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On Critical Massive (Super)Gravity in adS_3

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Abstract.
We review the status of three-dimensional "general massive gravity" (GMG) in its linearization about an anti-de Sitter (adS) vacuum, focusing on critical points in parameter space that yield generalizations of "chiral gravity". We then show how these results extend to $\mathscr{N}=1$ super-GMG, expanded about a supersymmetric adS vacuum, and also to the most general 'curvature-squared' $\mathscr{N}=1$ supergravity model.

Massive gravity models in three spacetime dimensions (3D) have been intensively investigated over the past few years because of the possibility of finding a consistent 'toy' model of quantum gravity from which we might learn something useful. The oldest and simplest of these massive gravity models is "topologically massive gravity" (TMG), which is defined by the addition to the usual Einstein-Hilbert (EH) action of a Lorentz Chern-Simons (LCS) term [1]. TMG propagates a single massive spin-2 mode but this is a ghost unless the EH term has the non-standard sign. It is useful to define a sign σ such that $\sigma = 1$ yields the standard EH term, so that "non-standard" means $\sigma = -1$. The addition of a cosmological term allows the possibility of anti-de Sitter (adS) vacua, but then $\sigma = -1$ implies a negative mass for (BTZ) black holes. In the context of the adS_3/CFT_2 correspondence, this problem with the bulk theory translates to a negative central charge of the boundary conformal field theory (CFT). Taking $\sigma = 1$ allows positive central charges but at the cost of non-unitary propagation of the bulk spin-2 modes, which again implies [2,3], although less directly, non-unitarity of the boundary CFT.

The classical equations of TMG depend on a mass parameter μ associated to the LCS term, and a length scale ℓ associated to the cosmological term, which we define such that ℓ is the radius of curvature in an adS vacuum. However, rescaling the metric is equivalent to rescaling μ and ℓ so the only parameter of the classical theory is the dimensionless product $\mu\ell$. Quantum corrections will introduce the additional dimensionless constant κ^2/ℓ, where $\kappa = \sqrt{16\pi G_3}$ is the gravitational coupling constant, but this appears classically only through an overall factor of ℓ/κ^2 in the on-shell action. In the equivalent approximation to the boundary CFT, the central charges take the form $c_\pm = (24\pi\ell/\kappa^2) f_\pm(\mu\ell)$ for dimensionless functions $f_\pm$. It was pointed out in [4] that the parameter $\mu\ell$ can be tuned to a critical value, the "chiral point", at which either c_+ or c_- is zero. The bulk massive graviton mode then disappears from the spectrum, which suggests that the problem of a non-unitary bulk graviton for $\sigma = 1$ is circumvented by this "chiral gravity" theory. The definition of such a theory depends crucially on a choice of boundary conditions [5, 6]. Boundary conditions weaker than the standard Brown-Henneaux boundary conditions lead to well-defined logarithmic boundary CFTs. They also allow new logarithmic bulk modes, which is related to the observation of [7] that the bulk spin-2 mode is replaced by a bulk spin-1 mode at the chiral point.

An alternative, parity-preserving 3D massive gravity is "new massive gravity" (NMG), which is defined by the addition to the EH plus cosmological terms of a curvature-squared term constructed from the scalar [8]

$$G^{\mu\nu}S_{\mu\nu} \equiv R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2 \tag{1}$$

where the tensors G, S, R are, respectively, the Einstein, Schouten and Ricci tensors. This involves the introduction of a mass-squared parameter m^2 (which is positive for NMG but which we allow to be negative in the most general curvature-squared model). In an expansion about a Minkowski vacuum one finds that two spin-2 modes of mass m are propagated and, as for TMG, that perturbative unitarity requires $\sigma = -1$. As required by parity, the two spin-2 modes have opposite sign "helicities" s; i.e. opposite sign of the 3D Pauli-Lubanski pseudo-scalar divided by the mass; we call $|s|$ the "spin".

