

Open M2-branes with Flux and Modified Basu-Harvey Equation

Chong-Sun Chu and Gurdeep S. Sehmbi

Centre for Particle Theory and Department of Mathematical Sciences, Durham University, Durham, DH1 3LE, UK

E-mail: chong-sun.chu@durham.ac.uk, g.s.sehmbi@durham.ac.uk

ABSTRACT: The supersymmetric actions of closed multiple M2 branes with flux for the BL and the ABJM theories have been constructed recently by Lambert and Richmond in [1]. In this paper we extend the construction to the case of open M2-branes with flux and derive the boundary conditions. This allows us to derive the modified Basu-Harvey equation in the presence of flux. As an example, we consider the Lorentzian BL model. A new feature of the fuzzy funnel solution describing a D2-D4 intersection is obtained as a result of the flux.

KEYWORDS: M-Theory, D-branes, M-branes.

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1. Introduction

The understanding of the physics of branes in M-theory is one of the most intriguing and mysterious tasks in string/M theory, see for example [2]. Recently tremendous progress has been made in the description of multiple M2-branes [3–8]. A major part of the excitement is due to the employment of a novel mathematical structure, the Lie 3-algebra, in the description of the gauge symmetry of the parallel M2-branes. Although the Lie 3-algebra is not essential in the $\mathcal{N} = 6$ description of the M2-branes [7], there is evidence [9–15] that it may have a deeper connection to M-theory in general.

In a recent paper, based on the earlier works [16–19], Lambert and Richmond [1] were able to construct a coupling of closed multiple M2-branes to specific configurations of background 3-form and 6-form gauge fields of eleven-dimensional supergravity. The coupling makes essential use of the underlying 3-algebra structure. The flux configuration considered in [1] is self-dual in the space transverse to the M2-branes and gives rise to a mass term and its supersymmetrization on the worldvolume theory of the M2-branes.

In this paper we extend the construction of [1] to the open case and derive the supersymmetric boundary conditions. As advocated in [10, 20], see also [21], the boundary condition can be interpreted as an equation of motion for the boundary fields and hence, in the case of the scalar fields, can be understood as describing the non-trivial shape of the boundary of the M2-branes. In particular, for a certain specific configuration, the boundary condition describes the M2-brane ending on an M5-brane and hence can be identified with the Basu-Harvey equation [22]. We show that this continues to be the case in the presence of the flux background.

The plan of the paper is as follows. In section 2, we analyse the Bagger-Lambert (BL) action coupled to flux and obtain the supersymmetric boundary condition. The boundary condition only has a trivial solution in general. However for specific configurations of the scalar fields we consider, the boundary condition is non-trivial. For example, one obtains a mass deformed Basu-Harvey equation which describes a system of M2-branes ending on an M5-brane in the presence of a background flux. The analysis of the boundary condition also asserts the absence of supersymmetric M2-M9 intersection in the presence of flux. In section 3, we perform the same analysis for the ABJM theory with flux. In section 4, we consider the flux modified BL theory with Lorentzian 3-algebras. In the closed case, the theory is equivalent to the $\mathcal{N} = 8$ supersymmetric Yang-Mills theory; thanks to the complete decoupling of one of the scalar fields, say X^{10} , of the eight scalar fields in the BL theory. This is no longer automatic in the open case and a boundary condition has to be chosen to achieve this. The resulting theory then describes multiple D2-branes in a mixed NS-NS and R-R flux background. The system is however not supersymmetric in general since the decoupling boundary condition generally breaks the supersymmetry. We consider a particular configuration of the scalar fields and show that a supersymmetric boundary condition can be obtained. We show that this describes a system of D2-branes ending on a D4-brane in the the background flux. New features of the fuzzy funnel solution are also discussed.

2. Boundary Condition for the BL Theory Coupled to Flux

2.1 $\mathcal{N} = 8$ closed M2-branes in flux background

In this subsection, we review the construction [1] of the supersymmetric action which describes the coupling of multiple closed M2-branes to a certain configuration of background gauge field. In the limit of large T_{M2} , the Lagrangian consists of a flux and a mass term modification to the Bagger-Lambert Lagrangian $\mathcal{L}_{\mathcal{N}=8}$:

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=8} + \mathcal{L}_{flux} + \mathcal{L}_{mass}, \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=8} = & -\frac{1}{2}\text{Tr}(D^\mu X^I, D_\mu X^I) + \frac{i}{2}\text{Tr}(\bar{\Psi}, \Gamma^\mu D_\mu \Psi) + \frac{i}{4}\text{Tr}(\bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]) \\ & -\frac{1}{12}\text{Tr}([X^I, X^J, X^K], [X^I, X^J, X^K]) + \mathcal{L}_{CS}, \end{aligned} \quad (2.2)$$

$$\mathcal{L}_{flux} = c\tilde{G}_{IJKL}\text{Tr}(X^I, [X^J, X^K, X^L]), \quad (2.3)$$

$$\mathcal{L}_{mass} = -\frac{1}{2}m^2\delta_{IJ}\text{Tr}(X^I, X^J) + b\text{Tr}(\bar{\Psi}\Gamma^{IJKL}, \Psi)\tilde{G}_{IJKL} \quad (2.4)$$

and $\text{Tr}(\cdot, \cdot)$ is the metric for the Lie 3-algebra. The background gauge field has the transverse components G_{IJKL} turned on and \tilde{G}_{IJKL} is defined by

$$\tilde{G}_{IJKL} = \frac{1}{4!}\epsilon_{IJKLMNPQ}G^{MNPQ}, \quad (2.5)$$

where $I, J, K, L = 3, 4, \dots, 10$. The supersymmetry transformation is given by $\delta = \delta_0 + \delta'$ where δ_0 is the supersymmetry transformation of the original $\mathcal{N} = 8$ theory,

$$\delta_0 X_a^I = i\bar{\epsilon}\Gamma^I\Psi_a, \quad (2.6)$$

$$\delta_0 \tilde{A}_\mu^b{}_a = i\bar{\epsilon}\Gamma_\mu\Gamma_I X_c^I \Psi_d f^{cdb}{}_a, \quad (2.7)$$

$$\delta_0 \Psi_a = D_\mu X_a^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X_b^I X_c^J X_d^K f^{bcd}{}_a \Gamma^{IJK} \epsilon, \quad (2.8)$$

and δ' is the additional contribution to the supersymmetry transformations due to the flux

$$\delta' X_a^I = 0, \quad (2.9)$$

$$\delta' \tilde{A}_\mu^b{}_a = 0, \quad (2.10)$$

$$\delta' \Psi_a = \omega \Gamma^{IJKL} \Gamma^M \epsilon X_a^M \tilde{G}_{IJKL}. \quad (2.11)$$

