Perturbative expansion of $\mathcal{N} < 8$ Supergravity

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Abstract

We characterise the one-loop amplitudes for $\mathcal{N} = 6$ and $\mathcal{N} = 4$ supergravity in four dimensions. For $\mathcal{N} = 6$ we find that the one-loop n-point amplitudes can be expanded in terms of scalar box and triangle functions only. This simplification is consistent with a loop momentum power count of $n-3$, which we would interpret as being $n+4$ for gravity with a further -7 from the $\mathcal{N} = 6$ superalgebra. For $\mathcal{N} = 4$ we find that, in contrast to previous studies, the amplitude is consistent with a loop momentum power count of n, which we would interpret as being $n + 4$ for gravity with a further -4 from the $\mathcal{N} = 4$ superalgebra. Specifically the $\mathcal{N} = 4$ amplitudes contain non-cut-constructible rational terms.

1. Introduction

Superficially the perturbative expansion of gravity scattering amplitudes [1] is extremely complicated and power counting suggests the theory is plagued with ultra-violet divergences. However, there is growing evidence that the ultra-violet behaviour of gravity theories is significantly softer than expected. The bulk of this evidence has arisen from studies of explicit on-shell scattering amplitudes rather than formal structures. The underlying drivers for this behaviour remain unclear. Even at tree level surprises have recently been noted: the large momentum behaviour of tree scattering amplitudes has a softer behaviour than expected [2–5] and a rich structure of relationships between the tree amplitudes has been uncovered [6–13], which go beyond the well known KLT relations [14].

At loop level, the softest theory is expected to be maximally supersymmetric $\mathcal{N} = 8$ supergravity [15] . Reexaminations of the perturbative expansion of $\mathcal{N} = 8$ have uncovered evidence that this theory has a softer UV structure than previously thought [16]. Explicit calculations of physical scattering amplitudes have shown that the four-graviton amplitude is finite at two [17], three [18, 19] and four loops [20]. In particular, the results indicate cancellations between diagrams beyond these explicit in any known formalism. At one-loop $\mathcal{N} = 8$ amplitudes for arbitrary numbers of external gravitons have been shown to have a very restricted form, to $O(\epsilon)$:

$$A = \sum_i c_i I_4^i \tag{1.1}$$

where I_4^i are scalar box-functions and c_i are rational coefficients [21–23]. This "no-triangle hypothesis" [24] must result from a much stronger cancellation within supergravity theories than previously thought and has been checked by explicit computations up to seven points [21–24] and proven within a string-based rules formalism [25] . Both of these calculations indicate that in the UV limit the behaviour of $\mathcal{N} = 8$ supergravity tracks that of $\mathcal{N} = 4$ super-Yang-Mills. This opens the possibility that $\mathcal{N} = 8$ supergravity is a finite quantum field theory of gravity. There is no evidence to the contrary at this point.

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In [26] and implicitly in [24] the source of these cancellations was examined. When calculating a one-loop amplitude in a general gravity theory we sum over diagrams. Let m be the number of legs attached to the loop, $m \leq n$. We expect loop momentum integrals of the form

$$I_m[P^{2m}[\ell]] \tag{1.2}$$

where $P^{2m}[\ell]$ is a polynomial of degree $2m$ in the loop momentum ℓ. Cancellations between diagrams can reduce the effective degree of the loop momentum polynomial. We denote this effective degree by d_{eff}. The traditional expectation within supergravity theories is that cancellation between particle types within a supermultiplet reduces the degree of the loop momentum polynomial from $2m$ to $d_{\text{eff}} = 2m - r$, where r depends upon the degree of supersymmetry. For maximal supergravity $r = 8$ [27, 28] is manifest within the "string-based rules" method. However the no-triangle hypothesis indicates that further cancellations arise, resulting in $d_{\text{eff}} = m - 4$. This suggests a degree of $m + 4$ (rather than $2m$) for pure gravity, reduced by 8 by the $\mathcal{N} = 8$ supersymmetry. In this article we explore the perturbative expansion of $\mathcal{N} = 6$ and $\mathcal{N} = 4$ supergravity theories to examine their UV behaviour. A starting hypothesis for the reduction in the degree of the loop momentum polynomial is

$$d_{\text{eff}} = (m + 4) - r \tag{1.3}$$

where $r = 4$ for $\mathcal{N} = 4$ supergravity and $r = 6$ for $\mathcal{N} = 6$ supergravity. To understand the implications of this for the structure of these amplitudes, we recall that a general one-loop amplitude in a theory of massless particles can be expressed, after a Passarino–Veltman reduction [29], in the form

$$A_n^{\text{one-loop}} = \sum_{i \in \mathcal{C}} a_i\, I_4^i + \sum_{j \in \mathcal{D}} b_j\, I_3^j + \sum_{k \in \mathcal{E}} c_k\, I_2^k + R_n\,, \tag{1.4}$$

where the I_f are f-point scalar integral functions and the a_i etc. are rational coefficients. R_n is a purely rational term. For $d_{\text{eff}} \geq n$ we expect this full generic form, while for $d_{\text{eff}} < n$ the rational term is absent, for $d_{\text{eff}} \leq n - 3$ the bubbles I_2 are also absent and for $d_{\text{eff}} \leq n - 4$ only the box functions appear.