In the adS context, there is a family of classical NMG models parametrized by the dimensionless constant $m^2\ell^2$. In any parity-preserving 3D gravity theory with an adS vacuum, the central charges of the boundary CFT are equal, $c_+ = c_- = c$, and may be computed by the formula [9–11]

$$c = \frac{8\pi\ell}{\kappa^2} g_{\mu\nu} \frac{\partial L_{3D}}{\partial R_{\mu\nu}}\bigg|_{\text{adS vacuum}} \tag{2}$$

where L_{3D} is the 3D gravity Lagrangian of mass dimension two, i.e. without the factor of κ^{-2}. For NMG this gives [12–14]

$$c = \frac{24\pi\ell}{\kappa^2}\,\hat\sigma\,, \qquad \hat\sigma = \sigma + \frac{1}{2\ell^2 m^2}\,. \tag{3}$$

The expansion of the NMG action about the adS vacuum is greatly facilitated by starting with an alternative second-order version of the action with an auxiliary tensor field [8]. Diagonalization of the quadratic term of this action yields the sum of a linearized EH action with coefficient $\hat\sigma$, which is therefore the 'effective' EH coefficient in the adS background, and a spin-2 Fierz-Pauli (FP) action with coefficient $-1/\hat\sigma$ [13]. The linearized EH action propagates no modes so we get two massive gravitons from the FP action, but these are propagated unitarily only if $\hat\sigma < 0$. Hence there is a clash between unitary bulk gravitons and positive CFT central charges, as in TMG. There is also an analog of chiral gravity because there is a critical value of $\ell^2 m^2$ at which $c = 0$, and at which the quadratic action becomes equivalent to a spin-1 Proca action [13]. This critical NMG model is associated to a logarithmic CFT [15].

Both TMG and NMG are special cases of "general massive gravity" (GMG) [8], defined by the addition of *both* the LCS and the NMG curvature-squared term to the EH plus cosmological terms. Expanded about a Minkowski vacuum, this model propagates two spin-2 modes of opposite helicity but with different masses. In an adS vacuum there is a two-parameter family of classical GMG models parametrized by the constants $\mu\ell$ and $m^2\ell^2$. The central charges of the boundary CFT can be computed by a generalization of (2) to allow for parity-odd terms [16]. The result is

$$c_\pm = (24\pi\ell/\kappa^2) f_\pm\,, \qquad f_\pm = \hat\sigma \pm \frac{1}{\mu\ell}\,. \tag{4}$$

The no-ghost conditions follow from an expansion of the GMG action about an adS vacuum, but this 'off-shell' analysis is more subtle here, in part because the auxiliary tensor field 'trick' reduces the order in derivatives to three rather than two when the LCS term is present. Recent results in [17] indicate that the parameter ranges with positive central charge and positive energy of the graviton modes are mutually exclusive in GMG, as in TMG and NMG. Here we shall review, with some simplifications, the results of the on-shell analysis of [18] and explain how it extends to supergravity.

To facilitate our task, it will be convenient to first review the connection between a unitary irreducible representation (UIR) of the $SO(2,2)$ isometry group of adS_3 and fields on adS_3. Let $\varphi^{(s)}$ be a rank-$|s|$ ($|s| \geq 1$) totally-symmetric and tracefree field on adS_3 subject to the 'divergence-free' condition

$$\bar{D}^{\mu} \varphi^{(s)}_{\mu\nu_1\cdots\nu_{s-1}} = 0\,, \tag{5}$$

where $\bar{D}$ is the covariant derivative with respect to a background adS_3 metric $\bar{g}$. Let $\mathscr{D}(\eta)$ be the family of first-order linear differential operators, parametrized by a dimensionless constant η, that act on the space of such tensors according to the definition

$$\left[\mathscr{D}(\eta)\varphi^{(s)}\right]_{\mu_1\cdots\mu_s} = \left[\mathscr{D}(\eta)\right]_{\mu_1}{}^{\rho}\varphi^{(s)}_{\rho\mu_2\cdots\mu_s}\,, \qquad \left[\mathscr{D}(\eta)\right]_{\mu}{}^{\nu} = \ell^{-1}\delta^{\nu}_{\mu} + \frac{\eta}{\sqrt{|\bar{g}|}}\varepsilon_{\mu}{}^{\tau\nu}\bar{D}_{\tau}\,. \tag{6}$$

Despite appearances, the rank-$|s|$ tensor $\mathscr{D}(\eta)\phi^{(s)}$ is also traceless, totally symmetric and 'divergence-free'. Using the identity

$$\mathscr{D}(\eta)\mathscr{D}(-\eta)\varphi^{(s)} \equiv -\eta^2\left[\bar{D}^2 + \ell^{-2}\left(|s| + 1 - \eta^{-2}\right)\right]\varphi^{(s)} \tag{7}$$

we see that eigenfunctions of the covariant D'Alembertian $\bar{D}^2$ acting on spin-$|s|$ fields are linear combinations of solutions to the first-order equations

$$\mathscr{D}(\eta)\varphi^{(s)} = 0\,, \qquad \mathscr{D}(-\eta)\varphi^{(s)} = 0\,. \tag{8}$$

