Localizability in de Sitter space

N. Yokomizo* and J. C. A. Barata
Instituto de Física, Universidade de São Paulo, C.P. 66318, 05315-970 São Paulo-SP, Brazil
(Dated: November 4, 2010)

An analogue of the Newton-Wigner position operator is defined for a massive neutral scalar field in de Sitter space. The one-particle subspace of the theory, consisting of positive-energy solutions of the Klein-Gordon equation selected by the Hadamard condition, is identified with an irreducible representation of de Sitter group. Postulates of localizability analogous to those written by Wightman for fields in Minkowski space are formulated on it, and a unique solution is shown to exist. A simple expression for the time-evolution of the operator is presented.

PACS numbers: 04.62.+v, 03.65.Sq, 03.65.Db

I. INTRODUCTION

The question of the existence and usefulness of a notion of localization for quantum particles moving at relativistic speed has a long history [1–4]. Although the idea of a position measurement is one of the most intuitive ideas of a quantum observable, there is no obvious mathematical counterpart to it in the relativistic domain. The conflict with intuition in such a fundamental subject is the main motivation to work on this problem. But there are also technical reasons for that. A quantum field theory is usually applied to the study of particle collision processes, so one needs to understand how to interpret the theory in terms of particles. The main question is how to assign probabilities for the detection of the produced particles in detectors placed at specific regions of space [5–8], what amounts to defining a position probability distribution. Besides that, there is the very fact that classical particles do exist, i.e., that a classical limit of the underlying quantum theory exists which describes particles. A position operator is the natural tool to deal with this limit [9, 10]. Now the current widespread interest in quantum effects in curved spacetimes boosted by experimental and theoretical discoveries in cosmology motivates the analysis of the problem of localizability in a more general context. In particular, the present accelerated expansion of the universe [11, 12] and the existence of an inflationary epoch in the very early universe [13–15] suggest that there should be eras in the beginning of the universe and in the distant future when the geometry of the universe is approximately a patch of de Sitter space, what justifies our interest in this special geometry. Local effects of de Sitter geometry on particle dynamics have been investigated at the classical and quantum level [16–18].

In flat Minkowski space, an early solution to the problem was provided by the work of Newton and Wigner [1], later reformulated in more rigorous terms by Wightman [2]. It was proved that a natural set of postulates defines a unique position operator, at least for massive fields. But the operator found is frame-dependent, and alternative covariant notions of localizability were put forward since then [3, 4]. The interpretation of these operators and the possibility of actually measuring them have been discussed in the context of Quantum Field Theory in terms of specific models of interaction between a detector and the quantum field (see [19, 20]). Now, if the particle moves in a curved spacetime, little is known. There are additional complications in the analysis, mainly due to the existence of multiple vacua. In fact, the concept of particle is not strictly necessary for Quantum Field Theory in Curved Spacetimes — the general theory can be formulated without introducing the notion of particles [21]. Only in special circumstances it still makes sense to speak of particles. In a flat Minkowski spacetime, for instance, that is certainly true, and it is also natural that in regions where the curvature is small one should be able to speak of particle states — high-energy experiments are actually performed in a slightly curved space, and particles are observed. However, there is no clear specification of the necessary conditions for a particle interpretation to be available.

We have studied the case of a neutral massive scalar field in 2d de Sitter space, and have showed that a particle interpretation of this theory is possible. In de Sitter space, it is possible to select a unique vacuum state — the Bunch-Davies vacuum — by requiring physical states to satisfy the Hadamard condition [22], which corresponds to the requirement that the averaged energy-stress tensor can be renormalized by a point-splitting prescription [21]. There is a unique Fock representation asssociated with the Bunch-Davies vacuum, on which we have written localizability conditions analogous to those of [1, 2], and a position operator which satisfies these conditions was found. This operator is the natural analogue in de Sitter space of the Newton-Wigner position operator.

II. THE QUANTIZED FIELD IN SPHERICAL COORDINATES

A. Normal modes

The simplest way of looking at 2d de Sitter space dS^2 is to consider it a submanifold embedded in a 3d Minkowski

* yokomizo@fma.if.usp.br

space M^3. Choosing a metric $\eta_{ab} = \mathrm{diag}(1, -1, -1)$ for M^3, one has

$$dS^2 = \{\mathbf{X} \in M^3 \mid \mathbf{X}^2 = X^a X^b g_{ab} = -\alpha^2\}\,,$$

where $\alpha > 0$ is the de Sitter radius. The space so obtained is a hyperboloid, with topology $S \times \mathbb{R}$. One may think of it as a spatial circle evolving in time. The symmetry group is the de Sitter group $O(2,1)$, i.e., the isometries are Lorentz transformations in the ambient space. Changing to the so-called spherical coordinates,

$$\begin{aligned} X^0 &= \alpha \sinh(t/\alpha)\,, \\ X^1 &= \alpha \cosh(t/\alpha) \cos\theta\,, \\ X^2 &= \alpha \cosh(t/\alpha) \sin\theta\,, \end{aligned}$$

the geometry of dS^2 is described by the metric coefficients:

$$g_{00} = 1\,, \quad g_{01} = 0\,, \quad g_{11} = -\alpha^2 \cosh^2(t/\alpha)\,.$$

The volume density is $\sqrt{-g} = \alpha \cosh(t/\alpha)$, and the D'Alembertian is

$$\Box = \partial_{tt} + \frac{1}{\alpha} \tanh(t/\alpha) \partial_t - \frac{1}{\alpha^2 \cosh^2(t/\alpha)} \partial_{\theta\theta}\,.$$

The Klein-Gordon equation reads

$$(\Box - \frac{m^2 + \xi R}{\hbar^2})\phi = 0\,.$$

The scalar curvature is related to the de Sitter radius by $R = -2/\alpha^2$. Put $\mu^2 = m^2 + \xi R$. After separation of variables, the Klein-Gordon equation becomes

$$\psi'' = -k^2 \psi \quad \Rightarrow \quad \psi_k(\theta) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{ik\theta}\,, \quad k \in \mathbb{Z}\,,$$

$$T'' + \frac{1}{\alpha} \tanh(t/\alpha) T' + \left(\frac{\mu^2}{\hbar^2} + \frac{k^2}{\alpha^2 \cosh^2(t/\alpha)} \right) T = 0\,.$$

In order to solve the time-dependence of the angular momentum modes described by the index k, put $x = i \sinh(t/\alpha)$, and get:

$$(1 - x^2) \frac{d^2 T}{dx^2} - 2x \frac{dT}{dx} + \left[-\frac{\alpha^2 \mu^2}{\hbar^2} - \frac{k^2}{1 - x^2} \right] T = 0\,. \quad (1)$$

This is an associated Legendre equation. The solutions are associated Legendre functions $P^k_\nu(x), Q^k_\nu(x)$, with $\nu(\nu+1) = -\alpha^2\mu^2/\hbar^2$. The coefficient ν is given by

$$\nu = \frac{-1 \pm \sqrt{1 - 4\alpha^2\mu^2/\hbar^2}}{2}\,. \quad (2)$$

If μ^2 is positive, then ν is either a real number in the interval $[-1, 0]$ or a complex number with real part equal to $-1/2$ and some nonzero imaginary part. If $\mu^2 = 0$, then $\nu = 0, 1$. If it is negative, then ν assumes real values.

We will first restrict to $\mu^2 > 0$. The squared mass is always positive, so this restriction corresponds in fact to not allowing a large negative coupling with the scalar curvature. In this case, a nice pair of linearly independent solutions of (1) is given by

$$T^k_\nu(i \sinh(t/\alpha))\,, \quad T^k_\nu(-i \sinh(t/\alpha))\,,$$

where

$$T^k_\nu(z) := \mathrm{e}^{-ik\pi/2} P^k_\nu(z) \theta(t) + \mathrm{e}^{ik\pi/2} P^k_\nu(z) \theta(-t)$$

is the Legendre function in 'Ferrer's notation' [23]. The function $T^k_\nu(z)$ is the analytic continuation of the Legendre function (defined by a convergent series in $|z-1| < 2$) to the complex plane cut along the real axis in the interval $[-1, 1]$. It doesn't matter which root ν of (2) is taken: both give the same function (that follows from the symmetry $T^k_\nu = T^k_{-\nu-1}$).

The functions $T^k_\nu(z)$ have the desired property that $[T^k_\nu(ix)]^* = T^k_\nu(-ix)$, i.e., the linearly independent solutions are complex conjugate (see Appendix A). Positive-energy solutions are identified with normal modes of the form

$$u_k(\theta, t) \propto \mathrm{e}^{ik\theta} T^k_\nu(i \sinh(t/\alpha))\,.$$

It must be checked that this choice is valid. Two out of the three required conditions [21] were already checked: (i) the space of solutions is decomposed in a direct sum of positive and negative energy solutions; (ii) the complex conjugate of a positive energy solution is a negative energy solution. The third condition is that the positive energy modes have a positive norm. The norm is defined with the help of the invariant sesquilinear form

$$\langle f | g \rangle = ia(t) \int_{S_t} \mathrm{d}\theta (f^* \partial_t g - \partial_t f^* g)\,, \quad (3)$$

where S_t is any spatial slice of constant time t, and $a(t) = \alpha \cosh(t/\alpha)$ is the corresponding scale factor in de Sitter space — the radius of the circle S_t. It is clear why condition (ii) is required: positive and negative energy modes are then automatically orthogonal.