Here Ψ and ϵ are eleven dimensional spinors satisfying the conditions

$$\Gamma_{012}\Psi = -\Psi, \quad (2.12)$$

$$\Gamma_{012}\epsilon = \epsilon. \quad (2.13)$$

Inclusion of the effects of the backreaction of the flux implies that $c = 2$ [1]. Supersymmetry requires the coefficients ω and b to be determined by the flux term

$$\omega = \frac{c}{8}, \quad b = -i\frac{c}{16}. \quad (2.14)$$

Moreover, the flux \tilde{G}_{IJKL} has to be self-dual, which implies that

$$\Gamma^{012}\tilde{\mathcal{F}} = \tilde{\mathcal{F}}, \quad (2.15)$$

where $\tilde{G} \equiv \Gamma^{IJKL}\tilde{G}_{IJKL}$. It also needs to satisfy the condition

$$\tilde{G}\tilde{G} = 8m^2(1 + \Gamma^{3456789(10)}), \quad (2.16)$$

which implies immediately that

$$G_{MN[IJG_{KL}]^{MN}} = 0 \quad \text{and} \quad m^2 = \frac{c^2}{32 \cdot 4!}G^2. \quad (2.17)$$

The self-duality condition is solved by \tilde{G} of the form $\tilde{G} = d(1 + \Gamma^{012})R$, where d is a constant coefficient and R is a sum of products of four transverse Γ^I 's, $I = 3, 4, \dots, 10$. The condition (2.16) then implies that

$$\tilde{G} = 2\mu \frac{1 + \Gamma^{012}}{2}R, \quad R^2 = 1, \quad (2.18)$$

for $\mu = \pm 2m$, $m \geq 0$. A simple solution is

$$R = \Gamma^{3456}. \quad (2.19)$$

This corresponds to the flux

$$\tilde{G} = \mu dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + \text{dual}, \quad (2.20)$$

and the Lagrangian (2.1) reproduces precisely the deformed Bagger-Lambert Lagrangian of [16] and [17].

2.2 Flux modified supersymmetric boundary condition

Next we want to consider the open case of the flux modified BL theory and derive the boundary condition. Note that in the above derivation of the supersymmetric invariance of the action, boundary contributions have been dropped due to the closedness of the M2-branes. These boundary terms have to be kept carefully in the presence of a boundary. It is easy to see that these contributions arise from the fermion and scalar kinetic terms in the Lagrangian $\mathcal{L}_{\mathcal{N}=8}$. We have

$$\delta\mathcal{L} = \frac{i}{2}\partial_\mu \text{Tr}(\bar{\Psi}\Gamma^\mu, \delta\Psi) - \partial_\mu \text{Tr}(\delta X^I, D^\mu X^I) + \text{bulk terms}, \quad (2.21)$$

where the ‘‘bulk terms’’ denote non-total derivative terms and are precisely equal to zero when the conditions (2.14)–(2.17) are satisfied. To proceed, let us consider the M2-branes to have a boundary at $\sigma_2 = 0$. We have

$$\delta \int d^3\sigma \mathcal{L} = \frac{i}{2} \int d^2\sigma (\text{Tr}(\bar{\Psi}\Gamma^2, \delta\Psi) - 2\text{Tr}(D_2 X^I, \bar{\Psi}\Gamma^I \epsilon)). \quad (2.22)$$

We obtain the boundary condition

$$0 = D_\alpha X^I \bar{\Psi} \Gamma^2 \Gamma^\alpha \Gamma^I \epsilon - \frac{1}{6} [X^I, X^J, X^K] \bar{\Psi} \Gamma^2 \Gamma^{JK} \epsilon + \frac{1}{4} X^M \bar{\Psi} \Gamma^2 \tilde{Q} \Gamma^M \epsilon - D_2 X^I \bar{\Psi} \Gamma^I \epsilon, \quad (2.23)$$

where $\alpha = 0, 1$ and the trace Tr is understood. This is the most general supersymmetric boundary condition one may have for a system of open M2-branes in our flux background. In general, due to the different number of Γ -matrices in each term, the equation (2.23) generically only has trivial solution. Non-trivial solutions can be obtained only when additional conditions are imposed on the matter fields and on the supersymmetry parameters.

The analysis of the boundary condition in the absence of flux was performed in [23], where the solutions to the boundary condition are classified according to the number of scalars obeying a Dirichlet condition (or more precisely being set equal to zero). In the following we perform a similar analysis for the boundary condition (2.23) with flux and determine what 1/2 BPS configurations are allowed as endpoint of the system of open M2-branes. To be specific, we will consider the flux configuration (2.20).

2.2.1 Half Dirichlet: flux modified Basu-Harvey equation

This case corresponds to an ansatz where half of the scalar fields are set to zero, for example,

$$X^{3,4,5,6} = 0. \quad (2.24)$$

This means that we have reduced the $SO(8)$ to an $SO(4)$ R-symmetry. Let us also impose the projection condition

$$\Gamma^{01789(10)} \epsilon = \epsilon. \quad (2.25)$$

It follows immediately that

$$\Gamma^{ijk} \epsilon = \varepsilon^{ijkl} \Gamma^2 \Gamma^l \epsilon, \quad i, j, k, l = 7, 8, 9, 10. \quad (2.26)$$

We also have

$$\Gamma^2 \tilde{Q} \epsilon = 2\mu \epsilon. \quad (2.27)$$

The boundary condition (2.23) is then reduced to

$$0 = D_\alpha X^i \bar{\Psi} \Gamma^2 \Gamma^\alpha \Gamma^i \epsilon - \frac{1}{6} \varepsilon^{ijkl} \bar{\Psi} \Gamma^l \epsilon [X^i, X^j, X^k] + \frac{\mu}{2} \bar{\Psi} \Gamma^i \epsilon X^i - D_2 X^i \bar{\Psi} \Gamma^i \epsilon. \quad (2.28)$$

We note that the first term in (2.28) is identically zero if we also impose the condition on the fermion

$$\Gamma^{01789(10)} \Psi = -\Psi. \quad (2.29)$$

This reduction in the degrees of freedom is compatible with the 1/2 BPS nature of the projector (2.25). As a result, we obtain the boundary equation of motion

$$D_2 X^i = -\frac{1}{6}\varepsilon^{ijkl}[X^j, X^k, X^l] + \frac{\mu}{2}X^i \quad (2.30)$$

for $i, j, k, l = 7, 8, 9, 10$. This is the Basu-Harvey equation modified by the flux (2.20).

