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# Eikonal equation of the Lorentz-violating Maxwell theory

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**Abstract.** We derive the eikonal equation of light wavefront in the presence of Lorentz invariance violation (LIV) from the photon sector of the standard model extension (SME). The results obtained from the equations of **E** and **B** fields respectively are the same. This guarantees the self-consistency of our derivation. We adopt a simple case with only one non-zero LIV parameter as an illustration, from which we find two points. One is that, in analogy with Hamilton-Jacobi equation, from the eikonal equation, we can derive dispersion relations which are compatible with results obtained from other approaches. The other is that, the wavefront velocity is the same as the group velocity, as well as the energy flow velocity. If further we define the signal velocity $v_s$ as the front velocity, there always exists a mode with $v_s > 1$, hence causality is violated classically. Thus our method might be useful in the analysis of Lorentz violation in QED in terms of classical causality .

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## 1 Introduction

The constant speed of light in vacuum is one of the basic assumptions of special relativity, and it serves as a crucial ingredient of Lorentz symmetry, which is a cornerstone of modern physics. However, currently there is a revival of interest in the possibility of varying speed of light, or in other words, a tiny deviation from exact Lorentz invariance, in the search of so-called quantum gravitational phenomena.

As the typical quantum gravitational scale, *i.e.*, the Planck scale $E_{\text{Planck}} = \sqrt{c\hbar/G}c^2$, is practically unattainable from conventional accelerator experiments, people turn to search for phenomenologically accessible effects from quantum gravity [1] at relatively low energy scales. One of them is the possibility of Lorentz invariance violation (LIV). The possibility that quantum gravity may leave a tiny imprint of LIV at relatively low energies was observed by many authors from various approaches to quantum gravity. These include string field theory, where the tachyon field may induce an instability to the naive Lorentz invariant vacuum and translate it into the potential of a tensor field. As a consequence, the tensor field acquires a vacuum expectation value [2] and breaks Lorentz invariance. Later, by incorporating various LIV coefficients into the standard model, the theory was developed into an effective field theory (EFT), called standard model extension (SME) [3]. Other approaches include spin network calculation in loop gravity [4], deformed special relativity [5], foamy structure of space-time [6], noncommutative field theory [7], emergent gravity [8], and the recently suggested Horava-Lifschitz gravity [9]. All of them suggest that tiny LIV may be a signature of new physics.

Fortunately, LIV not only resides in theoretical considerations, it also becomes experimentally testable, and has already been tested to very high accuracy in various sectors of the standard model (for a review, see Ref. [10]). From an experimental viewpoint, people have already been able to severely constrain the linear order LIV correction to the photon group velocity, which is suppressed by the ratio of experimental energy $E$ to a large mass scale, *e.g.*, $E/M_{\text{Planck}}$ (see *e.g.* Refs. [11,12,13,14,15]).

From the EFT viewpoint, conventionally non-renormalizable operators can be naturally suppressed by a large mass scale, where new physics might come in. So the mentioned tiny linear correction may come from dimension-5 operators [16]. However, if one assumes CPT symmetry and the anisotropic scaling between space and time [9] (also termed weighted power counting [17]), or alternatively imposes CPT invariance and the supersymmetric constraints [18], one can naturally expect that the leading order correction comes from dimension-6 operators, thus can evade current experimental constraints [19]. On the other side, there is no clear theoretical reason to pin down the values of dimension-3 and -4 LIV operators, though they have already been extensively studied and found to suffer severe constraints from experiments [10,20]. Hence it is still valuable to consider them both theoretically and experimentally, and it inspires us to revisit the renormalizable LIV operators [3]. In this paper, we mainly focus on the CPT-even part of the photon sector in the framework of SME.

The paper is organized as follows. In section 2, we give a brief review on the renormalizable photon sector of SME. With some *ad hoc* assumptions for simplification, we derive the modified Maxwell equations as our starting point. In section 3, following the derivation provided by Fock [21], we obtain the eikonal equation of the modified Maxwell equations implicitly. In section 4, by adopting a much simpler case with only one non-zero LIV parameter as an illustration, we give the eikonal equation explicitly. From the derivation, we find it is self-consistent, because one can obtain the same result from equations relevant to either an **E** or **B** field, respectively. It is also consistent with other approaches, as the dispersion relations obtained from our procedure and those from others are the same. Meanwhile, in this simple case, we find that the group velocity equals the wavefront velocity, as well as the energy flow velocity. As the derivation of eikonal equation is obviously not restricted to the simple model, the method may be useful in the classical causality analysis in LIV extension of quantum electrodynamics. Section 5 summarizes the results obtained in this paper and gives a further discussion on LIV topics.

## 2 A brief review of the photon part of SME

SME, including in principle all possible LIV terms in the framework of EFT [3], is a well-motivated testable approach to LIV physics, and it has been extensively tested in various sectors of the standard model [10]. Within SME, the renormalizable photon sector is one of the most tested parts [20,22]. Its Lagrangian reads

$$\mathcal{L}_{\text{photon}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}(k_F)_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu} + \frac{1}{2}(k_{AF})_{\kappa}\epsilon^{\kappa\lambda\mu\nu}A_{\lambda}F_{\mu\nu}, \tag{1}$$

where the coefficients $k_F$ and $k_{AF}$ characterize the violation of Lorentz symmetry. From Eq. (1), the equation of motion is deduced as

$$\partial^{\alpha}F_{\mu\alpha} + (k_F)_{\mu\alpha\beta\gamma}\partial^{\alpha}F^{\beta\gamma} + (k_{AF})^{\alpha}\epsilon_{\mu\alpha\beta\gamma}F^{\beta\gamma} = 0. \tag{2}$$

With the 3+1 decomposition, we define

$$(k_{DE})^{jk} \equiv -2(k_F)^{0j0k}, \quad (k_{HB})^{il} \equiv \frac{1}{2}(k_F)^{jkmn}\epsilon^{ijk}\epsilon^{lmn},$$
$$(k_{DB})^{jk} \equiv -(k_{HE})^{kj} \equiv \frac{1}{2}(k_F)^{0jmn}\epsilon^{kmn}, \tag{3}$$

where the Latin indices run over the three spatial coordinates, from 1 to 3. After subtracting the trace part of the first two matrices, we define further the following traceless matrices which are frequently used in this paper,

$${\beta_E}^{ij} = (k_{DE})^{ij} - \alpha\delta^{ij}, \quad {\beta_B}^{ij} = (k_{HB})^{ij} + \alpha\delta^{ij}, \quad \gamma^{ij} = (k_{DB})^{ij} = -(k_{HE})^{ij}, \tag{4}$$

where

$$\alpha = \frac{1}{3}\mathrm{tr}(k_{DE}) = -\frac{1}{3}\mathrm{tr}(k_{HB}). \tag{5}$$

Note that the $(k_F)_{\mu\alpha\beta\gamma}$ tensor has the same symmetry as the Riemann tensor, $R_{\mu\alpha\beta\gamma}$ [3,23]. Due to the double tracelessness of $k_F$ and the Bianchi identities $(k_F)_{\mu[\nu\rho\sigma]} = 0$, we can get Eq. (5) and $\mathrm{tr}[\gamma] = 0$, respectively. The first two matrices in Eq. (4) can be easily shown as symmetric, *i.e.*, $(\beta_E)^T = \beta_E$ and $(\beta_B)^T = \beta_B$. Since the $(k_{AF})_\mu$ term may cause theoretical instabilities [3] and has suffered severe astrophysical constraints [24], we abandon it from now on. Moreover, as $\gamma^{ij}$ mixes **E** and **B** fields and may cause further complications in calculation, we simply set $\gamma^{ij} = 0$. Further, we also set $\alpha = 0$ as an *ad hoc* assumption. Under these simplifications, there are only 10 parameters left, residing in the traceless symmetric matrices $\beta_E$ and $\beta_B$. Finally, we can write down the Lagrangian in terms of **E** and **B** as

$$\mathcal{L}_{\text{photon}} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) + \frac{1}{2}\left((\beta_E)^{jk}E^jE^k - (\beta_B)^{jk}B^jB^k\right), \tag{6}$$

where the electric field **E** and the magnetic field **B** are defined conventionally as

$$B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}, \quad E^i = -\partial_t A^i - \nabla_i\phi = F^{i0}. \tag{7}$$

From Eq. (6), we can deduce the modified Maxwell equations,

$$\nabla\cdot\mathbf{E} + \partial_\mathbf{i}(\beta_\mathbf{E})^{\mathbf{ij}}\mathbf{E}^\mathbf{j} = \mathbf{0}, \qquad -\dot{\mathbf{E}}^\mathbf{i} - (\beta_\mathbf{E})^{\mathbf{ij}}\dot{\mathbf{E}}^\mathbf{j} + \epsilon_{\mathbf{ijk}}\partial_\mathbf{j}\mathbf{B}^\mathbf{k} + \epsilon_{\mathbf{ijk}}\partial_\mathbf{j}(\beta_\mathbf{B})^{\mathbf{kl}}\mathbf{B}^\mathbf{l} = \mathbf{0}, \tag{8}$$

and the equations

$$\nabla\cdot\mathbf{B} = \mathbf{0}, \qquad \nabla\times\mathbf{E} + \dot{\mathbf{B}} = \mathbf{0} \tag{9}$$

come from the 3+1 decomposition of Bianchi identity $\partial_\mu{}^*F^{\mu\nu} = 0$, where ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\beta\gamma}F_{\beta\gamma}$. The above four equations are our starting points of derivation in the next section.

