Scalar Field in Non-commutative Curved Space time

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We will study the theory of scalar field in non-commutative curved space time (NCCST) with a percussionist who has been found by Mebaraki. In this paper, the motion equation of the scalar field will be found and the canonical energy-momentum tensor in NCCST without torsion will be found for $\theta^{0i} \neq 0$. The most important is that we assume the $\triangle$ is a constant where it is appearing in the deformed Moyal-product and the non-commutative parameter (θ) is x^{μ}-independent.

Keywords: Non-commutative Geometry, Non-commutative Field Theory

Introduction

We believe that the next physics will be built in Non-commutative Curved Space Time.

There are many approaches to non-commutative geometry and its use in physics, the operator algebra and $\mathcal{C}^{\star}$-algebra one, the deformation quantization one, the quantum group one, and the matrix algebra(fuzzy geometry) one. Julius Wess said: "Since Newton the concept of space time has gone through various changes. All stages, however, had in common the notion of a continues linear space. Today we formulate fundamental laws of physics, field theories, gauge field theories, and the theory of gravity on differentiable manifolds. That a change in the concept of space for very short distances might be necessary was already anticipated in 1854 by Riemann in his famous inaugural lecture. There are indications today that at very short distances we might have to go beyond differential manifolds. This is only one of several arguments that we have to expect some changes in physics for very short distances. Other arguments are based on the singularity problem in quantum field theory and the fact theory of gravity is non re-normalizable when quantized. Why not try an algebraic concept of space time that could guide us to changes in our present formulation of laws of physics? This is different from the discovery of quantum mechanics. There physics data forced us to introduce the concept of non-commutativity. We hope that it might solve some conceptual problems of physics that are still left at very short distances.

The idea of non-commutative coordinates is almost as old as quantum mechanics by first quantization introduced by Heisenberg. However, many physicists believe that Heisenberg proposed to solve the problem of divergent integrals in relativistic quantum field theory. Finally H. Snyder used systematic analysis of non-commutative spaces. From quantum spaces and quantum groups new mathematical concepts have emerged"Non-commutative Coordinates!"[1]. Physics Lecture Notes (physics series)

The formulation of classical and quantum field theories in NCCST has been a very active subject over the last few years. Most of these approaches focus on free or interacting QFTs on the Moyal-Weyl deformed or κ-deformed Minkowski space time. Now, many people are working in non-commutative field theory to find the details of the theory. It is a nonlocal theory and because of this, the NCFT has many ambiguous problems such as UV/IR divergence and causality. The causality is broken when $\theta^{0i} \neq 0$ so as a field theory it is not appealing[2]. In addition, some people thought the non-commutative field theory violets the Lorentz invariance, but a group of famous physicists [2] have shown the non-commutative field theory will keep the Lorentz invariance by Hopf algebra [2, 3]. However, many of physicists prefer to work in this field, because they think it is one of the best candidate for the brighter future of physics [2, 3, 4].

Here, by deforming the ordinary Moyal $\star$-product, we propose a new Moyal-Weyl $\triangleright$-product which takes into consideration the missing terms cited above, which generate gravitational terms to the order θ^2[5]. To any smooth function f, we associate the Weyl operator W(f) as

$$f(x) = (2\pi)^{-\frac{3}{2}} \int d^4\kappa \exp\left(-\imath\kappa x\right)\tilde{f}(\kappa) \longrightarrow W(f) = (2\pi)^{-\frac{3}{2}} \int d^4\kappa \exp\left(-\imath\kappa\hat{X}\right)\tilde{f}(\kappa) \tag{1}$$

where $\hat{X}$ are non-commuting operators associated with the following classical variables

$$X^\mu = x^\mu + \Gamma^\mu_{\alpha\beta}x^\alpha x^\beta - \frac{1}{2}\Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\lambda\kappa}x^\lambda x^\kappa x^\beta \tag{2}$$

where $\Gamma^\mu_{\alpha\beta}$ is the symmetric affine connection. The non-commuting operators $\hat{X}$ are defined by a symmetrization procedure:

$$\hat{X}^\mu = \hat{x}^\mu + \hat{\Gamma}^\mu_{\alpha\beta}\hat{x}^\alpha\hat{x}^\beta - \frac{1}{2}\hat{\Gamma}^\mu_{\alpha\beta}\hat{\Gamma}^\alpha_{\lambda\kappa}\hat{x}^\lambda\hat{x}^\kappa\hat{x}^\beta \tag{3}$$

