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Newtonian Gravity and the Bargmann Algebra

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ABSTRACT

We show how the Newton-Cartan formulation of Newtonian gravity can be obtained from gauging the Bargmann algebra, i.e., the centrally extended Galilean algebra. In this gauging procedure several curvature constraints are imposed. These convert the spatial (time) translational symmetries of the algebra into spatial (time) general coordinate transformations, and make the spin connection gauge fields dependent. In addition we require two independent Vielbein postulates for the temporal and spatial directions. In the final step we impose an additional curvature constraint to establish the connection with (on-shell) Newton-Cartan theory. We discuss a few extensions of our work that are relevant in the context of the AdS-CFT correspondence.

Contents

1 Introduction

It is well known that Einstein's formulation of gravity can be obtained by performing a formal gauging procedure of the Poincaré algebra. In this procedure one associates to each generator of the Poincaré algebra a gauge field. Next, one imposes constraints on the curvature tensors of these gauge fields such that the translational symmetries of the algebra get converted into general coordinate transformations. At the same time the gauge field of the Lorentz transformations gets expressed into (derivatives of) the Vierbein gauge field which is the only independent gauge field. One thus obtains an off-shell formulation of Einstein gravity. On-shell Einstein gravity is obtained by imposing the usual Einstein equations of motion.

One may consider the non-relativistic version of the Poincaré algebra and Einstein gravity independently. It turns out that the relevant non-relativistic version of the Poincaré algebra is a particular contraction of the Poincaré algebra trivially extended with a 1-dimensional algebra that commutes with all the generators. This contraction yields the so-called Bargmann algebra, which is the centrally extended Galilean algebra. On the other hand, taking the non-relativistic limit of general relativity leads to the well-known non-relativistic Newtonian gravity in flat space. The Newton-Cartan theory is a geometric re-formulation of this Newtonian theory, mimicking as much as possible the geometric formulation of general relativity [1, 2]. A notable difference with the relativistic case is the occurrence of a degenerate metric.

The question we pose in this note is: can we derive the Newton-Cartan formulation of Newtonian gravity directly from gauging the Bargmann algebra in the same way that Einstein gravity may be derived from gauging the relativistic Poincaré algebra as described above?[1] The answer will be yes, but there are some subtleties involved. This is partly due to the fact that the standard procedure leads to spin-connection fields that not only depend on the temporal and spatial Vielbeins but also on the gauge field corresponding to the central charge generator. These connections have to be fixed appropriately, via further curvature constraints, in order to obtain the correct non-relativistic Poisson equation as well as the geodesic equation for

[1] The gauging of the Bargmann algebra, from a somewhat different point of view, has been considered before in [3, 4].

a massive particle.

The outline of this note is as follows. In section 2 we first review how Einstein gravity may be obtained by gauging the Poincaré algebra. To keep the discussion in this section as general as possible we leave the dimension D of spacetime arbitrary. Next, we briefly review in section 3 the Newton-Cartan formulation of Newtonian gravity, since this is the theory we wish to end up with in the non-relativistic case. We next proceed, in section 4, with gauging the Bargmann algebra. In a first step we introduce a set of curvature constraints that convert the spatial (time) translational symmetries of the algebra into spatial (time) general coordinate transformations. We next impose a Vielbein postulate for the Vielbeins in the temporal and spatial directions. In a final step we impose further curvature constraints on the theory in order to recover the non-relativistic Poisson equation and the geodesic equation for a massive particle. Finally, our conclusions and suggestions for further work are presented in section 5.

2 Einstein Gravity and Gauging the Poincaré Algebra

In this section we briefly review how the basic ingredients of Einstein gravity may be obtained by applying a formal gauging procedure to the Poincaré algebra. We leave the dimension D of spacetime in this section arbitrary.

Our starting point is the D-dimensional Poincaré algebra $\mathfrak{iso}(D-1,1)$ with generators $P_a, M_{ab}\,(a = 0, 1, \cdots, D-1)$

$$\begin{aligned} [P_a, P_b] &= 0\,, \\ [M_{bc}, P_a] &= -2\eta_{a[b}P_{c]}\,, \\ [M_{cd}, M_{ef}] &= 4\eta_{[c[e}M_{f]d]}\,. \end{aligned} \tag{2.1}$$

Associating a gauge field $e_\mu{}^a$ to the local P-transformations with spacetime dependent parameters $\zeta^a(x)$, and a gauge field $\omega_\mu{}^{ab}$ to the local Lorentz transformations with spacetime dependent parameters $\lambda^{ab}(x)$, we obtain the following transformation rules

$$
\begin{aligned}
\delta e_\mu{}^a &= \partial_\mu \zeta^a - \omega_\mu{}^{ab}\zeta^b + \lambda^{ab} e_\mu{}^b \,, \\
\delta \omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]c} \,.
\end{aligned} \tag{2.2}
$$

In order to make contact with gravity we wish to replace the local P-transformations of all gauge fields by general coordinate transformations and to interpret $e_\mu{}^a$ as the Vielbein, with the inverse Vielbein field $e_a{}^\mu$ defined by

$$
e_\mu{}^a \, e_b{}^\mu = \delta_b{}^a \,, \qquad e_\mu{}^a \, e_a{}^\nu = \delta_\mu{}^\nu \,. \tag{2.3}
$$

To show how this can be achieved by imposing curvature constraints we first consider the following general identity for a gauge algebra:

$$
0 = \delta_{gct}(\xi^\lambda) B_\mu{}^A + \xi^\lambda R_{\mu\lambda}{}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda{}^C) B_\mu{}^A \,. \tag{2.4}
$$

The index A labels the gauge fields and corresponding curvatures of the gauge algebra. If we now set $A = a$ for the P-transformations and write the parameter ξ^λ as $\xi^\lambda = e_a{}^\lambda \zeta^a$ we can bring the contribution of $e_\mu{}^a$ in the sum in (2.4) to the left-hand side of the equation to obtain

$$
\delta_P(\zeta^b) e_\mu{}^a = \delta_{gct}(\xi^\lambda) e_\mu{}^a + \xi^\lambda R_{\mu\lambda}{}^a(P) - \delta_M(\xi^\lambda \omega_\lambda{}^{ab}) e_\mu{}^a \,. \tag{2.5}
$$

We see that the difference between a P-transformation and a general coordinate transformation is a curvature term and a Lorentz transformation. More generally, we deduce from the identity (2.4) that, whenever a gauge field transforms under a P-transformation, the P-transformations of this gauge field can be replaced by a general coordinate transformation plus other symmetries of the algebra by putting the curvature of the gauge field to zero. Since the Vielbein is the only field that transforms under the P-transformations, see (2.2), we are led to impose the following constraint:

$$
R_{\mu\nu}{}^a(P) = 0 \,. \tag{2.6}
$$

The same constraint allows us to solve for the Lorentz gauge field $\omega_\mu{}^{ab}$ in terms of (derivatives of) the Vielbein and its inverse:

$$
\omega_\mu{}^{ab}(e, \partial e) = -2 e^{\lambda[a} \partial_{[\mu} e_{\lambda]}{}^{b]} + e_\mu{}^c e^{a\lambda} e^{b\rho} \partial_{[\lambda} e_{\rho]}{}^c \,. \tag{2.7}
$$

What remains is a theory with the Vielbein $e_\mu{}^a$ as the only independent field transforming under local Lorentz transformations and general coordinate transformations and with $\omega_\mu{}^{ab}$ as the dependent spin connection field.

A Γ-connection may be introduced by imposing the Vielbein postulate:

$$\nabla_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - \Gamma^\rho_{\nu\mu} e_\rho{}^a - \omega_\mu{}^{ab} e_\nu{}^b = 0\,. \tag{2.8}$$

The anti-symmetric part of this equation, together with the curvature constraint (2.6), shows that the anti-symmetric part of the Γ-connection is zero, i.e. there is no torsion. From the Vielbein postulate (2.8) one may solve the Γ-connection in terms of the Vielbein and its inverse as follows:

$$\Gamma^\rho_{\nu\mu} = e^\rho{}_a D_\mu e_\nu{}^a\,. \tag{2.9}$$

Here D_μ is the Lorentz-covariant derivative. Finally, a non-degenerate metric and its inverse can be defined as:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}\,, \qquad\qquad g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab}\,. \tag{2.10}$$

This concludes our description of the basic ingredients of off-shell Einstein gravity and the Poincaré algebra. These basic ingredients are an independent non-degenerate metric $g_{\mu\nu}$ and a dependent Γ-connection $\Gamma^\rho_{\nu\mu}$ or, in the presence of flat indices, an independent Vielbein field $e_\mu{}^a$ and a dependent spin-connection field $\omega_\mu{}^{ab}$. The theory can be put on-shell by imposing the Einstein equations of motion.

3 Newton-Cartan Gravity

From now on we restrict the discussion to $D = 4$, i.e. one time and three space directions. We wish to review Newton-Cartan gravity as a geometric rewriting of Newtonian gravity [1, 2]. This geometric re-formulation is motivated by the following observation. First, consider the classical equations of motion of a massive particle,

$$\ddot{x}^i(t) + \frac{\partial\phi(x)}{\partial x^i} = 0\,, \tag{3.1}$$

where $x^i(t)\,(i = 1,2,3)$ are the spatial coordinates, t is the absolute time coordinate and a dot indicates differentiation with respect to t. Furthermore, $\phi(x^k)$ is the Newtonian potential which satisfies the Poisson equation

$$\partial_i\partial^i\phi = 4\pi G\rho\,, \tag{3.2}$$

where ρ is the mass density. The equations of motion (3.1) and (3.2) transform covariantly under the Galileo group

$$x^0 \to x^0 + \xi^0\,, \qquad\qquad x^i \to A^i{}_j x^j + v^i t + d^i\,, \tag{3.3}$$

where $A^i{}_j$ is a constant group element of SO(3) and $\{v^i, d^i\}$ are three-vectors. In addition, these equations are invariant under

$$x^i \to x^i + a^i(t)\,, \qquad\qquad \phi(x) \to \phi(x) - \ddot{a}^j(t)x^j\,, \tag{3.4}$$

where $a^i(t)$ is an arbitrary time-dependent shift vector which can give rise to an acceleration.