Note that Eq. (8) has the analogy of electrodynamics in a homogeneous anisotropic media [3] if we regard $(\beta_B)^{ij}$ as the inverse of magnetic permeability and $(\beta_E)^{ij}$ as the dielectric constant, *i.e.*, as defined in Ref. [23], $\mathbf{D}^\mathbf{i} = (\delta^{\mathbf{ij}} + {\beta_\mathbf{E}}^{\mathbf{ij}})\mathbf{E}^\mathbf{j}, \quad \mathbf{H}^\mathbf{i} = (\delta^{\mathbf{ij}} + {\beta_\mathbf{B}}^{\mathbf{ij}})\mathbf{B}^\mathbf{j}$. From this analogy, we expect that the differential equation satisfied by the light wavefront in LIV vacuo should have a similar form as the eikonal equation in an anisotropic medium. Actually, we will see that this is indeed the case in section 4.

Now we turn to the discussion of the energy-momentum tensor, which is useful for the discussion of causality in section 4. The conventional procedure to obtain a symmetric energy-momentum tensor is through the so-called Belinfante tensor [25]

$$\Theta^{\mu\nu} = T^{\mu\nu} - i\frac{1}{2}\partial_\kappa[\frac{\partial\mathcal{L}}{\partial(\partial_\kappa\Psi_l)}(\mathscr{I}^{\mu\nu})_l{}^m\Psi_m - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi_l)}(\mathscr{I}^{\kappa\nu})_l{}^m\Psi_m - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Psi_l)}(\mathscr{I}^{\kappa\mu})_l{}^m\Psi_m], \tag{10}$$

where

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi_l)}\partial^\nu\Psi_l - \eta^{\mu\nu}\mathcal{L}. \tag{11}$$

However, this does not work in the presence of LIV, as pointed out by Colladay and Kostelecky [3].

In fact, from Eq. (10), we can get the energy-momentum tensor corresponding to the Lagrangian in Eq. (1) without the CPT odd term,

$$\begin{aligned}\Theta^{\mu\nu} &= -[F^{\mu\alpha}F^{\nu}_{\ \alpha} + (k_F)^{\alpha\beta\mu\delta}F^{\nu}_{\ \delta}F_{\alpha\beta}] - \eta^{\mu\nu}\mathcal{L} \\ &= -[F^{\mu\alpha}F^{\nu}_{\ \alpha} + (k_F)^{\alpha\beta\mu\delta}F^{\nu}_{\ \delta}F_{\alpha\beta}] + \frac{1}{4}\eta^{\mu\nu}[(k_F)_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} + F^{\alpha\beta}F_{\alpha\beta}]. \end{aligned} \tag{12}$$

In the above derivation, we have used the matrix representation of the homogeneous Lorentz algebra for a covariant vector field,

$$(\mathscr{J}^{\mu\nu})_{\rho}^{\ \sigma} = i(\delta^{\mu}_{\ \rho}\eta^{\nu\sigma} - \delta^{\nu}_{\ \rho}\eta^{\mu\sigma}). \tag{13}$$

We find that due to the presence of the second asymmetric term, $-(k_F)^{\alpha\beta\mu\delta}F^{\nu}_{\ \delta}F_{\alpha\beta}$, one can no longer obtain a symmetric energy-momentum tensor. This may be one feature of LIV theories. One consequence is that the definition of the conserved 4-momentum density is $\Theta^{0\mu}$, instead of $\Theta^{\mu 0}$. Actually, with direct calculation it is easily checked that $\partial_\rho \Theta^{\rho\mu} = 0$, hence $d(\int d^3x \Theta^{0\mu})/dt = 0$ (for details, see Appendix A). In contrast, this is not valid for $\Theta^{\mu 0}$. The spatial part of the latter is defined as the generalized Poynting vector [3]. For the convenience of our discussions in section 4, we give those components of the energy-momentum tensor explicitly in terms of $\mathbf{E}$, $\mathbf{B}$ fields,

$$\begin{aligned}\Theta^{00} &= \frac{1}{2}(E^2 + B^2) - (k_F)^{0j0k}E^jE^k + \frac{1}{4}(k_F)^{ijkl}\epsilon^{ijm}\epsilon^{kln}B^mB^n \\ &= \frac{1}{2}(E^2 + B^2 + E^j(\beta_E)^{jk}E^k + B^m(\beta_B)^{mn}B^n), \end{aligned} \tag{14}$$

$$\begin{aligned}\Theta^{0i} &= -F^{0j}F^{i}_{\ j} - 2(k_F)^{0j0k}F^{i}\ jF_{0k} - (k_F)^{0jkl}F^{i}_{\ j}F_{kl} \\ &= \epsilon^{ijk}E^jB^k + \epsilon^{ijk}(\beta_E)^{jl}E^lB^k, \end{aligned} \tag{15}$$

$$\begin{aligned}\Theta^{i0} &= -F^{ij}F^{0}_{\ j} - (k_F)^{ijkl}F^{0}_{\ j}F_{kl} - 2(k_F)^{ij0l}F^{0}_{\ j}F_{0l} \\ &= \epsilon^{ijk}E^jB^k + \epsilon^{ijk}E^j(\beta_B)^{kl}B^l, \end{aligned} \tag{16}$$

where we have used the reparametrization of Eqs. (3) and (4), and the simplifications corresponding to the Lagrangian in Eq. (6). From Eqs. (15) and (16), it is apparent that due to the loss of Lorentz invariance, characterized by $\beta_E$ and $\beta_B$, generally the energy-momentum tensor is not symmetric, *i.e.*, $\Theta^{\mu\nu} - \Theta^{\nu\mu} = [(k_F)^{\alpha\beta\mu\delta}F^{\nu}_{\ \delta} - (k_F)^{\alpha\beta\nu\delta}F^{\mu}_{\ \delta}]F_{\alpha\beta} \neq 0$.

It has been well known that, in the Lorentz invariant theory, the energy-momentum tensor should be symmetric due to the following reasons [26]:

(1) When it couples with gravity in curved space-time, the symmetrical Einstein tensor $G_{\mu\nu}$ and metric tensor $g_{\mu\nu}$ automatically imply a symmetric $\Theta^{\mu\nu}$ due to the Einstein equation;
(2) The current conservation of $M^{\rho\mu\nu} = \frac{1}{2}(x^\nu\Theta^{\rho\mu} - x^\mu\Theta^{\rho\nu})$ of the angular momentum tensor $\Sigma^{\mu\nu} = \int d^3x M^{0\mu\nu}$ also requires a symmetric $\Theta^{\mu\nu}$.

