

A Study on Charged Neutron Star in AdS_5

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ABSTRACT: Motivated by an open question raised in recent times regarding the phase transition during the collapse of a neutron star to form a black hole and related stability issues, we have constructed charged neutron stars in AdS_5 and show that these stars become unstable at a particular value of their radius, regarded as the Chandrasekhar radius. We reproduced the calculations recently done in [20] in our AdS_5 charged star. The analysis shows that the non-Fermi liquid behavior found there in AdS_4 is still true in this higher dimensional case with the presence of Kosevich-Lifshitz oscillations.

KEYWORDS: .

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1. Introduction

In recent times there has been a flurry of activity in the interplay between condensed matter physics and AdS/CFT trying to bridge the gulf that exists in between them giving rise to several interesting original ideas in this frontier (For an introduction to this field see for exmple [1, 2, 3, 4]). Recently in a series of papers [5, 6] the authors addressed the issue of a holographic description of an astrophysical phenomena i.e. the collapse of a neutron star toward the formation of a black hole beyond the so-called Oppenheimer-Volkoff (OV) limit [7]. Near the very end of their article [5], they have proposed an open question regarding the CFT meaning of the OV limit under consideration. The Neutron Star to Black Hole formation gives a hunch of a second-order quantum phase transition from a condensed matter physicist's view point turning a high density baryonic state into a thermal QGP state and thus the issue needs proper investigation as suggested in [5, 6]. The main point of investigation is rather open ended and it would be really interesting to

settle the issue whether a reliable realization of a degenerate fermionic star can indeed be achieved conclusively in the AdS/CFT duality framework. We have tried to formulate the very problem on our own way rendering hope to give a partial, if not complete, explanation of the underlying phenomena which of course needs further study.

In this note we study charged neutron stars in AdS_5 space and focus on their collapse toward the formation of a black hole. This process is particularly interesting and worth studying from the holographic point of view (see [5, 6] and references therein). This is because in terms of the boundary theory this process might as well symbolize a phase transition in AdS space.

First we present a construction of a charged star in AdS space by adopting numerical methods. Then we find that such a star ceases to exist at a particular value of the mass, charge and radius. This fact implies that this star becomes unstable at this value of the mass and charge. This instability will trigger the formation of a new phase and (presumably) the star will collapse to form a black hole. Now the behavior of a test fermion in the black hole background has recently been analyzed in [8]. It is concluded from their analysis that in this black hole phase the boundary theory has excitations about the Fermi-surface which is unlike the Fermi liquid. Further research on this point [11] has shown that the Fermi vector depends exponentially on the scaling dimension if one uses the duality to extremal RN black holes while computing the non-Fermi liquid Green function. This proved the fact of violation of Luttinger theorem in case of non-Fermi liquids which is even true in case of the extremal AdS_5 RN Black Hole. All such computations reported in [8, 10, 12, 14] revealed the presence of a Fermi surface in the bulk of a non-Fermi liquid type. For a much better and clear understanding of their results and the underlying phenomena that occurs, we thought that it might be worthwhile to explore the boundary description of a simpler system, that of an ideal Fermi gas in AdS without the presence of a Black Hole.¹

We perform a similar kind of analysis by putting in a test fermion in the bulk with the charged neutron star serving as the background. From the dynamics of such a bulk fermionic field one can compute the two point function of the boundary composite operator that corresponds to the bulk fermion. Although, we have not written down the boundary correlation function explicitly in the star background, but we gave a possible direction how to do that in practice.² We hope that, from the pole structure of these two point function, it will be evident that in the hydrodynamic limit one gets either a Fermi liquid or a non-Fermi liquid type of behavior, nevertheless the explicit computation needs to be performed. Via the AdS/CFT dictionary for fermions [9] the conformal dimension of the dual operator in the boundary CFT can be controlled by the bulk fermionic mass. In [10], it has been already found that by tuning the mass one can match the conformal dimension of the boundary operator to that of the free fermion, and thus one obtains the spectral function exhibiting a peak which is consistent with a Fermi liquid behavior. It is quite clear

¹It is worth pointing out here that in [19] the authors evaluated the fermionic correlators on a background with fermions but without a Black Hole horizon.

²The computation is difficult in the sense that the boundary theory under consideration is strongly interacting, and the neutron star to black hole collapse process is time-dependent and hence the radial holographic dimension can't be easily constructed on the CFT side.

from their analysis that one can deviate away from the Fermi liquid behavior by tuning the mass away from the free value as the mass was interpreted as the proxy for coupling in [10].

However, we recalled the fact that if one computes the fermionic correlation functions in charged AdS_5 black hole backgrounds using probe fermion and analyze the spectral peak as done in [11], one finds that the Fermi momentum depends exponentially on the probe fermion mass interpreted as a proxy for the coupling and thereby violates Luttinger theorem indicating a non-Fermi liquid behavior as was found in [8]. We then show, following [20], the presence of Kosevich-Lifshitz (KL) kind of oscillation in our AdS_5 charged neutron star setting. One must emphasize the fact that, we are working with “global AdS” i.e. the spatial sections are S^3 rather than planes which implies that the field theory under consideration is on $time \times S^3$ instead of being on Minkowski space. In [18, 19, 20] the authors studied electron stars working with a field theory defined on the Minkowski space whereas the authors of [5, 6] adopted our point of view.

2. Construction of charged neutron stars

In this section we present the construction of the charged neutron star in AdS_5 . We will work with units in which the AdS radius is set to unity. We consider the following metric ansatz

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\omega_3^2. \quad (2.1)$$

Besides the metric we have a dynamical gauge field (H) and a fermionic field (ψ) in this background. We will work in the gauge where the radial component of the gauge field is zero (i.e. $H_r = 0$). With this choice of gauge we write the gauge field as

$$H(r) = h(r)dt. \quad (2.2)$$

Further we consider fermionic species with mass m , charged under this gauge field with charge q . However, we shall treat the fermions in a hydrodynamic approximation. For this we will take the limit ³ that the number of particles per AdS radius is infinite with ϵ_F/m fixed, ϵ_F being the Fermi energy. The ideal degenerate Fermi gas formed by these particles can then be described by the following hydrodynamic stress tensor for ideal fluids

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (2.3)$$

where the pressure p and the density ρ are related to each other by the flat space equation of state for the fermions i.e.

$$p + \rho = \mu n \quad (2.4)$$

³before taking this limit we must also ensure that the central charge of the boundary CFT is taken to infinity (something similar to the large N limit for $\mathcal{N} = 4$ Yang-Mills) with all other quantities keeping fixed. This is required to ensure that all the multi-trace operators are suppressed. One way to think of this limit in the bulk is to consider N species of fermions with the same mass and then take N to infinity keeping everything else fixed.

