Elko in 1+1 dimensions

Cheng-Yang Lee*
Department of Physics and Astronomy
University of Canterbury, Christchurch, New Zealand
(Dated: ; LaTeX-ed November 29, 2010)

It is well known that both the Dirac and Majorana quantum fields have Dirac spinors as their expansion coefficients. In 2005 Ahluwalia and Grumiller introduced a complete set of eigenspinors of the charge conjugation operator, and extended the notion of Majorana spinors to Elko. This work not only shed new light on Majorana field and Majorana spinors, but revealed new physical and mathematical contents resulting from the Elko extension. It is now known that Elko, and hence Majorana spinors, have a built-in violation of the Lorentz symmetry and carry a well defined element of non-locality. This has far reaching consequences for theories that rely on these spinors. All these results apply in 3+1 dimensions. Here, we show that, in 1+1 dimensions, the Elko fields constructed from Elko spinors are local and satisfy the symmetry of a subgroup of the Poincaré group. The fields in 1+1 dimensions are of mass dimension one-half and have Dirac-like Lagrangians with renormalisable self-coupling terms similar to those of the the Thirring model.

arXiv:1011.5519v1 [hep-th] 24 Nov 2010

Introduction.— Elko is a spin-half fermionic field proposed by Ahluwalia and Grumiller [1, 2]. The field in 3+1 dimensions has mass dimension one instead of three-half so allows renormalisable self-interaction. In addition, Elko has other properties that makes it a dark matter candidate.

Recently, Elko has attracted interests from cosmologists and mathematical physicists. In cosmology, it was shown by various authors that Elko has the properties to generate inflation [3–15]. In mathematics, the properties of Elko have been studied in detail by da Rocha et al. [16–19]. While in quantum field theory, Fabbri has shown that Elko does not violate causality [20–22]. Wunderle and Dick have used Elko to construct supersymmetric Lagrangians for fermionic fields with mass dimension one [23].

Our primary focus in this letter is on the symmetry of Elko. In Ref. [24, 25], it was shown that Elko is local only when the momentum of the particle is aligned to a particular axis. Therefore, Elko violates Lorentz symmetry. A possible solution was suggested by Ahluwalia and Horvath [26] claiming that Elko satisfies the symmetry of Very Special Relativity (VSR) proposed by Cohen and Glashow [27]. It is worth noting that Sheikh-Jabbari and Tureanu has pointed out the difficulties of constructing quantum fields with VSR symmetry. They have shown that these difficulties can be overcome via Drinfel'd twist which resulted in non-commutative space-time [28, 29]. Although Elko is not constructed under non-commutative space-time, due to its possible VSR connection, quantum field theory with non-commutative space-time may provide additional insights on Elko [30].

In this letter, following the formalism developed by Weinberg [31–33], we show that Elko, in 3+1 dimensions, violates rotation symmetry. While this result in itself is not new, its derivation within the quantum field theoretic context has not been attempted before. Subsequently, we show that Elko in 1+1 dimensions is local and satisfy the symmetry of a subgroup of the Poincaré group. Incidentally, this group is also a subgroup of VSR ($SIM(2)$). The propagator and field equation of Elko in 1+1 dimensions are derived. The resulting Elko Lagrangians are similar to the Dirac Lagrangian. The propagator shows Elko is of mass dimension one-half, therefore, it allows renormalisable self-coupling terms similar to the Thirring model [34].

Elko.— In the $(1/2,0)\oplus(0,1/2)$ representation space, the Elko spinors are defined as

$$\chi(\mathbf{p})=\sqrt{m}\begin{pmatrix}\eta\,\Theta\phi^*(\mathbf{p})\\ \phi(\mathbf{p})\end{pmatrix} \qquad (1)$$

where η is a phase and Θ is the Wigner spin-half time-reversal operator,

$$\Theta=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \qquad (2)$$

Here $\phi(\mathbf{0})$ is a left-handed Weyl spinor, under boost it transforms as

$$\phi(\mathbf{p})=\exp\left(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right)\phi(\mathbf{0}). \qquad (3)$$