From the Newtonian point of view the equations (3.1) describe a *curved* trajectory in a *flat* three-dimensional space. We now wish to re-interpret the same equations as a geodesic in a *curved* four-dimensional spacetime. Indeed, one may rewrite the equations (3.1) as the geodesic equations of motion

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0\,, \tag{3.5}$$

provided that one chooses coordinates $\{x^\mu\} = \{x^0, x^i\} = (t, x^i)$ and takes the following expression for the non-zero connection fields:

$$\Gamma^i_{00} = \delta^{ij}\partial_j\phi\,, \tag{3.6}$$

where we have used the Euclidean three-metric. At this point $\Gamma^\mu_{\nu\rho}$ is a symmetric connection independent of the metric. The coordinate choice $x^0 = t$ corresponds to choosing so-called adapted coordinates. The corresponding D-dimensional spacetime is called the Newton-Cartan spacetime $\mathscr{M}$. The only non-zero component of the Riemann tensor corresponding to the connection (3.6) is

$$R^i{}_{0j0} = \delta^{ik}\partial_k\partial_j\phi\,. \tag{3.7}$$

If one now imposes the equations of motion $R_{00} = 4\pi G\rho$ one obtains the Poisson equation (3.2). To write the Poisson equation in a covariant way we first must introduce a metric.

As it stands, the Γ-connection defined by (3.6) cannot follow from a non-degenerate four-dimensional metric. One way to see this is to consider the Riemann tensor that is defined by this Γ-connection. The Riemann tensor, defined in terms of a metric connection based upon a non-degenerate metric, satisfies certain symmetry properties. One may easily verify that these properties are not satisfied by the Riemann tensor (3.7). Another way to see that a degenerate metric is unavoidable is to consider the relativistic Minkowski metric and its inverse

$$\eta_{\mu\nu}/c^2 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_3/c^2 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} -1/c^2 & 0 \\ 0 & \mathbb{1}_3 \end{pmatrix}. \tag{3.8}$$

Taking the limit $c \to \infty$ naturally leads to a degenerate covariant temporal metric $\tau_{\mu\nu}$ with three zero eigenvalues and a degenerate contra-variant spatial metric $h^{\mu\nu}$ with one zero eigenvalue. We conclude that the Galilei group keeps invariant *two* metrics $\tau_{\mu\nu}$ and $h^{\mu\nu}$ which are degenerate, i.e. $h^{\mu\nu}\tau_{\nu\rho} = 0$. Since $\tau_{\mu\nu}$ is effectively a 1×1 matrix we will below use its Vielbein version which is defined by a covariant vector τ_μ defined by $\tau_{\mu\nu} = \tau_\mu \tau_\nu$.

A degenerate spatial metric $h^{\mu\nu}$ of rank 3 and a degenerate temporal Vielbein τ_μ of rank 1, together with a symmetric connection $\Gamma^\rho_{\mu\nu}$ on $\mathscr{M}$, that depends on these two degenerate metrics, can be introduced as follows [5]. First of all the degeneracy implies that

$$h^{\mu\nu}\tau_\nu = 0\,. \tag{3.9}$$

We next impose metric compatibility:

$$\nabla_\rho h^{\mu\nu} = 0\,, \qquad \nabla_\rho \tau_\mu = 0\,. \tag{3.10}$$

The covariant derivative ∇ is with respect to a connection $\Gamma^\rho_{\mu\nu}$. The second of these conditions indicates that

$$\tau_\mu = \partial_\mu f(x^\nu) \tag{3.11}$$

for a scalar function $f(x^\nu)$. In Newton-Cartan theory this scalar function is chosen to be the absolute time t which foliates $\mathscr{M}$:

$$f(x^\nu) \equiv t\,. \tag{3.12}$$

In general relativity metric compatibility allows one to write down the connection in terms of the metric and its derivatives in a unique way, see eq. (2.9).

In the present analysis, the connection $\Gamma^{\rho}_{\mu\nu}$ is not uniquely determined by the metric compatibility conditions (3.10). This can be seen from the fact that these conditions are preserved by the shift

$$\Gamma^{\rho}_{\mu\nu} \to \Gamma^{\rho}_{\mu\nu} + h^{\rho\lambda} K_{\lambda(\mu}\tau_{\nu)} \tag{3.13}$$

for an arbitrary two-form $K_{\mu\nu}$ [6]. Using this arbitrary two-form it is possible to write down the most general connection which is compatible with (3.10). In order to do this, one needs to introduce new tensors, the spatial inverse metric $h_{\mu\nu}$ and the temporal inverse Vielbein τ^{μ} which are defined by the following properties:

$$\begin{aligned} h^{\mu\nu} h_{\nu\rho} &= \delta^{\mu}_{\rho} - \tau^{\mu}\tau_{\rho}, \qquad & \tau^{\mu}\tau_{\mu} &= 1\,, \\ h^{\mu\nu}\tau_{\nu} &= 0, & h_{\mu\nu}\tau^{\nu} &= 0\,. \end{aligned} \tag{3.14}$$

Note that from these conditions it follows that

$$\nabla_{\rho} h_{\mu\nu} = -2\tau_{(\mu} h_{\nu)\sigma} \nabla_{\rho}\tau^{\sigma} \tag{3.15}$$

which is not zero in general. The most general connection compatible with (3.10) is then [6]

$$\Gamma^{\sigma}_{\mu\nu} = \tau^{\sigma}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\sigma\rho}\Big(\partial_{\nu} h_{\rho\mu} + \partial_{\mu} h_{\rho\nu} - \partial_{\rho} h_{\mu\nu}\Big) + h^{\sigma\lambda} K_{\lambda(\mu}\tau_{\nu)}\,. \tag{3.16}$$

We note that the original independent connection (3.6) is quite different from the metric connection defined in (3.16). Nevertheless, given extra conditions discussed below, the Newton-Cartan theory with the metric connection (3.16) reproduces Newtonian gravity. To see how this goes, it is convenient to use adapted coordinates $x^0 = t$. The conditions (3.11) and (3.14) then imply

$$\begin{aligned} \tau_{\mu} &= \delta^{0}_{\mu}, \qquad & \tau^{\mu} &= (1, \tau^{i})\,, \\ h^{\mu 0} &= 0, & h_{\mu 0} &= -h_{\mu i}\tau^{i}\,. \end{aligned} \tag{3.17}$$

These conditions are preserved by the coordinate transformations

$$\begin{aligned} x^0 &\to x^0 + \xi^0\,, \\ x^i &\to x^i + \xi^i(x^{\mu})\,, \end{aligned} \tag{3.18}$$

where ξ^0 is a constant. The finite spatial transformation generated by $\xi^i(x^\mu)$ is invertible. In adapted coordinates the connection coefficients (3.16) are given by [6]

$$\begin{aligned}
\Gamma^i_{00} &= h^{ij}(\partial_0 h_{j0} - \tfrac{1}{2}\partial_j h_{00} + K_{j0}) \equiv h^{ij}\Phi_j \,, \\
\Gamma^i_{0j} &= h^{ik}(\tfrac{1}{2}\partial_0 h_{jk} + \partial_{[j}h_{k]0} - K_{jk}) \equiv h^{ik}(\tfrac{1}{2}\partial_0 h_{jk} + \omega_{jk}) \,, \\
\Gamma^i_{jk} &= \left\{ {i \atop jk} \right\} , \qquad\qquad \Gamma^0_{\mu\nu} = 0 \,,
\end{aligned} \tag{3.19}$$

where $\left\{ {i \atop jk} \right\}$ are the usual Christoffel symbols with respect to the metric h_{ij} with inverse h^{ij}.

We now replace the original equations of motion $R_{00} = 4\pi G\rho$ by the covariant Ansatz

$$R_{\mu\nu} = 4\pi G\rho \, \tau_\mu \tau_\nu \tag{3.20}$$

and verify that this leads to Newtonian gravity. In adapted coordinates these equations imply that

$$R_{ij} = R_{i0} = 0 \,. \tag{3.21}$$

The condition $R_{ij} = 0$ implies that the spatial hypersurfaces are flat, i.e. one can choose a coordinate frame with $\Gamma^i_{jk} = 0$ such that the spatial metric is given by

$$h_{ij} = \delta_{ij}, \qquad h^{ij} = \delta^{ij} \,. \tag{3.22}$$

This implies

$$\begin{aligned}
\Gamma^i_{0j} &= h^{ik}\omega_{jk} \quad \leftrightarrow \quad \omega_{ij} = h_{k[j}\Gamma^k_{i]0} \,, \\
\Gamma^i_{00} &= h^{ij}\Phi_j \quad \leftrightarrow \quad \Phi_i = h_{ij}\Gamma^j_{00} \,.
\end{aligned} \tag{3.23}$$

The choice of a flat metric further reduces the allowed coordinate transformations (3.18) to

$$x^0 \to x^0 + \xi^0 \,, \qquad\qquad x^i \to A^i{}_j(t)x^j + a^i(t), \tag{3.24}$$

where $A^i{}_j(t)$ is an element of SO(3).

To derive the Poisson equation from the Ansatz (3.20) two additional conditions must be invoked. The first is the Trautman condition [7]:

$$h^{\sigma[\lambda} R^{\mu]}{}_{(\nu\rho)\sigma}(\Gamma) = 0 \,. \tag{3.25}$$

In adapted coordinates it implies

$$\partial_0 \omega_{mi} - \partial_{[m}\Phi_{i]} = 0\,, \qquad\qquad \partial_{[k}\omega_{mi]} = 0\,. \tag{3.26}$$

Although Φ_i and ω_{ij} are not tensors, both equations of (3.26) are separately covariant under (3.24) which can be checked explicitly. Using the definitions (3.23) of Φ_i and ω_{ij} one may verify that the conditions (3.26) are equivalent to the manifestly tensorial equation

$$\partial_{[\rho}K_{\mu\nu]} = 0 \quad \rightarrow \quad K_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}\,, \tag{3.27}$$

where m_μ is a vector field determined up to the derivative of some scalar field.