However, in our case, LIV undermines these two reasons;

(1) It is necessary to take LIV effects into account in gravity side for a consistent theory, however in this case, it is the local Lorentz invariance involved. Moreover, when coupled with gravity, LIV parameters should be promoted to dynamical fields instead of being constants [27], and the compatible framework to incorporate LIV in general relativity is the Riemann-Cartan geometry [28], where the extended Einstein tensor is no longer symmetric. The antisymmetric part of the Einstein tensor and the energy-momentum tensor is consistently linked through field equations [28]. Thus no inconsistency arises when an asymmetric $\Theta^{\mu\nu}$ couples with gravity in a spontaneous LIV theory [28].
(2) Due to the breaking of Lorentz invariance, the generator of the Lorentz group, *i.e.*, the angular momentum tensor, is no longer conserved in general, thus no constraints are imposed here. If one assumes that only boost invariance is broken, *i.e.*, rotational invariance is still preserved. It means that only $\alpha$ and $k^0_{AF}$ are non-zero in Eq. (1). Then in this case, one can prove $\Theta^{ij} = \Theta^{ji}$, which is required by the conservation of the spatial part of the angular momentum tensor, *i.e.*, $d\Sigma^{ij}/dt = 0$. However, in the presence of LIV, it is only valid in a particular reference frames, *e.g.*, CMB reference, as the transformation from one inertial frame, in which the LIV photon equation is isotropic, cannot remain isotropic in other inertial frames [29]. Here we present the proof in Appendix A.

## 3 A derivation of eikonal equation

In order to derive the eikonal equation of the modified electrodynamics, it is necessary to review how this is achieved in Lorentz invariant Maxwell theory. We follow the derivation of Fock [21]. The basic principle is that the fields on the wavefront, which is a space-like 2-dimension surface, must be noncontinuous in the normal direction of the surface. The reason is that the surface is an interface of two regions, one already influenced by the field perturbation and the other remaining intact. Thus the time derivatives of the fields on the wavefront should be singular.

Taking a right-moving plane wave as an example, the wavefront is an infinite plane perpendicular to its constant wavevector $\boldsymbol{k}$ at any instant. The left region of the plane is a region with non-vanishing electromagnetic fields, while the right side is a region with vanishing fields. This conveys the fact that light propagates with a finite speed, thus at any instant it cannot propagate further to the right of the wavefront plane. Since an electromagnetic field evolves dynamically, it is more helpful to analyze from a 4-dimensional point of view, rather than from the view at some fixed instant in 3-spatial dimensions.

For any instant $t$, the wavefront in the 3-spatial dimension can be written as

$$t = f(x, y, z). \tag{17}$$

This can also be viewed in a dynamical way as an evolving 2-dimensional space-like surface $f(x, y, z) - t = 0$. When we write Eq. (17) in an implicit form,

$$w(t; x, y, z) = 0, \tag{18}$$

the wavefront becomes a slice of 3-dimensional hyper-surface in flat space-time in the 4-dimensional point of view. So the 2-dimensional space-like wavefront is just a time slice of the null-surface (light-cone), and the non-continuity of fields on the wavefront is nothing more than the (classical) causality requirement. To be more specific, the fields inside the light-cone must be continuous since they are causally connected. Using Maxwell equations, one can obtain from the fields on any given surface in 3-spatial dimensions inside the light-cone, the fields on an infinitesimally close surface. However, this cannot be done for fields lying on the surface outside the light-cone. So the instant time slice of the hyper-surface $w(t; x, y, z) = 0$ becomes an interface between causally connected and non-connected regions, hence the derivatives of fields on the interface become singular. We use the above principle to derive the so-called eikonal equation with the same logic as that in the Lorentz invariant theory.

The Lorentz invariant eikonal equation reads

$$(\nabla f(x, y, z))^2 = n^2, \tag{19}$$

where in vacuum, $n$ equals 1 in our unit (*i.e.* $c = 1$). In a more explicit Lorentz invariant form, it reads

$$(\frac{\partial w}{\partial t})^2 - (\nabla w)^2 = 0. \tag{20}$$

We show in Appendix B how to get Eq. (20) from Eq. (19).

Now we turn to derive eikonal equation in the presence of LIV. Any field $u(x, y, z, t)$ on a 3-dimensional hyper-surface in the 4-dimensional Minkowski space is

$$u(t, x, y, z) = u(f(x, y, z); x, y, z) \equiv u_0(x, y, z). \tag{21}$$

Therefore, we have

$$\frac{\partial u_0}{\partial x^i} = \frac{\partial u}{\partial x^i} + \frac{\partial u}{\partial t}\frac{\partial f}{\partial x^i}, \tag{22}$$

where $x^i$ with Latin index $i$ running over 1 to 3, represents $x,\ y,\ z$ respectively. Thus the spatial derivatives of the electric and magnetic fields on the hyper-surface are

$$\frac{\partial \mathbf{E_0^i}}{\partial x^j} = \frac{\partial \mathbf{E^i}}{\partial x^j} + \partial_j f \dot{\mathbf{E}}^{\mathbf{i}}, \qquad \frac{\partial \mathbf{B_0^i}}{\partial \mathbf{x^j}} = \frac{\partial \mathbf{B^i}}{\partial \mathbf{x^j}} + \partial_{\mathbf{j}} \mathbf{f} \dot{\mathbf{B}}^{\mathbf{i}}. \tag{23}$$

From the linearity of Eq. (23), we have

$$\nabla \times \mathbf{E_0} = \nabla \times \mathbf{E} + \nabla \mathbf{f} \times \dot{\mathbf{E}}. \tag{24}$$

Then by using the second equation of the Bianchi identity (9), we get

$$\nabla \times \mathbf{E_0} = -\dot{\mathbf{B}} + \nabla \mathbf{f} \times \dot{\mathbf{E}}. \tag{25}$$

Similarly, one can obtain from Maxwell Eqs. (8) and (9) the set of equations below

$$\nabla \cdot ((1 + \beta_E) \cdot \mathbf{E_0}) = \nabla \mathbf{f} \cdot ((\mathbf{1} + \beta_{\mathbf{E}}) \cdot \dot{\mathbf{E}}), \qquad \nabla \times \mathbf{E_0} = -\dot{\mathbf{B}} + \nabla \mathbf{f} \times \dot{\mathbf{E}}; \tag{26}$$

$$\nabla \cdot \mathbf{B_0} = \nabla \mathbf{f} \cdot \dot{\mathbf{B}}, \qquad \nabla \times ((\mathbf{1} + \beta_{\mathbf{B}}) \cdot \mathbf{B_0}) = ((\mathbf{1} + \beta_{\mathbf{E}}) \cdot \dot{\mathbf{E}}) + \nabla \mathbf{f} \times ((\mathbf{1} + \beta_{\mathbf{B}}) \cdot \dot{\mathbf{B}}). \tag{27}$$

Then by multiplying the second equations of (26) and (27) respectively with $\nabla f$, we have

$$\nabla f \cdot \nabla \times \mathbf{E_0} = -\nabla \mathbf{f} \cdot \dot{\mathbf{B}}, \tag{28}$$
$$\nabla f \cdot \nabla \times ((1+\beta_B)\cdot \mathbf{B_0}) = \nabla \mathbf{f} \cdot ((\mathbf{1}+\beta_\mathbf{E})\cdot \dot{\mathbf{E}}). \tag{29}$$

Comparing the equations above with the first equations of (26) and (27), we find the following relations

$$\nabla f \cdot \nabla \times \mathbf{E_0} + \nabla \cdot \mathbf{B_0} = \mathbf{0}, \tag{30}$$
$$\nabla f \cdot \nabla \times ((1+\beta_B)\cdot \mathbf{B_0} - \nabla \cdot ((\mathbf{1}+\beta_\mathbf{E})\cdot \mathbf{E_0}) = \mathbf{0}. \tag{31}$$

When $f(x,y,z)$ is a constant, *i.e.*, by specifying a particular instant, Eqs. (30) and (31) become the first equations of the original field equations (8) and (9) respectively. This means that we need to specify proper initial conditions.