A physically acceptable[1] solution of either equation furnishes a UIR of $SO(2,2)$ labelled by its lowest weights (E_0, s), where $\ell^{-1}E_0$ is the lowest energy[2], which is real and satisfies the condition $E_0 \geq |s|$ [19–21]. The UIR furnished by a solution of (8) has lowest weights [22]

$$E_0 = 1 + \frac{1}{|\eta|}\,, \qquad s = \frac{|s|\eta}{|\eta|}\,. \tag{9}$$

We see that the sign of η gives the sign of the helicity s. The UIR's with $E_0 = |s|$ for $s \neq 0$ are called singleton irreps; their weight space is drastically reduced compared to a typical UIR, and they can be interpreted as describing modes that are confined to the 2D boundary of adS_3 [23]. More generally, the adS/CFT correspondence assigns to every UIR an operator in a dual 2D CFT with conformal weights $h_\pm = (E_0 \pm s)/2$, in which context the bound $E_0 \geq |s|$ translates into $h_\pm \geq 0$.

The above analysis applies for $|s| > 0$. For $s = 0$ we must consider the solutions of the second-order equation

$$\left(\bar{D}^2 - \mathscr{M}^2\right)\varphi = 0 \tag{10}$$

for constant $\mathscr{M}$. The solutions, if physically acceptable, furnish UIRs with $s = 0$ and

$$E_0 = 1 \pm \sqrt{1 + \ell^2\mathscr{M}^2}\,. \tag{11}$$

[1] i.e. nonsingular at the origin and normalizable with respect to the $SO(2,2)$ invariant measure [19,20].

[2] Or highest energy. It is not possible to distinguish between the two at the level of the equations of motion. This is why the unitarity of the irreducible representation is not sufficient for unitarity of the field theory.

Reality of E_0 implies the Breitenlohner-Freedman (BF) bound $\ell^2\mathscr{M}^2 \geq -1$. In addition, $E_0 \geq 0$ is required for a UIR. This condition allows both signs in (11) when $-1 \leq \ell^2\mathscr{M}^2 \leq 0$; otherwise only the plus sign is allowed.

Let us now see how all this applies to the 'Einstein' theory defined by the EH and cosmological terms alone. We write the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\left(1+\frac{1}{3}h\right) + H_{\mu\nu}\,, \qquad \bar{g}^{\mu\nu}H_{\mu\nu} = 0\,, \tag{12}$$

so that (H, h) are the tracefree and trace perturbations respectively. In the gauge

$$\bar{D}^\mu H_{\mu\nu} = 0\,, \tag{13}$$

the linearized field equations may now be written as

$$[\mathscr{D}(1)\mathscr{D}(-1)H]_{\mu\nu} = -\frac{1}{3}\left(\bar{D}_\mu\bar{D}_\nu - \frac{1}{3}\bar{g}_{\mu\nu}\bar{D}^2\right)h\,, \qquad \left(\ell^2\bar{D}^2 - 3\right)h = 0\,. \tag{14}$$

We have coupled equations for H and h but there is still a residual gauge invariance. If we write the vector gauge parameter of linearized diffeomorphisms as $\xi^T_\mu + \partial_\mu\xi$, where $\bar{D}^\mu\xi^T_\mu = 0$, then the gauge condition (13) is invariant provided that

$$\bar{D}_\mu\left(\ell^2\bar{D}^2 - 3\right)\xi = 0\,, \qquad \left(\ell^2\bar{D}^2 - 2\right)\xi^T_\mu = 0 \quad \Rightarrow \quad \left(\ell^2\bar{D}^2 + 2\right)\bar{D}_{(\mu}\xi^T_{\nu)} = 0\,. \tag{15}$$

The variation of h is $\delta h \sim \bar{D}^2\xi$ but this implies $\delta h \sim \xi$ for $\bar{D}^2\xi \sim \xi$. Since ξ is a residual gauge parameter, we may use it to set h to zero. This leaves us with the equation $\mathscr{D}(1)\mathscr{D}(-1)H = 0$, but the identity (7) implies that this is equivalent to

$$\left(\ell^2\bar{D}^2 + 2\right)H = 0\,. \tag{16}$$

This is clearly invariant under the residual gauge invariance, which suffices to remove all local degrees of freedom of H.