The second condition we need is that ω_{ij}, see (3.19), depends only on time, not on space coordinates [5, 6]. In [5] three possible conditions on the Riemann tensor are discussed that lead to the desired restriction on ω_{ij}:

$$h^{\rho\lambda}R^{\mu}{}_{\nu\rho\sigma}(\Gamma)R^{\nu}{}_{\mu\lambda\alpha}(\Gamma) = 0 \;\text{ or }\; \tau_{[\lambda}R^{\mu}{}_{\nu]\rho\sigma}(\Gamma) = 0 \;\text{ or }\; h^{\sigma[\lambda}R^{\mu]}{}_{\nu\rho\sigma}(\Gamma) = 0. \tag{3.28}$$

These are the so-called Ehlers conditions. Each condition separately leads to the condition $\partial_k\omega_{ij} = 0$ in adapted coordinates and thus $\omega_{ij} = \omega_{ij}(t)$. One can next set $\omega_{ij} \equiv 0$, or equivalently $\Gamma'^{i}_{0j} \equiv 0$, see (3.23), by a time-dependent rotation $x'^i = A^i{}_j(t)x^j$ [6]. The conditions (3.26) imply that in the new coordinate system $\partial'_{[i}\Phi'_{j]} = 0$ and hence that $\Phi'_i = \partial'_i\Phi$ for some scalar field Φ. This implies that

$$\Gamma'^{i}_{00} = \delta^{ij}\partial'_j\Phi \tag{3.29}$$

in this coordinate system. The equations (3.20) thus lead to the Poisson equation:

$$R_{00} = \partial_i\Gamma^i_{00} = \delta^{ij}\partial_i\partial_j\phi = 4\pi G\rho\,. \tag{3.30}$$

Finally, we should also recover the geodesic equation (3.5). Using adapted coordinates and performing the above time-dependent rotation indeed gives the desired equations:

$$\ddot{x}'^0(t) = 0, \qquad\qquad \ddot{x}'^i(t) + \partial'^i\Phi = 0\,. \tag{3.31}$$

This completes the proof that Newton-Cartan gravity, formulated in terms of two degenerate metrics (see eq. (3.9)), and supplied with the Trautman condition (3.25) and the Ehlers conditions (3.28), precisely leads to the equations of Newtonian gravity. In the next section we will show how the same Newton-Cartan theory, including the Trautman and Ehlers conditions, follows from gauging the so-called Bargmann algebra.

4 Gauging the Bargmann algebra

4.1 The Bargmann algebra

The Bargmann algebra is the Galilean algebra augmented with a central generator [2] M and can be obtained as follows. We first extend the Poincaré algebra $\mathfrak{iso}(D-1,1)$ to the direct sum of the Poincaré algebra and a commutative subalgebra $\mathfrak{g}_M$ spanned by M:

$$\mathfrak{iso}(D-1,1) \quad \to \quad \mathfrak{iso}(D-1,1) \oplus \mathfrak{g}_M\,. \tag{4.1}$$

We next perform the following contraction of this algebra:

$$P_0 \to \frac{1}{\omega^2}M + H\,, \qquad P_i \to \frac{1}{\omega}P_i\,, \qquad J_{i0} \to \frac{1}{\omega}G_i\,, \qquad \omega \to 0\,. \tag{4.2}$$

The contraction of P_0 is motivated by considering the non-relativistic approximation of P_0 for a massive free particle

$$P_0 = +\sqrt{c^2 P_i P^i + M^2 c^4} \;\approx\; Mc^2 + \frac{P_i P^i}{2M}\,, \tag{4.3}$$

where $c = \omega^{-1}$ is the speed of light. The contracted algebra is the so-called Bargmann algebra $\mathfrak{b}(D-1,1)$ which has the following non-zero commutation relations:

$$\begin{aligned} &[J_{ij}, J_{kl}] = 4\delta_{[i[k}J_{l]j]}\,, \qquad &&[J_{ij}, P_k] = -2\delta_{k[i}P_{j]}\,,\\ &[J_{ij}, G_k] = -2\delta_{k[i}G_{j]}\,, &&[G_i, H] = -P_i\,,\\ &[G_i, P_j] = -\delta_{ij}M\,, \end{aligned} \tag{4.4}$$

For $M = 0$ this is the Galilean algebra.

[2]In $D = 3$ dimensions three such central generators can be introduced [8, 9].

The M generator is needed to obtain massive representations of the Galilean algebra. This can be understood by considering the action for a non-relativistic free particle with mass M:[3]

$$S = \frac{1}{2}\int_{t_1}^{t_2} M\dot{x}^i\dot{x}^i dt\,. \tag{4.5}$$

This action is invariant under the Galilei transformations (3.3), but the Lagrangian L is not; it transforms as a total derivative under an infinitesimal Galilei boost $\delta x^i = v^i t$:

$$\delta L = \frac{d}{dt}\Big(Mx^i v^i\Big)\,. \tag{4.6}$$

Due to this the naive Noether charge $Q_{\text{naive}} = p^i\delta x^i = M\dot{x}^i v^i t$ gets modified by an additional boundary term such that the correct Noether charge corresponding to boosts becomes:

$$Q_G = M\dot{x}^i v^i t - Mx^i v^i\,. \tag{4.7}$$

Using this expression one may verify that the Poisson bracket of the Noether charge Q_G corresponding to infinitesimal boosts $\delta x^i = v^i t$ with the Noether charge Q_P corresponding to infinitesimal translations $\delta x^i = a^i$ indeed gives the central generator M:

$$\{Q_G, Q_P\}_{PB} = -Mv^k a^k\,, \tag{4.8}$$

in line with the $[G_i, P_j]$ commutator given in (4.4).

4.2 Gauging the Bargmann algebra

We now gauge the Bargmann algebra (4.4) following the same procedure we applied to the Poincaré algebra (2.1) in Section 2.

Compared to the Poincaré case the gauge fields and parameters corresponding to the Bargmann algebra split up into a spatial and temporal part:

$$\begin{aligned} e_\mu{}^a &\to \{e_\mu{}^0,\ e_\mu{}^i\}\,, \qquad & \omega_\mu{}^{ab} &\to \{\omega_\mu{}^{ij},\ \omega_\mu{}^{i0}\} \\ \zeta^a &\to \{\zeta^0,\ \zeta^i\}\,, \qquad & \lambda^{ab} &\to \{\lambda^{i0},\ \lambda^{ij}\}\,. \end{aligned} \tag{4.9}$$

[3]We thank J. Gomis for showing this argument to us.

The gauge field corresponding to the generator M will be called m_μ and its gauge parameter will be called σ. We label $e_\mu{}^0 = \tau_\mu$ and $\zeta^0 = \tau$. The variations of the gauge fields corresponding to the different generators are given by:

$$\begin{aligned} H: \quad \delta\tau_\mu &= \partial_\mu \tau\,, \\ P: \quad \delta e_\mu{}^i &= D_\mu \zeta^i + \lambda^{ij} e_\mu{}^j + \lambda^{i0}\tau_\mu - \tau\omega_\mu{}^{i0}\,, \\ G: \quad \delta\omega_\mu{}^{i0} &= D_\mu\lambda^{i0} + \lambda^{ij}\omega_\mu{}^{j0}\,, \\ J: \quad \delta\omega_\mu{}^{ij} &= D_\mu\lambda^{ij}\,, \\ M: \quad \delta m_\mu &= \partial_\mu\sigma - \zeta^i\omega_\mu{}^{i0} + \lambda^{i0}e_\mu{}^i\,. \end{aligned} \tag{4.10}$$

The derivative D_μ is covariant with respect to the J-transformations and as such only contains the $\omega_\mu{}^{ij}$ gauge field. The curvatures of the gauge fields read

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}\,, \tag{4.11}$$
$$R_{\mu\nu}{}^i(P) = 2(D_{[\mu}e_{\nu]}{}^i - \omega_{[\mu}{}^{i0}\tau_{\nu]})\,, \tag{4.12}$$
$$R_{\mu\nu}{}^{ij}(J) = 2(\partial_{[\mu}\omega_{\nu]}{}^{ij} + \omega_{[\mu}{}^{ki}\omega_{\nu]}{}^{jk})\,, \tag{4.13}$$
$$R_{\mu\nu}{}^{i0}(G) = 2D_{[\mu}\omega_{\nu]}{}^{i0}\,, \tag{4.14}$$
$$R_{\mu\nu}(M) = 2(\partial_{[\mu}m_{\nu]} + e_{[\mu}{}^j\omega_{\nu]}{}^{j0})\,. \tag{4.15}$$

Using the general formula (2.4) we convert the P and H transformations into general coordinate transformations in space and time. We write the parameter of the general coordinate transformations ξ^λ in (2.4) as

$$\xi^\lambda = e^\lambda{}_i\zeta^i + \tau^\lambda\tau\,. \tag{4.16}$$

Here we have used the inverse spatial Vielbein $e^\lambda{}_i$ and the inverse temporal Vielbein τ^λ defined by

$$e_\mu{}^i e^\mu{}_j = \delta^i_j, \qquad \tau^\mu\tau_\mu = 1\,, \tag{4.17}$$
$$\tau^\mu e_\mu{}^i = 0, \qquad \tau_\mu e^\mu{}_i = 0\,, \tag{4.18}$$
$$e_\mu{}^i e^\nu{}_i = \delta^\nu_\mu - \tau_\mu\tau^\nu\,. \tag{4.19}$$

These conditions are the Vielbein version of the conditions (3.14).