For $\mathcal{N} = 6$ our explicit calculations indicate $d_{\text{eff}} = n - 3$, i.e. $r = 7$. Compared with (1.3) there is an extra reduction in the power count by one for $\mathcal{N} = 6$ amplitudes, giving them a simplified expansion:

$$M_n^{\text{one-loop},\mathcal{N}=6} = \sum_{i \in \mathcal{C}} a_i\, I_4^i + \sum_{j \in \mathcal{D}} b_j\, I_3^j. \tag{1.5}$$

This is consistent with the expectations of [25, 26]. For $\mathcal{N} = 4$ we find amplitudes consistent with $d_{\text{eff}} = n$, implying that $r = 4$ and $R_n \neq 0$ in eq. (1.4). This contradicts previous expectations [25, 26]. The evidence for this, together with a discussion of the implications, will form the remainder of this article.

2. IR consistency and Choice of Integral Function Basis

For one-loop amplitudes IR consistency imposes a system of constraints on the rational coefficients of the integral functions. For the matter multiplets [30] there are in fact no IR singular terms in the amplitude, so the singular terms in the individual integral functions cancel. This gives enough information to fix the coefficients of the one- and two-mass triangles in terms of the box coefficients. The three-mass triangle is IR finite, so its coefficient is not determined by these constraints. It is convenient to combine the boxes and triangles in such a way that these infinities are manifestly absent. There are several ways to do this [31–34], here we choose to work with truncated box functions

$$I_4^{\text{trunc}} = I_4 - \sum_i \alpha_i \frac{(-s_i)^{-\epsilon}}{\epsilon^2} \tag{2.1}$$

where the α_i and s_i are chosen to make $I_4^{\rm trunc}$ IR finite. This effectively incorporates the one- and two-mass triangles together with the box integral functions. Using these truncated boxes, the coefficients of the one and two-mass triangles vanish and the amplitudes can be written as

$$M_n^{\text{one-loop}} = \sum_{i\in\mathcal{C}} a_i\, I_4^{i,\text{trunc}} + \sum_{j\in\mathcal{D}'} b_j\, I_3^{j,\text{3-mass}} + \sum_{k\in\mathcal{E}} c_k\, I_2^k + R_n\,, \tag{2.2}$$

with the single additional constraint $\sum c_k = 0$.

3. $\mathcal{N}=6$ one-loop amplitudes

At one-loop our $\mathcal{N}=6$ supergravity theory is specified by its particle content and tree amplitudes. There are two possible multiplets: the vector multiplet and the matter multiplet, with particle contents as follows:

Helicity	2	3/2	1	1/2	0	−1/2	−1	−3/2	−2
vector	1	6	16	26	30	26	16	6	1
matter	0	1	6	15	20	15	6	1	0

The contributions to the one-loop n-graviton scattering amplitude from the two $\mathcal{N}=6$ multiplets satisfy

$$M^{\mathcal{N}=6,\text{vector}} = M^{\mathcal{N}=8} - 2M^{\mathcal{N}=6,\text{matter}}. \tag{3.1}$$

As $M^{\mathcal{N}=8}$ is known, it is sufficient to compute the contribution from the matter multiplet alone.

3.1. MHV amplitudes

The one-loop n-point MHV amplitude[1] in $\mathcal{N}=8$ supergravity is [22]

$$\begin{aligned} &M_n^{\text{one-loop},\mathcal{N}=8}(1^-,2^-,3^+,\ldots,n^+) = \\ &\qquad \frac{(-1)^n}{8}\,\langle 1\,2\rangle^8 \sum_{\substack{1\le a<b\le n\\ M,N}} h(a,M,b)h(b,N,a)\,\mathrm{tr}^2[a\,M\,b\,N]\,I_4^{aMbN} \;+\; \mathcal{O}(\epsilon)\,, \end{aligned} \tag{3.2}$$

where $h(a,M,b)$ are the "half-soft" functions of ref. [22] and I_4^{aMbN} are the "two-mass-easy" scalar box functions with massless legs a and b and massive clusters M and N. The summation includes the degenerate cases where M or N reduce to a single massless leg. The half-soft functions have the explicit form

$$\begin{aligned} h(a,\{1,2,\ldots,n\},b) \equiv{}& \frac{[1\,2]}{\langle 1\,2\rangle}\frac{[3|K_{12}|a\rangle[4|K_{123}|a\rangle\cdots[n|K_{1\cdots n-1}|a\rangle}{\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle n-1,\,n\rangle\;\langle a\,1\rangle\,\langle a\,2\rangle\,\langle a\,3\rangle\cdots\langle a\,n\rangle\;\langle 1\,b\rangle\,\langle n\,b\rangle} \\ &+\mathcal{P}(2,3,\ldots,n), \end{aligned} \tag{3.3}$$

where we are using the usual spinor products $\langle j\,l\rangle \equiv \langle j^-|l^+\rangle = \bar u_-(k_j)u_+(k_l)$ and $[j\,l] \equiv \langle j^+|l^-\rangle = \bar u_+(k_j)u_-(k_l)$, and where $[i|K_{abc}|j\rangle$ denotes $\langle i^+|\not{K}_{abc}|j^+\rangle$ with $K^\mu_{abc} = k^\mu_a + k^\mu_b + k^\mu_c$ and $s_{ab} = (k_a+k_b)^2$, etc.

The $\mathcal{N}=6$ matter multiplet's contribution to one-loop n-point MHV amplitudes has vanishing three-mass triangle coefficients. The bubble coefficients also vanish as explicitly shown in appendix Appendix A. Considering the rational terms, R_n, the existence of an overall $d_{\rm eff}$ that ensures that the bubble coefficients vanish would also ensure the vanishing of R_n. Additionally,

[1]For clarity we suppress a factor of $i\kappa^{n-2}$ in each tree amplitude and $i\kappa^n/(4\pi^2)$ in each one-loop amplitude.

power counting in the string-based rules [27, 28] gives $R_4 = R_5 = 0$, and if we assume R_n could be recursively generated from R_{n-1}, this would be sufficient to ensure $R_n = 0$ for all n.