The rapidity parameter $\boldsymbol{\varphi}=\varphi\widehat{\mathbf{p}}$ is defined as

$$\cosh\varphi=E/m,\qquad \sinh\varphi=p/m \qquad (4)$$

and m is the mass of the particle. Equation (3) gives an explicit relation between rest spinors of $\mathbf{0}$ momentum and spinors of arbitrary momentum $\mathbf{p}$. It can be shown that $\eta\Theta\phi^*(\mathbf{0})$ transforms as a right-handed Weyl spinor [2]

$$\eta\Theta\phi^*(\mathbf{p})=\exp\left(\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right)\eta\Theta\phi^*(\mathbf{0}). \qquad (5)$$

Elko spinors of arbitrary momenta are then given by

$$\chi(\mathbf{p})=\kappa(\mathbf{p})\chi(\mathbf{0}) \qquad (6)$$

* cheng-yang.lee@pg.canterbury.ac.nz

where

$$\kappa(\mathbf{p}) = \begin{pmatrix} \exp\left(\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) & 0 \\ 0 & \exp\left(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) \end{pmatrix}. \quad (7)$$

One can also verify that the Elko spinors transform correctly under rotation.

The phase η is determined so that $\chi(\mathbf{p})$ is an eigenspinor of the finite-dimensional charge-conjugation operator $\mathcal{C}$,

$$\mathcal{C} = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix}\mathcal{K} \quad (8)$$

where $\mathcal{K}$ complex conjugates everything on its right. Therefore, we get

$$\mathcal{C}\chi(\mathbf{p})|_{\eta=\pm i} = \pm\chi(\mathbf{p})|_{\eta=\pm i}. \quad (9)$$

This holds for all momentum since $[\mathcal{C}, \kappa(\mathbf{p})] = 0$.

In 3+1 dimensions, the Elko field $\Lambda(x)$ takes the form

$$\Lambda(x) = (2\pi)^{-3/2}\int\frac{d^3p}{\sqrt{2E}}\sum_\sigma\left[e^{-ip\cdot x}\xi(\mathbf{p},\sigma)c(\mathbf{p},\sigma) + e^{ip\cdot x}\zeta(\mathbf{p},\sigma)d^\dagger(\mathbf{p},\sigma)\right] \quad (10)$$

where $c(\mathbf{p},\sigma)$ and $d(\mathbf{p},\sigma)$ are the annihilation operators for the particles and anti-particles respectively. They satisfy the standard anti-commutation relations

$$\{c(\mathbf{p},\sigma), c^\dagger(\mathbf{p}',\sigma')\} = \{d(\mathbf{p},\sigma), d^\dagger(\mathbf{p}',\sigma')\} = \delta_{\sigma\sigma'}\delta^3(\mathbf{p}-\mathbf{p}') \quad (11)$$

while all other anti-commutators identically vanish. Analogous to the Majorana fermions [35], we can construct another Elko field $\lambda(x)$,

$$\lambda(x) = \Lambda(x)\big|_{d^\dagger=c^\dagger} \quad (12)$$

where the particles are identical to the antiparticles. The coefficients $\xi(\mathbf{p},\sigma)$ and $\zeta(\mathbf{p},\sigma)$ are Elko spinors, so they take the form of Eq.(1).

We note that in most literature, only two of the Elko spinors with phase $\eta = i$ are noted, these spinors are called the Majorana spinors. However, for massive particles, this is not consistent with Lorentz symmetry. A massive spin-half field, by Lorentz symmetry, must have four degrees of freedom equally shared between particles and anti-particles distinguished by the spin-projection [37]. Constructing a field theory with only Majorana spinors (two Elko spinors) would be akin to projecting out the anti-particle spinors of the Dirac field. Hence we are forced to introduce $\xi(\mathbf{p},\sigma)$ and $\zeta(\mathbf{p},\sigma)$. It is important to note that here we do not treat the Elko spinors as Grassmann (anti-commuting) numbers. The fermionic properties of the particles are encoded in the standard anti-commutation relations in Eq.(11).

While the Elko spinors transform correctly under the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group, this does not imply the fields $\Lambda(x)$ and $\lambda(x)$ satisfy Lorentz symmetry. The work of Weinberg [31–33] shows that given a representation of the Lorentz group, the expansion coefficients of a quantum field can be determined by the demand of Lorentz symmetry up to a few caveats (Wigner type degeneracy [36]). The field equation can then be derived from the properties of the associated expansion coefficients and the propagator.

