partial_x, \partial_x) = \sum_{\mu=0}^{3} \eta_{\mu\mu} \left(\frac{\partial}{\partial x^\mu}\right)^2. \tag{2.17}$$

The distribution $D^c_{m^2}(x)$ is the fundamental solution of the Klein - Gordon equation

$$((\partial_x, \partial_x) + m^2)D^c_{m^2}(x) = \delta(x). \tag{2.18}$$

Hence the distribution $D^c_{m^2}(x) - D^-_{m^2}(x)$ satisfies the equation (2.18). Let us prove that the equation (2.18) has the unique solution in the class of the distributions with supports in the closed lower light cone. Let the equation (2.18) have two solutions $e^{(1)}(x)$, $e^{(2)}(x)$ with supports in the closed lower light cone. Since its supports lie in the closed lower light cone, the convolution is defined. The convolution commutativity implies the coincidence of these solutions

$$e^{(2)}(x) = ((\partial_x, \partial_x) + m^2) \int d^4y e^{(1)}(x-y)e^{(2)}(y) =$$
$$((\partial_x, \partial_x) + m^2) \int d^4y e^{(2)}(x-y)e^{(1)}(y) = e^{(1)}(x). \tag{2.19}$$

Therefore the distribution $e^{(1)}(x)$ coincides with the distribution

$$D^{ret}_{m^2}(-x) = \lim_{\epsilon \to +0} \frac{1}{(2\pi)^4} \int d^4k \frac{e^{i(k,x)}}{m^2 - (k^0 + i\epsilon)^2 + |\mathbf{k}|^2} \tag{2.20}$$

given by the relation (14.7) from the book [3] and

$$D^c_{m^2}(x) = D^{ret}_{m^2}(-x) + D^-_{m^2}(x). \tag{2.21}$$

The distribution (2.20) is fundamental for the quantum field theory. It seems natural to use the special notation instead the cumbersome notation $-D^{ret}_{m^2}(-x)$ of the book [3].

The substitution of the equality (2.21) into the relations (2.15) yields

$$< T(u_{\alpha(1)}(x_1); u_{\alpha(2)}(x_2)) >_0 = < u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2) >_0 + < u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2) >_c, \tag{2.22}$$

$$< \varphi(x)\varphi(y) >_c = \frac{1}{i} D^{ret}_{m^2}(y-x), \tag{2.23}$$

$$< \psi_\alpha(x)\bar{\psi}_\beta(y) >_c = \frac{1}{i}\left(i\sum_{\mu=0}^{3}\gamma^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu} + m\right) D^{ret}_{m^2}(y-x).$$

The distributions $< U^*_\lambda(x)U_\nu(y) >_c$ and $< A_\mu(x)A_\nu(y) >_c$ are similar to the distributions (2.23). The distributions $< u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2) >_c$ for other free fields are the derivatives of the distributions (2.23) or are equal to zero.

We substitute the relation (2.22) into the right-hand side of the equality (2.14). Let us take the distribution $< u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2) >_0$ in any sum (2.22) in the equality (2.14). Then we take the distribution $< u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2) >_c$ in one sum (2.22) in the equality (2.14), take the distribution $< u_{\alpha(1)}(x_1)u_{\alpha(2)}(x_2) >_0$ in all other sums (2.22) in the right-hand side of the equality (2.14) and so on. Therefore we have

$$\begin{aligned} &T(u_{\alpha(1)}(x_1);\cdots;u_{\alpha(n)}(x_n)) = \\ &\Big\{ : u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n) : + \sum_{1\le k<l\le n} < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_0 \times \\ &: u_{\alpha(1)}(x_1)\cdots \widehat{u_{\alpha(k)}(x_k)}\cdots\widehat{u_{\alpha(l)}(x_l)}\cdots u_{\alpha(n)}(x_n) : + \cdots \Big\} + \\ &\sum_{1\le k<l\le n} < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_c \times \\ &\Big\{ : u_{\alpha(1)}(x_1)\cdots \widehat{u_{\alpha(k)}(x_k)}\cdots\widehat{u_{\alpha(l)}(x_l)}\cdots u_{\alpha(n)}(x_n) : + \cdots \Big\} + \cdots. \end{aligned} \tag{2.24}$$

The following summings run over two pairs of the numbers from $1,..,n$, over three pairs of the numbers, etc. By making use of the relation (2.4) it is possible to rewrite the equality (2.24) as

$$\begin{aligned} T(u_{\alpha(1)}(x_1);\cdots;u_{\alpha(n)}(x_n)) &= u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n) + \\ &\sum_{1\le k<l\le n} < u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_c \times \\ u_{\alpha(1)}(x_1)\cdots &\widehat{u_{\alpha(k)}(x_k)}\cdots\widehat{u_{\alpha(l)}(x_l)}\cdots u_{\alpha(n)}(x_n) + \cdots. \end{aligned} \tag{2.25}$$

The following summings run over two pairs of the numbers from $1,..,n$, over three pairs of the numbers, etc. The relations (2.23), (2.25) yield the definition of the chronological product of free field operators. It is sufficient to use the distributions (2.23) for the definition (2.25) of the chronological product. The distributions (2.3) and (2.15) are needed for the definition (2.14) of the chronological product.