Consequently these contributions can be expressed purely as sums of truncated boxes with a single negative helicity leg in each massive corner, as shown in fig. 1. The box coefficients may be determined using unitarity methods [31] including quadruple cuts [35]. To use quadruple cuts we require the MHV tree amplitudes for $n-2$ gravitons and a pair of particles of helicity $\pm h$

$$M(1^-, 2^{-h}, 3^{+h}, 4^+ \cdots n^+) = \left(\frac{\langle 1\,3\rangle}{\langle 1\,2\rangle}\right)^{2h-4} M(1^-, 2^-, 3^+, 4^+ \cdots n^+) \tag{3.4}$$

where the MHV amplitudes from n-gravitons are given in [36]. We find the box-coefficients are related to the maximally supersymmetric case by simple factors, as in QCD [32],

$$\begin{aligned} &M_n^{\mathcal{N}=6,\mathrm{matter}}(1^-, 2^-, 3^+, \ldots, n^+) = \\ &\quad \frac{(-1)^n}{8} \langle 1\,2\rangle^8 \sum_{\substack{2<a<b\le n \\ 1\in M, 2\in N}} \left(\frac{\langle 1\,a\rangle \langle 2\,a\rangle \langle 1\,b\rangle \langle 2\,b\rangle}{\langle a\,b\rangle^2 \langle 1\,2\rangle^2}\right) h(a,M,b) h(b,N,a)\, \mathrm{tr}^2[a\,M\,b\,N]\, \mathcal{I}_4^{aMbN,\mathrm{trunc}} \end{aligned} \tag{3.5}$$

This gives an all-n expression for the amplitude consistent with a loop momentum power count of $n-3$ in agreement with previous results.

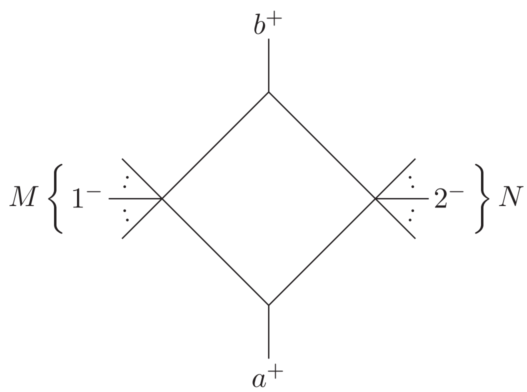

Figure 1: The box functions appearing in the $\mathcal{N}=6$ MHV one-loop amplitude

3.2. Six-point NMHV

The six-point next-to-MHV (NMHV) amplitude contains several features that are not present in the MHV amplitudes: in addition to the one-mass truncated boxes the amplitude also contains two-mass-hard truncated boxes and three-mass triangles.

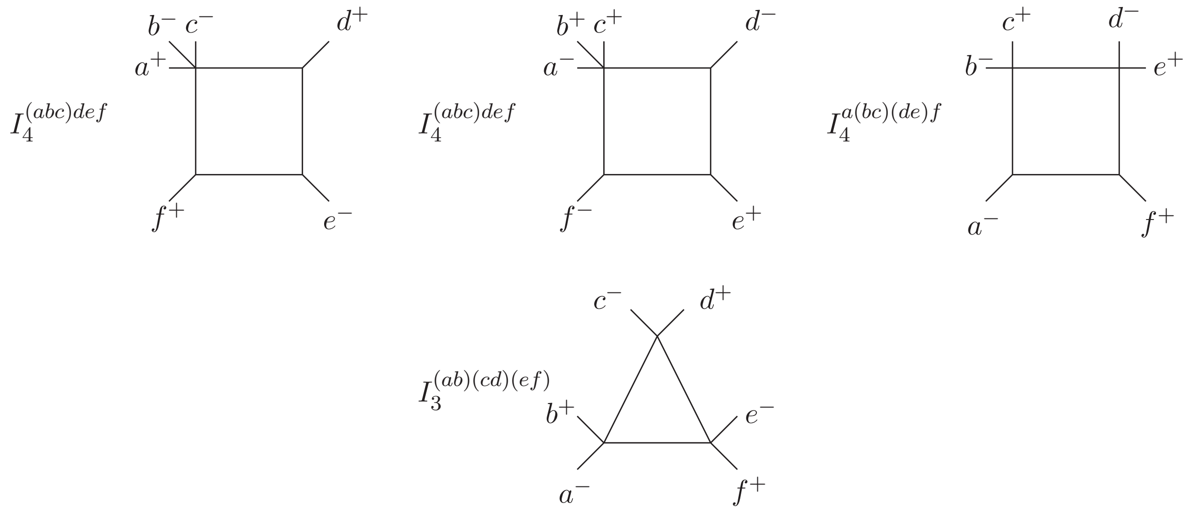

Figure 2: The box-functions appearing in the NMHV six-point one-loop amplitude

In terms of these integral functions the amplitude is,

$$
\begin{aligned}
&\mathcal{M}_6^{\mathcal{N}=6,matter}(1^-,2^-,3^-,4^+,5^+,6^+) = \\
&\sum_{(abd)\in P_3(123);(cef)\in P_3(456)} c^{a(bc)(de)f}\, I_4^{a(bc)(de)f,\mathrm{trunc}} + \sum_{(adf)\in P_3(456);(bcd)\in P_3(123)} c_{\mathcal{N}=6}^{(abc)def}\, I_4^{(abc)def,\mathrm{trunc}} \\
&+ \sum_{(adf)\in P_3(123);(bcd)\in P_3(456)} \bar{c}_{\mathcal{N}=6}^{(abc)def}\, I_4^{(abc)def,\mathrm{trunc}} + \sum_{(bde)\in P_3(456)} c_{\mathcal{N}=6}^{(1b),(2d),(3e)}\, I_3^{(1b)(2d)(3e)}\,.
\end{aligned}
\tag{3.6}
$$

The sums run over the permutations of indices $1,\ldots,6$, modulo symmetries of the integral functions $I_4^{(abc)def}$ and $I_4^{a(bc)(de)f}$.