The use of the flat space approximation is again justified as there are large number of fermions within a given AdS radius and therefore the fermions do not see the curvature of the space time. In order to be consistent with this approximation, the mass and charge of the fermions must be greater than the AdS radius (i.e. unity in our case). This equation of state is implicitly given by

$$p = \frac{(\mu - m)^3 (8m^2 + 3\mu(3m + \mu))}{1920\pi^2}, \rho = \frac{2m^5 - 5m^2\mu^3 + 3\mu^5}{480\pi^2}, \quad (2.5)$$

where μ is the chemical potential for the fermions, which can be expressed in terms of the Fermi momentum (k_F) by the relation

$$\mu = \sqrt{k_F^2 + m^2}.$$

In this flat space approximation for the fermions the particle number density (\tilde{n}) is given in terms of the volume of the Fermi surface and we have

$$\tilde{n} = \frac{(\mu^2 - m^2)^2}{128\pi^2}. \quad (2.6)$$

Then it immediately follows that the charge density (n) is given by

$$n = q\tilde{n} = q \frac{(\mu^2 - m^2)^2}{128\pi^2}. \quad (2.7)$$

Note that in our problem these relations are locally valid and generally μ, n, p and ρ are functions of r -coordinate ⁴ which we ultimately solve for. Here we would like to draw attention of the reader to a subtle point. u_μ as defined in (2.3) is a static velocity field: $u_\mu dx^\mu = A(r)dt$ and the radial profiles of p and ρ are determined by imposing the stress energy conservation which leads to the following condition:

$$\frac{dp}{dr} + \frac{1}{A} \frac{dA}{dr} (\rho + p) = 0 \quad (2.8)$$

This equation is very easy to solve. By making use of (2.4) one easily verifies the fact that (2.8) is satisfied while the chemical potential obeys

$$\mu(r) = \frac{\epsilon_F}{A(r)} \quad (2.9)$$

where at this stage ϵ_F is an arbitrary constant. Thus the radial dependence of the chemical potential μ is simply due to the gravitational redshift.

2.1 The Equations to be solved

In the above set up we now write down the dynamical equations which we must solve in order to obtain the Neutron star numerically.

⁴the symmetries of AdS prevent these quantities from becoming functions of the other coordinates

Firstly we have the Einstein equations. The two non-trivial Einstein equations are obtained from the rr-component and the tt-component and they are respectively given by

$$\begin{aligned} 3r (2A'(r) + rh'(r)^2) - 4A(r) (B(r) (r^2 p(r) + 6r^2 + 3) - 3) &= 0, \\ A(r) (6rB'(r) - 4B(r)^2 (r^2 \rho(r) - 6r^2 - 3) - 12B(r)) - 3r^2 B(r) h'(r)^2 &= 0, \end{aligned} \quad (2.10)$$

where as mentioned before we consider the pressure and the density as functions of the radial coordinate.

Then we consider the Maxwell equations. In this case the non-trivial equation stems out from the t-component (which is a mere generalization of Coulombs law). This equation is given by

$$\left(\frac{A'(r)}{A(r)} - \frac{6}{r} \right) h'(r) + \frac{B'(r)h'(r)}{B(r)} - 2B(r)n(r) - 2h''(r) = 0. \quad (2.11)$$

Finally we have to consider the equation of motion for the fermions. However, since the fermions are treated in a hydrodynamic approximation this equation is the conservation of the stress tensor⁵. In this case the radial component yields the non-trivial equation and is given by

$$\begin{aligned} A(r) (2rB(r)^2 A'(r)(p(r) + \rho(r)) + 3rB'(r)h'(r)^2 + \\ -6B(r)h'(r) (rh''(r) + 3h'(r))) + 3rB(r)A'(r)h'(r)^2 + 4rA(r)^2 B(r)^2 p'(r) &= 0. \end{aligned} \quad (2.12)$$

In these equations both $p(r)$ as well as $\rho(r)$ are present explicitly. However, we will eliminate both $p(r)$ and $\rho(r)$ in terms of $\mu(r)$ with the help of the equation of state (2.5). Now these equations are extremely non-linear and we have to resort to numerical means in order to solve them.

2.2 Solving the equations

One obvious solution to these equations is the charged black hole in AdS_5 with the chemical potential being constant throughout the space. In terms of the above mentioned functions this solution may be written as [13],

$$\begin{aligned} A(r) &= \left(1 + r^2 \left(1 - \frac{M}{r^4} + \frac{Q^2}{r^6} \right) \right), \\ B(r) &= \left(1 + r^2 \left(1 - \frac{M}{r^4} + \frac{Q^2}{r^6} \right) \right)^{-1}, \\ \mu(r) &= m, \\ h(r) &= \mu_B - \frac{Q}{r^2}. \end{aligned} \quad (2.13)$$

where m and μ_B are constants. The parameter m is related to the ADM mass of the Hole as $M = \frac{3\omega_3}{16\pi G} m$ [13] where in natural units $16\pi G = 1$ and ω_3 is the volume of the 3-sphere. Also, μ_B is the electrostatic potential difference between the horizon and infinity. Since the constant value of the chemical potential outside the star is m , the mass of the fermionic

⁵which is the relativistic version of the Navier-Stokes equation.

species, therefore we use the same notation here also. Similarly we use μ_B to denote the constant part of the gauge field as it ultimately turns out to be the boundary chemical potential.