Let us check that the modes are orthogonal and have positive norm (put now $y = \sinh(t/\alpha)$).

$$\langle u_k | u_l \rangle = -\delta_{kl} 2\pi \cosh^2(t/\alpha) \left[T^k_\nu(-iy)\, T^{k\,\prime}_\nu(iy) + \text{c.c.} \right]\,.$$

Invoke now the identity (from [23])

$$(1 - z^2) \left[T^k_\nu(z) \frac{d}{dz} T^k_\nu(-z) - \frac{d}{dz} T^k_\nu(z)\, T^k_\nu(-z) \right] = \frac{2}{\gamma_k}\,, \quad (4)$$

where $\gamma_k \equiv \Gamma(-\nu - k)\Gamma(\nu - k + 1)$, and Γ is Euler's Gamma function, to get

$$\langle u_k | u_l \rangle = \delta_{kl} \frac{4\pi}{\gamma_k}\,.$$

Now we show that this number is positive for any k. Consider first the case $k = 0$. Then

$$\frac{1}{\Gamma(-\nu)\Gamma(\nu+1)} = \frac{\sin[(\nu+1)\pi]}{\pi}\,,$$

where ν is either a real number in the interval $(-1, 0)$, or a complex number of the form $-1/2 + ix, x \in \mathbb{R}$. In the first case, $\nu + 1$ is in the interval $(0, 1)$, so $\sin[(\nu + 1)\pi]$ is positive. In the second case, $\sin[(\nu + 1)\pi] = \cosh(\pi x)$, positive too. For general k, first note that, for k positive,

$$\frac{1}{\gamma_k} = \frac{\prod_{l=1}^{k}(-\nu-l)(\nu-l+1)}{\Gamma(-\nu)\Gamma(\nu+1)} = \frac{\prod_{l=1}^{k}(\alpha^2\mu^2+l^2-l)}{\Gamma(-\nu)\Gamma(\nu+1)}\,.$$

It is clear that the product is positive ($l^2 \geq l$ when l is integer). A similar trick does the work for negative k. Thus all norms positive, and the normalized states are

$$u_k(t,\theta) = \sqrt{\frac{\gamma_k}{2}}\,T_\nu^k(i\sinh(t/\alpha))\frac{e^{ik\theta}}{\sqrt{2\pi}}\,. \qquad (5)$$

Until now, we have restricted to $\mu^2 > 0$. In this case, the index ν is either a real number or has the form $\nu = -1/2 + i\lambda$, with $\lambda \in \mathbb{R}$. But solutions with a real ν are purely real and not oscillatory, resembling little what one expects from particle wavefunctions. Later we will show that the dynamics of the position probability distribution depends on how the phases of the normal modes change with time, so that a purely real time-evolution is not interesting for our purposes. Therefore, we take $\nu = -1/2 + i\lambda$ in what follows, what is equivalent to requiring $\mu > \hbar/2\alpha$. We will keep the index ν in the Legendre functions, for convenience, but from now on we restrict to these values. In this case, the space of solutions of the wave equation corresponds to a representation of the de Sitter group in the so-called principal series (the continuous representations C_q^0 with $q > 1/4$ in [24]).

B. Canonical quantization and one-particle subspace

After fixing the positive-energy modes of the classical field, canonical quantization in de Sitter space follows pretty much the same steps as in Minkowski space [21, 25]. The quantized neutral massive scalar field is expanded in the form

$$\hat{\phi}(t,\theta) = \sum_{k=-\infty}^{\infty}(a_k u_k + a_k^* u_k^*)\,, \qquad (6)$$

where the u_k are the chosen orthonormal modes, and the coefficients a_k, a_k^* are annihilation and creation operators satisfying the commutation relations

$$[a_k, a_l^*] = \delta_{kl} \qquad [a_k, a_l] = [a_k^*, a_l^*] = 0\,.$$

The expansion in Eq. (6) can be seen as the analogue of a momentum space representation of the field. A Fock representation is obtained in the usual way: the vacuum $|\Omega\rangle$ is defined as the state annihilated by all annihilation operators, $a_k|\Omega\rangle = 0$, $\forall k$, and many-particle states are created by the application of creation operators to the vacuum. Of course, the choice of a specific set of modes as the positive-energy solutions is not innocuous — it is equivalent to the choice of a vacuum, as is well-known. Our choice leads to the Bunch-Davies vacuum, the unique representation selected by the Hadamard condition [26]. An explicit evaluation of the two-point function is presented in Appendix B, and the result is the same as obtained in [27], where flat coordinates are used.

The position operator will be defined in the one-particle subspace of the theory; call it $\mathcal{H}$. The vectors $\phi \in \mathcal{H}$ are normalized superpositions of positive-energy solutions, and can be represented explicitly as

$$\phi(t,\theta) = \sum_k \phi_k u_k(t,\theta)\,, \qquad \sum_k |\phi_k|^2 = 1\,, \qquad \phi_k \in \mathbb{C}\,. \qquad (7)$$

The scalar product in $\mathcal{H}$ is simply $\langle\phi|\psi\rangle = \sum \phi_k^*\psi_k$. We are going to think of the one-particle subspace as describing the quantum dynamics of a single relativistic particle in de Sitter space, following the usual physical interpretation of Fock space: $\phi(t, \theta)$ will be the spacetime representation of the wavefunction associated with the particle. Some problems with this interpretation might be expected — it has been repeatedly remarked that the concept of a particle for quantum fields in curved spacetimes is not well-defined. Nevertheless, it is just as clear that there are situations where a particle-like behavior is evident. As remarked in [28], particle physics experiments are actually performed in a curved spacetime, and we do see particle tracks in experiments. To understand how to deal with a quantum field theory in a curved spacetime under circumstances where a particle-like behavior is possible is one of the purposes of this paper.

C. Group action on the space of positive-energy solutions

The space $\mathcal{H}$ of positive-energy solutions was described in a given system of spherical coordinates (t, θ), but there is a whole family of systems (t', θ') related by isometries in the de Sitter group $O(2, 1)$. We want to prove that the definition of $\mathcal{H}$ is coordinate-independent, i.e., that the choice of positive-energy modes fixed by the Hadamard condition is preserved by the action of the group, as well as to find out how the group acts on these modes. Any element of $O(2, 1)$ is the product of an element of the restricted de Sitter group $O(2, 1)_+^\uparrow$ of Lorentz transformations of determinant 1 which do not reverse the direction of time, and possibly parity **P** and time reversal **T**. Next we describe the action of the generators of Lorentz transformations and discrete symmetries in $\mathcal{H}$.

There are three linearly independent generators in the algebra of $O(2, 1)$, which may be taken as the infinitesimal boosts along the rectangular axes, N_{1t} and N_{2t}, and the generator of rotations, N_{12}. The question is how these transformations act on the modes defined in Eq. (5). The case of rotations is quite simple. A transformation $U_{12}(\phi) = \exp(\phi N_{12})$ which rotates the space by an angle ϕ changes angles in spherical coordinates according to $\theta \mapsto \theta - \phi$, while the coordinate t remains unaffected. The generator of rotations is $N_{12} = -\partial/\partial\theta$. Its action on the basis vectors is just $N_{12}\, u_k = -ik\, u_k$, i.e., the basis $\{u_k\}$ is that of the eigenvectors of the Hermitian operator iN_{12}.

Now consider the case of N_{1t}. Since Lorentz transformations are naturally described in the flat coordinates of the ambient Minkowski space M, let us describe the modes u_k in the same coordinates:

$$u_k = \sqrt{\frac{\gamma_k}{4\pi}}\, T^k_\nu(-iX^0/\alpha)\, \frac{(X^1 + iX^2)^k}{[\alpha^2 + (X^0)^2]^{k/2}}\,.$$

An infinitesimal Lorentz transformation along the axis X^1 is given by

$$\begin{aligned}(X^0)' &= X^0 + \lambda X^1\,,\\ (X^1)' &= X^1 + \lambda X^0\,,\\ (X^2)' &= X^2\,,\end{aligned}$$

where λ is the infinitesimal parameter of the transformation (the transformation is Lorentz to first order in λ). Thus the variation of u_k is

$$N_{1t}\, u_k = \frac{\partial u_k}{\partial X^0}(-X^1) + \frac{\partial u_k}{\partial X^1}(-X^0)\,.$$

A similar equation holds for boosts along the axis X^2. Evaluating the derivatives and using a few relations between Legendre functions from [23], one finds that the action of the generators of de Sitter group is

$$\begin{aligned}N_{12}u_k &= -iku_k\,,\\ N_{1t}u_k &= -\frac{i}{2}|\nu + k|u_{k-1} - \frac{i}{2}|\nu - k|u_{k+1}\,,\\ N_{2t}u_k &= \frac{1}{2}|\nu + k|u_{k-1} - \frac{1}{2}|\nu - k|u_{k+1}\,.\end{aligned} \tag{8}$$

These equations show the space $\mathcal{H}$ of positive-energy solutions is closed under the action of the infinitesimal generators. Thus $\mathcal{H}$ is a representation space for $O(2,1)^{\uparrow}_{+}$, the action of a Lorentz transformation L on a wavefunction $\phi(x) \in \mathcal{H}$ being given by $\phi(x) \mapsto \phi(L^{-1}x)$. The Casimir operator which characterizes the irreducible representations is $C = N_{12}^2 - N_{1t}^2 - N_{2t}^2$, and is easily verified to be $C = -\nu(\nu+1)$ for the above expressions.