The two-mass-hard box coefficients are

$$
c_{\mathcal{N}=6}^{a^-(b^-c^+)(d^-e^+)f^+} = \frac{i}{2}\frac{s_{bc}s_{de}s_{af}^2(K_{abc}^2)[a|K_{abc}|d\rangle[c|K_{abc}|f\rangle[c|K_{abc}|d\rangle^6}{[a\,b]\,[b\,c]^2\,\langle d\,e\rangle^2\,\langle e\,f\rangle\,[a|K_{abc}|d\rangle[a|K_{abc}|e\rangle[b|K_{abc}|e\rangle[c|K_{abc}|f\rangle[a|K_{abc}|f\rangle^2}\,,
\tag{3.7}
$$

the one-mass box coefficients are

$$
\begin{aligned}
&c_{\mathcal{N}=6}^{(a^-b^+c^+)d^-e^+f^-} = \\
&\frac{i}{2}\frac{\langle d\,e\rangle^2\,\langle e\,f\rangle^2\,[d\,e]\,[e\,f]\,[e|K_{abc}|a\rangle^6\Big(\langle a\,b\rangle\,[b\,c]\,[f|K_{abc}|c\rangle\,[d\,a] + [a\,b]\,\langle b\,c\rangle\,[c\,d]\,[f|K_{abc}|a\rangle\Big)}{\langle a\,b\rangle\,\langle b\,c\rangle\,\langle a\,c\rangle\,[d\,f]^2\,[d|K_{abc}|b\rangle[f|K_{abc}|b\rangle[d|K_{abc}|c\rangle[f|K_{abc}|c\rangle K_{abc}^2}\,,
\end{aligned}
\tag{3.8}
$$

and

$$
\bar{c}_{\mathcal{N}=6}^{(a^+b^-c^-)d^+e^-f^+} = c_{\mathcal{N}=6}^{(a^-b^+c^+)d^-e^+f^-}|_{\lambda\leftrightarrow\bar{\lambda}}.
\tag{3.9}
$$

The three-mass triangle coefficients can be evaluated using analytic techniques [37–39] and are

$$
c_{\mathcal{N}=6}^{(a^-b^+)(c^-d^+)(e^-f^+)} = \frac{1}{s_{ab}s_{cd}s_{ef}\,\langle c\,d\rangle^2}\sum_{i=1}^{6} C_{A_i}\frac{\langle B_6|[K_{ab},K_{cd}]|A_i\rangle}{\langle A_i|K_{ab}K_{cd}|A_i\rangle},
\tag{3.10}
$$

where

$$
\{|A_i\rangle\} = \{|b\rangle, |f\rangle, K_{ef}|c], K_{ab}|d], K_{ab}K_{cd}|f\rangle, K_{ef}K_{cd}|b\rangle\},
\tag{3.11}
$$

with

$$|B_1\rangle = |B_2\rangle = |a\rangle[d|f|e\rangle + |e\rangle[d|b|a\rangle,$$
$$|B_3\rangle = |B_4\rangle = |c\rangle[f|b|a\rangle + |a\rangle[f|d|c\rangle, \tag{3.12}$$
$$|B_5\rangle = |B_6\rangle = |e\rangle[b|d|c\rangle + |c\rangle[b|f|e\rangle,$$

and

$$C_{A_i} = \frac{\prod_{j=1}^{5}\langle B_j|A_i\rangle}{\prod_{j\neq i}\langle A_j|A_i\rangle}. \tag{3.13}$$

This explicit six-point amplitude has all the correct cuts and is, again, consistent with a loop momentum power count of $n-3$. The absence of cut-constructible bubble terms can be seen from the $\mathcal{N}=6$ version of the analysis in section (3.3) of reference [24]. We have presented results for external gravitons: the box-coefficients for other external states may be obtained using supersymmetric Ward identities [40].

4. $\mathcal{N}=4$ one-loop amplitudes

The particle content multiplicities of $\mathcal{N}=4$ graviton and matter multiplets are as follows:

Helicity	2	3/2	1	1/2	0	−1/2	−1	−3/2	−2
graviton	1	4	6	4	2	4	6	4	1
matter	0	0	1	4	6	4	1	0	0

For convenience, we will calculate the one-loop amplitude using the $\mathcal{N}=4$ matter multiplet, which is related to the amplitude containing the graviton by

$$M^{\mathcal{N}=4,\text{graviton}} = M^{\mathcal{N}=8} - 4M^{\mathcal{N}=6,\text{matter}} + 2M^{\mathcal{N}=4,\text{matter}}. \tag{4.1}$$

To order ϵ^0, the four-point one-loop $\mathcal{N}=4$ amplitude is given by [28]

$$M^{\text{1-loop},\mathcal{N}=4}(1^-,2^-,3^+,4^+) = \frac{F}{2s^4}\Big((t-u)s\ln(-t/-u) - tu\big(\ln^2(-t/-u)+\pi^2\big) + s^2\Big) \tag{4.2}$$

where

$$F = \frac{1}{16}\left(\frac{st\,\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\right)^2 = \frac{stu}{4}M^{\text{tree}}(1^-,2^-,3^+,4^+) \tag{4.3}$$

and $s\equiv s_{12}$, $t\equiv s_{14}$ and $u\equiv s_{13}$, are the usual Mandelstam variables. In terms of integral functions this result can be expressed as

$$M^{\text{1-loop},\mathcal{N}=4}(1^-,2^-,3^+,4^+) = \frac{F}{2s^4}\Big((t-u)s(I_2(t)-I_2(u)) - (tu)^2 I_4^{\text{trunc}}(t,u) + s^2\Big). \tag{4.4}$$