Now it is expected that outside the neutron star our solutions should reduce to the black hole solution. Therefore, we shall obtain a solution inside the neutron star and then patch up our solution with this black hole solution outside. Thus we start with a boundary condition at the origin (which may be thought of as the centre of the star) and make a choice of the time coordinate inside the star such that at the boundary the first derivative of the field strengths match. In order to determine the consistent boundary conditions at the origin we solve the equations about $r = 0$ and find that,

$$\begin{aligned}
A(r) &= A_0 + \frac{1}{180}r^2 \left(-\frac{A_0m^5}{8\pi^2} + \frac{15A_0m^4\mu_0}{32\pi^2} - \frac{5A_0m^2\mu_0^3}{8\pi^2} + \frac{9A_0\mu_0^5}{32\pi^2} + 180A_0 \right) + O(r^3), \\
B(r) &= 1 + \frac{1}{90}r^2 \left(\frac{m^5}{16\pi^2} - \frac{5m^2\mu_0^3}{32\pi^2} + \frac{3\mu_0^5}{32\pi^2} - 90 \right) + O(r^3), \\
\mu(r) &= \mu_0 + \frac{r^2}{92160\pi^2 A_0} (32A_0m^5\mu_0 - 120A_0m^4\mu_0^2 + 160A_0m^2\mu_0^4 - 72A_0\mu_0^6 \\
&\quad - 46080\pi^2 A_0\mu_0 + 135m^4q^2 - 270m^2q^2\mu_0^2 + 135q^2\mu_0^4) + O(r^3), \\
h(r) &= \mu_B - \frac{qr^2(m^2 - \mu_0^2)^2}{1024\pi^2} + O(r^3),
\end{aligned} \tag{2.14}$$

solve the equations (2.10),(2.11) and (2.12) upto $O(r^2)$. Here A_0 and μ_0 are the values of $A(r)$ and $\mu(r)$ at the origin. These are the parameters of our problem and we have to choose values for these parameters which serve as initial values of our differential equations. Again m and q are the mass and charge of a single species of fermion. The most striking thing to note about this solution is that the value of $B(r)$ at the origin is fixed to be unity. We do not have the freedom to choose this value on independent grounds. Also the parameter q in (2.14) yield the charge of the black hole as $Q = \frac{2\sqrt{3}\omega_3}{8\pi G} q$ [13].

2.2.1 Numerical Solution of the equations

We proceed to solve these equations numerically in the following way. At first we fix the value of the chemical potential at the origin to be μ_0 . Then we fix the value of A_0 to be unity and then fix a scale for the time coordinate in the patch inside the star so as to meet the boundary conditions $A(r) = 1/B(r)$ at the radius of the star (R). The radius of the star is obtained from the value of r where the density $\rho(r)$ goes to zero or the chemical potential $\mu(r)$ goes to m as can be easily checked from the equations (2.5) and (2.13). Further from the value of $A(R)$ (which is the same as $1/B(R)$), together with the matching condition for the Electric field at $r = R$, we determine the mass and charge of the black hole solution with which we patch up outside the neutron star. Note that in this procedure, matching of the first derivative of $A(r)$ (which is the gravitational field strength in a rough sense) at $r = R$ is automatic. As a part of our choice of units we take the AdS radius to be unity.

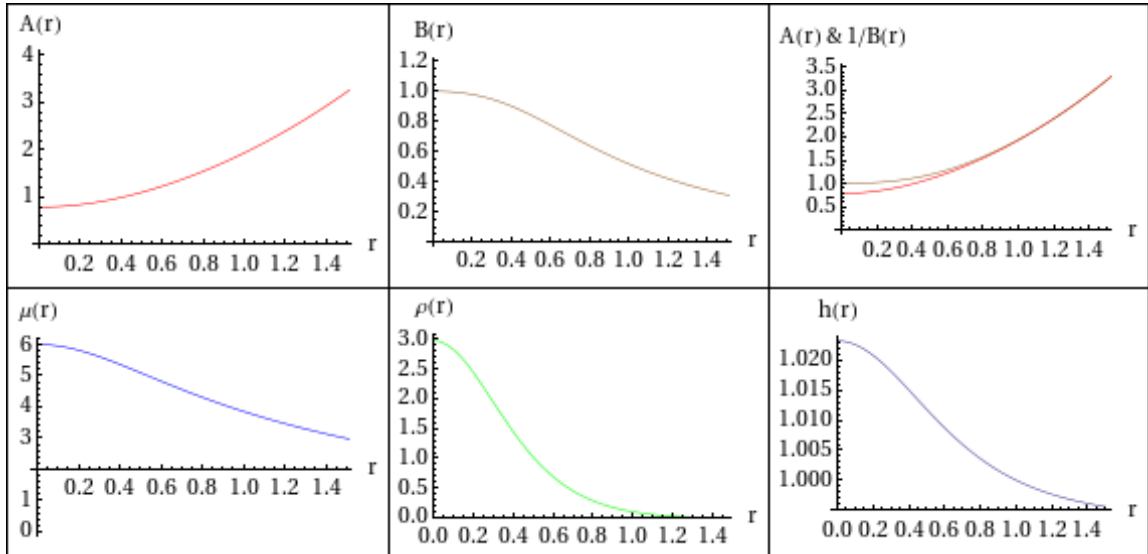


Figure 1: Solution of various functions inside the star. The values of the parameters are $m = 3$, $q = 1$ and $\mu_0 = 6$. We have shown the plots upto $r=1.48$ which is the critical radius of the star.

In fig:1, we present the solution when the parameters $m = 3$, $q = 1$ and $\mu_0 = 6$. This corresponds to the core density of 2.97. The Mass of the star is 0.073 and the charge of the star is 0.0083⁶. The radius of the star is 1.48. Note that the density function $\rho(r)$ goes to zero at the edge of the star. Also the values of $A(r)$ and $1/B(r)$ match at the end of the star.

2.2.2 A class of Neutron stars

Now we can vary the value of the chemical potential at the origin to obtain a large class of neutron stars. This amounts to obtaining neutron stars for various values of core density. Further under a variation of the core density (which varies from 0 to ∞) we can make a plot of the mass and radius of the star (see fig:2). In this plot we find there exists a maximum value of mass for the star sometimes termed as critical mass of the star.⁷ Further the solution ceases to exist at a particular value of the mass and radius. This is the signature of a critical behavior and occurs when the density at the origin approaches infinity. At this critical point the neutron star is expected to start collapsing into a black hole.

In a similar way, we can obtain a plot for the mass vs charge of the star (see fig:3). In this plot we see that for a given value of charge two solutions exist corresponding to two different masses. Presumably the one with more mass is the stable one, the other being unstable.

⁶These values of the charge and mass should not be directly compared with the mass and charge of a single fermion since in this case we should also consider the Newton's constant which we have not included in the present analysis.

⁷We can render a holographic interpretation to this critical mass, too. As found in [6] the limiting mass in the boundary CFT theory translates to a limiting conformal dimension of the composite operator made out of the fermionic primary fields which in the large N limit construct our model degenerate star.