We are now in a position to discuss GMG. Linearized about an adS vacuum, the GMG equations are

$$[\mathscr{D}(1)\mathscr{D}(-1)\mathscr{D}(\eta_+)\mathscr{D}(\eta_-)H]_{\mu\nu} = -\frac{1}{3\ell^2}\left(\bar{D}_\mu\bar{D}_\nu - \frac{1}{3}\bar{g}_{\mu\nu}\bar{D}^2\right)h\,, \tag{17}$$

$$\frac{\Omega}{m^2}\left(\ell^2\bar{D}^2 - 3\right)h = 0\,, \tag{18}$$

where

$$\Omega \equiv \ell^2 m^2\hat{\sigma} - 1 \tag{19}$$

and [18]

$$\eta_\pm = \Omega^{-1}\left(-\frac{\ell m^2}{2\mu} \pm \sqrt{\frac{\ell^2 m^4}{4\mu^2} - \Omega}\right)\,. \tag{20}$$

We have retained the factor of Ω in the h-equation because it allows the $\Omega = 0$ equations to be obtained by taking the $\Omega \to 0$ limit. The region in parameter space with $\Omega > 0$ is divided from the region with $\Omega < 0$ by the curve $\Omega = 0$. To see the significance of this division, we observe that

$$\eta_+\eta_- = \Omega^{-1}\,. \tag{21}$$

This shows that the parameters $\eta_\pm$ have opposite signs in the $\Omega < 0$ region and the same sign in the $\Omega > 0$ region. The reality of $\eta_\pm$ and the UIR condition $|\eta_\pm| \leq 1$ translate into $m^2 c_\pm \leq 0$ for $\Omega < 0$ and $m^2 c_\pm \geq 0$, $\ell^2 m^4 \geq 4\mu^2\Omega$ for $\Omega > 0$.

Provided that $\Omega \neq 0$, we may proceed as in the 'Einstein' case, eliminating h by a combination of its field equation and a residual gauge transformation, to arrive at the single 4th order equation

$$\mathscr{D}(1)\mathscr{D}(-1)\mathscr{D}(\eta_+)\mathscr{D}(\eta_-)H = 0\,. \tag{22}$$

As long as $(\eta_+ - \eta_-)\,(|\eta_+| - 1)\,(|\eta_-| - 1) \neq 0$ the general solution is a linear combination of the two singleton modes and solutions of the first-order equations $\mathscr{D}(\eta_\pm)H = 0$. This state of affairs applies in the NMG limit $|\mu| = \infty$ since we then have

$$\eta_+ = -\eta_- = -1/\sqrt{-\Omega}\,. \tag{23}$$

Clearly, we must be on the $\Omega < 0$ side of the parameter space divide to take this limit, in which the bound $|\eta_\pm| \leq 1$ becomes $m^2\hat{\sigma} < 0$. As confirmed by the computation of the quadratic action in [13], this bound is the condition for the absence of tachyons. This computation also shows that $\hat{\sigma} < 0$ is the no-ghost condition, i.e. for unitarity of the quantum field theory. As $m^2 > 0$ for NMG, the absence of tachyons implies the absence of ghosts. It seems likely to us that this will remain true for GMG as long as $\Omega < 0$. However, it also seems likely that GMG has ghosts when $\Omega > 0$, even though it is certainly possible to satisfy the UIR bound $|\eta_\pm| < 1$, and hence avoid tachyons. This is because solutions of the equation $\mathscr{D}(\eta)\mathscr{D}(\eta')H = 0$ with $\eta\eta' > 0$ could be expected to arise naturally in the context of 5th order 3D gravity models, which necessarily propagate ghosts in a Minkowski vacuum [24]. To settle this issue we would need an 'off-shell' analysis like that presented for NMG in [13].