As we can see, this $\mathcal{N}=4$ amplitude contains a rational term

$$R_4 = \frac{F}{2s^2} = \left(\frac{t\,\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\right)^2. \tag{4.5}$$

The presence of a rational term indicates that the power count is at least 4 in this case. Since higher-point amplitudes must reduce to the four-point amplitude in soft and factorisation limits, it appears inevitable that rational terms also appear in all n-point amplitudes, indicating that the power count for $\mathcal{N}=4$ supergravity and one loop is *at least* n.

The n-point MHV amplitude is

$$M_n^{\mathcal{N}=4}(1^-,2^-,3^+,\ldots,n^+) =$$
$$\frac{(-1)^n}{8}\,\langle 1\,2\rangle^8 \sum_{\substack{2<a<b\le n\\ 1\in M,2\in N}} \left(\frac{\langle 1\,a\rangle\,\langle 2\,a\rangle\,\langle 1\,b\rangle\,\langle 2\,b\rangle}{\langle a\,b\rangle^2\,\langle 1\,2\rangle^2}\right)^2 h(a,M,b)h(b,N,a)\,\mathrm{tr}^2[a\,M\,b\,N]\,\mathcal{I}_4^{aMbN,\mathrm{trunc}}$$
$$+\sum_{1\in A,2\in B} c_2(1,A;2,B)I_2(P_A^2)+R_n, \tag{4.6}$$

where the sets A and B, contain at least one positive helicity leg. The bubble coefficients $c_2(1,A;2,B)$ are derived and given explicitly in the appendix. Previously it has been suggested [25, 26] that $R_n = 0$ for $\mathcal{N} = 4$ amplitudes. Our analysis suggests otherwise: we have explicitly seen that $R_4 \neq 0$. As a further check we have evaluated R_5 at a specific kinematic point (given in the appendix) using string-based rules for gravity [27, 28, 41]. At this kinematic point we find

$$R_5 = -589.27 - 1180.37i \tag{4.7}$$

5. Beyond one-loop

All supergravity theories in $D = 4$ are one- and two-loop finite since there is no R^3 supersymmetric counterterm, but at three loops a potential R^4 counterterm exists [42]. Until recently is was widely believed that all supergravity theories would generate this counterterm at three loops [16]. (In higher dimensions multiple possible R^4 terms exist: for $D = 8, 10$ the dimensional "lifts" of $\mathcal{N} = 8$, $\mathcal{N} = 6$ and $\mathcal{N} = 4$ have different counterterm structures [43], but for $D = 4$ there is a unique R^4 counterterm consistent with supersymmetry.)

We can attempt to estimate the power counting of the multi-loop amplitudes by considering various cuts. In particular let us consider the "three-particle cut" of the three-loop four-point amplitude,

$$\int d\mathrm{LIPS}(l_i)\; M^{\text{one-loop}}(1,2,\ell_1,\ell_2,\ell_3)\times M^{\text{tree}}(3,4,\ell_1,\ell_2,\ell_3) \tag{5.1}$$

as shown in fig. 3.

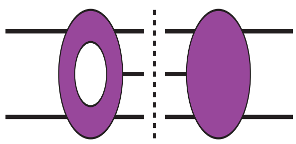

Figure 3: The three particle cut of a three-loop amplitude

We can estimate the overall power counting by looking at the power count of the uncut loop momenta. Note that we are looking at the amplitude rather than individual diagrams. For $\mathcal{N} = 8$ supergravity, examining individual diagrams suggests the three-loop power count is [17]

$$\int d\ell_i \frac{P_2(\ell_i,k_i)}{\prod_{j=1}^{10} D_j(\ell_i,k_i)} \tag{5.2}$$

for a diagram with propagators D_j and where P_2 is a polynomial in the the loop momenta of degree 2. However the "no-triangle" property suggest the power count in the indicated cut is only P_1. This gives a degree of divergence of

$$3D - 20 + 1 \tag{5.3}$$

making the amplitude divergent for

$$D \geq 6. \tag{5.4}$$

This is consistent with the explicit three-loop computation [19].

For $\mathcal{N}=6$ and $\mathcal{N}=4$ (assuming the degree of divergence can be inferred from this cut) we obtain

$$\begin{aligned} \mathcal{N}=6: &\quad 3D-20+2 \\ \mathcal{N}=4: &\quad 3D-20+5 \end{aligned} \tag{5.5}$$

Both degrees of divergence are less than -1 for $D=4$ and so we would predict that both theories remain finite at three-loops. These estimates must be taken with some caution: estimates of the power counting in supergravity theories have proven wrong on many occasions. Specifically, we cannot exclude further cancellations within integrands and we are not sensitive to all possible terms. Experience suggests that explicit calculations are required.

6. Conclusions

Explicit calculations of scattering amplitudes in $\mathcal{N}=6$ and $\mathcal{N}=4$ supergravity theories indicate loop momentum power counts of $n-3$ $(=n+4\ -7)$ and n $(=n+4\ -4)$ respectively. While the former is in agreement with previous expectations, the latter in not. In particular, the $\mathcal{N}=4$ amplitudes contain purely rational terms. We expect both these theories to remain finite up to three-loops.