We must now check that the boundary conditions (2.24), (2.29) and (2.30) are supersymmetric invariant. Indeed it is easy to verify that

$$\tilde{\delta}X^{i'} = 0, \quad i' = 3, 4, 5, 6 \quad (2.31)$$

and

$$(1 + \Gamma^{01789(10)})\tilde{\delta}\Psi = 0 \quad (2.32)$$

using the conditions (2.25) and (2.29). As for (2.30), supersymmetry requires the fermionic boundary equation

$$D_2\Psi + \frac{1}{2}\Gamma^2\Gamma_{ij}[\Psi, X^i, X^j] - \frac{\mu}{2}\Psi = 0. \quad (2.33)$$

Note that if instead of (2.25), we preserve the other half of supersymmetry

$$\Gamma^{01789(10)}\epsilon = -\epsilon, \quad (2.34)$$

then exactly the same analysis results in the other Basu-Harvey equation.

$$D_2 X^i = \frac{1}{6}\varepsilon^{ijkl}[X^j, X^k, X^l] - \frac{\mu}{2}X^i. \quad (2.35)$$

This is equivalent to (2.30) with $x^2 \rightarrow -x^2$.

The modified Basu-Harvey equation can also be derived as the Bogomoln'yi bound for the system of closed M2-branes [6]. In fact, consider static solutions that depend on one coordinate $\sigma_2 = s$, the energy can be written as

$$E = \frac{1}{2} \int d\sigma_1 ds \left[\text{Tr} \left(\frac{dX^i}{ds} \pm \partial^i W, \frac{dX^i}{ds} \pm \partial^i W \right) \mp \text{Tr} \left(2\partial^i W, \frac{dX^i}{ds} \right) \right], \quad (2.36)$$

where $W = -\frac{\mu}{4}(X^i)^2 + \frac{1}{24}\varepsilon^{ijkl}\text{Tr}(X^i, [X^j, X^k, X^l])$ and $\text{Tr}(T_a, T_b) = g_{ab}$ is the metric of the Lie 3-algebra. The “+” choice in the first term on (2.36) gives the modified Basu-Harvey equation (2.30), while the “-” choice gives the other BPS equation (2.35). Once again we have seen the power of utilizing the boundary system. For other applications of using the boundary system see, for example, [10, 20, 23].

The condition (2.30) (or (2.33)) represents a non-trivial boundary condition of the system of open M2-branes. For an \mathcal{A}_4 Lie 3-algebra, two kinds of solution were found [6]. One describes a domain wall interpolating between two vacua of the worldvolume theory,

the other is a fuzzy funnel solution that describes a system of M2-branes ending on a single M5-brane. For a more general Lie 3-algebra, the Basu-Harvey equation (2.30) still admits these solutions if \mathcal{A}_4 can be found as a subalgebra. For a Lie algebra, semisimplicity guarantees that a Lie algebra always admits an $su(2)$ subalgebra. The question of when a Lie 3-algebra has an \mathcal{A}_4 subalgebra is an interesting and open one [24].

2.2.2 No Dirichlet: absence of supersymmetric M9-brane

In this case we keep all the eight scalar fields and do not assume that any of them are zero. Consider imposing the projection condition

$$\Gamma^{013456789(10)}\epsilon = \epsilon, \quad (2.37)$$

which implies

$$\Gamma^2\epsilon = \epsilon = \Gamma^{01}\epsilon. \quad (2.38)$$

Also we impose the condition

$$\Gamma^{013456789(10)}\Psi = \Psi \quad (2.39)$$

on the fermion. The first term in (2.23) is again zero using these conditions. The second to last term in (2.23) is a linear combination of products of three or five transverse Gamma matrices Γ^I . One can see that the equation (2.23) contains different number of transverse Γ^I 's and linear independence of them implies that

$$D_2 X^I = 0 \quad (2.40)$$

and

$$X^I = 0. \quad (2.41)$$

i.e. There is no nontrivial solution.

Notice that if we turn off the flux ($m = 0$), then the boundary equation (2.23) can be solved non-trivially with

$$D_2 X^I = 0, \quad [X^I, X^J, X^K] = 0. \quad (2.42)$$

This has been interpreted as an M9-brane occupying the directions 013456789(10) where the M2-branes end on [23].

With the flux (2.20) turned on, however, we only get the trivial solution (2.41). This means, in the presence of flux, the system of M2-branes cannot end on an M9 brane supersymmetrically. This is a prediction of our open M2-branes analysis. One way to confirm its validity is to determine the supersymmetry projector of an M9-brane in the presence of flux and show that the preserved supersymmetry is incompatible with the M2-brane supersymmetry projector $\Gamma^{012}\epsilon = \epsilon$. To carry out this analysis, one needs to first construct the supergravity solution of M9-brane with a constant flux and then

determine the preserved supersymmetry as performed in [25] for the case without flux. One can also reduce the system down to 10 dimensions on x^{10} . This becomes a D2-D8 intersection. The D8-brane is endowed with a worldvolume NS-NS B -field in the 78, 79 or 89 directions and the preserved supersymmetry is determined by

$$e^{-a/2}\Gamma^{013456789(10)}e^{a/2}\epsilon = \epsilon, \quad (2.43)$$

where $a = \frac{1}{2}Y_{IJ}\Gamma^{IJ}\Gamma^{(10)}$ and Y is a nonlinear function of B whose explicit form can be found in [26]. What is important to us is that only the 78, 79 or 89 components are nonzero in our case. It is then clear that the supersymmetry preserved by the D8-brane is incompatible with $\Gamma^{012}\epsilon = \epsilon$ of the D2-brane. Therefore the D2-D8 system and the M2-M9 system are not supersymmetric.

2.2.3 All Dirichlet: M-wave

In this case we set all the eight scalars to zero at the boundary. As a result, all the modifications due to flux vanish and the boundary conditions read identically as in the flux-less case

$$D_2 X^I \bar{\Psi} \Gamma^I \epsilon = 0. \quad (2.44)$$

This can be solved immediately if one imposes the projection conditions

$$(1 - \Gamma^2)\epsilon = 0, \quad (1 + \Gamma^2)\Psi = 0. \quad (2.45)$$

The solution has been interpreted as an M-wave where the M2-branes end on [23].