When $\Omega = 0$ the h-equation drops out. This leaves us with the H-equation, that can be shown to reduce to

$$\left[\mathscr{D}(\eta)\mathscr{D}(1)\mathscr{D}(-1)\right]_\mu{}^\rho \varepsilon_\rho{}^{\alpha\beta}\bar{D}_\alpha H_{\beta\nu} = 0\,, \qquad \eta = -\frac{\mu}{\ell m^2}\,. \tag{24}$$

Unless $|\eta| = 1$, the solutions space is spanned by the singletons, the solution of $\mathscr{D}(\eta)H = 0$, and the solution of

$$\varepsilon_\rho{}^{\alpha\beta}\bar{D}_\alpha H_{\beta\nu} = 0\,. \tag{25}$$

By acting on this equation with the operator $\varepsilon_\lambda{}^{\tau\mu}\bar{D}_\tau$, we see that the solution of (25) also obeys $(\ell^2\bar{D}^2 + 3)H = 0$. Comparing this with (7), we might naively conclude that it requires $\eta = \infty$ and hence $E_0 = 1$, which violates the spin-2 bound $E_0 \geq 2$. The resolution of this puzzle is subtle because of an additional gauge invariance; the relevant solutions of (25) are "partially massless" [25, 26].

Our main interest here is in the 'critical' GMG cases. For finite non-zero Ω these are those for which

$$(\eta_+ - \eta_-)\,(|\eta_+| - 1)\,(|\eta_-| - 1) = 0\,. \tag{26}$$

When this condition holds, the solution space is no longer spanned by solutions of first-order equations of the form $\mathscr{D}(\eta)H = 0$. However, subject to appropriate boundary conditions, one finds that additional solutions appear. In the case that two of the η values coincide, these are logarithmic solutions of $\mathscr{D}^2(\eta)H = 0$ that do not solve $\mathscr{D}(\eta)H = 0$. In the case that three η values coincide we get additional doubly-logarithmic solutions of $\mathscr{D}^3(\eta)H = 0$ that solve neither $\mathscr{D}(\eta)H = 0$ nor $\mathscr{D}^2(\eta)H = 0$. A similar statement applies when $\Omega = 0$ but then the only critical case occurs when $\ell m^2 = |\mu|$, in which case we get a logarithmic solution of $\mathscr{D}^2(\pm 1)H = 0$. The catalog of critical points for $\Omega \neq 0$ is much richer[3]:

[3] The critical points of GMG have also been studied in [18]. Recently, properties of the logarithmic CFT dual to GMG at the critical points have been investigated in [17].

(i) $|\eta_+| = 1$ but $|\eta_-| \neq 1$ and $\Omega^{-1} \neq 0$. We find that $\eta_+ = \pm 1$ when $\ell\mu\hat{\sigma} = \mp 1$ and then $\eta_- = \mp 1/\left(1 \pm \ell m^2/\mu\right)$. We get no propagating graviton mode from $\mathscr{D}(\eta_+)H = 0$, just an extra logarithmic solution to $\mathscr{D}^2(\pm 1)H = 0$. In this case there is therefore a *single* propagating massive graviton of helicity $2\eta_-/|\eta_-|$. If $\Omega^{-1} \to 0$, then $m^2 \to \infty$, giving the TMG limit. In this case $\eta_- = 0$, and $|\ell\mu\sigma| = 1$.

(ii) $|\eta_-| = 1$ but $|\eta_+| \neq 1$ and $\Omega^{-1} \neq 0$. In this case $\eta_- = \pm 1$ when $\ell\mu\hat{\sigma} = \mp 1$ and then $\eta_- = \mp 1/\left(1 \pm \ell m^2/\mu\right)$. In this case there is a *single* propagating massive graviton of helicity $2\eta_+/|\eta_+|$. If $\Omega^{-1} \to 0$, then $m^2 \to \infty$, giving the TMG limit. In this case $\eta_+ = 0$, and $|\ell\mu\sigma| = 1$.

(iii) $\eta_+ = -\eta_-$ and $|\eta_\pm| = 1$. This case is realized by $\eta_- = -\eta_+ = 1$, which requires $|\mu| = \infty$, $m^2\hat{\sigma} = 0$. This is the critical limit of NMG. In this case (22) degenerates to

$$\mathscr{D}(1)^2\mathscr{D}(-1)^2 H = 0\,. \tag{27}$$

Although this propagates no gravitons, it does propagate two spin-1 modes [27]. One way to see this is to rewrite the fourth-order equation as the pair of second-order equations

$$\left(\ell^2\bar{D}^2 + 2\right) H = U\,, \qquad \left(\ell^2\bar{D}^2 + 2\right) U = 0\,, \tag{28}$$