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Appendix A. Bubbles in Supergravity MHV amplitudes

Here we present the bubble contributions to MHV amplitudes. Consider a cut in the momenta $P=k_a+\cdots k_b$. The coefficient of the bubble integral function $I_2(P^2)$ can be obtained from the cut,

$$C_{a,\ldots,b} \equiv \frac{i}{2}\sum_h \int d\mathrm{LIPS}\left[\mathcal{M}^{\mathrm{tree}}(-\ell_1^h,a,a+1,\ldots,b,\ell_2^{-h})\times\mathcal{M}^{\mathrm{tree}}(-\ell_2^h,b+1,b+2,\ldots,a-1,\ell_1^{-h})\right], \tag{A.1}$$

where $\int d\mathrm{LIPS}$ denotes integration over the on-shell phase space of the ℓ_i. We must sum over the states in the $\mathcal{N}=4$ matter multiplet. This cut vanishes unless we have a single negative helicity leg and at least one positive helicity leg on each side.

There are a variety of techniques available to determine the bubble coefficient from the cut: we will use the method of canonical forms [39]. We decompose the product of tree amplitudes appearing in a two-particle cut in terms of canonical forms $\mathcal{F}_i$,

$$\sum M^{\mathrm{tree}}(-\ell_1,\cdots,\ell_2)\times M^{\mathrm{tree}}(-\ell_2,\cdots,\ell_1)=\sum_i c_i\mathcal{F}_i(\ell_j), \tag{A.2}$$

where the c_i are coefficients independent of ℓ_j. We then use substitution rules to replace the $\mathcal{F}_i(\ell_j)$ by the evaluated forms $F_i(P)$ and obtain a bubble coefficient

$$\sum_i c_iF_i(P) \tag{A.3}$$

For example, the simplest canonical form we use is

$$\mathcal{H}_1(A;B;\ell)\equiv\frac{\langle\ell\, B\rangle}{\langle\ell\, A\rangle} \tag{A.4}$$

which, for $\ell = \ell_1$ or ℓ_2, evaluates to a contribution to the bubble coefficient of

$$H_1[A;B;P] = \frac{[A|P|B\rangle}{[A|P|A\rangle}. \tag{A.5}$$

It is convenient to define extensions,

$$\mathcal{H}_n(A_i;B_j;\ell) = \frac{\prod_{j=1}^n \langle B_j\,\ell\rangle}{\prod_{i=1}^n \langle A_i\,\ell\rangle} \longrightarrow H_n[A_i;B_j;P] = \sum_i \frac{\prod_{j=2}^n \langle B_j\,A_i\rangle}{\prod_{j\neq i} \langle A_j\,A_i\rangle} \frac{\langle B_1|P|A_i]}{\langle A_i|P|A_i]}\,, \quad \langle A_i\,A_j\rangle \neq 0. \tag{A.6}$$

We will also need the special cases where $A_1 = A_2 = A$,

$$\mathcal{H}_2^x(A,A;B_1,B_2;\ell_i) = \frac{\langle B_1\,\ell_1\rangle\,\langle B_2\,\ell_2\rangle}{\langle A\,\ell_1\rangle\,\langle A\,\ell_2\rangle} \longrightarrow H_2^x[A,A;B_1,B_2;P] = \frac{[A|P|B_1\rangle[A|P|B_2\rangle}{[A|P|A\rangle^2}. \tag{A.7}$$

and

$$\mathcal{H}_{2,1}^x = \frac{\langle B_1\,\ell_1\rangle\,\langle B_2\,\ell_2\rangle\,\langle B_3\,\ell_2\rangle}{\langle A\,\ell_1\rangle\,\langle A\,\ell_2\rangle\,\langle A_3\,\ell_2\rangle} \longrightarrow \left(\frac{\langle B_3\,A\rangle}{\langle A_3\,A\rangle} H_2^x[A,A;B_1,B_2;P] + \frac{\langle B_3\,A_3\rangle}{\langle A\,A_3\rangle} H_2[A,A_3;B_1,B_2;P]\right) \tag{A.8}$$

We now return to the cut $C_{a\cdots b}$. To be non-zero the set $a\cdots b$ must contain exactly one negative helicity graviton and at least one positive helicity graviton, i.e. must be of the form $\{a_1^+, a_2^+, \cdots, a_{n_L}^+, m_1^-\}$ and the legs on the other side must be $\{b_1^+, b_2^+, \cdots, b_{n_R}^+, m_2^-\}$. The product of tree amplitudes is then just a product of the two MHV trees.

When summing over the states in the multiplet, each tree amplitude is proportional to the tree amplitude with two scalars up to a simple factor. Summing over the tree amplitudes then yields

$$\begin{aligned}&\sum_h \Big[\mathcal{M}^{\rm tree}(-\ell_1^h, a_1^+, a_2^+, \cdots a_{n_L}^+, m_1^-, \ell_2^{-h}) \times \mathcal{M}^{\rm tree}(-\ell_2^h, b_1^+, b_2^+, \cdots b_{n_R}^+, m_2^-, \ell_1^{-h})\\ &= \mathcal{M}^{\rm tree}(-\ell_1^s, a_1^+, a_2^+, \cdots a_{n_L}^+, m_1^-, \ell_2^s) \times \mathcal{M}^{\rm tree}(-\ell_2^s, b_1^+, b_2^+, \cdots b_{n_R}^+, m_2^-, \ell_1^s)] \times \rho\end{aligned} \tag{A.9}$$

Where the ρ-factor is

$$\rho = \left(\frac{\langle m_1\,m_2\rangle^2\,\langle l_1\,l_2\rangle^2}{\langle m_1\,l_1\rangle\,\langle m_1\,l_2\rangle\,\langle m_2\,l_1\rangle\,\langle m_2\,l_2\rangle}\right)^A \tag{A.10}$$

where $A = 2$ for $\mathcal{N} = 4$ and $A = 3$ for $\mathcal{N} = 6$ ($A = 1$ for a $\mathcal{N} = 1$ matter multiplet).