If the Elko fields satisfy Lorentz symmetry, then we should be able to determine their expansions coefficients $\xi(\mathbf{p},\sigma)$ and $\zeta(\mathbf{p},\sigma)$. However, in 3+1 dimensions, this is not possible. To see this, let $\Psi(x)$ be a general spin-half quantum field in the $(1/2,0)\oplus(0,1/2)$ representation space

$$\Psi(x) = (2\pi)^{-3/2}\int\frac{d^3p}{\sqrt{2E(\mathbf{p})}}\sum_\sigma\left[e^{-ip\cdot x}u(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + e^{ip\cdot x}v(\mathbf{p},\sigma)b^\dagger(\mathbf{p},\sigma)\right].$$

Taking the Pauli matrices $\boldsymbol{\sigma}/2$ to be the rotation generator for the $(1/2,0)$ and $(0,1/2)$ representation spaces, rotation symmetry requires the spinors at rest to satisfy the following equations [33]

$$\sum_{\bar\sigma}\boldsymbol{\sigma}_{\sigma\bar\sigma}u_\ell(\mathbf{0},\bar\sigma) = \sum_{\bar\ell}\mathcal{J}_{\ell\bar\ell}\,u_{\bar\ell}(\mathbf{0},\sigma),$$

$$\sum_{\bar\sigma}\boldsymbol{\sigma}^*_{\sigma\bar\sigma}v_\ell(\mathbf{0},\bar\sigma) = -\sum_{\bar\ell}\mathcal{J}_{\ell\bar\ell}\,v_{\bar\ell}(\mathbf{0},\sigma) \quad (13)$$

where $\ell = 1,\cdots 4$ denote the components of the spinors and $\mathcal{J}$ is a 4×4 matrix that furnishes a representation of the Lorentz group under rotation. Without the loss of generality taking $\mathcal{J} = 1/2(\boldsymbol{\sigma}\oplus\boldsymbol{\sigma})$, we see that the Elko spinors cannot satisfy Eq.(13). Therefore, Elko violates rotation symmetry.

One should note, Lorentz violations for Elko in 3+1 dimensions are expected, since it was noted that the Elko spin-sums contain a preferred direction and the Elko spinors do not satisfy the Dirac equation in momentum space [1, 2, 24, 25]. Therefore, Elko inevitably violates Lorentz symmetry in 3+1 dimensions.

Elko in 1+1 dimensions.— The violation of rotation symmetry for $\Lambda(x)$ and $\lambda(x)$ can be resolved if we restrict the fields to 1+1 dimensions. The fields are then only subjected to the symmetries of a subgroup of the Poincaré group in 1+1 dimensions.

In 1+1 dimensions, the sub-algebra consists of a rotation, a boost, a translation generator along the spatial axis, and the time translation generator. Since there are no preferred axis, for our purpose, we choose the spatial axis to be the y-axis. The sub-algebra reads

$$[J_2, K_2] = [J_2, P_2] = [P_2, P_0] = [J_2, P_0] = 0, \quad (14a)$$

$$[P_2, K_2] = iP_0, \quad [P_0, K_2] = iP_2. \quad (14b)$$

where J_2, K_2, P_2 are the rotation, boost, translation generator respectively and P_0 is the time translation generator. In this case, the Elko field takes the form

$$\Lambda(x) = (2\pi)^{-1/2}\int \frac{dp}{\sqrt{2E}}\sum_\sigma \left[e^{-ip\cdot x}\xi(p,\sigma)c(p,\sigma) + e^{ip\cdot x}\zeta(p,\sigma)d^\dagger(p,\sigma)\right] \tag{15}$$

and Eq.(13) becomes

$$\sum_{\bar\sigma}(\sigma_2)_{\sigma\bar\sigma}\xi_\ell(0,\bar\sigma) = \sum_{\bar\ell}(\mathcal{J}_2)_{\ell\bar\ell}\,\xi_{\bar\ell}(0,\sigma),$$

$$\sum_{\bar\sigma}(\sigma_2)_{\sigma\bar\sigma}\zeta_\ell(0,\bar\sigma) = \sum_{\bar\ell}(\mathcal{J}_2)_{\ell\bar\ell}\,\zeta_{\bar\ell}(0,\sigma). \tag{16}$$

One can verify that both the $\xi(\mathbf{0},\sigma)$ and $\zeta(\mathbf{0},\sigma)$ Elko spinors are consistent with Eq.(16). We note that in 1+1 dimensions, the Elko spinors are not the unique solutions. They are special cases to a wider class of possible solutions of Eq.(16) that need to be studied.