The difference between the chronological product of the free quantum field operators $T(u_{\alpha(1)}(x_1);\cdots;u_{\alpha(n)}(x_n))$ and the usual product of these fields $u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n)$ is represented by the sum of the terms proportional to the distributions $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_c$, $1 \le k < l \le n$. The distribution $D^{ret}_{m^2}(y-x)$ vanishes except for the vectors $x-y$ lying in the closed lower light cone. Thus the chronological product of the free quantum field operators $T(u_{\alpha(1)}(x_1);\cdots;u_{\alpha(n)}(x_n))$ differs from the usual product of these free quantum field operators $u_{\alpha(1)}(x_1)\cdots u_{\alpha(n)}(x_n)$ in the only case when for some numbers $1 \le k < l \le n$ the argument difference $x_k - x_l$ lies in the closed lower light cone. In the definition of the chronological product the distributions $< u_{\alpha(k)}(x_k)u_{\alpha(l)}(x_l) >_c$, $1 \le k < l \le n$ define the delays:

$$D^{ret}_0(x) = (2\pi)^{-1}\theta(x^0)\delta((x,x)).$$

The Newton gravity law requires the instant propagation of the force action. The special relativity requires that the propagation speed does not exceed that of light. (We believe that the gravity propagation speed coincides with the light speed.) It requires also the gravity laws covariance under Lorentz transformation. Long ago Poincaré [4] tried to find such a modification of the Newton gravity law: ”First of all, it enables us to suppose that the gravity forces propagate not instantly, but at the light velocity”. The interaction force of two physical points should depend not on their simultaneous positions and speeds but on the positions and the speeds at the time moments which differ from each other in the interval needed for light covering the distance between the physical points. (The interaction force of two physical points should depend also on the acceleration of one physical point at the delayed time moment.) The delay is one of possible causality condition statements. The Lorentz covariance and the causality condition are the crucial points of the relativistic quantum field theory. These conditions were proposed by Poincaré [4] for the relativistic causal gravity law. These conditions should be valid for any interaction.

The distribution $D^c_{m^2}(x-y)$ defines the vacuum expectation of the chronological product (2.23), (2.25) of two free quantum fields. Due to Stueckelberg and Rivier [1] the classical ”causal action” is given by the distribution $D^{ret}_{m^2}(y-x)$ and the distribution $D^c_{m^2}(x-y) = D^c_{m^2}(y-x)$ defines the probability amplitude of the ”causal action”.

Let us prove that the chronological product definition (2.25) is in accordance with the correct definition (2.13). We rewrite the definition (2.25) in the recurrent way:

$$T(u_\alpha(x)) = u_\alpha(x),$$

$$\begin{aligned} T(u_{\alpha(1)}(x_1); \cdots ; u_{\alpha(n+1)}(x_{n+1})) = T(u_{\alpha(1)}(x_1); \cdots ; u_{\alpha(n)}(x_n)) u_{\alpha(n+1)}(x_{n+1}) + \\ \sum_{k=1}^{n} < u_{\alpha(k)}(x_k) u_{\alpha(n+1)}(x_{n+1}) >_c T(u_{\alpha(1)}(x_1); \cdots ; \widehat{u_{\alpha(k)}(x_k)}; \cdots ; u_{\alpha(n)}(x_n)). \end{aligned} \tag{2.26}$$

Let us consider the permutation j_1, j_2 of the numbers $1, 2$. The relation (2.1), the relation (14.8) from the book [3]

$$D^{ret}_{m^2}(x) = \begin{cases} D_{m^2}(x), & x^0 > 0, \\ 0, & x^0 < 0, \end{cases} \tag{2.27}$$

and the relations (2.25) for $n = 2$ imply that the relation (2.13) for $n = 2$ is valid for the chronological product (2.25). Let us consider the permutation $j_1, ..., j_m$ of the numbers $1, ..., m$. Suppose that for the coordinates $x^0_{j_1} > \cdots > x^0_{j_m}$ and for any number $m = 2, .., n$ the relation (2.13) is valid for the chronological product (2.25). Hence the definition (2.26) and the relations (2.1), (2.27) imply the relation (2.13) for the chronological product (2.25) in the case of $n+1$ fields for any permutation $j_1, ..., j_{n+1}$ of the numbers $1, ..., n+1$ and for the coordinates $x^0_{j_1} > x^0_{j_2} > \cdots > x^0_{j_{n+1}}$.