Now let us introduce the discrete symmetries of parity $\mathbf{P}$ and time-reversal $\mathbf{T}$. We represent parity as the reversal of the axis X^2 in the ambient Minkowski space. Then parity just reverses the sign of the angular coordinate of a wavefunction in $\mathcal{H}$, $\mathbf{P}\phi(t,\theta) = \phi(t,-\theta)$. In particular, for the basis vectors u_k, one may use the identity

$$\sqrt{\gamma_k}\, T^k_\nu(z) = \sqrt{\gamma_{-k}}\, T^{-k}_\nu(z) \tag{9}$$

(which is proved using the inversion formula for gamma functions and Eq. (A2)), in order to get

$$\mathbf{P}u_k = u_{-k}\,. \tag{10}$$

The action of $\mathbf{T}$ has a particularity connected with the restriction to the space of positive-energy states. The geometrical realization of the transformation is the reversal of the time coordinate in the ambient Minkowski space. But this cannot be represented as $\phi(t,\theta) \mapsto \phi(-t,\theta)$, since the result is a negative-energy state. In order that the transformation is closed in $\mathcal{H}$, we take the anti-unitary representation $\mathbf{T}\phi(t,\theta) = \phi^*(-t,\theta)$. But then the action of the operator on modes u_k is the same as that of parity, with the difference that the action is anti-linear,

$$\mathbf{T}u_k = u_{-k} \qquad \text{(anti-linear)}\,. \tag{11}$$

III. NEWTON-WIGNER LOCALIZATION

A. Definition of the localization system

The notion of localization of relativistic particles in Minkowski space provided by the Newton-Wigner (NW) position operator was introduced in [1]. In that paper, a list of properties is postulated, which are assumed to hold for any reasonable relativistic position operator, and it is proved that there is a unique operator satisfying them. A more direct way to understand this position operator is described in [29]. Let us review the basic argument. Consider a massive scalar field in Minkowski space. The one-particle subspace of the theory consists of vectors $\phi(p) \in L^2(\mathbb{R}, dp/\omega)$, with $\omega = \sqrt{p^2+m^2}$, i.e., the scalar product is

$$\langle\phi|\psi\rangle = \int \frac{\mathrm{d}p}{\omega}\phi^*(p)\psi(p)\,.$$

Now absorb a factor $\sqrt{\omega}$ in each wavefunction: i.e., consider the unitary transformation $M_\omega : L^2(\mathbb{R}, dp/\omega) \to L^2(\mathbb{R}, dp)$, whose action is $\phi(p) \mapsto \phi_{NW}(p) = \phi(p)/\sqrt{\omega}$. Then introduce a unitary operator of time-evolution $U_t : L^2(\mathbb{R}, dp) \to L^2(\mathbb{R}, dp)$, represented by the transformation $\phi_{NW}(p) \mapsto (U_t\,\phi_{NW})(p) = \exp(-i\omega t/\hbar)\phi_{NW}(p)$. Finally, Fourier transform the result in order to get a spatial representation,

$$\phi_{NW}(t,x) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p\, \mathrm{e}^{ipx/\hbar}\mathrm{e}^{-i\omega t/\hbar}\,\phi_{NW}(p)\,.$$

That gives the Newton-Wigner wavefunction. The probability density that the particle is detected at the point x in time t is $P(t,x) = |\phi_{NW}(t,x)|^2$. The position operator itself, at time t, is the multiplication operator in the spatial representation at the same time,

$$(q_t\phi)_{NW}(t,x) = x\phi_{NW}(t,x)\,.$$

Some difficulties show up if one tries to repeat the same steps in the case of de Sitter space. First, there is no canonical definition of a momentum space representation. We overcome this problem by looking at the mode expansion as a convenient (for our purposes) de Sitter analogue of the Fourier transform. It is clear that a mode expansion is a coordinate dependent concept, therefore the resulting position operator will depend on the choice of coordinates. But even in Minkowski space, the Newton-Wigner operator is not a covariant object: there is a distinct operator associated with each reference frame. The problem found in Minkowski space is just carried over into de Sitter space, and we do not attempt to solve it here.

The second point is the absence of a time-translation isometry in dS^2, what makes the time-evolution of individual modes much more complicated than in Minkowski space. Two aspects are relevant here: there is no definite frequency ω_k associated with each mode, so that time-evolution in momentum space is not just multiplication by varying phases $\exp(-i\omega t/\hbar)$ as before; and the oscillation of the field goes on together with a damping of the field amplitude, forced by the expansion of the universe (for increasing $|t|$). We will see that these effects can be isolated: the damping factor will be analogous to the factor $\sqrt{\omega}$ absorbed in the definition of the Newton-Wigner wavefunction, while the oscillating phases will be responsible for the time-evolution of the position operator.

Let us now proceed to the definition of the de Sitter version of Newton-Wigner localization. Later we will interpret the results drawing an analogy with the discussion above. We assume that a localization system in de Sitter space is:

I: A family of unitary transformations $W_t : \mathcal{H} \to L^2(S)$, $\phi \mapsto \phi_{NW}(t,\theta)$, where $L^2(S)$ is the Hilbert space of square-integrable functions on the circle S;

II: If $U_{12}(\alpha) \in \mathrm{SO}(1,2)$ is a rotation by an angle α, then $U_{12}(\alpha)\phi \mapsto \phi_{NW}(t,\theta-\alpha)$;

III: $\mathbf{P}\phi \mapsto \phi_{NW}(t,-\theta)$, and $\mathbf{T}\phi \mapsto \phi^*_{NW}(-t,\theta)$;

Let us discuss the intuitive content of the conditions above.

The Newton-Wigner wavefunction $\phi_{NW}(t,\theta)$ is interpreted, for each time t, as describing quantum amplitudes for finding the particle at position θ. In other words, the probability of finding the particle in a Lebesgue measurable set I is $P(I) = \int_I |\phi_{NW}(t,\theta)|^2 d\theta$. Condition I corresponds to the basic requirement that such a probability distribution exists for each time t.

The second condition is that the Newton-Wigner representation is well-behaved under rotations. A rotation in de Sitter group, when seen from the Newton-Wigner spatial representation, must rotate the probability amplitudes on the circle by the same angle. This condition can be reformulated as $W_t U_{12}(\varphi) W_t^* = R(\varphi)$, where R is the operator of rotation for square-integrable functions on the circle.

Condition III is the requirement that the discrete symmetries of parity and time-reversal act as geometrical transformations on the Newton-Wigner representations. The complex conjugation in the time-reversal condition is necessary because the image of an anti-unitary operator under a unitary equivalence must be anti-unitary too. A quantum symmetry is in general defined up to a phase, according to the celebrated Wigner's theorem; we are assuming here that the phases are equal to 1, avoiding complications with the possibility of a projective representation of the extended de Sitter group.

The consequences of the postulates can now be evaluated. Let us start with condition II. For each t, a suitable basis for $L^2(S)$ is that composed of eigenvectors of the Hermitian generator of rotations. That is the same as describing the Newton-Wigner wavefunction in its Fourier expanded form,

$$\phi_{NW}(t,\theta) = \sum_k q_k(t) \frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,, \qquad \sum_k |q_k(t)|^2 = 1\,,$$

with $q_k \in \mathbb{C}$. Consider the vector $u_k \in \mathcal{H}$. The action of a rotation $U_{12}(\alpha)$ on it is to multiply the state by a phase, $U_{12}(\alpha)u_k = \exp(-ik\alpha)u_k$. Since U_{12} is linear, the same must be true for its image in $L^2(S)$:

$$W_t\left(U_{12}(\alpha)u_k\right) = \mathrm{e}^{-ik\alpha} W_t(u_k)\,,$$

what implies

$$R(\alpha)W_t(u_k) = \mathrm{e}^{-ik\alpha} W_t(u_k)\,. \tag{12}$$

But then it must be

$$W_t(u_k) = \mathrm{e}^{-i\varphi_k(t)}\,\frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,, \tag{13}$$

where $\varphi_k(t)$ is some arbitrary phase. For, suppose the space V of solutions $W_t(u_k)$ of Eq. (12) has more than one dimension. Note that the action of the Hermitian generator J of rotations in V is multiplication by k. Then there would be at least two orthogonal vectors with the same eigenvalue k, what is impossible, since the eigenspaces of J are not degenerate. Therefore V is one-dimensional, the space of eigenvectors of J with eigenvalue k. Because the transformation W_t is unitary, and u_k has norm 1, there is just a phase freedom, what corresponds to Eq. (13).