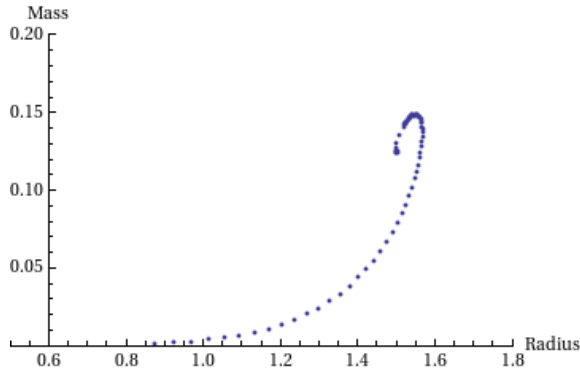


Figure 2: Plot for Mass vs the radius of the neutron star as the core density is varied. Here also we have taken $m = 3$ and $q = 1$.

We again remind the readers of the intriguing fact that working in “global AdS” we have S^3 as spatial section thus providing a scale in the field theory, which is the radius say R of S^3 . There is yet another scale in the problem, the chemical potential μ . The related phase transition from the star to a black hole occurs due to the competition between the two scales. In particular at the critical point of phase transition one would get $R_{S^3} \sim \frac{1}{\mu}$. So the very existence of R_{S^3} is crucial for our purpose.

In recent years, all the various works [8, 11, 18, 20] on Fermi surfaces consider Minkowski space while considering the boundary field theory. This was important in order to have a well-defined momentum k . On a sphere k is not a good quantum number, in particular if $R_{S^3} \sim \frac{1}{\mu}$ one might worry about whether the notion of a Fermi surface makes sense in the momentum space. The answer to this is, yes, indeed the notion can be justified. As we mentioned in Sec.2, that in our choice of units the charge and mass of the fermion are large, so that the boundary field theory fermions are locally in a flat space making sense of a Fermi momentum in approximate terms.

3. Holographic dual of the charged star

We now study the holographic description of this neutron star solution. By studying the dynamics of a probe fermion in this background one could get a hint at the boundary description of this charged neutron star.

3.1 Dirac Equation in Spherically symmetric space time

We first present the Dirac equations in spherically symmetric space times which we subsequently solve in our star background [15].⁸

We have to incorporate the interaction of the probe fermion with the fermions forming the star. This we accomplish by considering a new gauge (different from the existing one)

⁸One should pay attention to the fact that while the fermions are free in the bulk, they are not so on the boundary (i.e. they don’t obey Dirac equation on the boundary) as the boundary theory is strongly coupled.

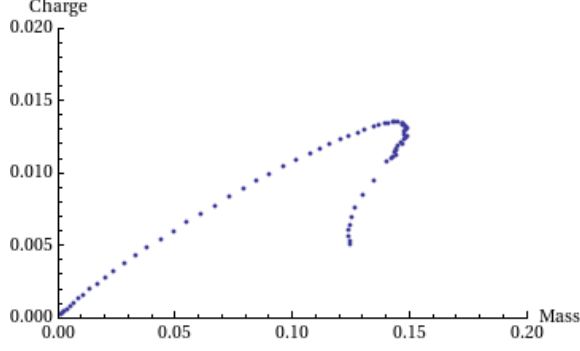


Figure 3: Plot for charge vs mass of the neutron star as the core density is varied. Here also we have taken $m = 3$ and $q = 1$

with only a non-zero time component being the chemical potential, due to the fermions forming the star. We justify this procedure in Appendix A.

We consider the metric to be of the form (as in (2.1))

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2 + \sin^2 \phi \sin^2 \theta d\lambda^2), \quad (3.1)$$

The natural choice of vielbein basis vectors are

$$e_{\tilde{t}} = \sqrt{A}; \quad e_{\tilde{r}} = \sqrt{B}; \quad e_{\tilde{\theta}} = r; \quad e_{\tilde{\phi}} = r \sin \theta; \quad e_{\tilde{\lambda}} = r \sin \theta \sin \phi. \quad (3.2)$$

where $\{\tilde{t}, \tilde{r}, \tilde{\phi}, \tilde{\theta}, \tilde{\lambda}\}$ are the tangent space coordinates. Then the non zero components of the spin connection are

$$\begin{aligned} \omega_{\tilde{t}, \tilde{r}} &= -\frac{1}{2} \frac{A'(r)}{\sqrt{A} B}; & \omega_{\tilde{r}, \tilde{\theta}} &= \frac{1}{\sqrt{B}}; \\ \omega_{\tilde{r}, \tilde{\lambda}} &= \frac{\sin \theta \sin \phi}{\sqrt{B}}; & \omega_{\tilde{\phi}, \tilde{\theta}} &= \frac{\sin \theta}{\sqrt{B}}; \\ \omega_{\tilde{\lambda}, \tilde{\lambda}} &= \cos \theta \sin \phi; & \omega_{\tilde{\phi}, \tilde{\lambda}} &= \cos \phi; & \omega_{\tilde{\theta}, \tilde{\phi}} &= \cos \theta. \end{aligned} \quad (3.3)$$

Using the above equations the Dirac equation

$$(\mathcal{D} - m)\psi = 0,$$

reduces to

$$\left(\Gamma^{\tilde{r}} \frac{1}{\sqrt{B}} \partial_r + \Gamma^{\tilde{t}} \frac{1}{\sqrt{A}} \nabla_t - \frac{1}{2} \frac{1}{\sqrt{B}} \left(\frac{A'(r)}{2A} + \frac{3}{r} \right) \Gamma^{\tilde{r}} + \frac{1}{r} \mathcal{D}^{(S^3)} - m \right) \psi = 0. \quad (3.4)$$

where $\mathcal{D}^{(S^3)}$ is the Dirac operator on the three sphere. The eigenvalues of this operator are [16]

$$\kappa = \pm i \left(\frac{3}{2} + n \right), \quad n \in \{0, 1, 2, \dots\}. \quad (3.5)$$

We shall denote the eigenfunction corresponding to these eigenvalues by f_κ . Also the ∇_t operator denote the gauge covariant derivative and contains both the gauge field and its

time derivative. All the gamma matrices in (3.4) have a tangent space index and hence are flat space gamma matrices. Here we use the following basis for the gamma matrices.