We observe that only the gauge fields $e_\mu{}^i\,, \tau_\mu$ and m_μ transform under the P and H transformations. These are the fields that should remain independent, while the spin connections should become dependent fields. This can be achieved with the following constraints:

$$R_{\mu\nu}{}^i(P) = R_{\mu\nu}(H) = R_{\mu\nu}(M) = 0\,. \tag{4.20}$$

The Bianchi identities then lead to additional relations between curvatures:

$$R_{[\lambda\mu}{}^{ij}(J)e_{\nu]}{}^j = -R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]}\,, \qquad\qquad e_{[\lambda}{}^i R_{\mu\nu]}{}^{i0}(G) = 0\,. \tag{4.21}$$

The constraint $R_{\mu\nu}(H) = 0$ gives the condition $\partial_{[\mu}\tau_{\nu]} = 0$ and hence we may take τ_μ as in (3.11). The other two constraints, $R_{\mu\nu}{}^i(P) = R_{\mu\nu}(M) = 0$, enable us to solve for the spin connection gauge fields $\omega_\mu{}^{ij}, \omega_\mu{}^{i0}$ in terms of the other gauge fields, so that indeed only $e_\mu{}^i\,, \tau_\mu$ and m_μ remain as independent fields.

To solve for $\omega_\mu{}^{ij}$, we write

$$R_{\mu\nu}{}^i(P)e_\rho{}^i + R_{\rho\mu}{}^i(P)e_\nu{}^i - R_{\nu\rho}{}^i(P)e_\mu{}^i = 0\,. \tag{4.22}$$

From this it follows that

$$\omega_\mu{}^{kl} = \partial_{[\mu}e_{\nu]}{}^k e^{\nu\, l} - \partial_{[\mu}e_{\nu]}{}^l e^{\nu\, k} + e_\mu{}^i \partial_{[\nu}e_{\rho]}{}^i e^{\nu\, k} e^{\rho\, l} - \tau_\mu e^{\rho\,[k}\omega_\rho{}^{l]0}\,. \tag{4.23}$$

Next we solve for $\omega_\mu{}^{i0}$. We substitute (4.23) into $R_{\mu\nu}{}^i(P) = 0$ and contract this with $e^\mu{}_j$ and τ^ν. This gives the condition

$$e^{\mu\,(i}\omega_\mu{}^{j)0} = 2\, e^{\mu\,(i}\partial_{[\mu}e_{\nu]}{}^{j)}\tau^\nu\,. \tag{4.24}$$

Furthermore, $R_{\mu\nu}(M) = 0$ can be contracted with $e^\mu{}_i$ and τ^μ to give the following conditions:

$$e^{\mu\,[i}\omega_\mu{}^{j]0} = e^{\mu\, i}e^{\nu\, j}\partial_{[\mu}m_{\nu]}\,, \qquad\qquad \tau^\mu\omega_\mu{}^{i0} = 2\tau^\mu e^{\nu\, i}\partial_{[\mu}m_{\nu]}\,. \tag{4.25}$$

Using the constraints (4.24) and (4.25) one arrives at the following solution for $\omega_\mu{}^{i0}$:

$$\omega_\mu{}^{i0} = e^{\nu i}\partial_{[\mu}m_{\nu]} + e^{\nu i}\tau^\rho e_\mu{}^j\partial_{[\nu}e_{\rho]}{}^j + \tau_\mu\tau^\nu e^{\rho i}\partial_{[\nu}m_{\rho]} + \tau^\nu\partial_{[\mu}e_{\nu]}{}^i. \tag{4.26}$$

At this point we are left with the independent fields $e_\mu{}^i$, τ_μ and m_μ. Furthermore, the theory is still off-shell; no equations of motion have been imposed.

4.3 Newton-Cartan Gravity

To make contact with the formulation of Newton-Cartan gravity presented in Section 3 we need to introduce a Γ-connection. In the gauge algebra approach this is most naturally done by imposing a Vielbein postulate for the spatial Vielbein

$$\partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_\nu{}^j - \omega_\mu{}^{i0}\tau_\nu - \Gamma^\rho_{\nu\mu} e_\rho{}^i = 0 \tag{4.27}$$

and a Vielbein postulate for the temporal Vielbein

$$\partial_\mu \tau_\nu - \Gamma^\lambda_{\nu\mu}\tau_\lambda = 0\,, \tag{4.28}$$

which is the second condition of (3.10). These Vielbein postulates imply

$$\Gamma^\rho_{\nu\mu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + e^\rho{}_i \Big(\partial_{(\mu} e_{\nu)}{}^i - \omega_{(\mu}{}^{ij} e_{\nu)}{}^j - \omega_{(\mu}{}^{i0}\tau_{\nu)} \Big)\,. \tag{4.29}$$

This connection is symmetric due to the curvature constraints $R_{\mu\nu}{}^i(P) = R_{\mu\nu}(H) = 0$, and satisfies (3.10). An important difference between the metric compatibility conditions given in (3.10) and in (4.27, 4.28) is that the latter define the connection Γ uniquely. From (3.16) and (4.29) we find that

$$K_{\mu\nu} = 2\omega_{[\mu}{}^{i0} e_{\nu]}{}^i\,, \tag{4.30}$$

with $\omega_\mu{}^{i0}$ given by (4.26). This implies via the $R(M) = 0$ constraint that

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]} \tag{4.31}$$

which solves the condition (3.27). The Riemann tensor corresponding to (4.29) can now be expressed in terms of the curvature tensors of the gauge algebra:

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma}(\Gamma) &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} \\ &= -e^\mu{}_i \Big(R_{\rho\sigma}{}^{i0}(G)\tau_\nu + R_{\rho\sigma}{}^{ij}(J) e^\mu{}_j \Big)\,. \end{aligned} \tag{4.32}$$

Here we have used (4.20). The Trautman condition (3.25), applied to (4.32), is equivalent to the first constraint of (4.21).

We know from the analysis in section 3 that, in order to make contact with the Newton-Cartan formulation, we must impose the Ehlers conditions (3.28).

One can show that each of the three Ehlers conditions (3.28) is equivalent to the single curvature constraint

$$R_{\mu\nu}{}^{ij}(J) = 0\,. \tag{4.33}$$

Substituting this result into (4.21) leads to the following constraints on $R^{i0}_{\mu\nu}(G)$:

$$R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]} = 0\,, \qquad\qquad e_{[\lambda}{}^{i}R_{\mu\nu]}{}^{i0}(G) = 0\,. \tag{4.34}$$

The contraction of (4.34) with $e^{\mu}{}_{i}$ and τ^{μ} gives

$$e^{\mu}{}_{i}e^{\nu}{}_{j}R_{\mu\nu}{}^{k0}(G) = 0\,, \qquad\qquad \tau^{\mu}e^{\nu\,[i}R_{\mu\nu}{}^{j]0}(G) = 0\,. \tag{4.35}$$

This implies that the only non-zero component of $R_{\mu\nu}{}^{i0}(G)$ is

$$\tau^{\mu}e^{\nu\;(i}R_{\mu\nu}{}^{j)0}(G) = \delta^{k(j}R^{i)}{}_{0k0}(\Gamma) \tag{4.36}$$

which is precisely the only non-zero component (3.7) of the Riemann tensor that occurs in the Newton-Cartan formulation.

At this point we have made contact with the Newton-Cartan gravity theory presented in Section 3. We have the same Γ-connection and (degenerate) metrics. It can be shown that these lead to the desired Poisson equation and geodesic equation of a massive free particle following the same steps as in Section 3. This concludes our discussion of the gauging procedure.

5 Conclusions

In this work we have shown how, just like Einstein gravity, the Newton-Cartan formulation of Newtonian gravity can be obtained by a gauging procedure. The Lie algebra underlying this procedure is the Bargmann algebra given in eq. (4.4). To obtain the correct Newton-Cartan formulation we need to impose constraints on the curvatures. In a first step we impose the curvature constraints (4.20). They enable us to convert the spatial (time) translational symmetries of the Bargmann algebra into spatial (time) general coordinate transformations. At the same time they enable us to solve for the spin-connection gauge fields $\omega_{\mu}{}^{i0}$ and $\omega_{\mu}{}^{ij}$ in terms of the remaining gauge fields ${e_\mu}^i\,, \tau_\mu$ and m_μ, see eqs. (4.23) and (4.26). For this to work it is essential that we work with a non-zero central element M in the algebra. Sofar, we work off-shell without comparing equations of motion.

In a second step we impose the Vielbein postulates (4.27) and (4.28). These enable us to solve for the Γ connection thereby solving the Trautman condition (3.25) automatically. In order to obtain the correct Poisson equation and geodesic equation of a massive free particle we impose in a third step the additional curvature constraints (4.33) which are equivalent to each of the three Ehlers conditions (3.28). The Poisson equation and the geodesic equation for a massive particle are obtained from the relation (4.36) between the curvature of the dependent field ${\omega_\mu}^{i0}$ and the Newton-Cartan Riemann tensor in the form (3.7). The independent gauge fields ${e_\mu}^i$ and τ_μ describe the degenerate metrics of Newton-Cartan gravity.

The present work can be extended in several directions. First of all, it would be interesting to see whether a supersymmetric version of the Bargmann algebra leads to the Newtonian version of a Poincaré supergravity model. Secondly, one could try to apply the gauging procedure developed in this paper to other algebras which have appeared in recent non-relativistic applications of the AdS-CFT correspondence. Examples of such algebras are the Galilean Conformal algebra, the Schrodinger algebra and the Lifshitz algebra. The gauging of the first algebra is expected to lead to a Newtonian version of conformal gravity. Irrespective of its of its role in the AdS/CFT correspondence it would be interesting to see whether this could lead to a non-relativistic version of the conformal tensor calculus.

One of the original motivations of this work was the possible role of Newton-Cartan gravity in non-relativistic applications of the AdS-CFT correspondence. In most applications the relativistic symmetries of the AdS bulk theory are broken by the vacuum solution one considers. This is the case if one considers the Schrodinger or Lifshitz algebras. The situation changes if one considers the Galilean Conformal Algebra instead. It has been argued that in that case the bulk gravity theory is given by an extension of the Newton-Cartan theory where the spacetime metric is degenerate with *two* zero eigenvalues corresponding to the time and the radial directions [10]. This leads to a foliation where the time direction is replaced by a two-dimensional AdS_2 space. This requires a contraction of the Poincaré algebra in which the

Bargmann algebra is replaced by a centrally extended string Galilean algebra or, if one includes the cosmological constant, by a string Newton-Hooke algebra [11, 12]. We expect that the systematic gauging procedure developed in this work will be essential to work out the non-relativistic theories corresponding to these new cases.