Define the mixed chronological and normal product of the free field operators. The sum for the chronological product

$$T\left(: \prod_{i=1}^{n_1} u_{\alpha(i)}(x_i) :; \cdots ; : \prod_{i=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} u_{\alpha(i)}(x_i) : \right)$$

is the sum (2.25) for the chronological product $T(u_{\alpha(1)}(x_1); \cdots ; u_{\alpha(n_1+\cdots+n_k)}(x_{n_1+\cdots+n_k}))$ where all distributions $< u_{\alpha(m)}(x_m) u_{\alpha(l)}(x_l) >_c$ with the arguments from the same group: $n_1 + \cdots +$

$n_{j-1} < m < l \leq n_1 + \cdots + n_j$ are replaced by the distributions $- < u_{\alpha(m)}(x_m) u_{\alpha(l)}(x_l) >_0$. Thus the chronological order is introduced only for the free field operators $u_{\alpha(m)}(x_m)$, $u_{\alpha(l)}(x_l)$ the arguments x_m, x_l of which are included into the different groups of the arguments. For the free field operators $u_{\alpha(m)}(x_m), u_{\alpha(l)}(x_l)$ the arguments x_m, x_l of which are included into the same group the normal product is supposed. For the chronological product we consider the operators $: u_{\alpha(1)}(x_1) \cdots u_{\alpha(n_i)}(x_{n_i}) :$, $i = 1, ..., k$ as the whole objects.

Let the groups of the time arguments $x_1^0, ..., x_{n_1}^0; x_{n_1+1}^0, ..., x_{n_1+n_2}^0; ...; x_{n_1+\cdots n_{k-1}+1}^0, ..., x_{n_1+\cdots n_k}^0$ are ordered due to the subdivision $i_1, ..., i_l$, $j_1, ..., j_{k-l}$ of the numbers $1, ..., k$: any argument from the first group exceeds any argument from the second group $x_m^0 > x_q^0$ for any numbers $n_1+\cdots+n_{i_s-1} < m \leq n_1+\cdots+n_{i_s}$, $s = 1, ..., l$, and $n_1+\cdots+n_{j_t-1} < q \leq n_1+\cdots+n_{j_t}$, $t = 1, .., k-l$. By making use of the definition of the mixed chronological and normal product of the free field operators it is easy to prove the following relation

$$T\left(: \prod_{s=1}^{n_1} u_{\alpha(s)}(x_s) :; \cdots ; : \prod_{s=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} u_{\alpha(s)}(x_s) :\right) =$$
$$(-1)^p T\left(: \prod_{m=n_1+\cdots+n_{i_1-1}+1}^{n_1+\cdots+n_{i_1}} u_{\alpha(m)}(x_m) :; \cdots ; : \prod_{m=n_1+\cdots+n_{i_l-1}+1}^{n_1+\cdots+n_{i_l}} u_{\alpha(m)}(x_m) :\right) \times$$
$$T\left(: \prod_{q=n_1+\cdots+n_{j_1-1}+1}^{n_1+\cdots+n_{j_1}} u_{\alpha(q)}(x_q) :; \cdots ; : \prod_{q=n_1+\cdots+n_{j_{k-l}-1}+1}^{n_1+\cdots+n_{j_{k-l}}} u_{\alpha(q)}(x_q) :\right) \quad (2.28)$$

where p is the parity of the Fermi operators permutation. The relation (2.28) for the mixed chronological and normal product of the free quantum fields is probably the differential form of the causality condition.

3 Scattering Matrix

Let us seek for a scattering matrix in the form

$$S(h) = 1 + \sum_{k=1}^{\infty} \sum_{n_1,...,n_k} \sum_{\alpha(1),...,\alpha(n_1+\cdots+n_k)}$$
$$\frac{1}{k!} K_{\alpha(1),...,\alpha(n_1)} \cdots K_{\alpha(n_1+\cdots+n_{k-1}+1),...,\alpha(n_1+\cdots+n_k)} \times$$
$$\int d^4x_1 \cdots d^4x_{n_1+\cdots+n_k} \left(h_{n_1}(x_1, ..., x_{n_1}) \cdots h_{n_k}(x_{n_1+\cdots+n_{k-1}+1}, ..., x_{n_1+\cdots+n_k})\right) \times$$
$$T\left(: \prod_{i=1}^{n_1} u_{\alpha(i)}(x_i) :; \cdots ; : \prod_{i=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} u_{\alpha(i)}(x_i) :\right) \quad (3.1)$$