Next, we rewrite the standard form of the MHV tree amplitude [36] so that the permutation is on the positive helicity gravitons,

$$\begin{aligned}M_n^{\rm tree}(1^s, 2^+, 3^+, \cdots, (n-2)^+, (n-1)^-, n^s) \;&=\; -i\,\langle 1\,n-1\rangle^4\,\langle n\,n-1\rangle^4 \times\\ \Bigg[\frac{[1\,2]\,[n-2\,n-1]}{\langle 1\,n-1\rangle\,N(n)}&\Big(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle i\,j\rangle\Big)\prod_{p=3}^{n-3}(-[p|K_{p+1\cdots n-1}|n\rangle) + \mathcal{P}(2,3,\cdots,n-2)\Bigg]\,,\end{aligned} \tag{A.11}$$

where $N(n) = \prod_{i<j}\langle i\,j\rangle$. Labelling the negative helicity leg $n-1$ as m and legs 2 to $n-2$ as $a_1\cdots a_{n'}$ and identifying legs 1 and n with ℓ_1 and ℓ_2 gives

$$\begin{aligned}M_n^{\rm tree}(l_1^s, a_1^+, a_2^+, \cdots, a_{n'}^+, m^-, l_2^s) &= -i\,\langle l_1\,m\rangle^4\,\langle l_2\,m\rangle^4 \Bigg[\frac{[l_1\,a_1]\,[a_{n'}\,m]\,\langle l_1\,m\rangle}{\langle l_1\,m\rangle^2\,\langle l_2\,m\rangle\,N_{n'}(\prod_i \langle l_1\,a_i\rangle\,\langle l_2\,a_i\rangle\,\langle a_i\,m\rangle\,\langle l_1\,l_2\rangle}\\ &\times\left(\prod_{j=2}^{n'}\langle l_1\,a_j\rangle\right)\left(\prod_{j=1}^{n'-1}\langle a_j\,m\rangle\right)\left(\prod_{i=1}^{n'-1}\prod_{j=i+2}^{n'}\langle a_i\,a_j\rangle\right)\prod_{p=2}^{n'-1}(-[a_p|\tilde{K}_p|l_2\rangle) + \mathcal{P}(a_1,a_2,\cdots,a_{n'})\Bigg]\\ &= -i\,\langle l_1\,m\rangle^3\,\langle l_2\,m\rangle^3\left[\frac{[a_{n'}\,m]\,[l_1\,a_1]\,(\prod_{i=1}^{n'-1}\prod_{j=i+2}^{n'}\langle a_i\,a_j\rangle)\prod_{p=2}^{n'-1}(-[a_p|\tilde{K}_{p+1}|l_2\rangle}{N_{n'}\,\langle a_{n'}\,m\rangle\,\langle l_1\,l_2\rangle\,\langle l_1\,a_1\rangle\,(\prod_{i=1}^{n'}\langle l_2\,a_i\rangle)} + \mathcal{P}(a_1,a_2,\cdots,a_{n'})\right]\end{aligned} \tag{A.12}$$

where $\tilde{K}_p = k_{a_p} + \cdots k_{a_{n'}} + k_m$ and $N_{n'} = \prod_{i<j} \langle a_i\, a_j \rangle$.

Counting each factor of the form $\langle A\, l_i \rangle$ or $[A\, l_i]$ as having a loop-momentum weight of $+\frac{1}{2}$, the power count on the cut momenta of a tree amplitude is of order $+2$. For the $\mathcal{N}=6$ multiplet, the ρ factor contributes -6 so the cut is of order ℓ_i^{-2} and thus [39] gives a bubble coefficient of zero.

For the $\mathcal{N}=4$ matter multiplet, the cut is

$$\sum_h \mathcal{M}^{\rm tree}(-\ell_1^h, \ldots, , \ell_2^{-h}) \times \mathcal{M}^{\rm tree}(-\ell_2^h, \ldots, \ell_1^{-h}) = \langle m_1\, m_2 \rangle^4 \sum_{P_l(a_i)} \sum_{P_r(b_i)} T_{(P_l;P_r)} \tag{A.13}$$

where

$$\begin{aligned} T_{(P_l;P_r)} &= C_{P_l} C_{P_r} \frac{\langle l_1\, l_2 \rangle^2 [l_1\, a_1]\, [l_1\, b_1] \langle m_1\, l_1 \rangle \langle m_1\, l_2 \rangle \langle m_2\, l_1 \rangle \langle m_2\, l_2 \rangle \prod_{l=2}^{n_L-1} \langle A_l\, l_2 \rangle \prod_{r=2}^{n_R-1} \langle B_r\, l_2 \rangle}{\langle l_1\, a_1 \rangle \langle l_1\, b_1 \rangle \prod_{x\in\{a_i,b_j\}} \langle x\, l_2 \rangle} \\ &= C_{P_l} C_{P_r} \frac{\langle m_1\, l_1 \rangle \langle m_2\, l_1 \rangle [a_1|P|l_2\rangle [b_1|P|l_2\rangle \langle m_1\, l_2 \rangle \langle m_2\, l_2 \rangle \prod_{l=2}^{n_L-1} \langle A_l\, l_2 \rangle \prod_{r=2}^{n_R-1} \langle B_r\, l_2 \rangle}{\langle l_1\, a_1 \rangle \langle l_1\, a_2 \rangle \prod_{x\in\{a_i,b_j\}} \langle x\, l_2 \rangle} \\ &= C_{P_l} C_{P_r} \frac{\langle m_1\, l_1 \rangle \langle m_2\, l_1 \rangle \prod_{i=1}^{n_L} \langle A_i\, l_2 \rangle \prod_{j=1}^{n_R} \langle B_j\, l_2 \rangle}{\langle l_1\, a_1 \rangle \langle l_1\, b_1 \rangle \prod_{x\in\{a_i,b_j\}} \langle x\, l_2 \rangle} \end{aligned} \tag{A.14}$$