The action of parity in $\mathcal{H}$ is given by Eq. (10). The first part of Postulate III, when applied to the general form of the solution of Postulate II described in Eq. (13), leads to

$$W_t(u_k) = W_t(\mathbf{P}u_{-k}) = \mathrm{e}^{-i\varphi_{-k}(t)}\,\frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,.$$

Comparing with Eq. (13), it follows that $\varphi_k(t) = \varphi_{-k}(t)$. The action of time-reversal in $\mathcal{H}$ is given by Eq. (11). The second part of Postulate III leads to

$$W_t(u_k) = W_t(\mathbf{T}u_{-k}) = \mathrm{e}^{i\varphi_{-k}(-t)}\,\frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,,$$

which, taken together with the previous conditions, corresponds to $\varphi_k(t) = -\varphi_k(-t)$.

Let us see how the relations restrict the form of the Newton-Wigner wavefunction. Put $t = 0$. Since $\varphi_k(t) = -\varphi_k(-t)$, it must be $\varphi_k(0) = -\varphi_k(0) = 0$. Thus the transformation W_0 is uniquely determined by conditions I–III. The transformation is simply:

$$\phi(t,\theta) = \sum_k \phi_k u_k(t,\theta) \mapsto \phi_{NW}(0,\theta) = \sum_k \phi_k \frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}} . \tag{14}$$

The time-evolution described by W_t in Eq. (13) is restricted, but not uniquely fixed by the axioms I—III. The varying phases must satisfy: $\varphi_k(t) = \varphi_{-k}(t)$ (parity), and $\varphi_k(t) = -\varphi_k(-t)$ (time-reversal), but there remains a lot of freedom after these conditions are imposed. Later we will discuss possible choices for the dynamics, but first we consider the transformation W_0 in more detail.

B. The case of $t = 0$

The Newton-Wigner wavefunction at time $t = 0$ of a state $\phi \in \mathcal{H}$ is given by Eq. (14). We want to discuss now the closely related concept of localized states, for which there is a natural definition when the localizability axioms are satisfied. These are the analogues in de Sitter space of the Dirac delta functions $\delta(x-a)$ in the configuration space of nonrelativistic quantum mechanics. The work of Newton and Wigner was written in terms of localized states, as well as an earlier discussion of localizability in de Sitter space [31], and we want to translate our results to this language for the sake of comparison with these works, and also because of the intuitive appeal of the concept. But first we have to discuss the relation between our representation of Sitter algebra in the space of Hadamard solutions of the Klein-Gordon equation, discussed in Section II C, and the more traditional representations described in Bargmann's work [24]. That will give us access to formulae available in papers devoted to Bargmann's representations, and is also a necessary step for us to compare our results with those of [31], which are described in terms of such representations.

The principal series Bargmann representation on $\mathcal{H}' := L^2(S)$ is briefly reviewed in Appendix C. Let $\{|k\rangle\}$ be the basis of $\mathcal{H}'$ composed of normalized eigenstates of the generator of rotations, $|k\rangle = \exp(ik\theta)/\sqrt{2\pi}$. The action of de Sitter algebra in this basis is given by:

$$\begin{aligned} N_{12}|k\rangle &= -ik|k\rangle , \\ N_{1t}|k\rangle &= \frac{\nu+k}{2}|k-1\rangle + \frac{\nu-k}{2}|k+1\rangle , \\ N_{2t}|k\rangle &= i\frac{\nu+k}{2}|k-1\rangle - i\frac{\nu-k}{2}|k+1\rangle , \end{aligned} \tag{15}$$

with $\nu = -1/2 + \lambda i$. These expressions are direct translations of Eqs. (C1) and (C2), discussed in more detail in [31]. On the other hand, the representation of de Sitter algebra in the space $\mathcal{H}$ of Hadamard solutions is described explicitly in Eq. (8). Comparing the expressions, it can be verified that a unitary equivalence $U_B^\dagger : \mathcal{H}' \to \mathcal{H}$ is given by $|k\rangle \mapsto \chi_k u_k$, where χ_k is a complex number defined by the recurrence relations:

$$\chi_0 = 1 , \qquad \chi_{k+1} = i\frac{\nu+k+1}{|\nu+k+1|}\chi_k .$$

The last relation is equivalent to

$$\chi_{k-1} = i\frac{\nu-k+1}{|\nu-k+1|}\chi_k .$$

These coefficients are, for $k > 0$,

$$\chi_k = \chi_{-k} = i^k \frac{\left(\frac{1}{2}+i\lambda\right)\left(\frac{3}{2}+i\lambda\right)\ldots\left(k-\frac{1}{2}+i\lambda\right)}{\left|\left(\frac{1}{2}+i\lambda\right)\left(\frac{3}{2}+i\lambda\right)\ldots\left(k-\frac{1}{2}+i\lambda\right)\right|} , \tag{16}$$

with $\chi_0 = 1$. Let $\mathcal{H}_{NW}^{(0)} := L^2(S)$ denote the space of Newton-Wigner wavefunctions at time $t = 0$, and build the composition $U_B^{(0)} := U_B \circ W_0^\dagger$. This transformation maps a NW-wavefunction to the corresponding state in Bargmann's representation. Choosing the basis $\{|k\rangle = \exp(ik\theta)/\sqrt{2\pi}\}$ for $\mathcal{H}_{NW}^{(0)}$, it follows that $W_0 = 1$, so that $U_B^{(0)} = U_B$. That is, a NW-wavefunction $\phi_{NW}(0,\theta) = \sum \phi_k|k\rangle$ corresponds to a vector $\phi_B(\theta) = \sum \chi_k^*\phi_k|k\rangle$ in Bargmann's representation.

A natural way to introduce the notion of an improper state $\psi_{NW}^{(t',\theta')}(t,\theta)$ localized at position θ' in time t' is to define it as a Dirac delta distribution in the corresponding Newton-Wigner representation. From such definition and Eq. (13) it follows that the coefficients of $\psi^{(t',\theta')}(t,\theta)$ in $\mathcal{H}$ are

$$\psi_k^{(t',\theta')} = \frac{1}{2\pi}\mathrm{e}^{i\varphi_k(t')}\mathrm{e}^{-ik\theta'} . \tag{17}$$

Let us compare these states with those of Philips and Wigner discussed in [31]. For $t = 0$, all φ_k are zero. Then a state $\psi^{(0,\pi/2)}$ localized at $\theta = \pi/2$ and $t = 0$ has coefficients $\psi_k^{(0,\pi/2)} = (-i)^k$. But a Philips-Wigner state localized at $\theta = \pi/2$ at $t = 0$ has Fourier coefficients l_k in $\mathcal{H}'$ given by Eq. (C5). That corresponds in $\mathcal{H}$ to a state $\phi^{(0,\pi/2)}$ with coefficients $\phi_k^{(0,\pi/2)} = l_k\chi_k$. From Eq. (16) and Eq. (C5), it follows that $l_k\chi_k = (-i)^k$. The result is the same as we found. Besides, both classes of states behave in the same way under rotations. So our definition of localized states allows one, for $t = 0$, to recover the results of [31]. The time-evolution of the position distributions is not discussed in [3]. Therefore, the localization schemes agree at the instant of time when both are well-defined.

Although the localized states are the same, there are some technical simplifications and a conceptual clarification that we would like to emphasize. Compare with what happens in Minkowski space. The work of Newton and Wigner was written in terms of distributions describing improper states localized at specific points. The results were latter reformulated by Wightman [2] in terms of projectors $E(S)$ in a Hilbert space associated with observables describing the property of the particle being in a region S of space. So the idea of localization at a point was replaced by localization in a finite region. These approaches are essentially equivalent in Minkowski space: the conditions used by Wightman were direct translations of the original conditions on distributions. In de Sitter space, the work of Philips and Wigner follows the original idea of looking for localized states, while we have studied the analogue of Wightman's localizability postulates. It turns out that now these approaches require distinct sets of basic axioms. In fact, one of the axioms of [31] (Axiom (c) in Appendix C) is not necessary in our approach, and can be discarded. Such simplification is related to some peculiarities involving distribution theory in compact spaces (see discussion in [31]), which would lead to the existence of spurious solutions if not ruled out by the extra axiom. These pathologies do not show up when one works from the start with quantum amplitudes in a space of square-integrable functions, describing probabilities of detecting the particle in measurable regions, as we have showed. We discuss this point further in Appendix C. We hope that our more compact set of axioms may be of practical convenience in eventual applications.