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Newtonian Gravity and the Bargmann Algebra

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ABSTRACT

We show how the Newton-Cartan formulation of Newtonian gravity can be obtained from gauging the Bargmann algebra, i.e., the centrally extended Galilean algebra. In this gauging procedure several curvature constraints are imposed. These convert the spatial (time) translational symmetries of the algebra into spatial (time) general coordinate transformations, and make the spin connection gauge fields dependent. In addition we require two independent Vielbein postulates for the temporal and spatial directions. In the final step we impose an additional curvature constraint to establish the connection with (on-shell) Newton-Cartan theory. We discuss a few extensions of our work that are relevant in the context of the AdS-CFT correspondence.

Contents

1 Introduction

It is well known that Einstein's formulation of gravity can be obtained by performing a formal gauging procedure of the Poincaré algebra. In this procedure one associates to each generator of the Poincaré algebra a gauge field. Next, one imposes constraints on the curvature tensors of these gauge fields such that the translational symmetries of the algebra get converted into general coordinate transformations. At the same time the gauge field of the Lorentz transformations gets expressed into (derivatives of) the Vierbein gauge field which is the only independent gauge field. One thus obtains an off-shell formulation of Einstein gravity. On-shell Einstein gravity is obtained by imposing the usual Einstein equations of motion.

One may consider the non-relativistic version of the Poincaré algebra and Einstein gravity independently. It turns out that the relevant non-relativistic version of the Poincaré algebra is a particular contraction of the Poincaré algebra trivially extended with a 1-dimensional algebra that commutes with all the generators. This contraction yields the so-called Bargmann algebra, which is the centrally extended Galilean algebra. On the other hand, taking the non-relativistic limit of general relativity leads to the well-known non-relativistic Newtonian gravity in flat space. The Newton-Cartan theory is a geometric re-formulation of this Newtonian theory, mimicking as much as possible the geometric formulation of general relativity [1, 2]. A notable difference with the relativistic case is the occurrence of a degenerate metric.

The question we pose in this note is: can we derive the Newton-Cartan formulation of Newtonian gravity directly from gauging the Bargmann algebra in the same way that Einstein gravity may be derived from gauging the relativistic Poincaré algebra as described above?[1] The answer will be yes, but there are some subtleties involved. This is partly due to the fact that the standard procedure leads to spin-connection fields that not only depend on the temporal and spatial Vielbeins but also on the gauge field corresponding to the central charge generator. These connections have to be fixed appropriately, via further curvature constraints, in order to obtain the correct non-relativistic Poisson equation as well as the geodesic equation for

[1] The gauging of the Bargmann algebra, from a somewhat different point of view, has been considered before in [3, 4].

a massive particle.

The outline of this note is as follows. In section 2 we first review how Einstein gravity may be obtained by gauging the Poincaré algebra. To keep the discussion in this section as general as possible we leave the dimension D of spacetime arbitrary. Next, we briefly review in section 3 the Newton-Cartan formulation of Newtonian gravity, since this is the theory we wish to end up with in the non-relativistic case. We next proceed, in section 4, with gauging the Bargmann algebra. In a first step we introduce a set of curvature constraints that convert the spatial (time) translational symmetries of the algebra into spatial (time) general coordinate transformations. We next impose a Vielbein postulate for the Vielbeins in the temporal and spatial directions. In a final step we impose further curvature constraints on the theory in order to recover the non-relativistic Poisson equation and the geodesic equation for a massive particle. Finally, our conclusions and suggestions for further work are presented in section 5.

2 Einstein Gravity and Gauging the Poincaré Algebra

In this section we briefly review how the basic ingredients of Einstein gravity may be obtained by applying a formal gauging procedure to the Poincaré algebra. We leave the dimension D of spacetime in this section arbitrary.

Our starting point is the D-dimensional Poincaré algebra $\mathfrak{iso}(D-1,1)$ with generators $P_a, M_{ab}\,(a = 0, 1, \cdots, D-1)$

$$\begin{aligned} [P_a, P_b] &= 0\,, \\ [M_{bc}, P_a] &= -2\eta_{a[b}P_{c]}\,, \\ [M_{cd}, M_{ef}] &= 4\eta_{[c[e}M_{f]d]}\,. \end{aligned} \tag{2.1}$$

Associating a gauge field ${e_\mu}^a$ to the local P-transformations with spacetime dependent parameters $\zeta^a(x)$, and a gauge field ${\omega_\mu}^{ab}$ to the local Lorentz transformations with spacetime dependent parameters $\lambda^{ab}(x)$, we obtain the following transformation rules

$$\begin{aligned}\delta e_\mu{}^a &= \partial_\mu \zeta^a - \omega_\mu{}^{ab}\zeta^b + \lambda^{ab} e_\mu{}^b\,,\\ \delta\omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]c}\,.\end{aligned} \tag{2.2}$$

In order to make contact with gravity we wish to replace the local P-transformations of all gauge fields by general coordinate transformations and to interpret $e_\mu{}^a$ as the Vielbein, with the inverse Vielbein field $e_a{}^\mu$ defined by

$$e_\mu{}^a\, e_b{}^\mu = \delta_b{}^a\,, \qquad e_\mu{}^a\, e_a{}^\nu = \delta_\mu{}^\nu\,. \tag{2.3}$$

To show how this can be achieved by imposing curvature constraints we first consider the following general identity for a gauge algebra:

$$0 = \delta_{gct}(\xi^\lambda) B_\mu{}^A + \xi^\lambda R_{\mu\lambda}{}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda{}^C) B_\mu{}^A\,. \tag{2.4}$$

The index A labels the gauge fields and corresponding curvatures of the gauge algebra. If we now set $A = a$ for the P-transformations and write the parameter ξ^λ as $\xi^\lambda = e_a{}^\lambda \zeta^a$ we can bring the contribution of $e_\mu{}^a$ in the sum in (2.4) to the left-hand side of the equation to obtain

$$\delta_P(\zeta^b) e_\mu{}^a = \delta_{gct}(\xi^\lambda) e_\mu{}^a + \xi^\lambda R_{\mu\lambda}{}^a(P) - \delta_M(\xi^\lambda \omega_\lambda{}^{ab}) e_\mu{}^a\,. \tag{2.5}$$

We see that the difference between a P-transformation and a general coordinate transformation is a curvature term and a Lorentz transformation. More generally, we deduce from the identity (2.4) that, whenever a gauge field transforms under a P-transformation, the P-transformations of this gauge field can be replaced by a general coordinate transformation plus other symmetries of the algebra by putting the curvature of the gauge field to zero. Since the Vielbein is the only field that transforms under the P-transformations, see (2.2), we are led to impose the following constraint:

$$R_{\mu\nu}{}^a(P) = 0\,. \tag{2.6}$$

The same constraint allows us to solve for the Lorentz gauge field $\omega_\mu{}^{ab}$ in terms of (derivatives of) the Vielbein and its inverse:

$$\omega_\mu{}^{ab}(e, \partial e) = -2e^{\lambda[a}\partial_{[\mu} e_{\lambda]}{}^{b]} + e_\mu{}^c e^{a\lambda} e^{b\rho} \partial_{[\lambda} e_{\rho]}{}^c\,. \tag{2.7}$$

What remains is a theory with the Vielbein $e_\mu{}^a$ as the only independent field transforming under local Lorentz transformations and general coordinate transformations and with $\omega_\mu{}^{ab}$ as the dependent spin connection field.

A Γ-connection may be introduced by imposing the Vielbein postulate:

$$\nabla_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - \Gamma^\rho_{\nu\mu} e_\rho{}^a - \omega_\mu{}^{ab} e_\nu{}^b = 0\,. \tag{2.8}$$

The anti-symmetric part of this equation, together with the curvature constraint (2.6), shows that the anti-symmetric part of the Γ-connection is zero, i.e. there is no torsion. From the Vielbein postulate (2.8) one may solve the Γ-connection in terms of the Vielbein and its inverse as follows:

$$\Gamma^\rho_{\nu\mu} = e^\rho{}_a D_\mu e_\nu{}^a\,. \tag{2.9}$$

Here D_μ is the Lorentz-covariant derivative. Finally, a non-degenerate metric and its inverse can be defined as:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}\,, \qquad\qquad g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab}\,. \tag{2.10}$$

This concludes our description of the basic ingredients of off-shell Einstein gravity and the Poincaré algebra. These basic ingredients are an independent non-degenerate metric $g_{\mu\nu}$ and a dependent Γ-connection $\Gamma^\rho_{\nu\mu}$ or, in the presence of flat indices, an independent Vielbein field $e_\mu{}^a$ and a dependent spin-connection field $\omega_\mu{}^{ab}$. The theory can be put on-shell by imposing the Einstein equations of motion.

3 Newton-Cartan Gravity

From now on we restrict the discussion to $D = 4$, i.e. one time and three space directions. We wish to review Newton-Cartan gravity as a geometric rewriting of Newtonian gravity [1, 2]. This geometric re-formulation is motivated by the following observation. First, consider the classical equations of motion of a massive particle,

$$\ddot{x}^i(t) + \frac{\partial\phi(x)}{\partial x^i} = 0\,, \tag{3.1}$$

where $x^i(t)\,(i = 1,2,3)$ are the spatial coordinates, t is the absolute time coordinate and a dot indicates differentiation with respect to t. Furthermore, $\phi(x^k)$ is the Newtonian potential which satisfies the Poisson equation

$$\partial_i \partial^i \phi = 4\pi G \rho \,, \tag{3.2}$$

where ρ is the mass density. The equations of motion (3.1) and (3.2) transform covariantly under the Galileo group

$$x^0 \to x^0 + \xi^0 \,, \qquad\qquad x^i \to A^i{}_j x^j + v^i t + d^i \,, \tag{3.3}$$

where $A^i{}_j$ is a constant group element of SO(3) and $\{v^i, d^i\}$ are three-vectors. In addition, these equations are invariant under

$$x^i \to x^i + a^i(t) \,, \qquad\qquad \phi(x) \to \phi(x) - \ddot{a}^j(t) x^j \,, \tag{3.4}$$

where $a^i(t)$ is an arbitrary time-dependent shift vector which can give rise to an acceleration.

