Minimal Length and Bouncing Particle Spectrum

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Abstract

In this paper we study the effects of the Generalized Uncertainty Principle (GUP) on the spectrum of a particle that is bouncing vertically and elastically on a smooth reflecting floor in the Earth's gravitational field (a quantum bouncer). We calculate energy levels and corresponding wave functions of this system in terms of the GUP parameter. We compare the outcomes of our study with the results obtained from elementary quantum mechanics. A potential application of the present study is discussed finally.

Keywords: Quantum Gravity; Generalized Uncertainty Principle; Bouncing Particle Spectrum.

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1 Introduction

The Generalized Uncertainty Principle (GUP) is a common feature of all promising candidates of quantum gravity. String theory, loop quantum gravity, black hole physics, and noncommutative geometry (based on a deeper insight to the nature of quantum spacetime at Planck scale) all support the need for a necessary modification of the standard Heisenberg principle. It has been shown that measurements in quantum gravity are governed by the generalized uncertainty principle [1–10]. There are some evidences from string theory and black holes physics (based on some gedanken experiments), that lead authors to re-examine the usual uncertainty principle of Heisenberg [11]. These evidences have origin on the quantum fluctuation of the background spacetime metric and are related to the very nature of spacetime in quantum gravity era. The introduction of this idea has drown attention and many authors considered

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various problems in the framework of the generalized uncertainty principle (see for instance [12–26]). These investigations have revealed some new features of the very nature of spacetime: spacetime is not commutative at Planck scale and it has a foam-like structure in this scale, it seems that gravity is not a fundamental interaction of the nature (it may be induced by the residual effects of fundamental quantum fields on the vacuum, with the Lagrangian playing the role of an elastic stress), constants of the nature are not really constant and the very notion of locality in position space representation breaks down in Planck scale. Therefore, it seems that a reformulation of quantum theory is required in order to incorporate gravitational effects in Planck scale phenomena. These issues have been the topics of a wide range of researches in recent years.

In this paper, we consider the problem of a particle of mass m that is bouncing vertically and elastically on a smooth reflecting floor in Earth's gravitational field. We solve this problem in the presence of a minimal length within the GUP framework. First, we give an overview to the GUP formalism and in this way we obtain a generalized Schrödinger equation. After studying this equation in the momentum space, we find the modified eigenstates and energy spectrum of this system. We compare the outcomes of our study with the results obtained from elementary quantum mechanics. A potential application of the present study in the spontaneous decay of an excited state for Ultra Cold Neutrons bouncing above a perfect mirror in the Earth's gravitational field is discussed finally. We note that modification to the decay rate due to existence of a minimal length studied here, becomes important at or above the Planck energy. Although this modification is too small to be measurable at present, we speculate on the possibility of extracting measurable predictions in the future.

2 A Generalized Uncertainty Principle

Quantum mechanics with modification of the usual canonical commutation relations has been investigated intensively in the last few years (see [27] and references therein). Such works which are motivated by several independent streamlines of investigations in string theory and quantum gravity, suggest the existence of a finite lower bound to the possible resolution ΔX of spacetime points. The following

deformed commutation relation has attracted much attention in recent years [27]

$$[X, P] = i\hbar(1 + \beta P^2), \tag{1}$$

and it was shown that it implies the existence of a minimal resolution length $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} \geq \hbar\sqrt{\beta}$ [27]. This means that there is no possibility to measure coordinate X with accuracy smaller than $\hbar\sqrt{\beta}$. Since in the context of the string theory the minimum observable distance is the string length, we conclude that $\sqrt{\beta}$ is proportional to this length. If we set $\beta = 0$, the usual Heisenberg algebra is recovered. The use of the deformed commutation relation (1) brings new difficulties in solving the quantum problems. A part of difficulties is related to the break down of the notion of locality and position space representation in this framework [27]. The above commutation relation results in the following uncertainty relation

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left(1 + \beta(\Delta P)^2 + \gamma\right), \tag{2}$$

where β and γ are positive quantities which depend on the expectation value of the position and the momentum operators. In fact, we have $\beta = \beta_0/(M_{Pl}c)^2$ where M_{Pl} is the Planck mass and β_0 is of the order of unity. We expect that these quantities are only relevant in the domain of the Planck energy $M_{Pl}c^2 \sim 10^{19}GeV$. Therefore, in the low energy regime, the parameters β and γ are irrelevant and we recover the well-known Heisenberg uncertainty principle. These parameters, in principle, can be obtained from the underlying quantum gravity theory such as string theory.