Substituting the Elko spinors into Eq.(16), we are able to determine them up to some proportionality constants. The demand of locality and CPT invariance on the fields along with orthonormality and completeness on the spinors determines the proportionality constants giving us the following solutions

$$\xi(0,1/2) = \sqrt{m}\begin{pmatrix}0\\ i\\ 1\\ 0\end{pmatrix},\quad \xi(0,-1/2) = \sqrt{m}\begin{pmatrix}-i\\ 0\\ 0\\ 1\end{pmatrix},$$

$$\zeta(0,1/2) = \sqrt{m}\begin{pmatrix}i\\ 0\\ 0\\ 1\end{pmatrix},\quad \zeta(0,-1/2) = \sqrt{m}\begin{pmatrix}0\\ i\\ -1\\ 0\end{pmatrix}. \tag{17}$$

The phases η in Eq.(1) for $\xi(\mathbf{p},\sigma)$ and $\zeta(\mathbf{p},\sigma)$ are determined to be $\eta = i$ and $\eta = -i$ respectively. The orthonormality and completeness relations obtained here are different to those associated with the Dirac spinors since the norms of the Elko spinors computed using the Dirac dual identically vanishes

$$\bar\xi(p,\sigma)\xi(p,\sigma') = \bar\zeta(p,\sigma)\zeta(p,\sigma') = 0 \tag{18}$$

where $\bar\xi(p,\sigma) = \xi^\dagger\gamma^0$ and

$$\gamma^0 = \begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix}. \tag{19}$$

Instead, we use the Elko dual introduced in [1, 2] and justified in [24, 25]

$$\vec{\xi}\,(0,\pm 1/2) \equiv \mp i\bar\xi(0,\mp 1/2),$$

$$\vec{\zeta}\,(0,\pm 1/2) \equiv \mp i\bar\zeta(0,\mp 1/2) \tag{20}$$

which gives the standard orthonormality and completeness relations

$$\vec{\xi}\,(0,\sigma)\xi(0,\sigma') = -\,\vec{\zeta}\,(0,\sigma)\zeta(0,\sigma') = 2m\delta_{\sigma\sigma'}$$

$$\frac{1}{2m}\sum_\sigma\left[\xi(p,\sigma)\,\vec{\xi}\,(p,\sigma) - \zeta(p,\sigma)\,\vec{\zeta}\,(p,\sigma)\right] = \mathbb{I}.$$

It is this sense, that Elko spinors are referred to as the complete set of Majorana spinors. A detailed account on the similarities and differences between the two can be found in Sec.5.1 of [26]. The dual Elko field is then defined as

$$\vec{\Lambda}\,(x) = (2\pi)^{-1/2}\int\frac{dp}{\sqrt{2E}}\sum_\sigma\left[e^{ip\cdot x}\,\vec{\xi}\,(p,\sigma)c^\dagger(p,\sigma) + e^{-ip\cdot x}\,\vec{\zeta}\,(p,\sigma)d(p,\sigma)\right]. \tag{21}$$

We derive the field equation for Elko by computing the propagator as vacuum-expectation value of the fermionic time-ordered product

$$S(y,t';x,t) = \langle\ |\mathcal{T}[\Lambda(y)\,\vec{\Lambda}\,(x)]|\ \rangle = i\int\frac{d^2q}{(2\pi)^2}e^{iq\cdot(y-x)}\frac{\Gamma^\mu q_\mu + m\mathbb{I}}{q^2-m^2+i\epsilon}. \tag{22}$$

Here the Γ-matrices are defined as

$$\Gamma^0 = \begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix},\qquad \Gamma^2 = \begin{pmatrix}0 & -\mathbb{I}\\ \mathbb{I} & 0\end{pmatrix} \tag{23}$$

and they satisfy the Clifford algebra $\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}$ in 1+1 dimensions. The propagator is a Green's function to the operator $(i\Gamma^\mu\partial_\mu - m\mathbb{I})$ with $\partial_\mu = \partial/\partial x^\mu$. Therefore, field equation for $\Lambda(x)$ is

$$(i\Gamma^\mu\partial_\mu - m\mathbb{I})\Lambda(x) = 0. \tag{24}$$

This can be verified explicitly by acting the operator on the field. The propagator and the field equation suggest the following Lagrangian densities for $\Lambda(x)$ and $\lambda(x)$

$$\mathscr{L}^\Lambda(x) = \vec{\Lambda}\,(x)(i\Gamma^\mu\partial_\mu - m\mathbb{I})\Lambda(x),$$

$$\mathscr{L}^\lambda(x) = \mathscr{L}^\lambda(x)\big|_{\Lambda\to\lambda}. \tag{25}$$