$$\Gamma^{\bar{r}} = \mathbb{I} \otimes (-\sigma_3); \quad \Gamma^{\bar{t}} = \mathbb{I} \otimes (i\sigma_1); \quad \Gamma^{\bar{i}} = \sigma^i \otimes \sigma_2. \quad (3.6)$$

where σ^i are the ordinary Pauli matrices. Further we consider the following separation of variables

$$\psi = f_\kappa \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} e^{-i\omega t}. \quad (3.7)$$

where ϕ_\pm are the eigenvectors of $\Gamma^{\bar{r}}$ with eigenvalues ± 1 . Using this separation of variables the Dirac equation (3.4) reduces to the following set of coupled first order equations

$$\begin{aligned} \left(\partial_r + \frac{1}{2} \left(\frac{A'(r)}{2\sqrt{A}} + \frac{3}{r} \right) + m\sqrt{B} \right) \phi_+ - \sqrt{B} \left(\frac{u(r)}{\sqrt{A}} - \frac{k}{r} \right) \phi_- &= 0, \\ \left(\partial_r + \frac{1}{2} \left(\frac{A'(r)}{2\sqrt{A}} + \frac{3}{r} \right) - m\sqrt{B} \right) \phi_- + \sqrt{B} \left(\frac{u(r)}{\sqrt{A}} + \frac{k}{r} \right) \phi_+ &= 0. \end{aligned} \quad (3.8)$$

where $k = i\kappa$, and $u(r) = \omega + \mu(r) - 3/2qh(r)$ ⁹.

3.2 The flow equation

In this subsection we will derive and analyze the so-called flow equation for the Dirac fermions in AdS_5 . Using the set of equations 3.8, one can define the wave function at small r to be of the form: (Note that m is the fermionic mass, and it enters in the expansion)

$$\begin{aligned} \phi_- &= \alpha (r^{mL} + \dots) + \beta (r^{-(mL+1)} + \dots), \\ \phi_+ &= \gamma (r^{-mL} + \dots) + \delta (r^{(mL-1)} + \dots), \end{aligned} \quad (3.9)$$

Here L is the AdS radius that we have taken to be unity $L = 1$. The coefficients α, β, γ and δ are what the authors of [11] call A, B, C, D in their article and they are related with one another. The retarded Green function could be written as [8, 11]

$$G_R = \epsilon^{-2m} \lim_{\epsilon \rightarrow 0} \left(\begin{array}{cc} \xi_+ & 0 \\ 0 & \xi_- \end{array} \right) \Big|_{1/\epsilon} \quad (3.10)$$

where ξ_- is defined as suitably defined ratio between ϕ_- and ϕ_+ .

$$\xi_- = -i \frac{\phi_-}{\phi_+} \quad (3.11)$$

From the system of equations, (3.8) the flow equation can be derived, dividing the second equation by ϕ_+ and inserting into it the first one, finally yielding:

$$i\partial_r \xi_- - 2i\xi_- m\sqrt{B} - \xi_-^2 \sqrt{B} \left(\frac{u(r)}{\sqrt{A}} - \frac{k}{r} \right) + \sqrt{B} \left(\frac{u(r)}{\sqrt{A}} + \frac{k}{r} \right) = 0 \quad (3.12)$$

The numerical solution of this equation has some singularity problems in r , due to the choice of boundary conditions, thus preventing its use in the calculation of the correlator in the neutron star background. This issue needs further careful investigation.

⁹Note that the unconventional factor of 3/2 is present because here we are using a different normalization of charge.

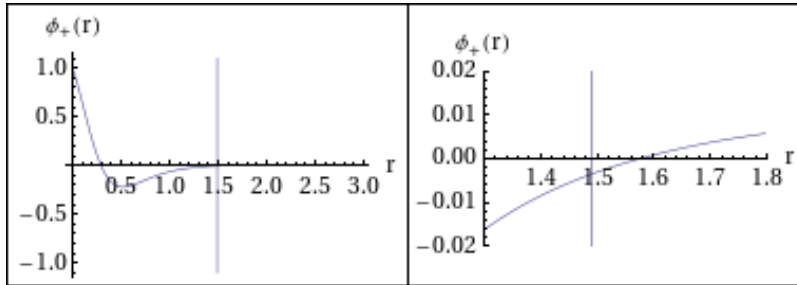


Figure 4: Plot of ϕ_+ in the star background constructed in the previous section. The vertical line denotes the value of r for the edge of the star. Here the values of the parameters are $m = 3$, $q = 1$, $\mu_B = 1$, $k = 3/2$, and $\omega = 3$. In the right image we focus on the edge where the first derivative of the function is continuous.

3.2.1 The initial conditions

Now we have to specify the boundary conditions for the equations (3.8). This is done by demanding regularity of the solution near the origin ($r = 0$). This regularity criterion in general depends on the value of k once we fix the mass and charge of the fermion. The lowest positive value is $k = 3/2$, in which we will focus for our present purpose. In this case the regularity at the origin demands, that if ϕ_+ is 1 at the origin then ϕ_- should be -1 i.e.

$$\phi_+(r = 0) = 1 \quad ; \quad \phi_-(r = 0) = -1$$

We shall use this boundary condition to solve the equations (3.8).

3.2.2 Numerical solution of the Dirac equation

The solution of the probe fermion in the neutron star background is obtained numerically for $m = 3$, $q = 1$ and $k = 3/2$. It is shown in figs:4 & 5. Here we use the boundary conditions as discussed in §3.2.1. We at first let the differential equation evolve to obtain a solution up to the end of the star and subsequently use the value of ϕ_{\pm} at this edge as the initial value for the subsequent evolution outside the star. The first derivative match of the solution in these two patches is automatic (see figs:4 & 5) and directly follows from the fact that all the functions that appear in the equations (3.8) are continuous at the edge of the star. One possible direction, one can take from here, is that, one might try to compute the boundary correlation function in the presence of the star and understand the behavior of the bulk Fermi surface in terms of the boundary correlators, taking the same route outlined in the formalism of [17]. A noteworthy point here, is that the only difference, we should care about, is that we need to evaluate the fermionic correlators in the neutron star background, whereas there exist large number of literature doing the same computation in the background of a charged black hole (See for example [8, 10, 11, 12, 14].)