where U is another symmetric traceless and 'divergence-free' tensor. Applying to the U-equation the same reasoning that we used above for the 'Einstein' case, we deduce that $U_{\mu\nu} = 2\bar{D}_{(\mu}A_{\nu)}$ for some vector field A satisfying $\left(\ell^2\bar{D}^2 - 2\right) A_\mu = 0$ and, since U is traceless, $\bar{D}^\mu A_\mu = 0$. In other words H satisfies

$$\left(\ell^2\bar{D}^2 + 2\right) H_{\mu\nu} = 2\bar{D}_{(\mu}A_{\nu)} \tag{29}$$

for a vector field such that

$$\left(\ell^2\bar{D}^2 - 2\right) A_\mu = 0\,, \qquad \bar{D}^\mu A_\mu = 0\,. \tag{30}$$

These equations for A, which also follow from taking the divergence and trace of (29), are Proca equations in the adS background. They are derivable from the Proca Lagrangian density

$$\mathscr{L}_{Proca} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\ell^{-2}A^\mu A_\mu\,, \qquad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,. \tag{31}$$

For singleton solutions the left-hand-side of (29) vanishes, implying that A is a Killing vector of the adS background. In contrast, the left-hand-side of (29) is non-vanishing for the logarithmic solutions of (27) and hence these must be related to non-trivial Proca modes. However, although a solution of the Proca equations furnishes a UIR of the adS_3 isometry group, with $(E_0, s) = (2, 1)$, a careful study of (29) shows that these solutions correspond to the first descendants of the logarithmic modes; see [28] for a detailed description of precisely such a descendant mode.

(iv) $\eta_+ = \eta_-$ but $|\eta_\pm| \neq 1$. This is possible only when $\Omega > 0$. In this case, $\eta_\pm = -2\mu/(m^2\ell)$. We get one propagating massive graviton from the solution to $\mathscr{D}(-2\mu/(m^2\ell)) = 0$ and an additional logarithmic solution of $\mathscr{D}^2(-2\mu/(m^2\ell)) = 0$.

(v) $\eta_+ = \eta_-$ *and* $|\eta_\pm| = 1$. This is possible only when $\Omega = 1$, which requires $2\mu = \mp m^2\ell$ according to whether $\eta_+ = \eta_- = 1$ or $\eta_+ = \eta_- = -1$. In this case we have, apart from singletons, only logarithmic and doubly logarithmic modes from $\mathscr{D}^3(1)H = 0$ or $\mathscr{D}^3(-1)H = 0$.

Observe that only cases (i)-(iv) are possible for $\Omega < 0$, which may be required if ghosts are to be avoided, for the reason given above. In these four cases we have

$$c_+ c_- = 0 \,. \tag{32}$$

In other words, at least one of the two central charges of the boundary CFT is zero. Unless both are zero, "critical" can be interpreted as "chiral". As pointed out in [18], the "chiral GMG" models interpolate between chiral TMG, which is case (iii), and critical NMG, which is case (iv).

In the remainder of this paper, we shall explain how these results extend to the supergravity case. The $\mathscr{N} = 1$ supersymmetric extensions of NMG and GMG have been found in recent work, along with all possible maximally supersymmetric vacua [27, 29]. These constructions are complicated by the fact that the off-shell graviton multiplet contains not just the metric but also an additional scalar field S, which is auxiliary in 3D 'Einstein' supergravity [30] and in super-TMG [31] but which propagates in generic curvature-squared models. Even when the kinetic term for S is absent it will still propagate a scalar mode in adS vacua unless the coefficients of the cubic equation of motion for S are constants[4], and this condition defines a "super-GMG" model for which the bosonic action is precisely of GMG-form after elimination of S. Remarkably, *all* adS vacua of GMG correspond to supersymmetric vacua of super-GMG, so the spectrum in such vacua is determined by the GMG spectrum. Of course, the boundary CFT must now be a boundary SCFT, but it will still have the same central charge [32]. For these reasons, the results summarized above for GMG extend with no essential modifications to super-GMG. Unfortunately, this makes it unlikely that supersymmetry can help to resolve the boundary/bulk unitarity "clash" explained above.