In order to extract more information, we multiply the curl of $E_0$ by the tensor $(1+\beta_B)$, then calculate its cross product with $\nabla f$. By utilizing the second equations of (26) and (27) successively, we get

$$\begin{aligned}&\nabla f \times [(1+\beta_B)\cdot \nabla \times \mathbf{E_0}] = \nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot (\nabla \mathbf{f} \times \dot{\mathbf{E}} - \dot{\mathbf{B}})]\\ &= (1+\beta_E)\cdot \dot{\mathbf{E}} - \nabla \times [(\mathbf{1}+\beta_\mathbf{B})\cdot \mathbf{B_0}] + \nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot (\nabla \mathbf{f} \times \dot{\mathbf{E}})].\end{aligned} \tag{32}$$

After arranging it in such a way that all time derivatives of $\mathbf{E}$ are on one side, and $E_0$ and $B_0$ on the other side, we have

$$(1+\beta_E)\cdot \dot{\mathbf{E}} + \nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot (\nabla \mathbf{f} \times \dot{\mathbf{E}})] = \nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot \nabla \times \mathbf{E_0}] + \nabla \times [(\mathbf{1}+\beta_\mathbf{B})\cdot \mathbf{B_0}]. \tag{33}$$

Now it is apparent that the right-hand side of (33) includes the wavefront function $f$ and the electromagnetic fields $E_0$ and $B_0$ with values on the wavefront, while the left-hand side includes only $\dot{\mathbf{E}}$ and $f$. With given wavefront function $f$ and the fields $E_0$ and $B_0$, $\dot{\mathbf{E}}$ on the interface must be singular, as we have already argued. Otherwise one can obtain the fields on an infinitesimal close surface outside the light-cone from field equations, which obviously violates the causality requirements. Thus, if we view the left-hand side of Eq. (33) as a matrix equation,

$$M_e \cdot \dot{\mathbf{E}} \equiv (\mathbf{1}+\beta_\mathbf{E})\cdot \dot{\mathbf{E}} + \nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot (\nabla \mathbf{f} \times \dot{\mathbf{E}})], \tag{34}$$

where $M_e$ is a tensor/matrix, then the determinant of $M_e$ must be zero, *i.e.*, $\mathrm{Det}(M_e)=0$. Otherwise, in principle, one would get a non-singular $\dot{\mathbf{E}}$ by solving Eq. (33).

Similarly, we can obtain the corresponding equation of the $\mathbf{B}$ field,

$$\dot{\mathbf{B}} + \nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{E})^{-\mathbf{1}}\cdot \{\nabla \mathbf{f} \times \Big((\mathbf{1}+\beta_\mathbf{B})\cdot \dot{\mathbf{B}}\Big)\}] = \nabla \mathbf{f} \times \{(\mathbf{1}+\beta_\mathbf{E})^{-\mathbf{1}}\cdot (\nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot \mathbf{B_0}])\} - \nabla \times \mathbf{E_0}, \tag{35}$$

where we have presumed the existence of the inverse of $(1+\beta_E)$, due to the fact that all of the 10 parameters, $(\beta_E)^{ij}$ and $(\beta_B)^{ij}$, must be very tiny compared to 1 to meet the stringent experimental constraints.

One can also define

$$M_b \cdot \dot{\mathbf{B}} \equiv \dot{\mathbf{B}} + \nabla \mathbf{f} \times \Big\{(\mathbf{1}+\beta_\mathbf{E})^{-\mathbf{1}}\cdot \Big(\nabla \mathbf{f} \times [(\mathbf{1}+\beta_\mathbf{B})\cdot \dot{\mathbf{B}}]\Big)\Big\}. \tag{36}$$

And for the same reason, $\mathrm{Det}(M_b)=0$. Actually, we find that generally $\mathrm{Det}(M_b) \propto \mathrm{Det}(M_e)$, hence the two equations, $\mathrm{Det}(M_b)=0$ and $\mathrm{Det}(M_e)=0$, give the same differential equations of $f(x,y,z)$. The former proportionality (in some cases, it even becomes an equality) holds as expected, as the differential equation of $f$ should be unique. This will be shown in detail in Appendix C. In the presence of LIV, $\mathrm{Det}(M_e)=0$ actually gives the eikonal equation implicitly. We will show it explicitly in a simple case in the following section.

## 4 A case study and discussions

In this section, we give explicitly the eikonal equation for a simple case, together with more detailed discussions. For this purpose, Eqs. (34) and (36) are our starting points.

First we try to extract the tensors/matrices $M_e$ and $M_b$ from Eqs. (34) and (36) respectively. By rewriting Eq. (34) in component form,

$$(M_e)^{ij}\dot{\mathbf{E}}^\mathbf{j} = (\mathbf{1}+\beta_\mathbf{E})^\mathbf{ij}\dot{\mathbf{E}}^\mathbf{j} + \epsilon^\mathbf{ijk}\mathbf{f_j}(\mathbf{1}+\beta_\mathbf{B})^\mathbf{kl}\epsilon^\mathbf{lmn}\mathbf{f_m}\dot{\mathbf{E}}^\mathbf{n}, \tag{37}$$

where $f_i \equiv \partial f/\partial x^i$, one can read

$$(M_e)^{ij} = [1-(\nabla f)^2]\delta^{ij} + f_i f_j + (\beta_E)^{ij} - \epsilon^{ink}\epsilon^{jml} f_n f_m (\beta_B)^{kl}. \tag{38}$$

Similarly, from Eq. (36), one can obtain

$$(M_b)^{ij} = \delta^{ij} - \epsilon^{ink}\epsilon^{jml} f_n f_m W^{kl} - \epsilon^{isk}\epsilon^{nml} f_s f_m W^{kl} (\beta_B)^{nj}, \tag{39}$$

where $W^{ij} = [(1+\beta_E)^{-1}]^{ij}$. Our task now is to calculate the determinants of $M_e$ and $M_b$, then one can obtain the eikonal equation satisfied by $f(x,y,z)$.

As **E**, **B** fields are simply the 3+1 decomposition of the electromagnetic field strength $F^{\mu\nu}$, and our working hypothesis relies only on the analysis of causality, we can reasonably expect that the eikonal equations obtained from the equations of **E** and **B** should be the same.

The explicit expressions of $M_e$ and $M_b$ in the matrix form are rather tedious, not to mention the calculation of their determinants. As an illustration, we need not to consider the full expression with all 10 parameters non-zero. We place the discussions of the full expression in Appendix C, where we show that the eikonal equations obtained from Eq. (38) and Eq. (39) are indeed the same. Instead, we assume here that only $(\beta_B)^{12} = (\beta_B)^{21} = \sigma \neq 0$ to simplify our discussions on $M_e$ and $M_b$. By virtue of this simplification, we can subsequently get and solve the eikonal equation explicitly.

However, here we derive it by restarting derivations from the simplified field equations,

$$\nabla\cdot\mathbf{E} = \mathbf{0}, \quad \epsilon_{\mathbf{ijk}}\partial_{\mathbf{j}}\mathbf{B^k} + \epsilon_{\mathbf{ijk}}\partial_{\mathbf{j}}(\beta_{\mathbf{B}})^{\mathbf{kl}}\mathbf{B^l} - \dot{\mathbf{E}}^{\mathbf{i}} = \mathbf{0}, \tag{40}$$

$$\nabla\cdot\mathbf{B} = \mathbf{0}, \quad \epsilon_{\mathbf{ijk}}\partial_{\mathbf{j}}\mathbf{E^k} + \dot{\mathbf{B}}^{\mathbf{i}} = \mathbf{0}, \tag{41}$$

where $(\beta_B)^{kl} = \sigma(\delta_1^k\delta_2^l + \delta_2^k\delta_1^l)$. Now the equations corresponding to Eqs. (35) and (33) are

$$\dot{\mathbf{B}} - (\nabla\mathbf{f})^{\mathbf{2}}[(\mathbf{1}+\beta_{\mathbf{B}})\cdot\dot{\mathbf{B}}] + \nabla\mathbf{f}[\nabla\mathbf{f}\cdot(\beta_{\mathbf{B}}\cdot\dot{\mathbf{B}})] = \nabla\mathbf{f}\times\{\nabla\times[(\mathbf{1}+\beta_{\mathbf{B}})\cdot\mathbf{B_0}]\} - \nabla\times\mathbf{E_0} - \nabla\mathbf{f}(\nabla\cdot\mathbf{B_0}), \tag{42}$$

$$[1-(\nabla f)^2]\dot{\mathbf{E}} + \nabla\mathbf{f}\times[\beta_{\mathbf{B}}\cdot(\nabla\mathbf{f}\times\dot{\mathbf{E}})] = \nabla\mathbf{f}\times[(\mathbf{1}+\beta_{\mathbf{B}})\cdot(\nabla\mathbf{f}\times\mathbf{E_0})] + \nabla\times[(\mathbf{1}+\beta_{\mathbf{B}})\cdot\mathbf{B_0}] - (\nabla\cdot\mathbf{E_0})\nabla\mathbf{f}. \tag{43}$$