3. Boundary Condition for the ABJM Theory Coupled to Flux

3.1 $\mathcal{N} = 6$ closed M2-branes in flux background

We now turn to the $\mathcal{N} = 6$ theory with mass and flux terms given by Lambert-Richmond [1] and discuss the boundary terms and their implications. The full Lagrangian of the flux deformed $\mathcal{N} = 6$ theory reads

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=6} + \mathcal{L}_{flux} + \mathcal{L}_{mass}. \quad (3.1)$$

where

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=6} = & -\text{Tr}(D^\mu \bar{Z}_A, D_\mu Z^A) - i\text{Tr}(\bar{\psi}^A, \gamma^\mu D_\mu \psi_A) - V + \mathcal{L}_{CS} \\ & - i\text{Tr}(\bar{\psi}^A, [\psi_A, Z^B; \bar{Z}_B]) + 2i\text{Tr}(\bar{\psi}^A, [\psi_B, Z^B; \bar{Z}_A]) \\ & + \frac{i}{2}\varepsilon_{ABCD}\text{Tr}(\bar{\psi}^A, [Z^C, Z^D; \psi^B]) - \frac{i}{2}\varepsilon^{ABCD}\text{Tr}(\bar{Z}_D, [\bar{\psi}_A, \psi_B; \bar{Z}_C]), \end{aligned} \quad (3.2)$$

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right), \quad (3.3)$$

$$\mathcal{L}_{flux} = \frac{c}{4}\text{Tr}(\bar{Z}_D, [Z^A, Z^B; \bar{Z}_C])\tilde{G}_{AB}{}^{CD}, \quad (3.4)$$

$$\mathcal{L}_{mass} = -m^2\text{Tr}(\bar{Z}_A, Z^A) + b\text{Tr}(\bar{\psi}^A, \psi_F)\tilde{G}_{AE}{}^{EF}. \quad (3.5)$$

The $\mathcal{N} = 6$ theory has a 3-algebra given by a matrix representation

$$[Z^A, Z^B, \bar{Z}_C] = \frac{2\pi}{k} (Z^A \bar{Z}_C Z^B - Z^B \bar{Z}_C Z^A), \quad (3.6)$$

which is only antisymmetric in the first two indices. Here V is defined in [8] and we also define

$$\tilde{G}_{AB}{}^{CD} = \frac{1}{4} \varepsilon_{ABEF} \varepsilon^{CDGH} G^{EF}{}_{GH}. \quad (3.7)$$

and $A, B, C, D = 1, \dots, 4$ are the $SU(4)$ R-symmetry indices. The supersymmetry transformations of the original $\mathcal{N} = 6$ theory is given by

$$\delta_0 Z^A = i\bar{\epsilon}^{AB} \psi_B, \quad (3.8)$$

$$\delta_0 A_\mu = \frac{2\pi}{k} [Z^B \bar{\psi}^A \gamma_\mu \epsilon_{AB} + \epsilon^{AB} \gamma_\mu \psi_A \bar{Z}_B], \quad (3.9)$$

$$\delta_0 \psi_A = \gamma^\mu \epsilon_{AB} D_\mu Z^B + N_A, \quad (3.10)$$

and their conjugates, where

$$N_A = \frac{2\pi}{k} [-\epsilon_{AB} (Z^C \bar{Z}_C Z^B - Z^B \bar{Z}_C Z^C) + 2\epsilon_{CD} Z^c \bar{Z}_A Z^D] \quad (3.11)$$

and δ' is the additional contribution to the supersymmetry transformations due to the flux

$$\delta' Z^A = 0, \quad (3.12)$$

$$\delta' A_\mu = 0, \quad \delta' \hat{A}_\mu = 0, \quad (3.13)$$

$$\delta' \psi_A = \omega \epsilon_{DF} Z^F \tilde{G}_{AE}{}^{ED}. \quad (3.14)$$

The symmetry transformation parameter satisfies the reality condition

$$\epsilon_{FP} = \frac{1}{2} \varepsilon_{IJFP} \epsilon^{IJ}. \quad (3.15)$$

For the action to be supersymmetric, the flux needs to take the form

$$\tilde{G}_{AB}{}^{CD} = \frac{1}{2} \delta_B^C \tilde{G}_{AE}{}^{ED} - \frac{1}{2} \delta_A^C \tilde{G}_{BE}{}^{ED} - \frac{1}{2} \delta_B^D \tilde{G}_{AE}{}^{EC} + \frac{1}{2} \delta_A^D \tilde{G}_{BE}{}^{EC}, \quad (3.16)$$

where the matrix $\tilde{G}_{AE}{}^{EB}$ has to be traceless $\tilde{G}_{AE}{}^{EA} = 0$ and squares to one

$$\tilde{G}_{AE}{}^{EB} \tilde{G}_{BF}{}^{FC} = \frac{m^2}{\omega^2} \delta_A^C. \quad (3.17)$$

Supersymmetry also relates the coefficients ω, b, m to the flux term:

$$\omega = \frac{c}{4}, \quad b = -i\frac{c}{4}, \quad m^2 = \frac{c^2}{32 \cdot 4!} G^2, \quad (3.18)$$

where $G^2 = 6G_{AB}{}^{CD}G^{AB}{}_{CD}$. As before, one finds $c = 2$ by a backreaction analysis.

Taking the flux

$$\tilde{G}_{AE}{}^{ED} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}, \quad (3.19)$$

for $\mu = \pm 2m$, $m \geq 0$, one obtains immediately the deformed theory in [18, 19].

3.2 Flux modified Basu-Harvey equation

We now proceed by finding the boundary contributions to the $\mathcal{N} = 6$ theory of open M2-branes probing the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. Again the contributions for the boundary come from total derivative terms in the ABJM theory, these arise from the scalar and fermionic terms once again. So we have

$$\delta\mathcal{L} = -2\partial_\mu \text{Tr}(\delta\bar{Z}_A, D^\mu Z^A) - i\partial_\mu \text{Tr}(\bar{\psi}^A, \gamma^\mu \delta\psi_A) + \text{bulk terms}, \quad (3.20)$$

Imposing the boundary condition $\sigma_2 = 0$ yields the boundary equations of motion

$$0 = -2i\text{Tr}\left(\bar{\psi}^B \epsilon_{AB}, D^2 Z^A\right) - i\text{Tr}\left[\bar{\psi}^A \gamma^2, \gamma^\mu \epsilon_{AB} D_\mu Z^B\right. \\ \left. + \frac{2\pi}{k}\left(-\epsilon_{AB}(Z^C \bar{Z}_C Z^B - Z^B \bar{Z}_C Z^C) + 2\epsilon_{CD} Z^C \bar{Z}_A Z^D\right) + \omega \epsilon_{DF} Z^F \tilde{G}_{AE}{}^{ED}\right]. \quad (3.21)$$

Now the flux (3.19) can be written compactly as

$$\tilde{G}_{AE}{}^{ED} = \mu \delta_A^D \eta_A, \quad (3.22)$$

where η_A is a sign defined as

$$\eta_A = \begin{cases} +1 & \text{if } A = 1, 2 \\ -1 & \text{if } A = 3, 4 \end{cases}. \quad (3.23)$$

Using this in (3.21), we obtain the boundary equation of motion

$$0 = \bar{\psi}^A \left[2D^2 Z^B - \gamma^2 \gamma^\mu D_\mu Z^B + \gamma^2 \frac{2\pi}{k} (Z^C \bar{Z}_C Z^B - Z^B \bar{Z}_C Z^C) \right. \\ \left. - \frac{\mu}{2} \eta_A \gamma^2 Z^B \right] \epsilon_{AB} - \frac{4\pi}{k} \bar{\psi}^F \gamma^2 \epsilon_{AB} Z^A \bar{Z}_F Z^B, \quad (3.24)$$

where we have now suppressed the traces and will imply them in the natural way henceforth. This is the most general supersymmetric boundary equation of motion for open M2-branes in the $\mathcal{N} = 6$ theory with our specific flux configuration.