Although the super-GMG model is the most interesting special case of the generic curvature squared 3D supergravity model constructed in [27], the results for the generic model provide us with the opportunity to explore some issues in the context of a model with more parameters. The bosonic Lagrangian is

$$e^{-1}\mathscr{L}_{bos} = -V(S) + f(S)R + \frac{1}{m^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right) + \frac{1}{8\tilde{m}^2}\left[R^2 - 16\,(\partial S)^2\right] + \frac{1}{\mu}e^{-1}\mathscr{L}_{LCS} \,, \tag{33}$$

where $\mathscr{L}_{LCS}$ is the Lorentz-Chern-Simons term, and

$$\begin{aligned} V(S) &= -MS + 2\sigma S^2 - \frac{1}{\check{\mu}}S^3 + \left(-\frac{3}{2m^2} + \frac{1}{\check{m}^2} + \frac{9}{2\tilde{m}^2}\right)S^4 \,, \\ f(S) &= \sigma + \frac{1}{2\check{\mu}}S + \left(-\frac{1}{2m^2} + \frac{3}{10\check{m}^2} + \frac{3}{2\tilde{m}^2}\right)S^2 \,. \end{aligned} \tag{34}$$

The model has six independent mass parameters $(M, m, \check{m}, \tilde{m}, \check{\mu}, \mu)$, which can be traded for dimensionless parameters using the gravitational coupling constant, and one discrete parameter $\sigma = \pm 1, 0$. Despite the notation, $(m^2, \tilde{m}^2, \check{m}^2)$ are allowed to take either sign.

The field equations following from (33) are given in [27]. The supersymmetric adS vacuum solutions of these equations have[5]

$$S = \bar{S} = -\ell^{-1} \,, \qquad M = -\ell^{-1}\left(4\sigma + \frac{2}{5\ell^2\check{m}^2}\right) \,. \tag{35}$$

[4] This condition may be relaxed slightly but we refer to [27] for a discussion of this point.

[5] We choose the sign of S so as to agree with the sign choice made in [27].

The central charges are given by (4) with $\hat\sigma$ given by

$$\hat\sigma = \sigma - \frac{1}{2\ell\breve\mu} + \frac{3}{10\ell^2\breve m^2}\,. \tag{36}$$

The adS/CFT correspondence suggests that the spectrum of propagating gravitons will be independent of $\tilde m$ and will depend on the $\breve\mu$ and $\breve m$ only through the parameter $\hat\sigma$. We shall verify this.

To expand the field equations about a supersymmetric adS vacuum, we write the metric as in (12) and we write

$$S = \bar S + \ell^{-1} s \tag{37}$$

for dimensionless scalar perturbation s. To present the results, we shall again make use of the parameters $\eta_\pm$ and Ω defined in (19) and (20) in terms of $\hat\sigma$, although it should be remembered that $\hat\sigma$ is now given by (36). In addition, it is useful to define the dimensionless parameter

$$a = m^2\left[-\frac{2}{m^2} + \frac{6}{5\breve m^2} + \frac{6}{\tilde m^2} - \frac{\ell}{\breve\mu}\right]. \tag{38}$$

This agrees with the definition in [27] when restricted to the models (with $\tilde m^2 = \infty$) for which this parameter was defined there. We will also need the following (dimensionless) linear differential operators:

$$L = \ell^2\bar D^2 - 3\ , \qquad \tilde L = \frac{m^2}{\tilde m^2}L - \Omega\,. \tag{39}$$

Using these definitions, one finds that the h and s equations are, respectively,

$$\frac{1}{m^2}L\left(\tilde L h + 3as\right) = 0\,, \qquad \frac{1}{m^2}\left(\tilde L + 2a\right)s + \frac{a}{12m^2}Lh = 0\,, \tag{40}$$

and that the linearized H-equation is

$$\mathscr{D}(1)\mathscr{D}(-1)\mathscr{D}(\eta_+)\mathscr{D}(\eta_-)H = \Omega^{-1}\,J\ , \tag{41}$$

where

$$J_{\mu\nu} = \frac{1}{3\ell^2}\left(\bar D_\mu\bar D_\nu - \tfrac{1}{3}\bar g_{\mu\nu}\bar D^2\right)\left[\tilde L h + 3as\right]. \tag{42}$$

The integrability condition $\bar D^\mu J_{\mu\nu} = 0$ is satisfied as a consequence of the h-field equation.