From (43) we obtain

$$(M_e) = [1-(\nabla f)^2]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sigma\begin{pmatrix} 0 & f_3^2 & -f_2f_3 \\ f_3^2 & 0 & -f_1f_3 \\ -f_2f_3 & -f_1f_3 & 0 \end{pmatrix}. \tag{44}$$

By direct calculation, we have

$$\mathrm{Det}(M_e) = [1-(\nabla f)^2]\left\{[1-(\nabla f)^2]^2 + 2f_1f_2\sigma[1-(\nabla f)^2] - f_3^2\sigma^2(\nabla f)^2\right\}. \tag{45}$$

Similarly for the **B** field, from (42) we have

$$(M_b) = [1-(\nabla f)^2]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sigma\begin{pmatrix} f_1f_2 & f_1^2-(\nabla f)^2 & 0 \\ f_2^2-(\nabla f)^2 & f_1f_2 & 0 \\ f_2f_3 & f_2f_3 & 0 \end{pmatrix}, \tag{46}$$

and

$$\mathrm{Det}(M_b) = [1-(\nabla f)^2]\{[1-(\nabla f)^2 + \sigma f_1f_2]^2 - (f_1^2+f_3^2)(f_2^2+f_3^2)\sigma^2\}. \tag{47}$$

One can easily verify that $\mathrm{Det}(M_e) = \mathrm{Det}(M_b)$. Generally, the equation $\mathrm{Det}(M_e) = 0$ has three solutions. One is the conventional $(\nabla f)^2 = 1$, and if written in terms of $w(x,y,z;t)$, it is Eq. (20). We will see below that this corresponds to the conventional dispersion relation $p^2 = 0$, which accompanies modified Lorentz-violating dispersion relations in many models, see *e.g.*, Ref. [12]. The other two solutions,

$$1-(\nabla f)^2 + \sigma f_1f_2 \pm \sigma\sqrt{(f_1^2+f_3^2)(f_2^2+f_3^2)} = 0, \tag{48}$$

manifest LIV explicitly, which is characterized by the non-zero LIV parameter $\sigma$. Treating space and time on the same footing, *i.e.*, rewriting Eq. (48) in terms of function $w(t,x,y,z)$, we can get

$$(\frac{\partial w}{\partial t})^2 - (\nabla w)^2 + \sigma(w_1w_2 \pm \sqrt{(w_1^2+w_3^2)(w_2^2+w_3^2)}) = 0, \tag{49}$$

where $w_i \equiv \partial w/\partial x^i$. When $\sigma = 0$, Eq. (49) reduces to the Lorentz invariant case, *i.e.*, Eq. (20).

In order to solve the eikonal equation, we use the Lorentz invariant case (20) as an illustration. Choosing the positive sign for convenience, we obtain a first order partial differential equation,

$$\frac{\partial w}{\partial t} + \sqrt{(\nabla w)^2} = 0. \tag{50}$$

Then from the well-known analogy between Hamilton-Jacobi equation and geometric optics [21], one can identify $w(x, y, z; t)$ as the Hamilton action $S(q, P; t)$, $\sqrt{(\nabla w)^2} = \sqrt{\sum_{i=1}^{3} w_i^2}$ as the Hamiltonian $H(q, \frac{\partial S}{\partial q}; t)$, and $w_i$ as the momentum $P_i = \partial S/\partial q^i$. So the question is transformed into solving the Hamilton equation,

$$\dot{x}^i = \{x^i, H\} = \frac{\partial \sqrt{\sum_{i=1}^{3} w_i^2}}{\partial w_i} = \frac{w_i}{\sqrt{\sum_{i=1}^{3} w_i^2}}. \tag{51}$$

We note that $\sum_{i=1}^{3}(\dot{x}^i)^2 = 1$, which can be written in an appropriate form, $d\tau^2 = dt^2 - (dx^i)^2 = 0$, characterizing the null geodesic of photons. If Eq. (50) is solved in the momentum space, we get the well-known dispersion relation $p^2 = 0$ for massless particles.

Next we use the same analogy to solve Eq. (49). In the presence of LIV, the formal Hamilton-Jacobi equation now reads

$$\frac{\partial w}{\partial t} + \sqrt{(\nabla w)^2 - \sigma(w_1 w_2 \pm \sqrt{(w_1^2 + w_3^2)(w_2^2 + w_3^2)})} = 0, \tag{52}$$

with

$$H(w_i) = \sqrt{(\nabla w)^2 - \sigma(w_1 w_2 \pm \sqrt{(w_1^2 + w_3^2)(w_2^2 + w_3^2)})}. \tag{53}$$

Since $H$ does not contain the canonical variables which conjugate to momentum $w_i$, it automatically implies that the momentum $k_i = w_i$ is conserved. From the absence of explicit dependence on time, we have $\partial S/\partial t = \partial w/\partial t = -E = -k_0$. Then based on the above observations, we can quickly read the dispersion relation

$${k^0}^2 = \boldsymbol{k}^2 - \sigma(k^1 k^2 \pm \sqrt{({k^1}^2 + {k^3}^2)({k^2}^2 + {k^3}^2)}), \tag{54}$$

which can also be confirmed by solving field equation (2) (for details, see Appendix D).

Note that the "$\pm$" signs in Eq. (54) imply that the photon dispersion relation depends on polarization, hence can lead to the so-called vacuum birefringence effect [3,24], which has already been used to place severe constraints on LIV parameters from astrophysical observations, like CMB [31], radio galaxies [32] and GRB [30,19].

Back to the Hamilton-Jacobi equation (52), by using the observer rotational invariance, we can rotate the observer frame to a particular one with $w_1 = w_2 = \rho$. This simplifies Eq. (52) to

$$\frac{\partial w}{\partial t} + \sqrt{(\nabla w)^2 - \sigma(\rho^2 \pm (\rho^2 + w_3^2))} = 0. \tag{55}$$

For the positive sign in (55), we can solve

$$\dot{x}^1 = \dot{x}^2 = \frac{1}{2}\frac{\partial H}{\partial \rho} = \frac{\rho\sqrt{1-\sigma}}{\sqrt{2\rho^2 + w_3^2}}, \quad \dot{x}^3 = \frac{\partial H}{\partial w_3} = \frac{w_3\sqrt{1-\sigma}}{\sqrt{2\rho^2 + w_3^2}}, \tag{56}$$

where the factor 1/2 comes from the symmetry $w_1 = w_2 = \rho$. We can read the group velocity from Eq. (56) as $v_g = \sqrt{\sum_{i=1}^{3} \dot{x}_i^2} = \sqrt{1-\sigma}$, or equivalently $d\tau^2 = (1-\sigma)dt^2 - d\boldsymbol{x}^2 = 0$. Accidentally, the phase velocity equals the group velocity, *i.e.*, $v_p = H/|\nabla w| = \sqrt{1-\sigma} = v_g$.

For the negative sign, we have

$$\dot{x}^1 = \dot{x}^2 = \frac{1}{2}\frac{\partial H}{\partial \rho} = \frac{\rho}{\sqrt{2\rho^2 + (1+\sigma)w_3^2}}, \quad \dot{x}^3 = \frac{\partial H}{\partial w_3} = \frac{w_3(1+\sigma)}{\sqrt{2\rho^2 + (1+\sigma)w_3^2}}. \tag{57}$$

Thus $d\tau^2 = dt^2 - [dx_1{}^2 + dx_2{}^2 + dx_3{}^2/(1+\sigma)] = 0$, and the group velocity reads

$$v_g = \sqrt{\frac{2\rho^2 + (1+\sigma)^2 w_3^2}{2\rho^2 + (1+\sigma) w_3^2}} = \sqrt{1 + \frac{\sigma(1+\sigma) w_3^2}{2\rho^2 + (1+\sigma) w_3^2}} \backsimeq 1 + \frac{\sigma w_3^2}{2|\nabla w|^2},$$

where in the last step we have approximated it to the first order of $\sigma$. Similarly, the phase velocity is

$$v_p = \frac{H}{|\nabla w|} = \sqrt{1 + \frac{\sigma w_3^2}{2\rho^2 + w_3^2}} \backsimeq 1 + \frac{\sigma w_3^2}{2|\nabla w|^2}.$$

We see that only in the first order approximation of $\sigma$, $v_p = v_g$, while considering higher orders, $v_p \neq v_g$, contrary to the case of positive sign. So in general $v_p \neq v_g$ in the presence of LIV, and this is easy to understand. As the presence of the background tensor $k_F$ breaks the Lorentz invariant vacuum, the vacuum is no longer an isotropic and dispersion free medium. In this case, just like the behavior of the electromagnetic wave in the conventional dispersive medium, generally $v_p$ depends on the photon momentum $k$, *i.e.*, $\partial v_p/\partial k \neq 0$. Consequently, $v_g = \partial k^0/\partial k = v_p + k\, \partial v_p/\partial k \neq v_p$, where $k = |\boldsymbol{k}|$.