3.3 Kosevich-Lifshitz oscillations in AdS_5 neutron star

It has been shown recently in [20] that, the charged star shares some important features

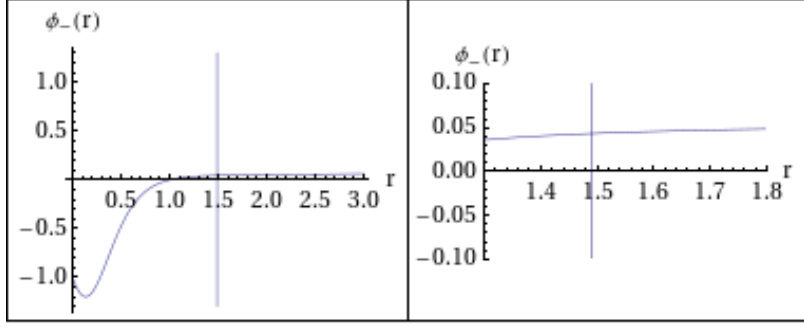


Figure 5: Plot of ϕ_- in the star background constructed in the previous section. Here the values of the parameters are also $m = 3$, $q = 1$, $\mu_B = 1$, $k = 3/2$, and $\omega = 3$. Again the first derivative of ϕ_- is continuous as can be seen from the plot on the left.

with the Fermi liquid. In particular, the presence of the Kosevich-Lifshitz oscillations [18] in magnetic field has been shown in the case of AdS_4 charged fermion star. As said in [20] the arguments used to demonstrate this feature could be generalized to higher dimensions, showing that even in AdS_5 Kosevich-Lifshitz oscillations are present, nevertheless the Luttinger theorem is violated. In order to calculate the magnetic susceptibility $\chi = -\frac{\partial^2 \Omega}{\partial B^2}$ we can use the relation between the free energy and the total charge in the dual CFT as in [19, 20]:

$$\hat{\Omega} \propto \hat{Q} = \int_0^{r_s} \sqrt{g(s)} r^3 \sigma(r) dr \quad (3.13)$$

where in general the charge density is

$$\sigma = \int_m^\mu g(E) \nu(E) dE \quad (3.14)$$

and the density of states in the full theory is

$$g(E) = \beta E (E^2 - m^2 - \ell B) \quad (3.15)$$

The easiest way to generalize the argument as presented in [20] to the higher dimensional case is to add an extra dimension, i.e. by considering the cylindrical-like metric

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\omega_2 + dz^2 \quad (3.16)$$

and then the integral expressing the charge density is generalized in AdS_5 just by adding an integration over p_z

$$\sigma = \frac{\beta B}{2} \int_{-\infty}^{\infty} dp_z \sum_l \int_{-\infty}^{\infty} dp \sum_l \frac{1}{1 + e^{\frac{E(l,p,p_z) - \mu}{T}}} \quad (3.17)$$

The energy, as in the AdS_4 case, is

$$E(l, p, p_z) = \sqrt{p^2 + p_z^2 + \ell B_{loc}} \quad (3.18)$$

The local physical quantities are related to the dual field theory quantities as [20]:

$$\begin{aligned} B_{loc} &\propto \frac{1}{r^2} \hat{B} \\ T_{loc} &= \frac{\hat{T}}{\sqrt{g_{tt}}} \end{aligned} \quad (3.19)$$

In the following calculation we will drop out this subscript, and restore the correct dependence on physical quantities back to position at a later point. The Poisson resummation over Landau levels gives

$$\sigma = \frac{\beta B}{2} \sum_k \int_0^\infty dl \int_{-\infty}^\infty dp_z \int_{-\infty}^\infty dp \sum_l \frac{1}{1 + e^{\frac{E(l,p,p_z) - \mu}{T}}} \quad (3.20)$$

Now expanding the Fermi distribution in terms of Matsubara frequencies and transforming the integral over ℓ into an integral over energies the integral (3.20) becomes

$$\sigma = -\frac{\beta T}{2} \sum_{k,n} \int_{-\infty}^\infty dp_z \int_{-\infty}^\infty dp e^{-\pi i k(p^2 + p_z^2 + m^2)/B} \int_{\sqrt{p^2 + p_z^2 + m^2}}^\infty dE E \frac{e^{\pi i k E^2/B}}{E - \mu(1 + i \frac{T}{\mu} 2\pi(n + \frac{1}{2}))} \quad (3.21)$$

This integral could be performed rotating the integration path so that the exponential in the numerator becomes real and decreasing, i.e. considering the rotation of the energy as $E \rightarrow E e^{i\pi/4}$ if $k > 0$ or $E \rightarrow E e^{-i\pi/4}$ if $k < 0$, and remembering that the lower integration limit is set to $\sqrt{p^2 + p_z^2 + m^2}$. In this analytical continuation one has to take into account the role of the poles given by Matsubara frequencies. These poles contribute only if $p^2 + p_z^2 \leq \mu^2 - m^2$ and in addition $n + 1/2 > 0$ if $k > 0$ and $n + 1/2 < 0$ if $k < 0$. The contribution of the poles is the one giving the oscillatory part of the charge density σ_{osc} , and thus it's the only one we will consider next, hence through the Residue theorem then the σ_{osc} is

$$\sigma_{osc} = 2\pi\beta T\mu \operatorname{Im} \sum_{k,n>0} e^{i\pi k(\mu^2 - m^2)/B} e^{-4\pi^2 k(n - \frac{1}{2})T\mu/B} \iint_D dp_z dp e^{-i\pi k(p^2 + p_z^2)/B} \quad (3.22)$$

while the last integration is made in the region D , being the circle of radius $\sqrt{\mu^2 - m^2}$ in the plane (p, p_z) . This expression could be inserted in (3.13), thus obtaining the oscillating part of the free energy

$$\begin{aligned} \hat{\Omega}_{osc} \propto \hat{Q}_{osc} &= 2\pi\beta T\mu \operatorname{Im} \sum_{k,n>0} \int_0^{r_s} dr r^3 \sqrt{g(r)} e^{i\pi k(\mu^2 - m^2)/B} e^{-4\pi^2 k(n - \frac{1}{2})T\mu/B} \times \\ &\times \iint_D dp_z dp e^{-i\pi k(p^2 + p_z^2)/B} \end{aligned} \quad (3.23)$$

Since $\mu \gg B$, a saddle point approximation could be done. The r dependence of the fast oscillating exponential is

$$e^{i\pi k(\mu^2 - m^2)/B} = \exp i\pi k \left(\frac{\hbar^2}{f} - m^2 \right) \frac{r^2}{\hat{B}} = \exp ik \frac{A(r)}{\hat{B}} \quad (3.24)$$