Observe that we recover the GMG equations that we have already analyzed on setting $\tilde m^2 = |\breve\mu| = \infty$ and $a = 0$. In that case, we argued that it was possible to set $h = 0$, and hence $J = 0$, by using a combination of the h-equation and a residual gauge transformation. It is no longer so clear that this argument still applies, but a simpler one is available, as long as $\Omega \neq 0$. We define the new symmetric traceless and 'divergence-free' perturbation

$$\tilde H = H - m^2\Omega^{-1}J\,. \tag{43}$$

Using the relation $\varepsilon_{(\mu}{}^{\alpha\beta}\bar D_{|\alpha}J_{\beta|\nu)} = 0$, we may deduce that

$$\mathscr{D}(1)\mathscr{D}(-1)\mathscr{D}(\eta_+)\mathscr{D}(\eta_-)\widetilde H_{\mu\nu} = 0\ . \tag{44}$$

The analysis of critical points for $\Omega \neq 0$ now proceeds *exactly* as for the GMG case. For $\Omega = 0$ the H-equation is given by

$$\left[\mathscr{D}(\eta)\mathscr{D}(1)\mathscr{D}(-1)\right]_\mu{}^\rho\varepsilon_\rho{}^{\alpha\beta}\bar D_\alpha H_{\beta\nu} = -\eta J_{\mu\nu}\,, \qquad \eta = -\frac{\mu}{\ell m^2}\,. \tag{45}$$

Note that by acting on this equation with the operator $\varepsilon_\lambda{}^{\tau\mu}\bar{D}_\tau$, we can arrive at the integrability condition $\mathscr{D}(\eta)\mathscr{D}(1)\mathscr{D}(-1)(\ell^2\bar{D}^2+3)H = 0$, an equation that also follows from (24). To summarize, the critical point structure of the generic curvature-squared supergravity model, expanded about a supersymmetric adS vacuum, is *identical* to that of the GMG model, with all dependence on the extra parameters absorbed into the effective EH coefficient $\hat{\sigma}$, as anticipated above.

The main difference of the generic case as compared to GMG is the possibility of additional scalar modes arising from the h and s equations (40). Here we shall consider some special cases.

- $\tilde{m}^2 = \infty$ *and* $a = 0$; this case was called "generalized GMG" in [27]; it reduces to the GMG case analysed above when $|\tilde{\mu}| = \infty$. For this case, and assuming non-zero finite m^2, the equations (40) become
$$\Omega L h = 0\,, \qquad (2a - \Omega)\, s = 0 \tag{46}$$
We see that $s = 0$ (unless $\Omega = 2a$ but then it is undetermined, implying an 'accidental' gauge invariance that allows one to choose $s = 0$). Provided $\Omega \neq 0$ we have $Lh = 0$ and residual gauge invariances allow us to set $h = 0$. There are therefore no propagating scalars. One can show that the generalized super-NMG limit ($|\mu| = \infty$) is ghost-free for $\hat{\sigma} \leq 0$ [27].
- $\tilde{m}^2 = \infty$ but $a \neq 0$. In this case the S-equation of motion is algebraic but it still propagates modes in adS, for reasons explained in [27]. Here we verify this for the generic $\Omega \neq 0$ case. Under these conditions, the equations (40) become equivalent to (for $m^2 \neq 0$ and finite)
$$L\left(h - 3a\Omega^{-1}s\right) = 0\,, \qquad Ls = \frac{4\Omega\,(\Omega - 2a)}{a^2}s \tag{47}$$
As the addition of a scalar to h does not change its residual gauge transformation, the first equation does not lead to propagating modes. The second one takes the form (10) with $\varphi = s$ and $\mathscr{M}^2 = 3 + 4\Omega\,(\Omega - 2a)\,/a^2$. This equation therefore propagates a scalar mode that is non-tachyonic. However there is no guarantee that it is not a ghost.
- $a = 0$ but $\tilde{m}^2 \neq \infty$. The equations (40) now reduce to
$$L\left(L - \frac{\Omega\tilde{m}^2}{m^2}\right)h = 0\,, \qquad \left(L - \frac{\Omega\tilde{m}^2}{m^2}\right)s = 0\,. \tag{48}$$
The first of these is a fourth-order equation for h that is not easy to analyse. The second equation takes the form (10) with $\varphi = s$ and $\ell^2\mathscr{M}^2 = 3 + \tilde{m}^2\Omega/m^2$. We therefore get a non-tachyonic scalar provided that $4 \geq -\tilde{m}^2\Omega/m^2$, although there is again no guarantee that it is not a ghost.

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