In this case, it is interesting to define an effective refractive index from the comparison of Eq. (48) with Eq. (19),

$$n_{\text{eff}}^2 = 1 + \sigma(f_1 f_2 \pm \sqrt{(f_1^2 + f_3^2)(f_2^2 + f_3^2)}). \tag{58}$$

Similarly, it is instructive to define an effective mass

$$m_{\text{eff}}^2 = -\sigma(k^1 k^2 \pm \sqrt{({k^1}^2 + {k^3}^2)({k^2}^2 + {k^3}^2)}), \tag{59}$$

by comparing Eq. (54) with that of the conventional massive particles.

Then from Eqs. (58) and (59), we see that, for $\sigma > 0$, the "+" sign gives $n_{\text{eff}}^2 > 1$ and $m_{\text{eff}}^2 < 0$. The analogy with optical media tells us that $v_p < 1$, which is confirmed in our special case where $v_p = \sqrt{1-\sigma}$. However, the analogy with massive particles by means of introducing an effective mass squared $m_{\text{eff}}^2$ breaks down. In the conventional case, when we have a negative mass squared, it means that we have expanded the theory at an unappropriate point, a point which is not a true vacuum. When we quantize the theory around this unstable vacuum, we get a tachyon with $v > 1$ and hence violate causality. But this analogy breaks down here, as the underlying physical mechanism is completely different from that of the false vacuum in a Lorentz invariant theory. For the "−" sign, we have $n_{\text{eff}}^2 < 1$ and $m_{\text{eff}}^2 > 0$. From the analogy with optical media, we expect $v > 1$, while naive analogy with conventional massive particles indicates $v < 1$. We see from the special case above that $v_p = v_g \backsimeq 1 + \sigma w_3^2 |\nabla w|^{-2}/2 > 1$, which is again consistent with the optical media analogy. For $\sigma < 0$, one can obtain similar results. Thus we conclude that the analogy between LIV vacua with optical medium is more reasonable and helpful.

In fact, our derivation of the eikonal equation is inspired by and starts from the analogy between the electromagnetic wave propagating in LIV vacuo and in anisotropic media. As the definition $n_{\text{eff}}^2$ is along the same line of the analogy, it is natural to expect that the qualitative results (of velocity) are consistent with direct calculations. However, this is not true for the analogy with conventional massive particles. The reason is that here the effective mass squared is no longer a free parameter in Lagrangian; instead, it is momentum-dependent. More importantly, now Lorentz invariance is broken, thus it is not necessary for the maximum attainable velocity of the effective massive particle to be $c = 1$.

Moreover, we see that, just as what happened in the conventional dispersive medium [34], in our case, the definition of light velocity is more involved than that in the conventional Lorentz invariant vacuo. Perhaps it is more complicated, for now it is the vacuum itself becoming anisotropic. Though in the dispersive medium, there already exist cases with $v_p > c$ or $v_g > c$ [34], they do not conflict with the requirement of causality. Since in practice, we can only measure directly the velocity of light signals, denoted as $v_s$. The only difficulty in that case is a proper definition of signal velocity. Once we have an appropriate definition of $v_s$, it can be shown that $v_s < 1$ [33]. Thus no causal problem arises there. Since in that case, our starting point, the wave equation, does indeed obey Lorentz invariance. Only the medium in which light propagates, is no longer the Lorentz invariant vacuum, but crystal or nontrivial QED vacuum. The medium then singles out a preferred direction, which leads to the so-called soft breaking of Lorentz invariance [35], with a superluminous phase/group velocity.

However, this is not the situation here. Now it is the basic wave equation, Maxwell equation, which is no longer Lorentz invariant. Hence there might be more difficulties in the proper definition of signal velocity and it may cause causal problems. For the simple case above, we can define $v_s$ as the wavefront velocity $v_f$. Then since our calculation deals directly with electromagnetic wavefront, we find $v_g = v_f$. Alternatively, we can also define energy transport velocity $\boldsymbol{v}_e^i = \Theta^{i0}/\Theta^{00}$ as the signal velocity. Then by choosing the appropriate ansatz $A_1(t, \boldsymbol{r})_\pm = W_\pm \cos[w(t - \boldsymbol{s} \cdot \boldsymbol{r}/v_{p\pm})]$ (where "±" sign refers to the two modes in Eq. (54), respectively) and the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, we can prove $|\boldsymbol{v}_e| = v_g$ for both "+" and "−" modes in Eq. (54) (for details of this issue, see Appendix D). Thus for the simple cases corresponding to Eqs. (56) and (57), and for either sign of $\sigma$, we see that there always exists one mode with $v_s > 1$, which indeed violates causality. However, note that all the treatments above are purely classical. Since LIV is believed to be a quantum gravitational phenomenon, the best way to treat the corresponding causality problem is to use quantum field theory. Hence it is more appropriate to discuss the microcausality from the calculation of 2-point correlation functions of observables at space-like separations. However, this is beyond the scope of the present paper.

## 5 Conclusion

Inspired by the analogy between the behavior of LIV electrodynamics and electrodynamics in macroscopic anisotropic medium, we derived the eikonal equation for Lorentz non-invariant vacuum from the modified Maxwell equations of SME. The results obtained from the equations of both **E** and **B** fields are the same, and the general case is given in Appendix C. This implies that the derivation is self-consistent. Then we use the well-known analogy between geometric optics and the Hamilton-Jacobi equation, and find that the solution of the eikonal equation (48) in the momentum space, *i.e.*, Eq. (54), turns out to be the modified dispersion relation of photons, which is the same as those obtained from other approaches. This fact confirms the consistency of our approach further.

With the definition of an effective refractive index $n_{\mathrm{eff}}$, we find that the dependence of the velocity on the LIV parameter is consistent with the velocity dependence on the refractive index $n$ of ordinary refractive medium. As our calculation deals directly with wavefront, the results show that the front velocity equals the group velocity for each LIV mode. From the analogy of the LIV vacuum with macroscopic anisotropic medium, we find that the presence of LIV makes the proper definition of signal velocity more complicated, especially when considering the existence of multi-definition of velocity. Therefore, the treatment of causality becomes much subtle. However, if we define the signal velocity as the wavefront velocity, *i.e.*, $v_s = v_f$, naive analysis shows that, for either sign of $\sigma$, causality is violated classically at least for one mode. Of course, to see whether or not tiny LIV might threaten microcausality, one should turn to the more trustable and rigorous treatment of quantum field theory to calculate the correlation functions at space-like separations. Actually, this was done for the Chern-Simons-like term [37] and it has caused extensive debates in the literature (see *e.g.*, Ref. [37]). Complete treatment of microcausality was also achieved for the massive fermion sector of SME [3,36], and it was shown that no crucial problems arise for a spontaneous Lorentz invariance breaking theory. If Lorentz violation happens intrinsically at high energies rather than spontaneously, as suggested in the so-called weighted dimensional model, Anselmi pointed out that it is Bogoliubov's criterion of causality, rather than the value of correlation functions, that makes sense [38].

As a by-product of this work, we find that the asymmetric energy-momentum tensor $\Theta^{\mu\nu}$ causes no inconsistency in principle. By assuming rotational invariance, the conservation of Noether charge, *i.e.*, $d\Sigma_{ij}/dt = 0$, requires that the spatial components of $\Theta^{\mu\nu}$ must be symmetric. Through inspection of the corresponding LIV parameters in the presence of rotational symmetry, we find that indeed $\Theta^{ij} = \Theta^{ji}$, though this is valid only in a particular reference frames, as already mentioned. While from the definitions of $\Theta^{\mu\nu}$ and the velocity of energy transport, in our simple case, explicit calculation shows that the group velocity also equals the energy transport velocity for each mode. Together with $v_g = v_f$, it makes the definition of signal velocity as wavefront velocity more reliable in the classical discussion of causality. The last point we would like to stress is that the derivation of the eikonal equation is not restricted to the simple model here. In fact, at least for gauge invariant Maxwell equations without **E**, **B** mixing terms, it should work as well. Thus the method might be helpful in the causality and phenomenological analysis of various LIV extensions of Maxwell equations.