where the mixed chronological and normal product of free quantum field operators is defined above; in order to guarantee the scalar character of the normal product Fermi operators must be included in the even combinations only; $K_{\alpha(1),...,\alpha(n)}$ are the constants; the switching functions $h_n(x_1, ..., x_n) \in D(\mathbf{R}^{4n})$. We consider that the natural numbers $n_1, ..., n_k$ and the indexes $\alpha(1), ..., \alpha(n_1 + \cdots + n_k)$ in the equality (3.1) run over the finite sets of values.

If we insert into the equality (3.1) the distributions

$$h_n(x_1, ..., x_n) = g(x_1)\delta(x_2 - x_1) \cdots \delta(x_n - x_1), \quad (3.2)$$

we get the expression similar to the scattering matrix expression in the paper ([2], relations (5), (17)) and in the book ([3], relation (18.32)). Bogoliubov[2] also believed that for the physical scattering matrix the switching function $g(x)$ is equal to 1 in the relations (3.2). Substituting the distributions (3.2) into the operator (3.1) we have the diverging integrals: the distributions should be integrated with the smooth functions rapidly decreasing at the infinity.

In the book ([3], relation (18.27)) the chronological product of the local operators is defined by means of the relation analogous to the relation (2.13). Due to the paper [2]: "Let us note as Stueckelberg did that the usual definition of T - product by means of introduction the chronological order for the operators is effective only without the coincidence of the arguments $x_1, ..., x_n$. In view of the corresponding coefficient functions singularity their "redefinition" in the domains of the arguments coincidence is not done explicitly and presents a special problem...

If we do not call attention to this difficulty and use the Wick theorem formally, then we get the expressions of the form:

$$\prod_{a<b} D^c_{m^2_{ab}}(x_a - x_b) \tag{3.3}$$

consisting of the causal D^c - functions products.

If we consider Fourier transform, then we get the integrals with the well-known "ultraviolet" divergences."

The local interaction Lagrangian in the scattering matrix (3.1), (3.2) implies the singularities (3.3). The local interaction Lagrangian (2.12) is the asymptotic value of the polylocal normal product (2.10): if the smooth function $h(x_1, .., x_n)$ tends to the distribution $g(x_1)\delta(x_2 - x_1) \cdots \delta(x_n - x_1)$ where the function $g(x_1) \in D(\mathbf{R}^4)$, then the polylocal normal product (2.10) tends to the local normal product (2.12). The chronological product for the local operators (2.12) is not correct. The interaction propagates not instantly but at the speed not exceeding the speed of light. We have to take into account the distance between the interacting particles. Every interacting particle needs its own delay. We need to consider the chronological product for the polylocal normal products (2.10).

Let us consider the scattering matrix (3.1) with the switching functions

$$h_n(x_1, ..., x_n) = h^{(1)}_n(x_1, ..., x_n) + h^{(2)}_n(x_1, ..., x_n). \tag{3.4}$$

The support of the function $h^{(i)}_n(x_1, ..., x_n)$ lies in the domain $G^{\times n}_i$, $i = 1, 2$, and all time points of the domain G_2 lie in the future relative to all time points of the domain G_1. The decomposition (3.4) is analogous to the decomposition (1.3).

Let the subdivision $i_1, ..., i_l,\ j_1, ..., j_{k-l}$ of the numbers $1, ..., k$ be given. The relation (2.28) implies

$$\int d^4x_1 \cdots d^4x_{n_1+\cdots+n_k} \left(\prod_{s=1}^{l} h^{(2)}_{n_{i_s}}(x_{n_1+\cdots+n_{i_s-1}+1}, ..., x_{n_1+\cdots+n_{i_s}}) \right) \times$$
$$\left(\prod_{t=1}^{k-l} h^{(1)}_{n_{j_t}}(x_{n_1+\cdots+n_{j_t-1}+1}, ..., x_{n_1+\cdots+n_{j_t}}) \right) \times$$
$$T\left(: \prod_{i=1}^{n_1} u_{\alpha(i)}(x_i) :; \cdots ; : \prod_{i=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} u_{\alpha(i)}(x_i) : \right) =$$