From the Newtonian point of view the equations (3.1) describe a *curved* trajectory in a *flat* three-dimensional space. We now wish to re-interpret the same equations as a geodesic in a *curved* four-dimensional spacetime. Indeed, one may rewrite the equations (3.1) as the geodesic equations of motion

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0 \,, \tag{3.5}$$

provided that one chooses coordinates $\{x^\mu\} = \{x^0, x^i\} = (t, x^i)$ and takes the following expression for the non-zero connection fields:

$$\Gamma^i_{00} = \delta^{ij} \partial_j \phi \,, \tag{3.6}$$

where we have used the Euclidean three-metric. At this point $\Gamma^\mu_{\nu\rho}$ is a symmetric connection independent of the metric. The coordinate choice $x^0 = t$ corresponds to choosing so-called adapted coordinates. The corresponding D-dimensional spacetime is called the Newton-Cartan spacetime $\mathscr{M}$. The only non-zero component of the Riemann tensor corresponding to the connection (3.6) is

$$R^i{}_{0j0} = \delta^{ik} \partial_k \partial_j \phi \,. \tag{3.7}$$

If one now imposes the equations of motion $R_{00} = 4\pi G \rho$ one obtains the Poisson equation (3.2). To write the Poisson equation in a covariant way we first must introduce a metric.

As it stands, the Γ-connection defined by (3.6) cannot follow from a non-degenerate four-dimensional metric. One way to see this is to consider the Riemann tensor that is defined by this Γ-connection. The Riemann tensor, defined in terms of a metric connection based upon a non-degenerate metric, satisfies certain symmetry properties. One may easily verify that these properties are not satisfied by the Riemann tensor (3.7). Another way to see that a degenerate metric is unavoidable is to consider the relativistic Minkowski metric and its inverse

$$\eta_{\mu\nu}/c^2 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_3/c^2 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} -1/c^2 & 0 \\ 0 & \mathbb{1}_3 \end{pmatrix}. \tag{3.8}$$

Taking the limit $c \to \infty$ naturally leads to a degenerate covariant temporal metric $\tau_{\mu\nu}$ with three zero eigenvalues and a degenerate contra-variant spatial metric $h^{\mu\nu}$ with one zero eigenvalue. We conclude that the Galilei group keeps invariant *two* metrics $\tau_{\mu\nu}$ and $h^{\mu\nu}$ which are degenerate, i.e. $h^{\mu\nu}\tau_{\nu\rho} = 0$. Since $\tau_{\mu\nu}$ is effectively a 1×1 matrix we will below use its Vielbein version which is defined by a covariant vector τ_μ defined by $\tau_{\mu\nu} = \tau_\mu \tau_\nu$.

A degenerate spatial metric $h^{\mu\nu}$ of rank 3 and a degenerate temporal Vielbein τ_μ of rank 1, together with a symmetric connection $\Gamma^\rho_{\mu\nu}$ on $\mathscr{M}$, that depends on these two degenerate metrics, can be introduced as follows [5]. First of all the degeneracy implies that

$$h^{\mu\nu}\tau_\nu = 0\,. \tag{3.9}$$

We next impose metric compatibility:

$$\nabla_\rho h^{\mu\nu} = 0\,, \qquad \nabla_\rho \tau_\mu = 0\,. \tag{3.10}$$

The covariant derivative ∇ is with respect to a connection $\Gamma^\rho_{\mu\nu}$. The second of these conditions indicates that

$$\tau_\mu = \partial_\mu f(x^\nu) \tag{3.11}$$

for a scalar function $f(x^\nu)$. In Newton-Cartan theory this scalar function is chosen to be the absolute time t which foliates $\mathscr{M}$:

$$f(x^\nu) \equiv t\,. \tag{3.12}$$

In general relativity metric compatibility allows one to write down the connection in terms of the metric and its derivatives in a unique way, see eq. (2.9).

In the present analysis, the connection $\Gamma^\rho_{\mu\nu}$ is not uniquely determined by the metric compatibility conditions (3.10). This can be seen from the fact that these conditions are preserved by the shift

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + h^{\rho\lambda} K_{\lambda(\mu}\tau_{\nu)} \tag{3.13}$$

for an arbitrary two-form $K_{\mu\nu}$ [6]. Using this arbitrary two-form it is possible to write down the most general connection which is compatible with (3.10). In order to do this, one needs to introduce new tensors, the spatial inverse metric $h_{\mu\nu}$ and the temporal inverse Vielbein τ^μ which are defined by the following properties:

$$\begin{aligned} h^{\mu\nu} h_{\nu\rho} &= \delta^\mu_\rho - \tau^\mu \tau_\rho, \qquad & \tau^\mu \tau_\mu &= 1\,, \\ h^{\mu\nu}\tau_\nu &= 0, & h_{\mu\nu}\tau^\nu &= 0\,. \end{aligned} \tag{3.14}$$

Note that from these conditions it follows that

$$\nabla_\rho h_{\mu\nu} = -2\tau_{(\mu} h_{\nu)\sigma} \nabla_\rho \tau^\sigma \tag{3.15}$$

which is not zero in general. The most general connection compatible with (3.10) is then [6]

$$\Gamma^\sigma_{\mu\nu} = \tau^\sigma \partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\sigma\rho}\Big(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}\Big) + h^{\sigma\lambda} K_{\lambda(\mu}\tau_{\nu)}\,. \tag{3.16}$$

We note that the original independent connection (3.6) is quite different from the metric connection defined in (3.16). Nevertheless, given extra conditions discussed below, the Newton-Cartan theory with the metric connection (3.16) reproduces Newtonian gravity. To see how this goes, it is convenient to use adapted coordinates $x^0 = t$. The conditions (3.11) and (3.14) then imply

$$\begin{aligned} \tau_\mu &= \delta^0_\mu, \qquad & \tau^\mu &= (1, \tau^i)\,, \\ h^{\mu 0} &= 0, & h_{\mu 0} &= -h_{\mu i}\tau^i\,. \end{aligned} \tag{3.17}$$

These conditions are preserved by the coordinate transformations

$$\begin{aligned} x^0 &\to x^0 + \xi^0\,, \\ x^i &\to x^i + \xi^i(x^\mu)\,, \end{aligned} \tag{3.18}$$

where ξ^0 is a constant. The finite spatial transformation generated by $\xi^i(x^\mu)$ is invertible. In adapted coordinates the connection coefficients (3.16) are given by [6]

$$
\begin{aligned}
\Gamma^i_{00} &= h^{ij}(\partial_0 h_{j0} - \tfrac{1}{2}\partial_j h_{00} + K_{j0}) \equiv h^{ij}\Phi_j\,,\\
\Gamma^i_{0j} &= h^{ik}(\tfrac{1}{2}\partial_0 h_{jk} + \partial_{[j}h_{k]0} - K_{jk}) \equiv h^{ik}(\tfrac{1}{2}\partial_0 h_{jk} + \omega_{jk})\,,\\
\Gamma^i_{jk} &= \left\{{}^{\,i}_{jk}\right\}, \qquad\qquad \Gamma^0_{\mu\nu} = 0\,,
\end{aligned}
\tag{3.19}
$$

where $\left\{{}^{\,i}_{jk}\right\}$ are the usual Christoffel symbols with respect to the metric h_{ij} with inverse h^{ij}.

We now replace the original equations of motion $R_{00} = 4\pi G\rho$ by the covariant Ansatz

$$R_{\mu\nu} = 4\pi G\rho\,\tau_\mu\tau_\nu \tag{3.20}$$

and verify that this leads to Newtonian gravity. In adapted coordinates these equations imply that

$$R_{ij} = R_{i0} = 0\,. \tag{3.21}$$

The condition $R_{ij} = 0$ implies that the spatial hypersurfaces are flat, i.e. one can choose a coordinate frame with $\Gamma^i_{jk} = 0$ such that the spatial metric is given by

$$h_{ij} = \delta_{ij}, \qquad h^{ij} = \delta^{ij}\,. \tag{3.22}$$

This implies

$$
\begin{aligned}
\Gamma^i_{0j} &= h^{ik}\omega_{jk} \quad\leftrightarrow\quad \omega_{ij} = h_{k[j}\Gamma^k_{i]0}\,,\\
\Gamma^i_{00} &= h^{ij}\Phi_j \quad\;\leftrightarrow\quad \Phi_i = h_{ij}\Gamma^j_{00}\,.
\end{aligned}
\tag{3.23}
$$

The choice of a flat metric further reduces the allowed coordinate transformations (3.18) to

$$x^0 \to x^0 + \xi^0\,, \qquad\qquad x^i \to A^i{}_j(t)x^j + a^i(t), \tag{3.24}$$

where $A^i{}_j(t)$ is an element of SO(3).

To derive the Poisson equation from the Ansatz (3.20) two additional conditions must be invoked. The first is the Trautman condition [7]:

$$h^{\sigma[\lambda}R^{\mu]}{}_{(\nu\rho)\sigma}(\Gamma) = 0\,. \tag{3.25}$$

In adapted coordinates it implies

$$\partial_0\omega_{mi} - \partial_{[m}\Phi_{i]} = 0\,, \qquad\qquad \partial_{[k}\omega_{mi]} = 0\,. \tag{3.26}$$

Although Φ_i and ω_{ij} are not tensors, both equations of (3.26) are separately covariant under (3.24) which can be checked explicitly. Using the definitions (3.23) of Φ_i and ω_{ij} one may verify that the conditions (3.26) are equivalent to the manifestly tensorial equation

$$\partial_{[\rho}K_{\mu\nu]} = 0 \quad \to \quad K_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}\,, \tag{3.27}$$

where m_μ is a vector field determined up to the derivative of some scalar field.