With this approximation the integral (3.13) becomes

$$\hat{Q}_{osc} = 2\pi\beta \operatorname{Im} \sum_{k,n>0} r^* \sqrt{g(r^*)} \frac{m^* e^{i\pi/4} \sqrt{\hat{B}}}{\sqrt{A''(r^*)}} e^{i\pi k \frac{A(r^*)}{B}} e^{-4\pi^2 k(n-\frac{1}{2})m^*T/B} \times \iint_D dp_z dp e^{-i\pi k(p^2+p_z^2)/B} \quad (3.25)$$

where, as usual, r^* is the value of the radius where the argument of the exponential has an extremum. The last integration could be done passing to circular coordinates in the (p, p_z) plane inserting the radial coordinate $\rho = \sqrt{p^2 + p_z^2}$ as

$$\iint_D dp_z dp e^{-i\pi k(p^2+p_z^2)/B} = \int_0^b \rho d\rho e^{-i\pi k\rho^2/B} = e^{-i\frac{b^2\pi}{2B}} \frac{\sin \frac{\pi b^2}{2B}}{\pi} \quad (3.26)$$

where $b = \sqrt{\mu^2 - m^2}$ has been defined. The oscillatory part of the total charge is then obtained introducing this result in (3.25). The summation over the index n could be done as usual, giving an hyperbolic sine, moreover the sum over k could be suppressed considering only the term $k = 1$, since the other terms are negligible. In the end after all dusts get settled the oscillating free energy is

$$\begin{aligned} \hat{\Omega} \propto \hat{Q}_{osc} &= 2\pi\beta \operatorname{Im} \left[\frac{1}{r^*} \sqrt{g(r^*)} \frac{m^* \hat{B}^{3/2}}{\sqrt{A''(r^*)}} e^{i\pi \frac{A(r^*)}{B}} e^{-i\frac{b^2\pi}{2B}} \frac{\sin \frac{\pi b^2}{2B}}{\pi} \sum_{n>0} e^{-4\pi^2(n-\frac{1}{2})m^*T/B} \right] = \\ &= F(r^*) \frac{\hat{B}^{\frac{5}{2}}}{\sqrt{A''(r^*)}} \frac{2\pi m^*T/B}{\sinh 2\pi m^*T/B} \operatorname{Im} \left[e^{i\pi/4} e^{i\pi \frac{A(r^*)}{B}} e^{-i\frac{A(r^*)\pi}{2B}} \right] \sin \frac{\pi A(r^*)}{2B} = \\ &= F(r^*) \frac{\hat{B}^{\frac{5}{2}}}{\sqrt{A''(r^*)}} \frac{2\pi m^*T/B}{\sinh 2\pi m^*T/B} \left[\frac{\sqrt{2}}{2} - \cos \left(\frac{\pi A(r^*)}{B} - \frac{\pi}{4} \right) \right] \end{aligned} \quad (3.27)$$

The function $F(r^*)$ contains all the factors coming from the integration over the radial coordinates, like $\sqrt{g(r^*)}$ and the numerical prefactors. As can be seen, the expression found agrees with the standard result for three dimensional systems, apart from an additional and unimportant offset of the oscillations. In particular the usual scaling dimension with the magnetic field is found, i.e. $\hat{\Omega} \propto B^{5/2}$ [21] and the same frequency dependence of oscillations on the extremal section of the Fermi volume is observed.

4. Discussions

In a nutshell the findings of our paper is as follows: We built, using numerical means, a charged neutron star in the AdS_5 and analyzed in detail the related gravitational collapse to form a black hole. We depicted the critical behavior of the degenerate star with few plots and then solved the Dirac equation in the spherically symmetric star geometry. We wrote down a flow equation governing the dynamics of fermions constructing the charged neutron star in AdS_5 . We encountered few problems while computing the boundary correlation function in the presence of the star, we list the difficulties that arise in finding the fermionic

correlator in the star background. In a sense, according to [5, 6], it is an open question to find a reliable realization of the degenerate star using AdS/CFT duality and find a CFT interpretation of the OV limit [7] in the context of collapse phenomena. Here we note that, if one computes the retarded Green function 3.10 in the charged AdS_5 extremal RN Black Hole background then, according to [11],

$$k_F \approx k_F(0) \exp(-\sigma m L) \quad (4.1)$$

i.e. one observes that the Fermi momentum fall off exponentially from the value it takes at zero fermion mass with $k_F(0) \simeq 0.8155$ and $\sigma \simeq 0.80$ for AdS_5 geometry, clearly indicating a violation of Luttinger's theorem and hence proving a non-Fermi liquid behavior in the black hole phase. This initially prompted us to make a conclusion that one possible holographic interpretation one could dub to this collapse process is that when a neutron star collapse to form a black hole, seeing from a boundary point of view one finds a second order phase transition from an ideal degenerate Fermi gas to a non-Fermi liquid. Recently in a paper [20] the authors have found that the Kosevich-Lifshitz oscillations still persist in the charged neutron star phase. In the same article the authors also explain how to reconcile the violation of Luttinger theorem in the case of the charged fermion star and the existence of the KL oscillation and the argument is still true in our AdS_5 case : only fermions in the spherical shell of radius r^* contribute to the oscillations, and thus it's not possible to reconstruct the whole Fermi volume from the analysis of oscillations. From the boundary field theory point of view this means that not all the degrees of freedom are taken into account through quantum oscillations, and this results in the violation of Luttinger theorem. Reproducing the calculations done in [20] in our AdS_5 charged star, we have found again the appearance of the KL oscillations due to magnetic field consistent with the standard results for three dimensional systems (see [21]). After all these, it is still not clear how to realize a holographic dual of the AdS_5 star, although it seems to us that both the star and the black hole are non-Fermi liquid states of matter.

Finally, despite trying to give a concrete holographic picture of this underlying phenomena we tried to rethink from a different perspective, all the existing results on this subject and did a coherent study of all of them reaching a conclusion that this elusive phenomena still require further studies.

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A. The zero temperature and finite chemical potential two point function in flat space

In our analysis above we have used the crucial fact that the effect of finite chemical potential

at zero temperature is captured by introducing a gauge field whose time component is the chemical potential (all other components being zero). In this section we shall try to justify this statement by considering fermions in flat space with a constant potential. We shall do this by considering the two point function of the fermions. At first we shall view, the introduction of the chemical potential, as a redefinition of the vacuum state of theory of free fermions. Then from there we shall demonstrate that the two point function computed in this new vacuum state is the same as that computed for the case where the chemical potential is introduced through the time component of the gauge field. The later method more elegant and easy to generalize. In fact, we have used this above in more complicated settings where the chemical potential has a spatial variation. Therefore, in order to be certain,¹⁰ we present an analysis of the situation in this simple setting and verify the equivalence of the two methods.