$$\int d^4x_1 \cdots d^4x_{n_1+\cdots+n_k} \left(\prod_{s=1}^{l} h^{(2)}_{n_{i_s}}(x_{n_1+\cdots+n_{i_s-1}+1}, \ldots, x_{n_1+\cdots+n_{i_s}}) \right) \times$$
$$\left(\prod_{t=1}^{k-l} h^{(1)}_{n_{j_t}}(x_{n_1+\cdots+n_{j_t-1}+1}, \ldots, x_{n_1+\cdots+n_{j_t}}) \right) \times$$
$$T\left(: \prod_{m=n_1+\cdots+n_{i_1-1}+1}^{n_1+\cdots+n_{i_1}} u_{\alpha(m)}(x_m) :; \cdots ; : \prod_{m=n_1+\cdots+n_{i_l-1}+1}^{n_1+\cdots+n_{i_l}} u_{\alpha(m)}(x_m) : \right) \times$$
$$T\left(: \prod_{q=n_1+\cdots+n_{j_1-1}+1}^{n_1+\cdots+n_{j_1}} u_{\alpha(q)}(x_q) :; \cdots ; : \prod_{q=n_1+\cdots+n_{j_{k-l}-1}+1}^{n_1+\cdots+n_{j_{k-l}}} u_{\alpha(q)}(x_q) : \right). \tag{3.5}$$

The relations (3.5) imply the equality

$$\sum_{n_1,\ldots,n_k} \sum_{\alpha(1),\ldots,\alpha(n_1+\cdots+n_k)} \int d^4x_1 \cdots d^4x_{n_1+\cdots+n_k}$$
$$\left(\prod_{s=1}^{k} K_{\alpha(n_1+\cdots+n_{s-1}+1),\ldots,\alpha(n_1+\cdots+n_s)} h_{n_s}(x_{n_1+\cdots n_{s-1}+1}, \ldots, x_{n_1+\cdots n_s}) \right) \times$$
$$T\left(: \prod_{i=1}^{n_1} u_{\alpha(i)}(x_i) :; \cdots ; : \prod_{i=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} u_{\alpha(i)}(x_i) : \right) =$$
$$\sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \Bigg\{ \sum_{n_1,\ldots,n_l} \sum_{\alpha(1),\ldots,\alpha(n_1+\cdots+n_l)} \int d^4x_1 \cdots d^4x_{n_1+\cdots+n_l}$$
$$\left(\prod_{s=1}^{l} K_{\alpha(n_1+\cdots+n_{s-1}+1),\ldots,\alpha(n_1+\cdots+n_s)} h^{(2)}_{n_s}(x_{n_1+\cdots n_{s-1}+1}, \ldots, x_{n_1+\cdots n_s}) \right) \times$$
$$T\left(: \prod_{i=1}^{n_1} u_{\alpha(i)}(x_i) :; \cdots ; : \prod_{i=n_1+\cdots+n_{l-1}+1}^{n_1+\cdots+n_l} u_{\alpha(i)}(x_i) : \right) \Bigg\} \times$$
$$\Bigg\{ \sum_{n_1,\ldots,n_{k-l}} \sum_{\alpha(1),\ldots,\alpha(n_1+\cdots+n_{k-l})} \int d^4y_1 \cdots d^4y_{n_1+\cdots+n_{k-l}}$$
$$\left(\prod_{s=1}^{k-l} K_{\alpha(n_1+\cdots+n_{s-1}+1),\ldots,\alpha(n_1+\cdots+n_s)} h^{(1)}_{n_s}(y_{n_1+\cdots n_{s-1}+1}, \ldots, y_{n_1+\cdots n_s}) \right) \times$$
$$T\left(: \prod_{i=1}^{n_1} u_{\alpha(i)}(y_i) :; \cdots ; : \prod_{i=n_1+\cdots+n_{k-l-1}+1}^{n_1+\cdots+n_{k-l}} u_{\alpha(i)}(y_i) : \right) \Bigg\}. \tag{3.6}$$

Inserting the equalities (3.6) into the right-hand side of the equality (3.1) we get the equality

$$S(h^{(1)} + h^{(2)}) = S(h^{(2)})S(h^{(1)}) \tag{3.7}$$

similar to the equality (1.4).

It is possible to choose the constants in the equality (3.1) such that the relation analogous to the relation (1.1) is valid

$$S(Lh) = U(L)S(h)U^*(L). \tag{3.8}$$

Here $Lh_n(x_1, \ldots, x_n) = h_n(L^{-1}x_1, \ldots, L^{-1}x_n)$ and $U(L)$ is a unitary operator by means of which the quantum wave functions transform under the transformations L from the Lorentz group and the group of translations.

In order to conserve the amplitude state norm under the transformation from the initial state to the final state the operator $S(h)S^{*}(h)$ has to be the identity operator.

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