The second condition we need is that ω_{ij}, see (3.19), depends only on time, not on space coordinates [5, 6]. In [5] three possible conditions on the Riemann tensor are discussed that lead to the desired restriction on ω_{ij}:

$$h^{\rho\lambda}R^{\mu}{}_{\nu\rho\sigma}(\Gamma)R^{\nu}{}_{\mu\lambda\alpha}(\Gamma) = 0 \;\text{ or }\; \tau_{[\lambda}R^{\mu}{}_{\nu]\rho\sigma}(\Gamma) = 0 \;\text{ or }\; h^{\sigma[\lambda}R^{\mu]}{}_{\nu\rho\sigma}(\Gamma) = 0. \tag{3.28}$$

These are the so-called Ehlers conditions. Each condition separately leads to the condition $\partial_k\omega_{ij} = 0$ in adapted coordinates and thus $\omega_{ij} = \omega_{ij}(t)$. One can next set $\omega_{ij} \equiv 0$, or equivalently $\Gamma'^{i}_{0j} \equiv 0$, see (3.23), by a time-dependent rotation $x'^i = A^i{}_j(t)x^j$ [6]. The conditions (3.26) imply that in the new coordinate system $\partial'_{[i}\Phi'_{j]} = 0$ and hence that $\Phi'_i = \partial'_i\Phi$ for some scalar field Φ. This implies that

$$\Gamma'^{i}_{00} = \delta^{ij}\partial'_j\Phi \tag{3.29}$$

in this coordinate system. The equations (3.20) thus lead to the Poisson equation:

$$R_{00} = \partial_i\Gamma^i_{00} = \delta^{ij}\partial_i\partial_j\phi = 4\pi G\rho\,. \tag{3.30}$$

Finally, we should also recover the geodesic equation (3.5). Using adapted coordinates and performing the above time-dependent rotation indeed gives the desired equations:

$$\ddot{x}'^0(t) = 0, \qquad\qquad \ddot{x}'^i(t) + \partial'^i\Phi = 0\,. \tag{3.31}$$

This completes the proof that Newton-Cartan gravity, formulated in terms of two degenerate metrics (see eq. (3.9)), and supplied with the Trautman condition (3.25) and the Ehlers conditions (3.28), precisely leads to the equations of Newtonian gravity. In the next section we will show how the same Newton-Cartan theory, including the Trautman and Ehlers conditions, follows from gauging the so-called Bargmann algebra.

4 Gauging the Bargmann algebra

4.1 The Bargmann algebra

The Bargmann algebra is the Galilean algebra augmented with a central generator [2] M and can be obtained as follows. We first extend the Poincaré algebra $\mathfrak{iso}(D-1,1)$ to the direct sum of the Poincaré algebra and a commutative subalgebra $\mathfrak{g}_M$ spanned by M:

$$\mathfrak{iso}(D-1,1) \quad \rightarrow \quad \mathfrak{iso}(D-1,1) \oplus \mathfrak{g}_M \,. \tag{4.1}$$

We next perform the following contraction of this algebra:

$$P_0 \rightarrow \frac{1}{\omega^2}M + H\,, \qquad P_i \rightarrow \frac{1}{\omega}P_i\,, \qquad J_{i0} \rightarrow \frac{1}{\omega}G_i\,, \qquad \omega \rightarrow 0\,. \tag{4.2}$$

The contraction of P_0 is motivated by considering the non-relativistic approximation of P_0 for a massive free particle

$$P_0 = +\sqrt{c^2 P_i P^i + M^2 c^4} \;\approx\; Mc^2 + \frac{P_i P^i}{2M}\,, \tag{4.3}$$

where $c = \omega^{-1}$ is the speed of light. The contracted algebra is the so-called Bargmann algebra $\mathfrak{b}(D-1,1)$ which has the following non-zero commutation relations:

$$\begin{aligned}
&[J_{ij}, J_{kl}] = 4\delta_{[i[k}J_{l]j]}\,, \qquad [J_{ij}, P_k] = -2\delta_{k[i}P_{j]}\,,\\
&[J_{ij}, G_k] = -2\delta_{k[i}G_{j]}\,, \qquad [G_i, H] = -P_i\,,\\
&[G_i, P_j] = -\delta_{ij}M\,,
\end{aligned} \tag{4.4}$$

For $M = 0$ this is the Galilean algebra.

[2]In $D = 3$ dimensions three such central generators can be introduced [8, 9].

The M generator is needed to obtain massive representations of the Galilean algebra. This can be understood by considering the action for a non-relativistic free particle with mass M:[3]

$$S = \frac{1}{2}\int_{t_1}^{t_2} M\dot{x}^i\dot{x}^i dt\,. \tag{4.5}$$

This action is invariant under the Galilei transformations (3.3), but the Lagrangian L is not; it transforms as a total derivative under an infinitesimal Galilei boost $\delta x^i = v^i t$:

$$\delta L = \frac{d}{dt}\Big(M x^i v^i\Big)\,. \tag{4.6}$$

Due to this the naive Noether charge $Q_{\text{naive}} = p^i \delta x^i = M\dot{x}^i v^i t$ gets modified by an additional boundary term such that the correct Noether charge corresponding to boosts becomes:

$$Q_G = M\dot{x}^i v^i t - M x^i v^i\,. \tag{4.7}$$

Using this expression one may verify that the Poisson bracket of the Noether charge Q_G corresponding to infinitesimal boosts $\delta x^i = v^i t$ with the Noether charge Q_P corresponding to infinitesimal translations $\delta x^i = a^i$ indeed gives the central generator M:

$$\{Q_G, Q_P\}_{PB} = -M v^k a^k\,, \tag{4.8}$$

in line with the $[G_i, P_j]$ commutator given in (4.4).

4.2 Gauging the Bargmann algebra

We now gauge the Bargmann algebra (4.4) following the same procedure we applied to the Poincaré algebra (2.1) in Section 2.

Compared to the Poincaré case the gauge fields and parameters corresponding to the Bargmann algebra split up into a spatial and temporal part:

$$\begin{aligned} &{e_\mu}^a \to \{{e_\mu}^0,\ {e_\mu}^i\}\,, \qquad && {\omega_\mu}^{ab} \to \{{\omega_\mu}^{ij},\ {\omega_\mu}^{i0}\} \\ &\zeta^a \to \{\zeta^0,\ \zeta^i\}\,, && \lambda^{ab} \to \{\lambda^{i0},\ \lambda^{ij}\}\,. \end{aligned} \tag{4.9}$$

[3] We thank J. Gomis for showing this argument to us.

The gauge field corresponding to the generator M will be called m_μ and its gauge parameter will be called σ. We label $e_\mu{}^0 = \tau_\mu$ and $\zeta^0 = \tau$. The variations of the gauge fields corresponding to the different generators are given by:

$$
\begin{aligned}
H: &\quad \delta\tau_\mu = \partial_\mu \tau\,,\\
P: &\quad \delta e_\mu{}^i = D_\mu \zeta^i + \lambda^{ij} e_\mu{}^j + \lambda^{i0}\tau_\mu - \tau\omega_\mu{}^{i0}\,,\\
G: &\quad \delta\omega_\mu{}^{i0} = D_\mu\lambda^{i0} + \lambda^{ij}\omega_\mu{}^{j0}\,,\\
J: &\quad \delta\omega_\mu{}^{ij} = D_\mu\lambda^{ij}\,,\\
M: &\quad \delta m_\mu = \partial_\mu\sigma - \zeta^i\omega_\mu{}^{i0} + \lambda^{i0}e_\mu{}^i\,.
\end{aligned}
\tag{4.10}
$$

The derivative D_μ is covariant with respect to the J-transformations and as such only contains the $\omega_\mu{}^{ij}$ gauge field. The curvatures of the gauge fields read

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}\,, \tag{4.11}$$

$$R_{\mu\nu}{}^i(P) = 2(D_{[\mu}e_{\nu]}{}^i - \omega_{[\mu}{}^{i0}\tau_{\nu]})\,, \tag{4.12}$$

$$R_{\mu\nu}{}^{ij}(J) = 2(\partial_{[\mu}\omega_{\nu]}{}^{ij} + \omega_{[\mu}{}^{ki}\omega_{\nu]}{}^{jk})\,, \tag{4.13}$$

$$R_{\mu\nu}{}^{i0}(G) = 2D_{[\mu}\omega_{\nu]}{}^{i0}\,, \tag{4.14}$$

$$R_{\mu\nu}(M) = 2(\partial_{[\mu}m_{\nu]} + e_{[\mu}{}^j\omega_{\nu]}{}^{j0})\,. \tag{4.15}$$

Using the general formula (2.4) we convert the P and H transformations into general coordinate transformations in space and time. We write the parameter of the general coordinate transformations ξ^λ in (2.4) as

$$\xi^\lambda = e^\lambda{}_i\zeta^i + \tau^\lambda\tau\,. \tag{4.16}$$

Here we have used the inverse spatial Vielbein $e^\lambda{}_i$ and the inverse temporal Vielbein τ^λ defined by

$$e_\mu{}^i e^\mu{}_j = \delta^i_j, \qquad \tau^\mu\tau_\mu = 1\,, \tag{4.17}$$

$$\tau^\mu e_\mu{}^i = 0, \qquad \tau_\mu e^\mu{}_i = 0\,, \tag{4.18}$$

$$e_\mu{}^i e^\nu{}_i = \delta^\nu_\mu - \tau_\mu\tau^\nu\,. \tag{4.19}$$

These conditions are the Vielbein version of the conditions (3.14).

We observe that only the gauge fields $e_\mu{}^i\,, \tau_\mu$ and m_μ transform under the P and H transformations. These are the fields that should remain independent, while the spin connections should become dependent fields. This can be achieved with the following constraints:

$$R_{\mu\nu}{}^i(P) = R_{\mu\nu}(H) = R_{\mu\nu}(M) = 0\,. \tag{4.20}$$

The Bianchi identities then lead to additional relations between curvatures:

$$R_{[\lambda\mu}{}^{ij}(J) e_{\nu]}{}^j = -R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]}\,, \qquad e_{[\lambda}{}^i R_{\mu\nu]}{}^{i0}(G) = 0\,. \tag{4.21}$$

The constraint $R_{\mu\nu}(H) = 0$ gives the condition $\partial_{[\mu}\tau_{\nu]} = 0$ and hence we may take τ_μ as in (3.11). The other two constraints, $R_{\mu\nu}{}^i(P) = R_{\mu\nu}(M) = 0$, enable us to solve for the spin connection gauge fields $\omega_\mu{}^{ij}, \omega_\mu{}^{i0}$ in terms of the other gauge fields, so that indeed only $e_\mu{}^i\,, \tau_\mu$ and m_μ remain as independent fields.