A second remark concerns the connection between representations of de Sitter group and wave equations in de Sitter space. As widely known, the irreducible representations of de Sitter group were classified by Bargmann in [24]. Yet, when one considers applications to quantum field theory, it is natural to ask for an interpretation of the representations in spaces of solutions of wave equations in de Sitter space. More than that, one wants to restrict to positive-energy solutions. Here we have used the Hadamard condition in order to select a suitable space of positive-energy solutions of the Klein-Gordon equation, and displayed such an interpretation for the principal series representations. In so doing, we have identified the one-particle subspace of the quantized massive scalar field theory with a particular (principal series) irreducible representation of the de Sitter group. In an intuitive sense, that identification provides a spacetime representation for vectors in the more abstract (from a physicist's perspective) Bargmann's representations: it allows one to see these vectors as wavefunctions in de Sitter space. In particular, it becomes possible to determine how the localized states are spread in spacetime, i.e., Eq. (17) (remember that relativistic localized states are not strictly localized, but "as localized as possible" states.). This question could not be dealt with without a prescription for the choice of the positive-energy states, and was not investigated in [31].

In Minkowski space, if one wants to see how a localized state defined in momentum space looks like in spacetime, one just goes to configuration space, using the well-known transformation $\phi(\mathbf{p}) \mapsto \phi(\mathbf{x}, t)$, the relativistic Fourier transform. The result is a Hankel function, with an exponential decay for large spatial distances [1]. It might be surprising, but so familiar a transformation has no natural analogue in curved spacetimes. A Bargmann's representation can be seen as a sort of 'momentum representation', but the transformation to configuration space — that of wavefunctions in de Sitter space — is not unique: it depends on the choice of a vacuum, or equivalently, of the positive-energy states. That is, it is necessary to combine purely group theoretical results with the modern specification of positive-energy states given by the Hadamard condition in order to find the spacetime representation of states of interest.

C. Time-evolution of the Newton-Wigner wavefunction

The postulates I—III determine uniquely the form of the Newton-Wigner wavefunction at time $t = 0$. They also impose restrictions on the time-evolution of the wavefunction, but do not fix it uniquely. In this section we discuss a solution of these conditions, suggested by an analogy with the definition of the position operator in Minkowski space discussed in the beginning of Section III A. It is natural that in generalizing structures defined in Minkowski space to the context of curved spacetimes some non-uniqueness might be met with. Nevertheless, one would certainly like to restrict it as much as possible, and in de Sitter space there is the advantage of dealing with a maximally symmetric spacetime. We discuss later in this section the possibility of using the group symmetry operations in de Sitter space in order to fix the Newton-Wigner dynamics, but we will answer this question in the negative, at least for a simple implementation of these symmetries.

So let us describe a solution of the time-evolution W_t of the Newton-Wigner wavefunction. Keep in mind the discussion in Section III A. From Eq. (7) and the explicit form of the normal modes given in Eq. (5), a generic state in $\mathcal{H}$ can be written as

$$\phi(t,\theta) = \sum_k \phi_k \sqrt{\frac{\gamma_k}{2}}\, T^k_\nu(i\sinh(t/\alpha))\, \frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,, \tag{18}$$

with $\sum_k |\phi_k|^2 = 1$. The scalar product is given by Eq. (3), which reduces to $\langle\phi|\psi\rangle = \sum \phi_k^* \psi_k$ in this representation. Now introduce

$$N_k(t) := \frac{1}{\gamma_k\ |T^k_\nu(i\sinh(t/\alpha))|^2}\,,$$

and

$$\varphi_k(t) := -\arg\Big(T^k_\nu(i\sinh(t/\alpha))\Big)\,. \tag{19}$$

Then define a time-dependent unitary transformation W_t given for each t by

$$\phi(t,\theta) \mapsto \phi_{NW}(t,\theta) = \sum_k \phi_k \mathrm{e}^{-i\varphi_k(t)}\, \frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,. \tag{20}$$

i.e., a factor $[2N_k(t)]^{-1/2}$ is absorbed in each coefficient ϕ_k, and a time-evolution $\exp[-i\varphi_k(t)]$ is associated with each mode. The analogy with the definition of the position operator in Minkowski space should be clear. The absorption of the factor $[N_k(t)]^{-1/2}$ is a consequence of Postulates I—III; what is added is the choice of the phases $\varphi_k(t)$ prescribed by Eq. (19). Note that for $t=0$, it follows from the identity $[T^k_\nu(iy)]^* = T^k_\nu(-iy)$ proved in Appendix A and Eq. (19) that $\varphi_k(0)=0$. Substituting that in Eq. (20), we get the operator W_0 displayed in Eq. (14). So we have the right transformation at $t=0$. Moreover, it is easy to check that the Postulates I — III are satisfied. It must be verified that $\varphi_k(t) = \varphi_{-k}(t)$ (parity), and $\varphi_k(t) = -\varphi_k(-t)$ (time-reversal). But the identity $[T^k_\nu(i\sinh(t/\alpha))]^* = T^k_\nu(-i\sinh(t/\alpha))$ together with Eq. (19) imply that $\varphi_k(t) = -\varphi_k(-t)$. And the identity in Eq. (A2) shows that $T^{-k}_\nu(z)$ and $T^k_\nu(z)$ are proportional, with a real proportionality factor, leading to $\varphi_k(t) = \varphi_{-k}(t)$.

There is a simple physical interpretation for of the prescribed choice of the phases $\varphi_k(t)$. The Newton-Wigner wavefunction at time t is described in Eq. (20) as a square-integrable function on a circle of radius 1. Its squared value gives the probability of finding the particle in an infinitesimal interval of angles. But the actual spatial radius of the corresponding time slice is $R = \alpha\cosh(t/\alpha)$, so if one wants to get the probability density in the spatial slice itself, a factor of $\sqrt{R}$ must be included, leading to

$$\tilde{\phi}_{NW}(t,\theta) = \frac{1}{\sqrt{\alpha\cosh(t/\alpha)}} \sum_k \phi_k \mathrm{e}^{-i\varphi_k(t)}\, \frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}}\,.$$

In this case, the transformation which defines $\tilde{\phi}_{NW}(t,\theta)$ involves the absorption of a factor $[2\omega^{dS}_k(t)]^{-1/2}$, with

$$\omega^{dS}_k(t) := \frac{1}{\gamma_k\, |T^k_\nu(i\sinh(t/\alpha))|^2\, \alpha\cosh(t/\alpha)}\,, \qquad (21)$$

There is an interesting relation between the derivative of the phases $\varphi_k(t)$ and the factors $\omega^{dS}_k(t)$. Pick Eq. (4) and consider it on the imaginary axis, with $z = iy$. Divide it by $\left|T^k_\nu(-iy)\right|^2 = T^k_\nu(iy)\, T^k_\nu(-iy)$:

$$\begin{aligned} \frac{-2}{\gamma_k(1+y^2)\,|T^k_\nu(-iy)|^2} &= \left[\frac{T^{k\,\prime}_\nu(-iy)}{T^k_\nu(-iy)} + \frac{T^{k\,\prime}_\nu(iy)}{T^k_\nu(iy)}\right] \\ &= 2\mathrm{Re}\left[\frac{T^{k\,\prime}_\nu(iy)}{T^k_\nu(iy)}\right]\,. \qquad (22)\end{aligned}$$

The last identity follows from the fact that the derivative of a Legendre function can be written as a (real) linear combination of Legendre functions, which are complex conjugated by the inversion $iy \to -iy$. Taken together with Eq. (19), that implies

$$\varphi'_k(t) = -\frac{1}{\alpha}\cosh(t/\alpha)\,\mathrm{Re}\left[\frac{T^{k\,\prime}_\nu(i\sinh(t/\alpha))}{T^k_\nu(i\sinh(t/\alpha)}\right]\,,$$

what in turn, using Eq. (21) and Eq. (22), leads to

$$\varphi'_k(t) = \omega^{dS}_k(t)\,.$$

Therefore, the dynamics described by Eq. (20) corresponds to that generated by normal modes k with a time-dependent energy $\omega^{dS}_k(t)$, with the time $t=0$ representation fixed by Postulates I—III. In other words, we are looking at Eq. (21) as a time-dependent dispersion relation giving the energy of a mode k as a function of time.

Now let us discuss the possibility of deriving the dynamics of the NW-wavefunction from the action of the de Sitter group on the NW-wavefunctions at time $t=0$. We are going to show that this idea does not work, at least for what seems to us the most natural way of implementing it. It is well-known that there is no time-translation isometry in dS^2. Thus there is no unitary operator of time-translation associated with the symmetry group, and the best one can do is to find a local notion of time-evolution in terms of isometries. That can be done in the following way. Let $U_{nt}(\beta)$ be a boost along the direction $\hat{n}$ at an angle θ' with the axis X^1. These boosts move points (t,θ') along their geodesics (with a vanishing angular velocity), thus acting as time-evolution operators, at least for points in the direction $\hat{n}$. So there is a distinct time-evolution operator associated with each direction $\hat{n}$. The quantum representation of this action can be written in the form:

IV: If $U_{nt}(\beta)$ is a boost along the direction $\hat{n}$ at an angle θ' with the axis X^1, which sends the point (t,θ') to (t',θ') in dS^2, then $\left(U_{nt}(\beta)\phi\right)_{NW}(t,\theta') = f(t,t')\,\phi_{NW}(t',\theta')$, where $f(t,t')$ is some real function.