To analyse the boundary condition, it is convenient to introduce the following notation $A = (a, i)$ and denote $Z^A = (X^a, Y^i)$, $\psi_A = (\chi_a, \xi_i)$, where $a = 1, 2$ corresponds to

the directions 3456 and $i = 1, 2$ corresponds to the directions 789(10). The supersymmetry parameter ϵ_{AB} is in the **6** representation of $SU(4)$, and it decomposes as [23]

$$\epsilon_{AB} = \begin{pmatrix} \epsilon_{ab}\epsilon & \epsilon_{ai} \\ -\epsilon_{ai} & \epsilon_{ij}\tilde{\epsilon} \end{pmatrix}. \quad (3.25)$$

Using the new notation, the boundary condition (3.24) splits into four equations

$$0 = \bar{\chi}^a \epsilon_{ab} \left[2D^2 X^b - \gamma^2 \gamma^\mu D_\mu X^b + \gamma^2 \frac{2\pi}{k} (Z^C \bar{Z}_C X^b - X^b \bar{Z}_C Z^C) - \frac{\mu}{2} \gamma^2 X^b \right] \epsilon - \frac{4\pi}{k} \epsilon_{ab} \bar{\psi}^F \gamma^2 \epsilon X^a \bar{Z}_F X^b, \quad (3.26)$$

$$0 = \bar{\chi}^a \left[2D^2 Y^i - \gamma^2 \gamma^\mu D_\mu Y^i + \gamma^2 \frac{2\pi}{k} (Z^C \bar{Z}_C Y^i - Y^i \bar{Z}_C Z^C) - \frac{\mu}{2} \gamma^2 Y^i \right] \epsilon_{ai} - \frac{4\pi}{k} \bar{\psi}^F \gamma^2 \epsilon_{ai} X^a \bar{Z}_F Y^i, \quad (3.27)$$

$$0 = \bar{\xi}^i \left[2D^2 X^a - \gamma^2 \gamma^\mu D_\mu X^a + \gamma^2 \frac{2\pi}{k} (Z^C \bar{Z}_C X^a - X^a \bar{Z}_C Z^C) + \frac{\mu}{2} \gamma^2 X^a \right] \epsilon_{ai} - \frac{4\pi}{k} \bar{\psi}^F \gamma^2 \epsilon_{ai} Y^i \bar{Z}_F X^a, \quad (3.28)$$

$$0 = \bar{\xi}^i \epsilon_{ij} \left[2D^2 Y^j - \gamma^2 \gamma^\mu D_\mu Y^j + \gamma^2 \frac{2\pi}{k} (Z^C \bar{Z}_C Y^j - Y^j \bar{Z}_C Z^C) + \frac{\mu}{2} \gamma^2 Y^j \right] \tilde{\epsilon} - \frac{4\pi}{k} \epsilon_{ij} \bar{\psi}^F \gamma^2 \tilde{\epsilon} Y^i \bar{Z}_F Y^j. \quad (3.29)$$

Let us consider the half Dirichlet case by setting half of the scalars zero, in particular, $Y^i = 0$. This condition reduces the R-symmetry from $SU(4)$ to $SU(2)$. We first analyze (3.26) and (3.28). It turns out that the second term “ $\gamma^\mu D_\mu X^b$ ” in these equations vanishes for $\mu = 0, 1$. We will come back to this later. For the moment, assuming that this is true, then (3.26) and (3.28) becomes

$$0 = \bar{\chi}^a \epsilon_{ab} \left[D^2 X^b + \gamma^2 \frac{2\pi}{k} (X^c \bar{X}_c X^b - X^b \bar{X}_c X^c) - \frac{\mu}{2} \gamma^2 X^b \right] \epsilon - \frac{4\pi}{k} \epsilon_{cd} \bar{\chi}^a \gamma^2 \epsilon X^c \bar{X}_a X^d \quad (3.30)$$

and

$$0 = \bar{\xi}^i \left[D^2 X^a + \gamma^2 \frac{2\pi}{k} (X^c \bar{X}_c X^a - X^a \bar{X}_c X^c) + \frac{\mu}{2} \gamma^2 X^a \right] \epsilon_{ai}. \quad (3.31)$$

The two equations are not compatible with each other in general. However it is possible to impose a suitable supersymmetry projection conditions on the spinors ϵ and ϵ_{ai} so that these two equations become equivalent. The needed conditions are

$$(1 + \gamma^2)\epsilon = 0 = (1 + \gamma^2)\tilde{\epsilon} \quad (3.32)$$

$$(1 - \gamma^2)\epsilon_{ai} = 0, \quad (3.33)$$

or

$$(1 - \gamma^2)\epsilon = 0 = (1 - \gamma^2)\tilde{\epsilon} \quad (3.34)$$

$$(1 + \gamma^2)\epsilon_{ai} = 0, \quad (3.35)$$

As a result, (3.30) and (3.31) are identical since

$$X^c \bar{X}_c X^b - X^b \bar{X}_c X^c = \varepsilon^{ba} \varepsilon^{cd} X^c \bar{X}_a X^d. \quad (3.36)$$

And we obtain the modified Basu-Harvey equation with mass,

$$D_2 X^a \pm \frac{2\pi}{k} (X^c \bar{X}_c X^a - X^a \bar{X}_c X^c) \pm \frac{\mu}{2} X^a = 0, \quad (3.37)$$

where the + sign corresponds to the choice (3.32), (3.33) and the - sign corresponds to the choice (3.34), (3.35). This can also be written in terms of the 3-bracket

$$D_2 X^a \pm [X^c, X^a; \bar{X}_c] \pm \frac{\mu}{2} X^a = 0. \quad (3.38)$$

This is the mass deformed Basu-Harvey equation for the flux modified ABJM theory. In the following, we consider the choice of projectors (3.32), (3.33) and the Basu-Harvey equation

$$D_2 X^a + \frac{2\pi}{k} (X^c \bar{X}_c X^a - X^a \bar{X}_c X^c) + \frac{\mu}{2} X^a = 0. \quad (3.39)$$

The analysis for the other choice is exactly the same.