A.1 The free Fermi sea: The operator calculation

In this subsection we consider free massive fermions in flat space with the fermions being filled upto the fermi level with fermi momentum k_F . Thus the chemical potential¹¹ of the system is non-zero and is given by

$$\mu = \sqrt{k_F^2 + m^2},$$

with m being the mass of the fermion. We consider the system to be at zero temperature. We will be interested to calculate the Feynman propagator or the time ordered two point correlator of this system. We shall perform this through an operator calculation in which the two point function is the expectation value of a product of two field operators in a state. This state is the one in which all the single particle states upto the Fermi momentum is filled up.

We consider free fermions in flat space with the system being described by the Dirac Lagrangian [22] This theory is a quadratic theory and can be solved exactly. The fields ψ and $\bar{\psi}$ can be expanded in terms of the creation and annihilation operators as follows

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) \exp(-ip \cdot x) + b_p^{s\dagger} v^s(p) \exp(ip \cdot x) \right), \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger} \bar{u}^s(p) \exp(ip \cdot x) + b_p^s \bar{v}^s(p) \exp(-ip \cdot x) \right). \end{aligned} \tag{A.1}$$

Now we shall incorporate the presence of a finite chemical potential by constructing a state in which the fermions are filled upto the energy equal to the chemical potential,

¹⁰This fact is certainly true and is very well known for finite temperature. In case of finite temperature the chemical potential can be introduced but putting a twisted boundary condition for the fermion (instead of a mere anti-periodic one) along the compactified time circle. It is very well known that such twist can be undone by a gauge field whose time component is the chemical potential. This implies that the chemical potential can also be introduced through such a gauge field without putting the twist. What we verify here is that this fact continues to hold even at zero temperature.

¹¹Here we shall define the chemical potential to be the energy required to add one more particle at the fermi momentum

characterized by the fermi momentum. Let us denote the state in which the fermions are filled upto the fermi level by $|k_F\rangle$ such that,

$$|k_F\rangle = \prod_p^{k_F} \sqrt{2E_p} a_p^{s\dagger} |0\rangle. \quad (\text{A.2})$$

Now for the time ordered correlation function we are required to calculate the quantities $\langle k_F | \psi(x) \bar{\psi}(y) | k_F \rangle$ and $\langle k_F | \bar{\psi}(y) \psi(x) | k_F \rangle$. These quantities evaluate to

$$\begin{aligned} \langle k_F | \psi(x) \bar{\psi}(y) | k_F \rangle &= \int_{k_F}^{\infty} \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} + m) \exp(-p \cdot (x - y)), \\ \langle k_F | \bar{\psi}(y) \psi(x) | k_F \rangle &= \int_{k_F}^{\infty} \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} - m) \exp(-p \cdot (y - x)) \\ &\quad + \int_0^{k_F} \frac{d^3 p}{(2\pi)^3} ((\not{p} + m) \exp(-p \cdot (x - y)) \\ &\quad \quad + (\not{p} - m) \exp(-p \cdot (y - x))). \end{aligned} \quad (\text{A.3})$$

Now the time-ordered two point correlation function is given by

$$S_F(x - y) = \begin{cases} \langle k_F | \psi(x) \bar{\psi}(y) | k_F \rangle, & \text{for } x_0 > y_0, \text{ (close the contour below),} \\ -\langle k_F | \bar{\psi}(y) \psi(x) | k_F \rangle, & \text{for } y_0 > x_0, \text{ (close the contour above).} \end{cases} \quad (\text{A.4})$$

Now the above two point function can be captured in the contour integral

$$S_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2} \exp -ip \cdot (x - y). \quad (\text{A.5})$$

where the contour for p_0 is chosen as shown in fig:6. Nevertheless, the same answer can be obtained with the usual contour prescription (the dotted red line in fig:6) if we include a real shift of μ in p_0 . Therefore it is convenient to define the shifted variable,

$$\tilde{p}_0 = p_0 - \mu$$

and make this substitution in (A.4). Then in terms of this shifted variable \tilde{p}_0 the contour prescription is the usual one. Now in terms of this shifted variables, the effect of μ is completely captured as we introduce a gauge field whose time component is μ . This is true if we consider the eigenvalues of the ∂_t operator to be \tilde{p}_0 , instead of p_0 . This justifies our use chemical potential as the time component of the gauge field.

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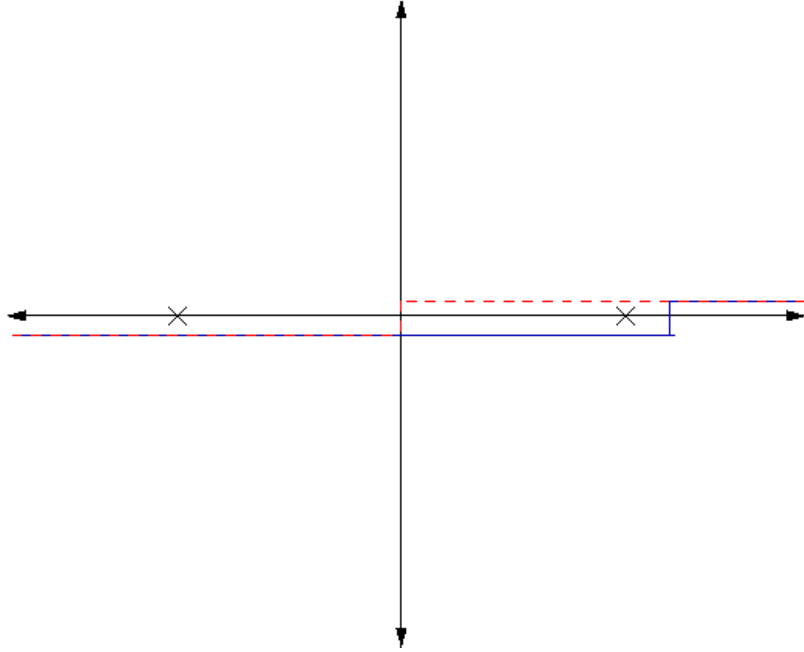


Figure 6: The contour prescriptions. The blue line is the new contour prescription in the presence of a finite constant chemical potential. The red dotted line represents the usual contour prescription.

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