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## Appendix A

This appendix gives the proof that, even for a generic asymmetric energy-momentum tensor, the requirement of rotational invariance implies the symmetric relation $\Theta^{ij} = \Theta^{ji}$, which ensures the consistency with the corresponding Noether current. Here, the conserved charge corresponding to the Noether current is the angular momentum.

For a Lorentz invariant theory, the conservation of generators of Lorenz group is manifested through the Noether current,

$$M^{\rho}{}_{\mu\nu} = \frac{1}{2}(\Theta^{\rho}{}_{\nu}x_{\mu} - \Theta^{\rho}{}_{\mu}x_{\nu}). \tag{60}$$

One can construct the corresponding generators of Lorentz group as

$$\Sigma_{\mu\nu} = \int d^3x M^{0}{}_{\mu\nu}. \tag{61}$$

Then from the current conservation,

$$\partial_{\rho} M^{\rho}{}_{\mu\nu} = 0, \tag{62}$$

it is easy to show that $d\Sigma_{\mu\nu}/dt = 0$. Hence the six generators of Lorentz group are conserved. Meanwhile, combined with the conservation of energy-momentum tensor, *i.e.*, $\partial_{\rho}\Theta^{\rho\mu} = 0$, the current conservation implies that,

$$\begin{aligned} 0 = \partial_{\rho} M^{\rho\mu\nu} &= \frac{1}{2}[x^{\mu}\partial_{\rho}\Theta^{\rho\nu} - x^{\nu}\partial_{\rho}\Theta^{\rho\mu} + (\Theta^{\rho\nu}\delta^{\mu}_{\rho} - \Theta^{\rho\mu}\delta^{\nu}_{\rho})] \\ &= \frac{1}{2}(\Theta^{\mu\nu} - \Theta^{\nu\mu}), \end{aligned} \tag{63}$$

*i.e.*, the energy-momentum tensor is symmetric. However, as mentioned in the main text, generally this is not valid in the presence of LIV.

Here we note that the same reasoning above can also be applied to the rotational symmetry. If we only break boost invariance, then the infinitesimal symmetry transformation

$$\Psi^{l}(x) \to \Psi^{l}(x) - \frac{i}{2}w^{\mu\nu}(\mathscr{J}_{\mu\nu})^{l}{}_{m}\Psi^{m}(x) \tag{64}$$

under Lorentz invariance is replaced with

$$\Psi^{l}(x) \to \Psi^{l}(x) - \frac{i}{2}w^{ij}(\mathscr{J}_{ij})^{l}{}_{m}\Psi^{m}(x) \tag{65}$$

under rotational invariance. It is nearly the same as (64) except that $w^{\mu\nu}$ is replaced by $w^{ij}$ (where as before, Latin indices $i$, $j$ run over the three spatial coordinate labels, usually taken as 1, 2, 3; and Greek indices $\mu$, $\nu$ run over the four space-time coordinate labels, 1, 2, 3, 0).

With the above observation, it is apparent that from the Noether theorem, we can obtain the corresponding Noether current $M^{\rho}{}_{ij}$ and Noether charge $\Sigma_{ij}$. With the same reasoning, $\partial_{\rho}M^{\rho ij} = 0$ also implies $\Theta^{ij} = \Theta^{ji}$.

The symmetrizability of $\Theta^{ij}$ is valid for a generic LIV theory with rotational symmetry SO(3) unbroken, like the photon Lagrangian with an anisotropic scaling in Ref. [12]. However, we stress again that this symmetric property and SO(3) invariance is valid only in a specific inertial reference frames, because LIV implies the existence of a preferred direction. While for a specific Lagrangian (1) without CPT odd terms, we have

$$\Theta^{\mu\nu} = -[F^{\mu\alpha}F^{\nu}{}_{\alpha} + (k_F)^{\alpha\beta\mu\delta}F^{\nu}{}_{\delta}F_{\alpha\beta}] - \eta^{\mu\nu}\mathcal{L}, \tag{66}$$

so

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = -[(k_F)^{\alpha\beta\mu\delta}F^{\nu}{}_{\delta} - (k_F)^{\alpha\beta\nu\delta}F^{\mu}{}_{\delta}]F_{\alpha\beta} \tag{67}$$

and

$$\begin{aligned} \Theta^{[ij]} &= \frac{1}{2}(\Theta^{ij} - \Theta^{ji}) \\ &= \frac{1}{2}F_{\alpha\beta}[(k_F)^{\alpha\beta j\delta}F^{i}{}_{\delta} - (k_F)^{\alpha\beta i\delta}F^{j}{}_{\delta}] \\ &= \frac{1}{2}F_{kl}[((k_F)^{kljm}F^{i}{}_{m} - (k_F)^{klim}F^{j}{}_{m}) + ((k_F)^{klj0}F^{i}{}_{0} - (k_F)^{kli0}F^{j}{}_{0})] \\ &\quad + F_{k0}[((k_F)^{k0jl}F^{i}{}_{l} - (k_F)^{k0il}F^{j}{}_{l}) + ((k_F)^{k0j0}F^{i}{}_{0} - (k_F)^{k0i0}F^{j}{}_{0})]. \end{aligned} \tag{68}$$

Since rotational invariance requires only $\alpha \neq 0$, from (3) and (4), we have

$$(k_F)^{kljm} = \frac{\alpha}{2}(\delta^{kj}\delta^{lm} - \delta^{km}\delta^{lj}), \quad (k_F)^{kli0} = 0, \quad (k_F)^{k0i0} = -\frac{\alpha}{2}\delta^{ki}. \tag{69}$$

Substituting (69) into (68), one can easily find that

$$\Theta^{[ij]} = \frac{\alpha}{2}[(F_{im}F^{j}{}_{m} - F_{jm}F^{i}{}_{m}) + (F_{i0}F^{j}{}_{0} - F_{j0}F^{i}{}_{0})] = 0, \tag{70}$$

which is consistent with the requirement of rotational invariance. The same property of $\Theta^{ij}$ can also be checked for more general theories with rotational invariance, *e.g.*, the Horava-Lifschitz theory.

## Appendix B

This appendix shows how to get (20) from (19).

From (17), we have

$$dt = f_i dx^i \quad \Rightarrow \frac{\partial t}{\partial x^i} = f_i. \tag{71}$$

While from (18), we get

$$0 \equiv dw = w_i dx^i + w_t dt \quad \Rightarrow \quad \frac{\partial t}{\partial x^i} = -\frac{w_i}{w_t}, \tag{72}$$

where $w_i \equiv \partial w/\partial x^i$ and $w_t \equiv \partial w/\partial t$. Then from (71) and (72), one can easily obtain $\triangledown_i f = -w_i/w_t$. So we have

$$1 = (\triangledown f)^2 = \sum_{i=1}^{3}(w_i/w_t)^2 = (\triangledown w)^2/(w_t)^2. \tag{73}$$

## Appendix C

In this appendix we show that the eikonal equations we got from (38) and (39) are the same, and hence are consistent with the general arguments in the main text: as the derivation roots in the classical causality analysis, and **E**, **B** fields are components of the $3+1$ decomposition of $F^{\mu\nu}$, hence being connected with each other through the Maxwell equations, the eikonal equations derived from the equations of **E** and **B** must be the same.