Next we note that since $\delta Y^i = i\epsilon^{ia}\chi_a + i\varepsilon^{ij}\tilde{\epsilon}\xi_j$, the boundary condition $Y^i = 0$ is supersymmetric invariant, after imposing (3.32) and (3.33), if

$$(1 + \gamma^2)\chi_a = 0, \quad (3.40)$$

$$(1 - \gamma^2)\xi_i = 0. \quad (3.41)$$

As for the conditions (3.27) and (3.29), which read

$$\bar{\chi}^a \epsilon_{ai} D_2 Y^i = 0, \quad (3.42)$$

and

$$\tilde{\xi}^i \varepsilon_{ij} \tilde{\epsilon} D_2 Y^j = 0. \quad (3.43)$$

These are satisfied immediately as a result of the projection conditions (3.32), (3.33), (3.40), (3.41). It is also easy to see that these projection conditions are supersymmetric invariant. Moreover, supersymmetry on (3.37) requires the fermionic boundary equations

$$D_2 \chi_c - [\chi_c, X^a; \bar{X}_a] + 2[\chi_d, X^d; \bar{X}_c] - \frac{\mu}{2} \chi_c = 0, \quad (3.44)$$

$$D_2\xi_j + [\xi_j, X^a; \bar{X}_a] - \varepsilon_{jk}\varepsilon_{ab}[X^a, X^b; \xi_k] - \frac{\mu}{2}\xi_j = 0. \quad (3.45)$$

As for the above assumption of the vanishing of the terms of the form “ $\Gamma^\mu D_\mu X^b$ ” in the equations (3.26) and (3.28), one can see that it follows immediately from the projection conditions (3.32) and (3.40), and respectively (3.33) and (3.41).

The Basu-Harvey equation (3.39) can be readily solved by employing the ansatz

$$X^a(s) = f(s)R^a, \quad (3.46)$$

where $s = x_2$ and R^a are $N \times N$ matrices satisfying the relation

$$R^c R_c^\dagger R^a - R^a R_c^\dagger R^c = -R^a. \quad (3.47)$$

Then we obtain

$$f' - \frac{2\pi}{k}f^3 + \frac{\mu}{2}f = 0. \quad (3.48)$$

The equation (3.47) has been solved in [18] and the irreducible solution is

$$(R^1)_{mn} = \delta_{m,n}\sqrt{m-1}, \quad (R^2)_{mn} = \delta_{m-1,n}\sqrt{N-m+1}, \quad m, n = 1, \dots, N. \quad (3.49)$$

A direct sum of such blocks is also a solution. The equation (3.48) is the same equation as in the $\mathcal{N} = 8$ theory. This is how the domain wall solution and the M2-M5 intersection are represented in the $\mathcal{N} = 6$ theory.

Finally, let us comment briefly on the no Dirichlet and all Dirichlet cases. For the no Dirichlet case, we find only the trivial solution $X^a = Y^i = 0$ as in the $\mathcal{N} = 8$ theory. As for the all Dirichlet case, since the flux modifications all go away when all the scalars are set to zero at the boundary, hence the boundary conditions (3.26)-(3.29) reduce to exactly the same form as in flux-less case and one gets an M-wave [23].

4. Lorentzian 3-Algebras and a Reduction to D2-Branes

In the original construction of the BL theory [3,5,6], the Lie 3-algebra \mathcal{A}_4 was employed. The use of \mathcal{A}_4 was motivated by the studies of Basu and Harvey [22] whose main objective was to construct a generalization of the Nahm equation for describing intersecting M-branes. The next simplest example of a Lie 3-algebra is the Lorentzian algebra. It has been shown that when one considers a Lorentzian 3-algebra, the Bagger-Lambert Lagrangian reduces to the $\mathcal{N} = 8$ SYM theory of multiple D2-branes [27–33], as opposed to the nontrivial reduction for the original BL theory [34]. In this section, we consider the Lorentzian BL theory with flux and analyse its reduction. We will also derive the supersymmetric boundary condition and obtain from it the corresponding mass deformed Nahm equation.

The Lorentzian 3-algebra is defined by a set of generators $T^a = \{T^+, T^-, T^i\}$, where T^i are the generators of a Lie algebra \mathcal{G} of the compact gauge group G with the structure constant f^{ijk} and Killing metric δ^{ij} . The 3-bracket is specified by

$$[T^-, T^a, T^b] = 0, \quad a = +, -, i, \quad (4.1)$$

$$[T^+, T^i, T^j] = f^{ij}_k T^k, \quad (4.2)$$

$$[T^i, T^j, T^k] = f^{ijk} T^-. \quad (4.3)$$

The invariant metric on this algebra is

$$\begin{aligned} \text{Tr}(T^-, T^+) &= -1, \\ \text{Tr}(T^i, T^j) &= \delta^{ij}. \end{aligned} \quad (4.4)$$

Expanding all the fields with respect to the generators

$$X^I = X^I_a T^a = X^I_- T^- + X^I_+ T^+ + \hat{X}^I, \quad (4.5)$$

where $\hat{X}^I = X^I_i T^i$ are the modes corresponds to the Lie algebra \mathcal{G} , one obtain the action for a Lorentzian BL theory [32],

$$\begin{aligned} \mathcal{L}_{Lorentz} &= -\frac{1}{2} \text{Tr}(\hat{D}_\mu \hat{X}^I - B_\mu X^I_+)^2 + \partial_\mu X^I_+ (\partial_\mu X^I_- - \text{Tr}(B_\mu, \hat{X}^I)) + \frac{1}{2} \varepsilon^{\mu\nu\lambda} \text{Tr}(B_\lambda F_{\mu\nu}) \\ &+ \frac{i}{2} \text{Tr}(\hat{\Psi} \Gamma^\mu, (\hat{D}_\mu \hat{\Psi} - B_\mu \Psi_+)) - \frac{i}{2} \hat{\Psi}_+ \Gamma^\mu (\partial_\mu \Psi_- - \text{Tr}(B_\mu, \hat{\Psi})) - \frac{i}{2} \Psi_- \Gamma^\mu \partial_\mu \Psi_+ \\ &+ \frac{i}{2} \text{Tr}(\hat{\Psi} \Gamma^{IJ} X^I_+ [\hat{X}^J, \hat{\Psi}]) + \frac{i}{4} \text{Tr}(\hat{\Psi} \Gamma^{IJ} [\hat{X}^I, \hat{X}^J] \Psi_+) - \frac{i}{4} \text{Tr}(\hat{\Psi}_+ \Gamma^{IJ} [\hat{X}^I, \hat{X}^J] \hat{\Psi}) \\ &- \frac{1}{12} \text{Tr} \left(X^I_+ [\hat{X}^J, \hat{X}^K] + X^J_+ [\hat{X}^K, \hat{X}^I] + X^K_+ [\hat{X}^I, \hat{X}^J] \right)^2, \end{aligned} \quad (4.6)$$