Note that X and P are symmetric operators on the dense domain S_∞ with respect to the following scalar product [27]

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{dP}{1 + \beta P^2}\psi^*(P)\phi(P). \tag{3}$$

Moreover, the comparison between Eqs. (1) and (2) shows that $\gamma = \beta\langle P\rangle^2$. Now, let us define

$$\begin{cases} X = x, \\ P = p\left(1 + \frac{1}{3}\beta\, p^2\right), \end{cases} \tag{4}$$

where x and p obey the canonical commutation relations $[x, p] = i\hbar$. One can check that using Eq. (4), Eq. (1) is satisfied to $\mathcal{O}(\beta)$. Also, from the above equation we can interpret p as the momentum operator

at low energies ($p = -i\hbar\partial/\partial x$) and P as the momentum operator at high energies. Now, consider the following form of the Hamiltonian:

$$H = \frac{P^2}{2m} + V(x), \tag{5}$$

which using Eq. (4) can be written as

$$H = H_0 + \beta H_1 + \mathcal{O}(\beta^2), \tag{6}$$

where $H_0 = \frac{p^2}{2m} + V(x)$ and $H_1 = \frac{p^4}{3m}$.

In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in the quasi-position representation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + \beta\frac{\hbar^4}{3m}\frac{\partial^4\psi(x)}{\partial x^4} + V(x)\psi(x) = E\psi(x), \tag{7}$$

where the second term is due to the generalized commutation relation (1). This equation is a 4th-order differential equation which in principle admits 4 independent solutions. Therefore, solving this equation in x space and separating the physical solutions is not an easy task. In the next section, for the case of a Bouncing Particle, we find the energy spectrum and the corresponding eigenstates up to the first order of the GUP parameter.

3 Spectrum of a Quantum Bouncer in the GUP Scenario

Consider a particle of mass m which is bouncing vertically and elastically on a reflecting hard floor so that

$$V(X) = \begin{cases} mgX & \text{if} \quad X > 0, \\ \\ \infty & \text{if} \quad X \leq 0, \end{cases} \tag{8}$$

where g is the acceleration in the Earth's gravitational field. The Hamiltonian of the system is

$$H = \frac{P^2}{2m} + mgX, \tag{9}$$

which results in the following generalized Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + \beta\frac{\hbar^4}{3m}\frac{\partial^4\psi(x)}{\partial x^4} + mgx\psi(x) = E\psi(x). \tag{10}$$

This equation, for $\beta = 0$ is exactly solvable and the solutions can be written in the form of the Airy functions. Moreover, the energy eigenvalues are related to the zeros of the Airy function. However, for $\beta \neq 0$, the situation is quite different. Because, we need to solve a forth order differential equation and eliminate the unphysical solutions. On the other hand, because of the linear form of the potential, this equation can be cast into a first order differential equation in the momentum space. Since the later form is much easier to handle, we define a new variable $z = x - \frac{E}{mg}$ and rewrite above equation in the momentum space, namely

$$\frac{p^2}{2m}\phi(p) + \beta\frac{p^4}{3m}\phi(p) + i\hbar mg\phi'(p) = 0, \tag{11}$$

where $\phi(p)$ is the inverse Fourier transform of $\psi(z)$ and the prime denotes the derivative with respect to p. It is straight forward to check that this equation admits the following solution

$$\phi(p) = \phi_0 \exp\left[\frac{i}{6m^2g\hbar}\left(p^3 + \frac{2\beta}{5}p^5\right)\right]. \tag{12}$$