and $[\hat{x}^\mu,\ \hat{x}^\nu]_\triangleright = \imath\theta^{\mu\nu}$ where, after the corresponding expression substitutions, reads

$$\hat{X}^\mu = \hat{x}^\mu + \imath^2\frac{\theta^{ab}\theta^{\alpha\beta}}{2\sqrt{-g}}\partial_b\hat{R}^\mu_{a\alpha\beta} \tag{4}$$

here $\hat{R}^\mu_{b\alpha\beta}$ stands for the Riemann curvature tensor and $\sqrt{-g}$ is determinant of metric. For the effect of gravitation on the non-commutative space time we might use from $\hat{X}$ so the non-commutative space time can be realized by the coordinate operators and one can write

$$\begin{aligned}\exp\left(\imath p\hat{X}\right)\exp\left(\imath k\hat{X}\right) &= \exp\left(\imath pX + \imath kX + \frac{1}{2}\imath^2 k_\mu[\hat{X}^\mu,\ \hat{X}^\nu]_\triangleright p_\nu + ...\right) \\ &= \exp\left(\imath k\hat{x} + \imath p\hat{x} - \frac{\imath}{2}\theta^{\mu\nu}k_\mu p_\nu + \frac{1}{2}\imath\hat{\triangle}^\mu(k_\mu + p_\mu) + ...\right)\end{aligned} \tag{5}$$

with $\hat{\triangle}^\mu = \frac{\imath^2}{\sqrt{-g}}\theta^{\alpha\beta}\theta^{ab}\partial_\beta\hat{R}^\mu_{\alpha ab}$. However, we will continue with $\hat{X}$ and for when $\hat{\triangle}$ will be a constant we define a canonical transformation. As for phase space we choose

$$\begin{aligned}[\hat{x}^\mu,\ \hat{x}^\nu] &= \imath\theta^{\mu\nu} \\ [\hat{x}^\mu,\ \hat{p}^\nu] &= \imath\hbar\delta^{\mu\nu} \\ [\hat{p}^\mu,\ \hat{p}^\nu] &= 0\end{aligned} \tag{6}$$

If $\triangle$ will be a constant the $\triangleright$-product is [5]

$$\triangleright \equiv e^{\triangle x^{\mu}\partial_{\mu}(\Im\otimes\Im)+\frac{\imath\theta^{\mu\nu}}{2}\partial_{\mu}\otimes\partial_{\nu}}$$

where

$$(A\triangleright B)(\hat{x}) = A\ e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial}_{\mu}\otimes\overrightarrow{\partial}_{\nu}+\imath\triangle^{\mu}\partial_{\mu}\ (\Im_A\otimes\Im_B)}\ B$$

where $\Im$ stands for the identity of spaces (for example $\Im_A$ is identity of "A"). If the $\triangle$ is x^{μ}-independent we can write

$$\begin{aligned}
\int d^dx\ (A\triangleright B)(\hat{x}) &= \int d^dx \int \mathbf{dkdq}\tilde{A}(k)\tilde{B}(q)\ e^{\imath(k+q)\cdot\hat{x}-\frac{\imath}{2}\theta^{\mu\nu}k_{\mu}p_{\nu}+\imath\triangle^{\mu}(k+q)_{\mu}} \\
&= \int d^dx\ e^{\triangle^{\mu}\partial_{\mu}} \int \mathbf{dkdq}\tilde{A}(k)\tilde{B}(q)\ e^{\imath(k+q)\cdot\hat{x}-\frac{\imath}{2}\theta^{\mu\nu}k_{\mu}p_{\nu}} \\
&= \int d^dx(A(x)\star B(x)+\triangle^{\mu}\partial_{\mu}(A(x)B(x))+0(\theta^4)) \\
&= \int d^dxA(x)\star B(x) \qquad (7)
\end{aligned}$$

where $\mathbf{dq} = \frac{1}{(2\pi)^{\frac{d}{2}}}dq$

Physics Notes and The Motion Equation of Fields

We start by showing how to construct an action for a real scalar field in NCCST. If one direct follows the general rule of transforming usual theories in non-commutative ones by replacing product of fields by star product[7] and we believe that this change should be done on the Lagrangian density. In fact $\mathbf{S}_{Cm}(\mathcal{L}_{Cm}) \rightarrow \mathbf{S}_{Nc}(\mathcal{L}^{Sym}_{Nc})$. or $\mathbf{S} = \int d^dx\ \sqrt{-g}\triangleright\mathcal{L}^{\ Sym}_{\triangleright Nc}$ where $\triangleright$ is a new product. The classical Lagrangian density for scalar field is