Using the Lagrangian densities, the fields $\Lambda(x)$ and $\lambda(x)$ are local, satisfying the following equal-time anti-commutation relations

$$\{\Lambda(x,t),\Pi(y,t)\} = \{\lambda(x,t),\pi(y,t)\} = i\delta(x-y),$$

$$\{\Lambda(x,t),\Lambda(y,t)\} = 0,\quad \{\lambda(x,t),\lambda(y,t)\} = i\gamma^0\delta(x-y),$$

$$\{\Pi(x,t),\Pi(y,t)\} = 0,\quad \{\pi(x,t),\pi(y,t)\} = -i\gamma^0\delta(x-y).$$

where $\Pi(x,t)$ and $\pi(x,t)$ are the conjugate momenta of $\Lambda(x)$ and $\lambda(x)$ respectively. The resulting normal-ordered Hamiltonians for $\Lambda(x)$ and $\lambda(x)$ are bounded from below, in agreement with the standard Hamiltonian for a free field theory.

While Elko in 1+1 dimensions share many similar features to the Dirac field, they are not the same since they violate Lorentz symmetry and do not satisfy the Dirac equation in 3+1 dimensions. Instead, their symmetry group is a subgroup of the Poincaré group given by Eqs.(14a,14b) in 1+1 dimensions. Since there is no preferred direction, Elko in 1+1 dimensions are special cases to the class of physically equivalent fields whose symmetry differs only by the chosen axis.

One important feature of the theory must be noted here. In 1+1 dimensions, the propagator in Eq.(22) shows that Elko is of mass dimension one-half. Therefore, it allows renormalisable self-coupling terms of the form $g(\overset{\neg}{\Lambda}\,\Lambda)^2$ and $g(\overset{\neg}{\Lambda}\,\Gamma^{\mu}\Lambda)(\overset{\neg}{\Lambda}\,\Gamma_{\mu}\Lambda)$ similar to the Thirring model [34].

Conclusions.—We have shown that the Elko fields satisfy the symmetry of a subgroup of the Poincaré group in 1+1 dimensions. The Lagrangians of Elko in 1+1 dimensions are consistent with the propagators and yield Lorentz invariant field equations. The fields have mass dimension one-half, so it allow self-interactions similar to the Thirring model.

It is important to note that in this letter, we did not construct the Elko fields in 1+1 dimensions. Instead, what we have shown is that the fields are local when confined to 1+1 dimensions and its expansion coefficients are consistent with Eq.(16). It is in this sense Elko satisfies the symmetry given by Eqs.(14a,14b). One of the pressing issue is then to find the symmetry group of Elko. Towards this end, recent results obtained by Ahluwalia and Horvath suggest the symmetry group to be $SIM(2)$ given by VSR [26].

If one takes the view that the underlying space-time symmetry is a reflection of the properties of the matter and gauge fields [38], then it is possible that the symmetry of the dark matter sector may not be dictated by the Poincaré group since they do not interact with the Standard Model particles [39]. Although the entire structure of Elko remains to be fully understood, results obtained from various fields ranging from astro-particle physics, cosmology, mathematical physics and quantum field theory suggest that Elko is an example in support of this paradigm.

Acknowledgement.—I thank to D. V. Ahluwalia and S. P. Horvath for discussions on the subject. I am also grateful to D.V. Ahluwalia for the critical remarks and suggestions on this letter. This work is supported by the UC Doctoral Scholarship.