where $I = 3, \dots, 10$. Here A_μ is a gauge field for the compact gauge group G . The gauge field B_μ is defined by $B_\mu = A_{\mu ij} f^{ijk} T^k$ and the theory is invariant under an extra non-compact gauge symmetry associated with B_μ :

$$\begin{aligned} \delta B_\mu &= D_\mu \zeta, \quad \delta \hat{X}^I = \zeta X^I_+, \quad \delta X^I_- = \text{Tr}(\zeta, \hat{X}^I), \\ \delta \hat{\Psi} &= \zeta \Psi_+, \quad \delta \Psi_- = \text{Tr}(\zeta, \hat{\Psi}). \end{aligned} \quad (4.7)$$

The supersymmetry transformations read:

$$\delta_0 X^I_- = i\bar{\epsilon} \Gamma^I \Psi_-, \quad \delta_0 X^I_+ = i\bar{\epsilon} \Gamma^I \Psi_+, \quad \delta_0 \hat{X}^I = i\bar{\epsilon} \Gamma^I \hat{\Psi}, \quad (4.8)$$

$$\delta_0 \hat{A}_\mu = \frac{i}{2} \bar{\epsilon} \Gamma_\mu \Gamma^I (X^I_+ \hat{\Psi} - \hat{X}^I \Psi_+), \quad \delta_0 B_\mu = i\bar{\epsilon} \Gamma_\mu \Gamma^I [\hat{X}^I, \hat{\Psi}], \quad (4.9)$$

$$\begin{aligned} \delta_0 \Psi_- &= (\partial_\mu X^I_- - \text{Tr}(B_\mu X^I)) \Gamma^\mu \Gamma^I \epsilon - \frac{1}{3} \text{Tr}(\hat{X}^I \hat{X}^J \hat{X}^K) \Gamma^{IJK} \epsilon, \\ \delta_0 \Psi_+ &= \partial_\mu X^I_+ \Gamma^\mu \Gamma^I \epsilon, \end{aligned} \quad (4.10)$$

$$\delta_0 \hat{\Psi} = \hat{D}_\mu \hat{X}^I \Gamma^\mu \Gamma^I \epsilon - \frac{1}{2} X^I_+ [\hat{X}^J, \hat{X}^K] \Gamma^{IJK} \epsilon. \quad (4.11)$$

A special feature of the Lagrangian (4.6) is that the fields X_-^I, Ψ_- appear linearly. For convenience, let us collect the terms containing X_-^I, Ψ_- . It is

$$\mathcal{L}_{gh} = \partial_\mu X_+^I \partial^\mu X_-^I - \bar{\Psi}_+ \Gamma^\mu \partial_\mu \Psi_- . \quad (4.12)$$

We have called it a ghost term since \mathcal{L}_{gh} has an indefinite metric and is hence non-unitary. One can integrate out X_-^I, Ψ_- and obtain the equations of motion:

$$\partial^2 X_+^I = 0, \quad (4.13)$$

$$\Gamma^\mu \partial_\mu \Psi_+ = 0. \quad (4.14)$$

A solution to (4.13) and (4.14) which preserves gauge symmetry and supersymmetry is given by

$$X_+^I = v_0 \delta_{10}^I, \quad (4.15)$$

$$\Psi_+ = 0, \quad (4.16)$$

where $v_0 \in \mathbb{R}$. Substituting this into the Lagrangian (4.6) and integrating out the field B_μ gives us the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(\hat{D}_\mu \hat{X}^I)^2 + \frac{i}{2} \text{Tr}(\hat{\Psi}, \Gamma^\mu D_\mu \hat{\Psi}) - \frac{1}{4v_0^2} \text{Tr}(F_{\mu\nu}, F^{\mu\nu}) - \frac{v_0^2}{4} \text{Tr}([\hat{X}^I, \hat{X}^J], [\hat{X}^I, \hat{X}^J]) \\ & + \frac{iv_0}{2} \text{Tr}(\hat{\Psi} \Gamma^{(10)I} [\hat{X}^I, \hat{\Psi}]) + \partial_\lambda \left(\frac{\varepsilon^{\mu\nu\lambda} \hat{F}_{\mu\nu} \hat{X}^{10}}{2v_0} \right), \end{aligned} \quad (4.17)$$

where we have kept the boundary term for later discussions. For a closed theory, the Lagrangian (4.17) is the maximally supersymmetric $\mathcal{N} = 8$ SYM theory in 2+1 dimensions. For an open theory, one will need an appropriate boundary condition in order to decouple the field \hat{X}^{10} at the boundary. Moreover one gets additional boundary conditions from requiring supersymmetry of the Lagrangian. For these boundary conditions to be supersymmetric, ϵ must be further restricted. This will be the subject of section 4.2.

We remark that apart from integrating out the fields X_-^I, Ψ_- , one can also keep them and perform a BRST analysis by promoting a certain global shift symmetry to a local one [31, 32]. The analysis for the open case can be performed similarly. In the following we will concentrate on the “integrating out” approach for our analysis.

4.1 Multiple D2-branes in a background flux

With the Lorentzian 3-algebra, the flux term and the mass term read

$$\begin{aligned}\mathcal{L}_{flux} &= 2\tilde{G}_{IJKL}\text{Tr}(X^I, [X^J, X^K, X^L]) \\ &= -8\tilde{G}_{IJKL}X_+^I\text{Tr}(\hat{X}^L, [\hat{X}^J, \hat{X}^K]),\end{aligned}\quad (4.18)$$

$$\begin{aligned}\mathcal{L}_{mass} &= -\frac{1}{2}m^2\text{Tr}(X^I, X^I) - \frac{i}{8}\text{Tr}(\bar{\Psi}, \Gamma^{IJKL}\Psi)\tilde{G}_{IJKL} \\ &= -\frac{1}{2}m^2\text{Tr}(\hat{X}^I, \hat{X}^I) + m^2X_+^IX_-^I \\ &\quad - \frac{i}{8}\text{Tr}(\hat{\Psi}, \Gamma^{IJKL}\hat{\Psi})\tilde{G}_{IJKL} + \frac{i}{4}\bar{\Psi}_+\Gamma^{IJKL}\Psi_-\tilde{G}_{IJKL}.\end{aligned}\quad (4.19)$$

As an aside, we note it is easy to check that the gauge symmetry (4.7) extends to the flux and mass Lagrangians (4.18), (4.19). We will take $\mu = 2m$ in the following analysis.