What the condition above states is that the boosts which generate the local time-evolution just carry probability amplitudes along the related geodesics. The factor $f(t,t')$ is introduced so that there is the possibility of normalizing the states related by the transformation. The intended role of Condition IV is clear: it should fix the form of the Newton-Wigner wavefunction for all times t from the knowledge of the wavefunction at time $t=0$, according to:

$$\phi_{NW}(t,\theta') = \left(U_{nt}(\beta)\phi\right)_{NW}(0,\theta')\,,$$

where $U_{nt}(\beta)$ is the boost which maps the point $(0,\theta')$ into (t,θ').

There is a technical matter here: since we are working with square-integrable functions, the value of a function $\phi_{NW}(t,\theta')$ at a single point θ' is not a well-defined quantity — the so-called function is in fact an equivalence class of functions which differ at some set of measure zero. In order to avoid this problem, we require condition IV to be valid only for smooth Newton-Wigner functions, in which case the probability amplitude is well-defined at each point. More precisely: we are trying to find the unitary operator W_t from the knowledge of W_0 and Postulate IV. Recall that W_t is a unitary operator,

$W_t : \mathcal{H} \to L^2(S)$. Thus it is a bounded transformation, and as a consequence of that, continuous. But then it is sufficient to determine its action in a dense subset of $\mathcal{H}$: the continuity of W_t fixes its action on the whole space. We are taking as a suitable dense subset the set of vectors in $\mathcal{H}$ whose images under W_0 are smooth NW-wavefunctions in $L^2(S)$.

The physical content of Postulate IV is perhaps clearer when one looks at it from another point of view, that of localized states. In this language, what is required is that a state localized at (t, θ') is sent by the boost $U_{nt}(\beta)$ to a new localized state (up to a normalizing factor) situated at the point (t', θ'). This covariance postulate is meant to emulate the corresponding fact in Minkowski space, where the time-evolution operator maps localized states in localized states.

Now let us prove that Postulate IV is incompatible with Postulates I—III. Let $\phi \in \mathcal{H}$ be such that its NW-wavefunction $\phi_{NW}(0, \theta)$ at $t = 0$ is a smooth function, and let $\phi_{NW}(t, \theta) = W_t \phi$ be its NW-wavefunction for some t. Write the dynamics in Fourier expanded form,

$$\phi_{NW}(t, \theta) = \sum_k q_k(t) \frac{\mathrm{e}^{ik\theta}}{\sqrt{2\pi}} \,. \tag{23}$$

The general solution of Postulates I — III is given in Eq. (13),

$$q_k(t) = \mathrm{e}^{-i\varphi_k(t)} q_k(0) \,, \tag{24}$$

where $\varphi_k(t)$ is some by now undetermined phase (we are not assuming here the prescription in Eq. (19)), restricted only by $\varphi_k(t) = \varphi_{-k}(t)$, and $\varphi_k(t) = -\varphi_k(-t)$. Let $\beta \in \mathbb{R}$ be a parameter such that the boost $U_{1t}(\beta)$ along the axis X^1 sends the point $(0, 0)$ to $(t, 0)$ in de Sitter space. Then, from postulate IV,

$$(U_{1t}(\beta)\phi)_{NW}(0, 0) = f(t)\, \phi_{NW}(t, 0) \,, \tag{25}$$

where we have written $f(t)$ instead of $f(0, t)$, for short. This identity must be valid for all t, in which case $\beta := \beta(t)$ is a function of t. Consider the case of the normal modes, i.e., take $\phi = u_l$. In this case, $q_k(t) = \delta_{kl} \exp[-i\varphi_k(t)]$, where δ is the Kronecker delta. The matrix elements of the operator $U_{1t}(\beta)$ can be written as:

$$D_{ml}(\beta) := [U_{1t}(\beta)]_{ml} = \sum_{j=0}^{\infty} \frac{\beta^j}{j!} \langle m | (N_{1t})^j | l \rangle \,. \tag{26}$$

Then Eq. (25), taken together with Eq. (23), leads to

$$\sum_m D_{ml}(\beta) = \mathrm{e}^{-i\varphi_k(t)} f(t) \,. \tag{27}$$

This relation must hold for all l, with the same $f(t)$. But then the modulus of the sum on the left side in the above equation must be independent of l. Or, equivalently, $M_k(\beta) = M_l(\beta)$, with $M_l(\beta) = \left|\sum_m D_{ml}(\beta)\right|^2$ (it will be easier to work with the squared modulus). A power expansion for $M_k(\beta)$ can be written from Eq. (26). If the identity $M_k(\beta) = M_l(\beta)$ holds for all β, then all terms in the expansion are independent of k. But that is not true. Keeping terms up to second order in β, what one gets is:

$$M_k(\beta) \simeq 1 + \frac{\beta^2}{4} \left[2|(\nu + k)(\nu - k)| - |(\nu + k)(\nu + k - 1)| - |(\nu - k)(\nu - k - 1)| \right] .$$

The value of this expression depends on k. Thus it is impossible to satisfy the Postulate IV together with Postulates I—III.

IV. PERSPECTIVES

We have showed that a notion of localization exists for massive neutral scalar fields in de Sitter space compatible with the prescription for the choice of positive energy modes encoded in the Hadamard condition. In de Sitter space, this condition is equivalent to the choice of the Bunch-Davies vacuum as the "physical vacuum" among the family of α-vacua. Therefore, we have proved that localizability is compatible with this choice of vacuum. A natural question arises whether other choices of vacuum are compatible or not with localizability. If they are not, that would be another argument in favor of the Bunch-Davies vacuum. We expect to investigate this problem in a future work.

Another direction of research is related to the problem of understanding the classical limit of quantum field theories in curved spacetimes. Following the general procedure for studying classical limits introduced by Hepp in [9], we have proved in a previous work [10] that the quantum theory of the free neutral massive scalar field in Minkowski space has two distinct kinds of classical limits: one of them describing a classical field theory, the other one a classical particle dynamics. The Newton-Wigner position operator is used in order to prove the existence of the latter. We expect that the same problem can be investigated in de Sitter space along similar lines, with the position probability distributions discussed herein playing the role of the Newton-Wigner operator.

ACKNOWLEDGMENTS

This work was supported by FAPESP under Grant 2007/55450-0. NY thanks the Erwin Schrödinger Institute, Vienna, for support and hospitality during the program *"Quantum Field Theory on Curved Spacetimes and Curved Target Spaces"*. He also thanks U. Moschella for discussions during the initial stages of this work.

Appendix A: Proof of $[T_\nu^k(iy)]^* = T_\nu^k(-iy)$

The functions $T_\nu^k(z)$ are defined for $|z-1| < 2$ in terms of hypergeometric functions by

$$T_\nu^k(z) := \frac{(1-z^2)^{k/2}\Gamma(\nu+k+1)}{2^k\Gamma(k+1)\Gamma(\nu-k+1)}\, f(z)\,, \tag{A1}$$

with

$$f(z) := F\left(-\nu+k, \nu+k+1, k+1; \frac{1-z}{2}\right)\,.$$

Let us see what happens to the function under complex conjugation. For $k \geq 0$, the hypergeometric function can be represented as a convergent power series in the radius $|z| < 1$. The $(j+1)$-th term in the expansion of f in powers of $(1-z)/2$ has a coefficient of the form

$$\frac{\prod_{l=0}^{j}(-\nu+k+l)(\nu+k+1+l)}{\prod_{l=0}^{j}(k+1+l)}\,.$$

The denominator is real, so ignore it. Recall that ν is a root of the quadratic equation $\nu(\nu+1) = -\alpha^2\mu^2$, so whenever $\nu(\nu+1)$ makes an appearance, it is a real number. It follows that every factor in the product is real. Thus the power function has real coefficients, and $[f(z)]^* = f(z^*)$.

Now, the gamma functions. The factor $\Gamma(k+1)$ is real. The part that matters is

$$\frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)} = (\nu+k)(\nu+k-1)\cdots(\nu-k+2)(\nu-k+1)\,.$$

This can be rewritten as

$$\prod_{l=0}^{k-1}(\nu+k-l)(\nu+1-k+l) = \prod_{l=0}^{k-1}\left[\nu(\nu+1)-(k-l)^2+(k-l)\right],$$

which is also real. Besides that, each factor in the product is a negative number: $\nu(\nu+1) = -\alpha^2\mu^2$, and $(k-l)^2 \geq (k-l)$, since $k-l$ is an integer. Thus the product is negative for odd k, and positive for even k. This result will be needed somewhere else.