[1] D. V. Ahluwalia and D. Grumiller, JCAP **0507**, 012 (2005) [arXiv:hep-th/0412080].
[2] D. V. Ahluwalia and D. Grumiller, Phys. Rev. D **72** (2005) 067701 [arXiv:hep-th/0410192].
[3] C. G. Boehmer, J. Burnett, D. F. Mota and D. J. Shaw, JHEP **1007**, 053 (2010) [arXiv:1003.3858 [hep-th]].
[4] C. G. Boehmer and J. Burnett, arXiv:1001.1141 [gr-qc].
[5] C. G. Boehmer and J. Burnett, Mod. Phys. Lett. A **25**, 101 (2010) [arXiv:0906.1351 [gr-qc]].
[6] C. G. Boehmer and J. Burnett, Phys. Rev. D **78**, 104001 (2008) [arXiv:0809.0469 [gr-qc]].
[7] C. G. Boehmer, Phys. Rev. D **77**, 123535 (2008) [arXiv:0804.0616 [astro-ph]].
[8] C. G. Boehmer and D. F. Mota, Phys. Lett. B **663**, 168 (2008) [arXiv:0710.2003 [astro-ph]].
[9] C. G. Boehmer, Annalen Phys. **16**, 325 (2007) [arXiv:gr-qc/0701087].
[10] C. G. Boehmer, Annalen Phys. **16**, 38 (2007) [arXiv:gr-qc/0607088].
[11] G. Chee, arXiv:1007.0554 [gr-qc].
[12] H. Wei, arXiv:1002.4230 [gr-qc].
[13] S. Shankaranarayanan, arXiv:1002.1128 [astro-ph.CO].
[14] S. Shankaranarayanan, Int. J. Mod. Phys. D **18**, 2173 (2009) [arXiv:0905.2573 [astro-ph.CO]].
[15] D. Gredat and S. Shankaranarayanan, JCAP **1001**, 008 (2010) [arXiv:0807.3336 [astro-ph]].
[16] J. M. Hoff da Silva and R. da Rocha, Int. J. Mod. Phys. A **24**, 3227 (2009) [arXiv:0903.2815 [math-ph]].
[17] R. da Rocha and J. M. Hoff da Silva, arXiv:0811.2717 [math-ph].
[18] R. da Rocha and J. M. Hoff da Silva, J. Math. Phys. **48**, 123517 (2007) [arXiv:0711.1103 [math-ph]].
[19] R. da Rocha and W. A. J. Rodrigues, Mod. Phys. Lett. A **21**, 65 (2006) [arXiv:math-ph/0506075].
[20] L. Fabbri, arXiv:1008.0334 [gr-qc].
[21] L. Fabbri, Mod. Phys. Lett. A **25**, 2483 (2010) [arXiv:0911.5304 [gr-qc]].
[22] L. Fabbri, Mod. Phys. Lett. A **25**, 151 (2010) [Erratum-ibid. A **25**, 1295 (2010)] [Mod. Phys. Lett. A **25**, 1295 (2010)] [arXiv:0911.2622 [gr-qc]].
[23] K. E. Wunderle and R. Dick, arXiv:1010.0963 [hep-th].
[24] D. V. Ahluwalia, C. Y. Lee, D. Schritt and T. F. Watson, Phys. Lett. B **687** (2010) 248 [arXiv:0804.1854 [hep-th]].
[25] D. V. Ahluwalia, C. Y. Lee and D. Schritt, arXiv:0911.2947 [hep-ph].
[26] D. V. Ahluwalia and S. P. Horvath, JEHP **11**, 078 (2010) [arXiv:1008.0436 [hep-ph]].
[27] A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. **97**, 021601 (2006) [arXiv:hep-ph/0601236].
[28] M. M. Sheikh-Jabbari, A. Tureanu, Phys. Atom. Nucl. **73**, 230-236 (2010). [arXiv:0811.3670 [hep-ph]].
[29] M. M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. **101**, 261601 (2008). [arXiv:0806.3699 [hep-th]].
[30] I am grateful to D.V. Ahluwalia for bringing my attention to this work.
[31] S. Weinberg, Phys. Rev. **133**, B1318 (1964).
[32] S. Weinberg, Phys. Rev. **134**, B882 (1964).
[33] S. Weinberg, *Cambridge, UK: Univ. Pr. (1995) 609 p.*
[34] W. E. Thirring, Annals Phys. **3** (1958) 91.
[35] E. Majorana, Nuovo Cim. **14**, 171 (1937).
[36] E. P. Wigner, Annals Math. **40**, 149 (1939) [Nucl. Phys. Proc. Suppl. **6**, 9 (1989)].

[37] D. V. Ahluwalia, Int. J. Mod. Phys. A **11**, 1855 (1996) [arXiv:hep-th/9409134].
[38] H. R. Brown, "Physical Relativity: Space-time structure from a dynamical perspective," *Oxford, UK: Univ. Pr. (2005) 225 p.*
[39] D. V. Ahluwalia, Int. J. Mod. Phys. **D18**, 2311-2316 (2009). [arXiv:0904.0066 [gr-qc]].