Since β is a small quantity, we can expand the above solution up to the first order of β as

$$\phi(p) \simeq \phi_0 \exp\left(\frac{ip^3}{6m^2g\hbar}\right)\left(1 + \frac{i\beta p^5}{15m^2g\hbar} + \mathcal{O}(\beta^2)\right). \tag{13}$$

Now, using the Fourier transform, we can find the solution in the position space up to a normalization factor

$$\begin{aligned}\psi(x) &= \mathrm{Ai}\left[\alpha\left(x - \frac{E}{mg}\right)\right] + \frac{4}{15}\beta m^2 g\left(x - \frac{E}{mg}\right) \\ &\times \left\{4\mathrm{Ai}\left[\alpha\left(x - \frac{E}{mg}\right)\right] + \left(x - \frac{E}{mg}\right)\right. \\ &\times \left.\mathrm{Ai}'\left[\alpha\left(x - \frac{E}{mg}\right)\right]\right\},\end{aligned} \tag{14}$$

where $\alpha = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}$ and the prime denotes derivative with respect to x. Finally, since the potential is infinite for $x \leq 0$, we demand that the wave function should vanish at $x = 0$. This condition results in the quantization of the particle's energy, namely

$$\begin{aligned}\mathrm{Ai}\left(-\frac{\alpha E_n}{mg}\right) - \frac{4}{15}\beta m E_n &\left[4\mathrm{Ai}\left(-\frac{\alpha E_n}{mg}\right)\right. \\ &\left. -\frac{E_n}{mg}\mathrm{Ai}'\left(-\frac{\alpha E_n}{mg}\right)\right] = 0.\end{aligned} \tag{15}$$

To proceed further and for the sake of simplicity, let us work in the units of $g = 2\hbar = 4m = 2$. In this set of units, the energy eigenvalues are the minus of the roots of the following algebraic equation

$$\mathrm{Ai}\,(x) + \frac{2}{15}\beta x\left[4\mathrm{Ai}\,(x) + x\mathrm{Ai}'\,(x)\right] = 0. \tag{16}$$

So, the energy eigenvalues will be quantized and result in the following eigenfunctions

$$\begin{aligned}\psi_n(x) &= \mathrm{Ai}\,(x - E_n) + \frac{2}{15}\beta(x - E_n)\left[4\mathrm{Ai}\,(x - E_n)\right.\\ &+ \left.(x - E_n)\mathrm{Ai}'\,(x - E_n)\right],\end{aligned} \tag{17}$$

where E_n s should satisfy Eq. (16). Figure 1 shows the resulting normalized ground state and first excited state eigenfunctions for $\beta = 0\,, 0.1,\, 0.2$. Moreover, the calculated values of the energy eigenvalues for the first ten states are also shown in Table 1. These results show that the presence of β increases the energy levels in agreement with the functional form of H_1. In other words, existence of a minimal length results in a positive shift in the energy levels of quantum bouncer.

4 A potential Application: Transition Rate of a Quantum Bouncer

As a potential application of our analysis, we note that quantization of the energy of *Ultra Cold Neutrons* bouncing above a mirror in the Earth's gravitational field has been demonstrated in an experiment few years ago [28]. This effect demonstrates quantum behavior of the gravitational field if we consider the spontaneous decay of an excited state in this experiment as a manifestation of the Planck-scale effect [29]. Since the spectrum of a quantum bouncer changes in the presence of the minimal length, we expect the rate of this decay will change as a trace of quantum gravitational effects via existence of a minimal length scale. In fact, as we have shown, the energy levels of a quantum bouncer in the GUP framework attain a positive shift as given by equation (15). The energy levels of a quantum bouncer in ordinary quantum mechanics are given by zeros of Airy function as $\lambda_n \approx \left(\frac{3\pi}{8}(4n-1)\right)^{2/3}$. In the presence of a minimal length, the locations of zeros are given by $\lambda_n^{(\mathrm{GUP})} = \lambda_n + \Delta\lambda_n$. Within a semi-classical analysis, we evaluate the rate for a bouncer to make a transition $k \longrightarrow n$. The quantum quadrupole moment for the transition $k \longrightarrow n$ for a quantum bouncer of mass m is given by $Q_{kn} = m\langle k|X^2|n\rangle$ [29]. The quantum