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\partial_{\mu}\phi(x)\partial_{\nu}\phi(x) - (m^2+\xi R(x))\phi^2) \qquad (8)$$

where ϕ is the scalar field and m the mass of the field. The coupling between the field and gravitational field represented by the term $\xi R\phi^2$ where ξ is a numerical factor and $R(x)$ is the Ricci scalar curvature. In non-commutative curved space time with $\triangleright$-product the action is more difficult because symmetric ordering must also considered. We are in particular interested in the deformation of the canonical action

$$\mathbf{S} = \int d^dx\ \ \frac{1}{2}\sqrt{-g}\triangleright(S_{\triangleright}(\partial_{\mu}\phi, g^{\mu\nu}, \partial_{\nu}\phi) - m^2(\phi\triangleright\phi+\xi S_{\triangleright}(\phi,\phi,R(x)))) \qquad (9)$$

where the $S_{\triangleright}(f,g,h)$ means that we must make of the symmetric structures of f and g and h and ξ is a dimensionless constant. As for multiplication of functions we can write $\int\ (f \triangleright g)(x) = \int\ (f \star g)(\bar{x})$ where $\bar{x}^{\mu} = x^{\mu} + \triangle^{\mu}$ where $\bar{f} = f(\bar{x})$ and $(f \star g)(\bar{x}) = \bar{f} \star \bar{g}$ and the earlier star product ($\star$-product) will be an associative by the Drinfel'd map. Using these tools, and hermitical structure we have

$$S_{\triangleright}(\nabla^{\star}_{\mu}\phi, g^{\mu\nu}, \nabla^{\star}_{\nu}\phi) = S_{\star}(\nabla^{\star}_{\mu}\bar{\phi}, \bar{g}^{\mu\nu}, \nabla^{\star}_{\nu}\bar{\phi}) = \nabla^{\star}_{\mu}\bar{\phi} \star \bar{g}^{\mu\nu} \star \nabla^{\star}_{\nu}\bar{\phi} \tag{10}$$

so we can deform the classical action to the global expression

$$\mathbf{S} = \int d^{d}x\ \sqrt{-\bar{g}} \star \mathcal{L}_{\star} = \int d^{d}x\ \sqrt{-g} \star \mathcal{L}_{\star} + total\ \ derivative \tag{11}$$

but when the $\triangle$ is x-independent the last term is vanish and the Lagrangian density is

$$\mathcal{L}_{\star} \ = \ \frac{1}{2}((\partial_{\mu}\phi \star g^{\mu\nu} \star \partial_{\nu}\phi) - m^{2}(\phi \star \phi + \xi\phi \star R \star \phi) \tag{12}$$

we know the one of the $\star$'s can be removed also $\int f \star g \star h = \int h \star g \star f$ also for research of the motion equation of fields we start from the principle of least action $\frac{\delta \mathbf{S}}{\delta\phi(z)} = 0$

$$\frac{\delta \mathbf{S}}{\delta\phi(z)} = \frac{\delta}{\delta\phi(z)} \int\ d^{d}x\ \ \frac{\sqrt{-g}}{2} \star ((\partial_{\mu}\phi \star g^{\mu\nu} \star \partial_{\nu}\phi) - m^{2}(\phi \star \phi + \xi\phi \star R \star \phi)) \tag{13}$$

so with these tools and with $\Phi^{\mu} = g^{\mu\nu} \star \partial_{\nu}\phi \star \sqrt{-g} + \sqrt{-g} \star \partial_{\nu}\phi \star g^{\mu\nu}$ we get to

$$-\partial_{\mu}\Phi^{\mu} - m^{2}\{\ \sqrt{-g}\ ,\ \phi\}_{\star} - \xi m^{2}(\sqrt{-g} \star \phi \star R + R \star \phi \star \sqrt{-g}) = 0 \tag{14}$$

or

$$\Box_{\star} \star \phi + m^{2}\{\ \sqrt{-g}\ ,\ \phi\}_{\star} + \xi m^{2}\{\ \sqrt{-g} \mid \phi \mid R\} = 0 \tag{15}$$

where $\Box_{\star} \star \phi = \partial_{\mu}\{g^{\mu\nu} \mid \partial_{\nu}\phi \mid \sqrt{-g}\}$ and $\{A \mid B \mid C\} = A \star B \star C + C \star B \star A$