Finally, take $z = iy$, $y \in \mathbb{R}$, and consider the factor $(1-z^2)^{k/2}$. Here one must be careful. The functions T_μ^k are defined with square roots cut along distinct lines: the factor $\sqrt{1-z}$ is cut along $x > 1$ on the real axis, while the factor $\sqrt{1+z}$ has a cut along $x < -1$. With these choices, $\sqrt{1-(ix)^2} = |1+iy|$. Then it follows that $\{[1-(ix)^2]^{k/2}\}^* = [1-(-ix)^2]^{k/2}$, so that $[T_\nu^k(iy)]^* = T_\nu^k((iy)^*) = T_\nu^k(-iy)$, at least in the radius $|z| < 1$ and for $k \geq 0$.

In order to extend the result to the domain of T_ν^k, introduce an auxiliary analytic function $[T_\nu^k(-x,y)]^*$. This function coincides with $T_\nu^k(-z)$ along the imaginary axis inside the radius $|z| < 1$. Moreover, both functions are defined on the same domain: $T_\nu^k(z)$ is single-valued on a domain invariant both under inversion $z \to -z$, and inversion of the real part $(x,y) \to (-x,y)$. Thus $[T_\nu^k(-x,y)]^* = T_\nu^k(-z)$. Restricting to the imaginary axis, $[T_\nu^k(iy)]^* = T_\nu^k(-iy)$. The result is extended to negative k using the relation

$$T_\nu^{-k}(z) = (-1)^k\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}T_\nu^k(z)\,. \tag{A2}$$

It was already proved that the factor with the Γ's is real.

Appendix B: Two-point function

The two-point function, $G := \langle 0|\phi(t,\theta), \phi(t',0)|0\rangle$, is given by:

$$\begin{aligned}
G &= \sum_k u_k(t,\theta)u_k^*(t',0) \\
&= \sum_k \frac{\Gamma(-\nu-k)\Gamma(\nu-k+1)}{4\pi}\mathrm{e}^{ik\theta}T_\nu^k(iy)T_\nu^k(-iy') \\
&= \frac{1}{4|\sin\nu\pi|}\sum_k(-)^k\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}\mathrm{e}^{ik\theta}T_\nu^k(iy)T_\nu^k(-iy'),
\end{aligned}$$

with $y = \sinh(t/\alpha)$. Call the sum in the last line S. Using (A2), it can be written as

$$\begin{aligned}
S &= \sum_{k=-\infty}^{0}\left[\mathrm{e}^{ik\theta}T_\nu^{-k}(iy)T_\nu^k(-iy')\right] - T_\nu(iy)T_\nu(-iy') \\
&\quad + \sum_{k=0}^{\infty}\left[\mathrm{e}^{ik\theta}T_\nu^k(iy)T_\nu^{-k}(-iy')\right] \\
&= 2\sum_{k=0}^{\infty}\cos(k\theta)T_\nu^k(iy)T_\nu^{-k}(-iy') - T_\nu(iy)T_\nu(-iy') \\
&= 2\sum_{k=0}^{\infty}\epsilon_k\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}\cos(k(\pi-\theta))T_\nu^k(iy)T_\nu^k(-iy')\,,
\end{aligned}$$

where $\epsilon_k = 1-\delta_{k0}/2$, i.e., ϵ_k is 1 except for $k=0$, when it is 1/2. There is a nice summation theorem for Legendre functions [23],

$$P_\nu\left(z_1z_2 + \sqrt{1-z_1^2}\sqrt{1-z_2^2}\cos\theta\right) = 2\sum_{k=0}^{\infty}\epsilon_k\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}\cos(k\theta)T_\nu^k(z_1)T_\nu^k(z_2)\,,$$

which leads to

$$S = P_\nu\left(yy' - \sqrt{1-(iy)^2}\sqrt{1-(-iy')^2}\cos\theta\right) = P_\nu(Z)\,.$$

The argument in the function $P_\nu(Z)$ can be written in invariant form:

$$\begin{aligned}
Z &= \sinh(t/\alpha)\sinh(t'/\alpha) - \cosh(t/\alpha)\cosh(t'/\alpha)\cos\theta \\
&= \alpha^{-2}X\cdot X'\,,
\end{aligned}$$

where X is the vector in the Minkowski space M^3 corresponding to the point (t, θ) in de Sitter space, while X' corresponds to the point $(t', 0)$. Collecting the calculations,

$$G = G(Z) = \frac{1}{4|\sin(\nu\pi)|} P_\nu(Z) .$$

The Legendre function is singular at $Z = -1$, where a cut begins which extends along the real axis to $-\infty$. This value has a simple geometric interpretation. Recall that the causality relations on de Sitter hyperboloid are inherited form the Minkowski ambient space: two points x, x' are space (light,time) related if their corresponding vectors are space (light,time)-like. In particular, light-like related vectors satisfy $(X - X')^2 = 0 \Rightarrow X \cdot X' = -\alpha^2$, so that $Z = -1$ in this case. In other words, the two-point function is singular on the light cone. This property is characteristic of the Bunch-Davies vacuum: any other choice of modes would lead to an additional singularity at the antipodal points of the light-cone. Besides that, one can write the Legendre function in terms of a hypergeometric function to get

$$G(Z) = \frac{1}{4|\sin(\nu\pi)|} F\left(\nu + 1, -\nu, 1; \frac{1 - Z}{2}\right) .$$

Compare with the original Bunch and Davies work [27]. In their notation, a coefficient μ is introduced:

$$\mu = \sqrt{\frac{1}{4} - 2\xi - m^2\alpha^2} ,$$

in terms of which our ν becomes $\nu = -1/2 + \mu$. It is easy to check this relation. Our definition (2) of ν can be rewritten using $R = -2/\alpha^2$ as

$$\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2\xi - m^2\alpha^2} .$$

Moreover, $\sin(\nu\pi) = \sin((-1/2 + \mu)\pi) = (-1)\cos(\mu\pi)$, where $\mu \in (-1/2, 1/2)$ or is purely imaginary, so that $\cos(\mu\pi)$ is positive either way. Thus

$$G(Z) = \frac{1}{4} \sec(\mu\pi) F\left(\nu + 1, -\nu, 1; \frac{1 - Z}{2}\right) .$$

That is just the expression in Eq. (2.13) for the two-point function in [27].

Appendix C: Philips and Wigner localized states

Our work has a close relation with that of Philips and Wigner [31], so here we present a brief review of their little known article. Their purpose was to investigate how the existence of localized states is related to the condition of positivity of the energy. But it was not known at that time how to define positive energy states in curved spaces, so in order to check the sign of the energy of a given state it was necessary to study the limit where the geometry of de Sitter approached that of Minkowski space, what was done invoking a contraction of the group representation. Although the problem remains unsolved in general, it is now known that, at least in the case of spacetimes with a compact Cauchy surface, the Hadamard condition is sufficient to fix the ambiguity in the choice of the positive energy solutions [21].

Let us describe the unitary representation of the de Sitter group used in [31]. We restrict to the case of $O(2,1)$ which is relevant here. Let $\mathcal{H}'$ be the set of square-integrable functions $\psi(\theta)$ on the unitary circle S on the Euclidean plane $\mathbb{R}^2$. Extend these functions to the whole plane: $\psi(\theta) \mapsto f(\rho)\psi(\theta)$, where ρ is the radius $\rho = \sqrt{(X^1)^2 + (X^2)^2}$, and $f(\rho)$ is a fixed function, smooth and square-integrable on the plane. Rotations are realized as rotations on the circle, i.e.,

$$N_{12} = -\partial/\partial\theta , \tag{C1}$$

and infinitesimal boosts are represented by

$$N_{1t} = -\sin\theta \frac{\partial}{\partial\theta} + \nu\cos\theta ,$$
$$N_{2t} = \cos\theta \frac{\partial}{\partial\theta} + \nu\sin\theta , \tag{C2}$$

where $\nu = -1/2 + i\lambda$. The generators can be integrated to give finite boosts and rotations, so that there are unitary operators $U(S)$ corresponding to each element S of the restricted de Sitter group. The parity operator $\mathbf{P}$, understood as the representation of the geometric operation p of reversing the axis X^1 in the ambient Minkowski space, must satisfy the group relations up to some projective factor,

$$\mathbf{P}U(S) = \omega(S)U(pSp)\mathbf{P} ,$$

where S is any Lorentz transformation in the restricted de Sitter group. But it can be proved that $\omega(S) = 1$, and that

$$\mathbf{P}\psi(X^1, X^2) = \pm\psi(-X^1, X^2) ,$$

where the choice of the sign must be the same for all ψ. This choice is physically irrelevant, so just pick the sign $+1$. For the time-reversal operator $\mathbf{T}$, the group relations lead to essentially two possibilities, corresponding to a unitary $\mathbf{T}_u$ or an anti-unitary $\mathbf{T}_a$, given by

$$\mathbf{T}_u\psi(X^1, X^2) = \pm\psi(-X^1, -X^2) ,$$

and

$$\mathbf{T}_a\psi(\theta) = \int K(\theta - \theta')\psi^*(\theta')d\theta' , \tag{C3}$$

where the kernel K is given in Fourier expanded form by

$$K(\theta - \theta') = \sum a_k e^{ik(\theta - \theta')} , \quad \frac{a_{k+1}}{a_k} = -\frac{\frac{1}{2} + k - i\lambda}{\frac{1}{2} + k + i\lambda} , \tag{C4}$$

with $a_0 = 1/(2\pi)$. The coefficients automatically satisfy $a_k = a_{-k}$. In order that $\mathbf{T}_a$ is uniquely defined, it is assumed that $\mathbf{T}^2 = 1$ (it could be -1), and that $\mathbf{TP} = \mathbf{PT}$ (there could be a phase difference).