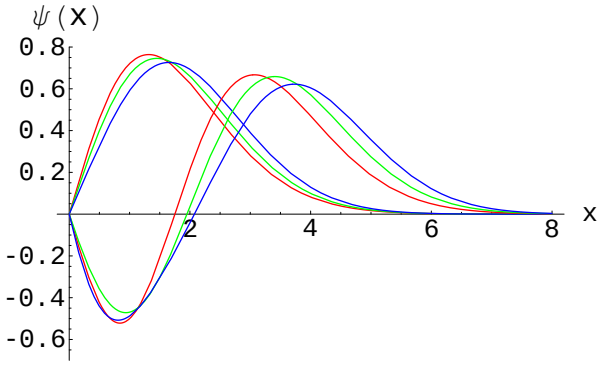

Figure 1: The normalized ground state and first excited state eigenfunctions of a bouncing particle in the framework of the generalized commutation relation (1) for $\beta = 0$ (red), $\beta = 0.1$ (green), and $\beta = 0.2$ (blue).

mechanical transition rate is (in the quadrupole approximation)

$$\Gamma_{k\longrightarrow n} = \frac{4}{15}\frac{\omega_{kn}^5}{M_{Pl}^2 c^4} Q_{kn}^2, \tag{18}$$

where $\omega_{kn} = (E_k - E_n)/\hbar$. Now, the quadrupole matrix element for quantum bouncer in the presence of minimal length can be calculated using the generalized Airy function zeros given by equation (15). The transition probability in our framework is therefore

$$\begin{aligned} \Gamma_{k\longrightarrow n}^{(\mathrm{GUP})} &= \frac{512}{5}\frac{\left(\lambda_k^{(\mathrm{GUP})} - \lambda_n^{(\mathrm{GUP})}\right)^5}{\left(\lambda_k - \lambda_n\right)^8}\left(\frac{m}{M_{Pl}}\right)^2 \frac{E_0^5 c}{\alpha^4(\hbar c)^5}, \\ &= (\Gamma_{k\longrightarrow n})\left(1 + \frac{5\Delta\lambda_{kn}}{\lambda_k - \lambda_n}\right) \end{aligned} \tag{19}$$

where $E_0 = mg/\alpha$, $\Delta\lambda_{kn} = \Delta\lambda_k - \Delta\lambda_n$, and $\Gamma_{k\longrightarrow n} = \frac{512}{5}\left(\frac{m}{M_{Pl}}\right)^2 \frac{E_0^5 c}{\alpha^4(\hbar c)^5}$. So, there will be an essentially measurable difference in the transition rate of a quantum bouncer due to the presence of the extra $\Delta\lambda_{kn}$ in comparison with the case that we consider just the ordinary Heisenberg uncertainty relation.

5 Conclusions

In this paper, we considered the problem of a bouncing particle in a constant gravitational field in the framework of the generalized uncertainty principle. We found the modified Hamiltonian and the generalized Schrödinger equation as a forth order differential equation. We solved this equation in the momentum space and obtained the corresponding energy eigenvalues and eigenstates up to the first order

Table 1: The first ten quantized energies of a bouncing particle in GUP formalism.

n	$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$
0	2.338	2.428	2.570
1	4.088	4.380	4.644
2	5.521	5.947	6.107
3	6.787	7.257	7.352
4	7.944	8.420	8.483
5	9.023	9.493	9.536
6	10.040	10.499	10.532
7	11.008	11.456	11.481
8	11.936	12.371	12.391
9	12.829	13.253	13.269

of the GUP parameter. As we have expected, we found a positive shift in the energy spectrum due to the generalized commutation relation. A potential application of this analysis for transition rate of an Ultra Cold Neutron bouncing above a mirror in the Earth's gravitational field has been explained. We emphasize that modification to the transition rate of a quantum bouncer due to existence of a minimal length becomes important at or above the Planck energy. Although this modification is too small to be measurable at present, we speculate on the possibility of extracting measurable predictions in the future. At that case, this may provide a direct test of underlying quantum gravity scenario.

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