The Formal Canonical Energy – Momentum Tensor

It is useful to study the derivative the different pieces of the action with respect to $g^{\mu\nu}$. In view of the physical interpretations to be given later we shall introduce the tensor $T^{\mu\nu}$ via

$$\frac{\delta \mathbf{S}^{\theta^{0i}\neq 0}}{\delta g_{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\ T^{\mu\nu}$$

as the field symmetric energy-momentum tensor. Consider the action of field for $\theta^{0i} \neq 0$. In this case, we will remove all $\star$'s related to metric tensor, because we do not search the momentum of metric[6, 7]. In order to exhibit the dependence on the metric tensor we can write

$$\mathbf{S}^{\theta^{0i}\neq 0} = \int d^d x \sqrt{-g} \mathcal{L}_\star \tag{16}$$

variation with respect to $g_{\mu\nu}$ gives

$$\delta \mathbf{S}^{\theta^{0i}\neq 0} = \int d^d x (\delta\sqrt{-g}\ \mathcal{L}_\star(x) + \sqrt{-g}\ \delta\mathcal{L}_\star(x)) \tag{17}$$

we first perform the variation of $\sqrt{-g}$ with respect to $\delta g_{\mu\nu}$. For this we write $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g$ and observe that by varying $g_{\mu\nu}(\bar{x})$, the variation of determinant g involves the co-factors, which in fact are equal to g times the inverse, $g^{\mu\nu}$

$$\delta g = g\ g^{\mu\nu}\delta g_{\mu\nu} \tag{18}$$

moreover, due to $g^{\mu\nu}\ g_{\nu\lambda} = \delta^\mu_\lambda$, we have

$$g^{\lambda\mu}\delta g_{\mu\nu} = -g_{\nu\alpha}\delta g^{\lambda\alpha} \tag{19}$$

so that $\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$ and

$$\delta\ \sqrt{-g} = \frac{1}{2}\sqrt{-g}\ g^{\mu\nu}\ \delta g_{\mu\nu} \tag{20}$$

notice that (19) implies a change of sign[8]. Therefore, we can write the variation of $\mathbf{S}^{\theta^{0i}\neq 0} = \int d^d x\sqrt{-g}\ \mathcal{L}_\star$ as

$$\delta \mathbf{S}^{\theta^{0i}\neq 0} = \int d^d x ((\frac{1}{2}\sqrt{-g} g^{\lambda\kappa}\delta g_{\lambda\kappa})\ \mathcal{L}_\star + \sqrt{-g}\ \delta\mathcal{L}_\star) \tag{21}$$

Consider now the variation of the Lagrangian density, $\delta\mathcal{L}_\star = \frac{1}{2}\partial_\mu\phi \star \delta g^{\mu\nu} \star \partial_\nu\phi$ and write this as

$$\delta \mathbf{S}^{\theta^{0i}\neq 0} = \int d^d x ((-\frac{1}{2}\sqrt{-g}\ g_{\lambda\kappa}\delta g^{\lambda\kappa})\ \mathcal{L}_\star + \frac{1}{2}\sqrt{-g}(\partial_\mu\phi \star \delta g^{\mu\nu} \star \partial_\nu\phi)) \tag{22}$$

also the one of $\star$'s can be removed so we get to

$$\mathbf{T}_{\lambda\kappa} = g_{\lambda\kappa}\mathcal{L}_\star - \partial_\lambda\phi \star \sqrt{-g} \star \partial_\kappa\phi \tag{23}$$

and we can see the $\mathbf{T}_{\mu\nu}$ is not a symmetric tensor.

Discussion

We have considered a non-commutative theory developed in a curved background and we have studied field theory in non-commutative curved space time and we have found the motion equation of fields. We have got to the new motion equation of field that it is reduced to the motion equation of field in non-commutative flat space time when curvature will be a zero and θ is x^μ-independent. However, in continue we could construct the typical energy-momentum tensor, from general way and we have shown that it is not a symmetric tensor. In next paper, we must research for find the motion equation of new stress tensor.

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