To solve for $\omega_\mu{}^{ij}$, we write

$$R_{\mu\nu}{}^i(P) e_\rho{}^i + R_{\rho\mu}{}^i(P) e_\nu{}^i - R_{\nu\rho}{}^i(P) e_\mu{}^i = 0\,. \tag{4.22}$$

From this it follows that

$$\omega_\mu{}^{kl} = \partial_{[\mu} e_{\nu]}{}^k e^{\nu\, l} - \partial_{[\mu} e_{\nu]}{}^l e^{\nu\, k} + e_\mu{}^i \partial_{[\nu} e_{\rho]}{}^i e^{\nu\, k} e^{\rho\, l} - \tau_\mu e^{\rho\,[k} \omega_\rho{}^{l]0}\,. \tag{4.23}$$

Next we solve for $\omega_\mu{}^{i0}$. We substitute (4.23) into $R_{\mu\nu}{}^i(P) = 0$ and contract this with $e^\mu{}_j$ and τ^ν. This gives the condition

$$e^{\mu\,(i} \omega_\mu{}^{j)0} = 2\, e^{\mu\,(i} \partial_{[\mu} e_{\nu]}{}^{j)} \tau^\nu\,. \tag{4.24}$$

Furthermore, $R_{\mu\nu}(M) = 0$ can be contracted with $e^\mu{}_i$ and τ^μ to give the following conditions:

$$e^{\mu\,[i} \omega_\mu{}^{j]0} = e^{\mu\, i} e^{\nu\, j} \partial_{[\mu} m_{\nu]}\,, \qquad \tau^\mu \omega_\mu{}^{i0} = 2\tau^\mu e^{\nu\, i} \partial_{[\mu} m_{\nu]}\,. \tag{4.25}$$

Using the constraints (4.24) and (4.25) one arrives at the following solution for $\omega_\mu{}^{i0}$:

$$\omega_\mu{}^{i0} = e^{\nu i} \partial_{[\mu} m_{\nu]} + e^{\nu i} \tau^\rho e_\mu{}^j \partial_{[\nu} e_{\rho]}{}^j + \tau_\mu \tau^\nu e^{\rho i} \partial_{[\nu} m_{\rho]} + \tau^\nu \partial_{[\mu} e_{\nu]}{}^i\,. \tag{4.26}$$

At this point we are left with the independent fields $e_\mu{}^i$, τ_μ and m_μ. Furthermore, the theory is still off-shell; no equations of motion have been imposed.

4.3 Newton-Cartan Gravity

To make contact with the formulation of Newton-Cartan gravity presented in Section 3 we need to introduce a Γ-connection. In the gauge algebra approach this is most naturally done by imposing a Vielbein postulate for the spatial Vielbein

$$\partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_\nu{}^j - \omega_\mu{}^{i0}\tau_\nu - \Gamma^\rho_{\nu\mu} e_\rho{}^i = 0 \tag{4.27}$$

and a Vielbein postulate for the temporal Vielbein

$$\partial_\mu \tau_\nu - \Gamma^\lambda_{\nu\mu}\tau_\lambda = 0\,, \tag{4.28}$$

which is the second condition of (3.10). These Vielbein postulates imply

$$\Gamma^\rho_{\nu\mu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + e^\rho{}_i\Big(\partial_{(\mu} e_{\nu)}{}^i - \omega_{(\mu}{}^{ij} e_{\nu)}{}^j - \omega_{(\mu}{}^{i0}\tau_{\nu)}\Big)\,. \tag{4.29}$$

This connection is symmetric due to the curvature constraints $R_{\mu\nu}{}^i(P) = R_{\mu\nu}(H) = 0$, and satisfies (3.10). An important difference between the metric compatibility conditions given in (3.10) and in (4.27, 4.28) is that the latter define the connection Γ uniquely. From (3.16) and (4.29) we find that

$$K_{\mu\nu} = 2\omega_{[\mu}{}^{i0} e_{\nu]}{}^i\,, \tag{4.30}$$

with $\omega_\mu{}^{i0}$ given by (4.26). This implies via the $R(M) = 0$ constraint that

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]} \tag{4.31}$$

which solves the condition (3.27). The Riemann tensor corresponding to (4.29) can now be expressed in terms of the curvature tensors of the gauge algebra:

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma}(\Gamma) &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} \\ &= -e^\mu{}_i\Big(R_{\rho\sigma}{}^{i0}(G)\tau_\nu + R_{\rho\sigma}{}^{ij}(J) e^\mu{}_j\Big)\,. \end{aligned} \tag{4.32}$$

Here we have used (4.20). The Trautman condition (3.25), applied to (4.32), is equivalent to the first constraint of (4.21).

We know from the analysis in section 3 that, in order to make contact with the Newton-Cartan formulation, we must impose the Ehlers conditions (3.28).

One can show that each of the three Ehlers conditions (3.28) is equivalent to the single curvature constraint

$$R_{\mu\nu}{}^{ij}(J) = 0\,. \tag{4.33}$$

Substituting this result into (4.21) leads to the following constraints on $R^{i0}_{\mu\nu}(G)$:

$$R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]} = 0\,, \qquad\qquad e_{[\lambda}{}^{i}R_{\mu\nu]}{}^{i0}(G) = 0\,. \tag{4.34}$$

The contraction of (4.34) with $e^{\mu}{}_{i}$ and τ^{μ} gives

$$e^{\mu}{}_{i}e^{\nu}{}_{j}R_{\mu\nu}{}^{k0}(G) = 0\,, \qquad\qquad \tau^{\mu}e^{\nu\,[i}R_{\mu\nu}{}^{j]0}(G) = 0\,. \tag{4.35}$$

This implies that the only non-zero component of $R_{\mu\nu}{}^{i0}(G)$ is

$$\tau^{\mu}e^{\nu\;(i}R_{\mu\nu}{}^{j)0}(G) = \delta^{k(j}R^{i)}{}_{0k0}(\Gamma) \tag{4.36}$$

which is precisely the only non-zero component (3.7) of the Riemann tensor that occurs in the Newton-Cartan formulation.

At this point we have made contact with the Newton-Cartan gravity theory presented in Section 3. We have the same Γ-connection and (degenerate) metrics. It can be shown that these lead to the desired Poisson equation and geodesic equation of a massive free particle following the same steps as in Section 3. This concludes our discussion of the gauging procedure.

5 Conclusions

In this work we have shown how, just like Einstein gravity, the Newton-Cartan formulation of Newtonian gravity can be obtained by a gauging procedure. The Lie algebra underlying this procedure is the Bargmann algebra given in eq. (4.4). To obtain the correct Newton-Cartan formulation we need to impose constraints on the curvatures. In a first step we impose the curvature constraints (4.20). They enable us to convert the spatial (time) translational symmetries of the Bargmann algebra into spatial (time) general coordinate transformations. At the same time they enable us to solve for the spin-connection gauge fields $\omega_{\mu}{}^{i0}$ and $\omega_{\mu}{}^{ij}$ in terms of the remaining gauge fields $e_\mu{}^i\,, \tau_\mu$ and m_μ, see eqs. (4.23) and (4.26). For this to work it is essential that we work with a non-zero central element M in the algebra. Sofar, we work off-shell without comparing equations of motion.

In a second step we impose the Vielbein postulates (4.27) and (4.28). These enable us to solve for the Γ connection thereby solving the Trautman condition (3.25) automatically. In order to obtain the correct Poisson equation and geodesic equation of a massive free particle we impose in a third step the additional curvature constraints (4.33) which are equivalent to each of the three Ehlers conditions (3.28). The Poisson equation and the geodesic equation for a massive particle are obtained from the relation (4.36) between the curvature of the dependent field $\omega_\mu{}^{i0}$ and the Newton-Cartan Riemann tensor in the form (3.7). The independent gauge fields $e_\mu{}^i$ and τ_μ describe the degenerate metrics of Newton-Cartan gravity.

The present work can be extended in several directions. First of all, it would be interesting to see whether a supersymmetric version of the Bargmann algebra leads to the Newtonian version of a Poincaré supergravity model. Secondly, one could try to apply the gauging procedure developed in this paper to other algebras which have appeared in recent non-relativistic applications of the AdS-CFT correspondence. Examples of such algebras are the Galilean Conformal algebra, the Schrodinger algebra and the Lifshitz algebra. The gauging of the first algebra is expected to lead to a Newtonian version of conformal gravity. Irrespective of its of its role in the AdS/CFT correspondence it would be interesting to see whether this could lead to a non-relativistic version of the conformal tensor calculus.

One of the original motivations of this work was the possible role of Newton-Cartan gravity in non-relativistic applications of the AdS-CFT correspondence. In most applications the relativistic symmetries of the AdS bulk theory are broken by the vacuum solution one considers[4]. This is the case if one considers the Schrodinger or Lifshitz algebras. The situation changes if one considers the Galilean Conformal Algebra instead. It has been argued that in that case the bulk gravity theory is given by an extension of the Newton-Cartan theory where the spacetime metric is degenerate with *two* zero eigenvalues corresponding to the time and the radial directions [10]. This leads to a foliation where the time direction is replaced by a two-dimensional

[4]For other aspects of Newton-Cartan gravity, see, e.g., [13, 14]

AdS_2 space. This requires a contraction of the Poincaré algebra in which the Bargmann algebra is replaced by a centrally extended string Galilean algebra or, if one includes the cosmological constant, by a string Newton-Hooke algebra [11, 12][5]. We expect that the systematic gauging procedure developed in this work will be essential to work out the non-relativistic theories corresponding to these new cases.

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