First, let us focus on the case $(\beta_E)^{ij} = 0$. The consistent check of this simpler case will be a little different from the case $(\beta_E)^{ij} \neq 0$ . For $(\beta_E)^{ij} = 0$, we denote the corresponding matrices of (38) and (39) as $M_{eB}$ and $M_{bB}$ respectively, *i.e.*,

$$(M_{eB})^{ij} = [1 - (\triangledown f)^2]\delta^{ij} - \epsilon^{ink}\epsilon^{jml} f_n f_m (\beta_B)^{kl}, \tag{74}$$

$$(M_{bB})^{ij} = [1 - (\triangledown f)^2]\delta^{ij} - (\triangledown f)^2(\beta_B)^{ij} - f_i f_k (\beta_B)^{kj}. \tag{75}$$

If we assign

$$\beta_B = \begin{pmatrix} b_4 & b_1 & b_2 \\ b_1 & b_5 & b_3 \\ b_2 & b_3 & -(b_4+b_5) \end{pmatrix}, \tag{76}$$

the matrices above take the explicit matrix forms as below

$$(M_{eB}) = \begin{pmatrix} [1-(\triangledown f)^2] + f_2^2(b_4+b_5) - f_3^2 b_5 + 2f_2f_3b_3 & f_3^2 b_1 - f_2f_1(b_4+b_5) - f_3(b_3f_1+b_2f_2) & f_1f_3b_5 + b_2f_2^2 - f_2(b_3f_1+b_1f_3) \\ f_3^2 b_1 - f_2f_1(b_4+b_5) - f_3(b_3f_1+b_2f_2) & [1-(\triangledown f)^2] + f_1^2(b_4+b_5) - f_3^2 b_4 + 2f_1f_3b_2 & f_2f_3b_4 + b_3f_1^2 - f_1(b_2f_2+b_1f_3) \\ f_1f_3b_5 + b_2f_2^2 - f_2(b_3f_1+b_1f_3) & f_2f_3b_4 + b_3f_1^2 - f_1(b_2f_2+b_1f_3) & [1-(\triangledown f)^2] - (f_2^2b_4 + b_5f_1^2) + 2f_2f_1b_1 \end{pmatrix}$$

$$(M_{bB}) = \begin{pmatrix} [1-(1+b_4)(\triangledown f)^2] + f_1(b_4f_1+b_1f_2+b_2f_3) & f_1(b_1f_1+b_5f_2+b_3f_3) - b_1(\triangledown f)^2 & f_1(b_2f_1+b_3f_2-(b_4+b_5)f_3) - b_2(\triangledown f)^2 \\ f_2(b_4f_1+b_1f_2+b_2f_3) - b_1(\triangledown f)^2 & [1-(1+b_5)(\triangledown f)^2] + f_2(b_1f_1+b_5f_2+b_3f_3) & f_2(b_2f_1+b_3f_2-(b_4+b_5)f_3) - b_3(\triangledown f)^2 \\ f_3(b_4f_1+b_1f_2+b_2f_3) - b_2(\triangledown f)^2 & f_3(b_1f_1+b_5f_2+b_3f_3) - b_3(\triangledown f)^2 & [1-(1-b_4-b_5)(\triangledown f)^2] + f_3(b_2f_1+b_3f_2-(b_4+b_5)f_3) \end{pmatrix}.$$

Note that $(M_{eB})^T = M_{eB}$, *i.e.*, $M_{eB}$ is a symmetric matrix, while $M_{bB}$ is not. So it is not a trivial check that $\mathrm{Det}[M_{eB}] = \mathrm{Det}[M_{bB}]$ by direct calculation. Imposing the requirements that their determinants be equal to zero, one can obtain the same equations, as expected. Since the results are tedious, we do not present them here. On the other hand, if we allow only $(\beta_B)^{21} = (\beta_B)^{12} \neq 0$, one can easily get back to (45) and (47). This can be regarded as another consistent check.

For the case $(\beta_E)^{ij} \neq 0$, as the calculation involves the inverse of $(1+\beta_E)^{ij}$, *i.e.*, $W^{ij} = [(1+\beta_E)^{-1}]^{ij}$, the results will be more tedious than the previous ones. So we also do not present the details here. We just make two remarks.

(1) As the calculation of (39) involves the matrix $W^{ij}$, while that of (38) does not, the determinants of (38) and (39) are indeed not the same, contrary to the previous case. However, note that the requirement of the same eikonal equations is derived from the vanishing of the determinants of matrices (38) and (39). This requires only that the determinants of these two matrices are proportional to each other, not necessarily being the same. In fact, one can show that they differ only by a constant, *i.e.*,

$$\mathrm{Det}[M_e] = \mathrm{Det}[M_b] \cdot \mathrm{Det}[1 + \beta_E]. \tag{77}$$

So as long as $\mathrm{Det}[1 + \beta_E] \neq 0$, the differential equations obtained from $\mathrm{Det}[M_e] = 0$ and $\mathrm{Det}[M_b] = 0$ must be the same.

(2) By taking $\beta_E = 0$, the solution reduces to those obtained from (74) or (75), except for a missing factor $1 - (\nabla f)^2$. This can be traced back to solving field equations, and our calculation is implicitly equivalent to that procedure. In solving Maxwell equations, in order to obtain the independent equations of the two physical degrees of freedom, we need to choose a gauge, *e.g.*, Coulomb gauge, to eliminate the gauge degrees of freedom (see Appendix D). In this process, we eliminate one polarization and leave a constraint, which in turn gives an identity. Eventually, we are left with only two independent equations, corresponding to the two physical polarizations. The eliminated identity in this process then corresponds to the lacking factor in our method. In our derivation, it is the $f_i f_j$ factor in (38) (or the $-\epsilon^{ink}\epsilon^{jml} f_n f_m W^{kl}$ factor in (39)) playing the role of gauge constraints to remove $1 - (\nabla f)^2$ in (77).

In conclusion, we see that, as expected, the calculation from either (38) or (39) indeed leads to the same eikonal equations, and the derivation is also consistent with solving field equations. Thus it is natural to get from the eikonal equations two independent dispersion relations, corresponding to two physical polarizations, see *e.g.*, (54).

## Appendix D

In this appendix we give an explicit calculation of the energy-momentum flow velocity $\boldsymbol{v}_e^i = \Theta^{i0}/\Theta^{00}$.

Using the ansatz $A^\mu(x) = \epsilon^\mu(p)\exp[-i(p^0 t - \boldsymbol{p}\cdot\boldsymbol{x})]$ and choosing the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, we get $\phi = 0$ as a special solution from $\nabla \cdot \boldsymbol{E} = 0$ (*i.e.*, the first equation of (40)). Then by substituting Eq. (7) into Eqs. (40) and (41), we find that (41) is satisfied automatically. Then from the left equation of (40), we get

$$\begin{aligned}
&-p^2 A_1 + \sigma p_3(p_2 A_3 - p_3 A_2) = 0,\\
&-p^2 A_2 + \sigma p_3(p_1 A_3 - p_3 A_1) = 0,\\
&-(p^2 + 2\sigma p_1 p_2) A_3 + \sigma(p_1 A_2 + p_2 A_1) p_3 = 0.
\end{aligned} \tag{78}$$

By imposing the Coulomb gauge $p_i A_i = 0$, the equations above can be reduced to

$$\begin{pmatrix} p^2 + \sigma p_1 p_2 & \sigma(p_3^2 + p_2^2) \\ \sigma(p_3^2 + p_1^2) & p^2 + \sigma p_1 p_2 \end{pmatrix} \left\{ \begin{matrix} A_1 \\ A_2 \end{matrix} \right\} = 0. \tag{79}$$

Note that, in this appendix, we do not distinguish the upper and lower indices, *e.g.*, $A_1 = A^1$. From the existence of non-zero solutions of (79), one can easily get the dispersion relation (54). To calculate the energy-momentum flux velocity, we use the real component of our previous ansatz instead, *i.e.*, $A_1(t, \boldsymbol{r})_\pm = W_\pm \cos[w(t - \boldsymbol{s}\cdot\boldsymbol{r}/v_{p\pm})]$, where $\boldsymbol{s}$ denotes the unit vector pointing to the direction of propagation and $v_{p\pm}$ denote the two independent phase velocities of the two modes in (54). With the help of (79), one can express $\boldsymbol{A}$ in terms of $A_1$, *i.e.*,

$$A_2 = -\frac{p^2 + \sigma p_1 p_2}{\sigma(p_3^2 + p_2^2)} A_1, \quad A_3 = -\frac{(p_2 A_2 + p_1 A_1)}{p_3}. \tag{80}$$

Substituting these back to (7), we can obtain the explicit forms of $\mathbf{E}$ and $\mathbf{B}$, where we have assigned

$$\boldsymbol{s} = (\frac{1}{\sqrt{2}}\cos[\theta],\ \frac{1}{\sqrt{2}}\cos[\theta],\ \sin[\theta]). \tag{81}$$

Then by substituting the explicit forms of $\mathbf{E}$ and $\mathbf{B}$ into (16) and (14), we find that indeed $|\boldsymbol{v}_{e\pm}^i| = v_{g\pm}$ for each mode.

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