and

$$|A_i\rangle = \begin{cases} \tilde{K}_i|a_i] & i \le n_L - 1 \\ |m_1\rangle & i = n_L \end{cases} \qquad |B_j\rangle = \begin{cases} \tilde{K}'_j|b_j] & j \le n_R - 1 \\ |m_2\rangle & j = n_R \end{cases} \tag{A.15}$$

$$C_{P_L} = \frac{\left(\prod_{i=1}^{n_L-1} \prod_{j=i+2}^{n_L} \langle a_i\, a_j \rangle\right)}{N(n_L) \langle n_L\, m_1 \rangle} = \frac{1}{\langle n_L\, m_1 \rangle \prod_{i=1}^{n_L-1} \langle a_i\, a_{i+1} \rangle} \tag{A.16}$$

We can rearrange the ℓ_2 dependant part,

$$\frac{\langle m_1\, l_1 \rangle \langle m_2\, l_1 \rangle \prod_{i=1}^{n_L} \langle A_i\, l_2 \rangle \prod_{j=1}^{n_R} \langle B_j\, l_2 \rangle}{\langle l_1\, a_1 \rangle \langle l_1\, b_1 \rangle \prod_{x\in\{a_i,b_j\}} \langle x\, l_2 \rangle} = \sum_{x\in\{a_i,b_j\}} D_x \frac{\langle m_1\, l_1 \rangle \langle m_2\, l_1 \rangle \langle m_1\, l_2 \rangle}{\langle l_1\, a_1 \rangle \langle l_1\, b_1 \rangle \langle x\, l_2 \rangle} \tag{A.17}$$

with

$$D_x = \frac{\langle m_2\, x \rangle \prod_{l=1}^{n_L-1} [a_l|\tilde{K}_{l+1}|x\rangle \prod_{k=1}^{n_R-1} [b_k|\tilde{K}'_{k+1}|x\rangle}{\prod_{y\neq x} \langle x\, y \rangle} = \frac{\prod_{l=1}^{n_L-1} [a_l|\tilde{K}_{l+1}|x\rangle \prod_{k=1}^{n_R} [b_k|\tilde{K}'_{k+1}|x\rangle}{[b_{n_R}\, m_2] \prod_{y\neq x} \langle x\, y \rangle} \tag{A.18}$$

Now for $x \neq a_1, b_1$ the term in eq. (A.17) just gives H_3 canonical forms. For $x = a_1$ or b_1 we get $H_{2,1}^x$ terms, Putting the pieces together, we have a bubble coefficient of

$$\begin{aligned} c(m_1, \{a_i\}; m_2, \{b_i\}) = \langle m_1\, m_2 \rangle^4 \sum_{P_L, P_R} C_{P_L} C_{P_R} \Bigg(&\sum_{x\neq a_1, b_1} D_x H_3(x, a_1, b_1; m_1, m_2, m_1; P) \\ &+ D_{a_1} H_{2,1}^x(a_1, a_1, b_1; m_1, m_2, m_1; P) + D_{b_1} H_{2,1}^x(b_1, b_1, a_1; m_1, m_2, m_1; P) \Bigg) \end{aligned} \tag{A.19}$$

Appendix B. Kinematic Point

We use a kinematic point defined in terms of the following spinors:

$$\lambda_\alpha^{(1)} = \mu^2 \begin{pmatrix} 46 + i \\ 14 + 18i \end{pmatrix},$$

$$\lambda_\alpha^{(2)} = \mu^2 \begin{pmatrix} 54 + 39i \\ 39 + 53i \end{pmatrix},$$

$$\lambda_\alpha^{(3)} = \mu^2 \begin{pmatrix} 9 + 46i \\ 16 + 13i \end{pmatrix},$$

$$\lambda_\alpha^{(4)} = \mu^2 \sqrt{\frac{42\,331}{4\,181\,993}} \begin{pmatrix} 540 + 480i \\ 200 + 170i \end{pmatrix},$$

$$\lambda_\alpha^{(5)} = \mu^2 \begin{pmatrix} \sqrt{\frac{14\,499\,838\,743}{4\,181\,993}} \\ (5\,099\,005\,787 + 2\,200\,443\,816i)\sqrt{\frac{3}{20\,212\,741\,374\,784\,933}} \end{pmatrix},$$

and the conjugate spinors $\tilde{\lambda}^{(i)}$ are given by

$$\tilde{\lambda}_{\dot{\alpha}}^{(i)} = \begin{cases} (\lambda_\alpha^{(i)})^* & \text{for } i = 1, 2, 3, \\ -(\lambda_\alpha^{(i)})^* & \text{for } i = 4, 5. \end{cases}$$

The numerical complexity of this point comes from the requirements that it is real in Minkowski space and free from any coplanarities. Momenta 4 and 5 have negative energy. Additionally, we set the renormalisation scale $\mu^2 = 10^{-2}$.

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