The definition of the localized states is based on a set of three postulates, which represent the de Sitter version of the postulates of Newton and Wigner adopted in the case of Minkowski space [1]. The postulates are:

(a): A localized state is invariant under reflections that leave the point of localization invariant.

(b): A rotation applied to a localized state gives a new localized state — the point of localization is just rotated accordingly.

(c): A boost which keeps the point of localization invariant changes the state as little as possible.

The first result is that the postulates cannot be satisfied with a unitary time-reversal operator. Thus the existence of localized states implies that $\mathbf{T}$ is anti-unitary — it must be the $\mathbf{T}_a$ defined in Eqs. (C3), (C4). In this case, the postulates are satisfied by two distinct sets of localized states.

Consider a state $\psi_1(\theta)$ localized at $\theta = \pi/2$ at $t = 0$. It must be invariant under parity and time-reversal. Writing a Fourier expansion

$$\psi_1(\theta) = \sum l_k \mathrm{e}^{ik\theta} ,$$

invariance under parity implies

$$l_{-k} = (-1)^k l_k ,$$

while invariance under time-reversal leads to

$$2\pi a_k l_{-k}^* = l_k .$$

Combining these results, and using (C4), it follows that

$$\frac{l_{k+1}}{l_k} = \zeta_{k+1/2} \frac{\frac{1}{2} + k - i\lambda}{\left[(\frac{1}{2} + k)^2 + \lambda^2\right]^{1/2}} ,$$

where the ζ's are real numbers satisfying

$$\zeta_{k+1/2}\zeta_{-k-1/2} = 1 .$$

Then condition (b), together with (c), which reduces here to minimal deformation under boosts along X^1, fixes $\zeta = 1$ or $\zeta = -1$. The first possibility is ruled out by looking what happens in the contraction of the de Sitter group representation to a representation of the inhomogeneous Lorentz group. The choice $\zeta = 1$ corresponds to a state of negative-energy in Minkowski space in this limit. So it must be $\zeta = -1$. The Fourier coefficients of ψ_1 are then completely determined, being given by $(k > 0)$,

$$\begin{aligned} l_k &= (-1)^k \frac{\left(\frac{1}{2} - i\lambda\right)\left(\frac{3}{2} - i\lambda\right)\ldots\left(k - \frac{1}{2} - i\lambda\right)}{\left|\left(\frac{1}{2} - i\lambda\right)\left(\frac{3}{2} - i\lambda\right)\ldots\left(k - \frac{1}{2} - i\lambda\right)\right|}, \\ l_0 &= 1 \qquad \text{(C5)} \\ l_{-k} &= (-1)^k \frac{\left(-\frac{1}{2} + i\lambda\right)\left(-\frac{3}{2} + i\lambda\right)\ldots\left(-k + \frac{1}{2} + i\lambda\right)}{\left|\left(-\frac{1}{2} + i\lambda\right)\left(-\frac{3}{2} + i\lambda\right)\ldots\left(-k + \frac{1}{2} + i\lambda\right)\right|} . \end{aligned}$$

States localized at other angles are obtained with the application of rotations.

It is curious that in this approach the condition (b) that localized states are well-behaved under rotations is not as important as its counterpart in Minkowski space. It is necessary to supplement it here with the auxiliary condition (c), which has a more obscure interpretation — it is not an invariance condition, nor a mapping of one localized state into another, corresponding to the geometrical action. What one would really like to require was that the boost kept the state invariant; since that is impossible, the condition is relaxed to that of minimal deformation. In our approach, this axiom is not necessary, and the axiom of covariance under rotations is restored to its central position.

[1] T. D. Newton and E. P. Wigner, *"Localized states for elementary systems"*, Rev. Mod. Phys. **21**, 400 (1949)
[2] A. S. Wightman, *"On the localizability of quantum mechanical systems"*, Rev. Mod. Phys. **34**, 845 (1962)
[3] T. O. Philips, *"Lorentz invariant localized states"*, Phys. Rev. **136**, B893 (1964)
[4] G. Kaiser, *"Phase-space approach to relativistic quantum mechanics II: Geometrical aspects"*, J. Math. Phys. **19**, 502 (1978)
[5] R. Haag and J. A. Swieca, *"When does a quantum field theory describe particles?"*, Commun. Math. Phys. **1**, 308 (1965)
[6] D. Buchholz and E. Wichmann, *"Causal independence and the energy level density of states in local quantum field theory"*, Commun. Math. Phys. **106**, 321 (1986)
[7] B. Schroer, *"Localization and the interface between quantum mechanics, quantum field theory and quantum gravity I."*, Stud. Hist. Philos. Mod. Phys. **41**, 104 (2010)
[8] H. Araki and R. Haag, *"Collision cross sections in terms of local observables"*, Commun. Math. Phys. **4**, 77 (1967).
[9] K. Hepp, *"The classical limit of quantum mechanical correlation functions"*, Commun. Math. Phys. **35**, 265 (1974)
[10] N. Yokomizo and J. C. A. Barata, *"Multiple classical limits in relativistic and nonrelativistic quantum mechanics"*, J. Math. Phys. **50**, 123512 (2009)
[11] A. G. Riess *et al*, *"Observational evidence form Supernovae for an accelerating universe and a cosmological constant"*, Astron. J. **116**, 1009 (1998)
[12] S. Perlmutter et al., *"Measurements of Omega and*

Lambda from 42 high redshiftsupernovae", Astrophys. J. **517**, 565 (1999)
[13] S. Weinberg, *"Cosmology"*, (Oxford, 2008)
[14] A. D. Linde, *"Particle physics and inflationary cosmology"*, (Harwood, 1990)
[15] A. R. Liddle and D. H. Lyth, *"Cosmological inflation and large-scale structure"*, (Cambridge, 2000)
[16] S. Cacciatori, V. Gorini, A. Kamenshchik and U. Moschella, "Conservation laws and scattering for de Sitter classical particles", Class. Quant. Grav. **25**, 075008 (2008)
[17] J. Bros, H. Epstein and U. Moschella, *"Lifetime of a massive particle in a de Sitter universe"*, JCAP **0802**, 3 (2008)
[18] R. Aldrovandi, J. P. Beltrán Almeida, and J. G. Pereira, *"Cosmological term and fundamental physics"*, Int. J. Mod. Phys. D **13**, 2241 (2004)
[19] W. G. Unruh and R. M. Wald, ''What happens when an accelerating observer detects a Rindler particle", Phys. Rev. D **29**, 1047 (1984)
[20] D. Marolf and C. Rovelli, *"Relativistic quantum measurement"*, Phys. Rev. D **65**, 023510 (2002)
[21] R. Wald, ''Quantum field theory in curved spacetime and black hole thermodynamics", (Chicago, 1994)
[22] R. Brunetti, K. Fredenhagen and S. Hollands, ''A remark on alpha-vacua for quantum field theories on de Sitter space", JHEP **0505**, 063 (2005)
[23] C. Snow, ''Hypergeometric and Legendre functions with applications to integral equations and potential theory", N.B.S. Applied Math. Series **19** (U.S. Govt. Printing Office, 1952)
[24] V. Bargmann, ''Irreducible unitary representations of the Lorentz group", Ann. Math. **48**, 568 (1947)
[25] N. D. Birrell and P. C. W. Davies, ''Quantum fields in curved space", (Cambridge, 1982)
[26] R. Brunetti, K. Fredenhagen and S. Hollands, ''A remark on alpha vacua for quantum field theories on de Sitter space", JHEP **0505**, 063 (2005)
[27] T. S. Bunch and P. C. W. Davies, ''Quantum field theory in de Sitter space: renormalization by point-splitting", Proc. R. Soc. Lond. A **360**, 117 (1978)
[28] C. Fewster, "Lectures on quantum field theory in curved spacetime", Lecture Note **39** (2008), Max Planck Institute for Mathematics in the Natural Sciences.
[29] R. Haag, ''Local quantum physics", (Springer, 1980)
[30] I. S. Gradshteyn and I. M. Ryzhik, ''Table of integrals, series and products", (Academic, 2007), 7a. Ed.
[31] T. O. Philips and E. P. Wigner, ''de Sitter space and positive energy", in ''Group theory and its applications", ed. E. M. Loebl (Academic, 1968)