Due to the presence of new terms linear in the fields X_-^I and Ψ_- , after integrating out these fields, we get the modified equations of motion

$$\partial^2 X_+^I - m^2 X_+^I = 0, \quad (4.20)$$

$$i\Gamma^\mu\partial_\mu\Psi_+ - \frac{i}{4}\Gamma^{IJKL}\Psi_+\tilde{G}_{IJKL} = 0. \quad (4.21)$$

These are the Klein-Gordon and Dirac equations respectively. We will next show that one is able to pick solutions to X_+^I and Ψ_+ which preserves gauge invariance and supersymmetry.

The supersymmetry transformations remains the same as for the bosons as in (4.8) but the flux modifies the supersymmetry transformations for the fermions:

$$\delta\Psi_- = (\partial_\mu X_-^I - \text{Tr}(B_\mu, X^I))\Gamma^\mu\Gamma^I\epsilon - \frac{1}{3}\text{Tr}(\hat{X}^I\hat{X}^J\hat{X}^K)\Gamma^{IJK}\epsilon + \frac{1}{4}\tilde{G}\Gamma^M\epsilon X_-^M, \quad (4.22)$$

$$\delta\Psi_+ = \partial_\mu X_+^I\Gamma^\mu\Gamma^I\epsilon + \frac{1}{4}\tilde{G}\Gamma^M\epsilon X_+^M, \quad (4.23)$$

$$\delta\hat{\Psi} = \hat{D}_\mu\hat{X}^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{2}X_+^I[\hat{X}^J, \hat{X}^K]\Gamma^{IJK}\epsilon + \frac{1}{4}\tilde{G}\Gamma^M\epsilon\hat{X}^M. \quad (4.24)$$

The simplest solution to (4.20) and (4.21) is

$$\Psi_+ = 0, \quad (4.25)$$

$$X_+^I = \begin{cases} v_0 e^{m\sigma_1} \delta_{10}^I, & \text{or} \\ v_0 e^{imt} \delta_{10}^I, \end{cases} \quad (4.26)$$

where v_0 is a real constant and the real part is assumed for the second solution in (4.26). As an illustration, we will consider below the first solution with the σ_1 dependence and for convenience we will denote $v = v_0 e^{m\sigma_1}$ below. It is easy to see that the solution

(4.25) is supersymmetrically invariant. In fact for the flux (2.18), we have $\delta\Psi_+ = mv\Gamma^{2(10)}(1 - \Gamma^2 R')\epsilon$, where R' is defined by $R\Gamma^{10} = \Gamma^{(10)}R'$. Therefore the configuration (4.25) is supersymmetrically invariant for ϵ satisfying

$$(1 - \Gamma^2 R')\epsilon = 0. \quad (4.27)$$

Since the projectors (2.13), (4.27) commute, 8 supersymmetries are preserved.

Substituting the solution (4.25), (4.26) and integrating out the B_μ fields, we finally obtain

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 - \frac{1}{4v^2}\text{Tr}(F_{\mu\nu}, F^{\mu\nu}) - \frac{v^2}{4}\text{Tr}([\hat{X}^A, \hat{X}^B B], [\hat{X}^A, \hat{X}^B]) \\ & + \frac{i}{2}\text{Tr}(\hat{\Psi}, \Gamma^\mu D_\mu \hat{\Psi}) + \frac{iv}{2}\text{Tr}(\hat{\Psi}, \Gamma^{(10)A}[\hat{X}^A, \hat{\Psi}]) \\ & - 8v\tilde{G}_{(10)ABC}\text{Tr}(\hat{X}^A, [\hat{X}^B, \hat{X}^C]) - \frac{i}{2}\text{Tr}(\hat{\Psi}, \Gamma^{(10)ABC}\Psi)\tilde{G}_{(10)ABC} \\ & - \frac{1}{2}m^2\text{Tr}(\hat{X}^A, \hat{X}^A) - \frac{i}{8}\text{Tr}(\hat{\Psi}, \Gamma^{ABCD}\Psi)\tilde{G}_{ABCD} \\ & + \partial_\lambda \left(\frac{\varepsilon^{\mu\nu\lambda}}{2v}\text{Tr}(\hat{F}_{\mu\nu}, \hat{X}^{10}) \right) - \frac{m}{2}\partial_1\text{Tr}(\hat{X}^{10}, \hat{X}^{10}), \end{aligned} \quad (4.28)$$

where the indices $A, B, C, D = 3, \dots, 9$. In the closed case, one can drop the last two terms in (4.28). Since the Lambert-Richmond action is supersymmetric and the solution (4.26) is 1/2-BPS, by construction our action (4.28) is supersymmetric and preserves 8 supersymmetries:

$$\delta\hat{X}^A = i\bar{\epsilon}\Gamma^A\hat{\Psi}, \quad (4.29)$$

$$\delta\hat{\Psi} = \hat{D}_\mu \hat{X}^A \Gamma^\mu \Gamma^A \epsilon - \frac{1}{2v}\epsilon_{\mu\nu\lambda}\hat{F}^{\nu\lambda}\Gamma^\mu\Gamma^{10}\epsilon - \frac{v}{2}[\hat{X}^A, \hat{X}^B]\Gamma^{AB(10)}\epsilon + \frac{1}{4}\tilde{G}_{\#}\Gamma^A\epsilon\hat{X}^A, \quad (4.30)$$

$$\delta\hat{A}_\mu = \frac{iv}{2}\bar{\epsilon}\Gamma_\mu\Gamma^{10}\hat{\Psi}. \quad (4.31)$$

The Lagrangian (4.28) can be understood as the worldvolume theory of D2-branes with a space(time) dependent coupling $g_{\text{YM}} = v$ and coupled to NS-NS and R-R fluxes. In 10 dimensions, the flux \tilde{G}_{ABCD} is identified with the R-R 4-form flux of the 3-form potential C_3 and $\tilde{G}_{(10)ABC}$ is identified with the NS-NS 3-form flux of the 2-form potential B_2 . The term in (4.28) proportional to $\tilde{G}_{(10)ABC}$ can be traced back as the low energy limit of the Myers' action [35], together with its superpartner. The terms proportional to m^2 and \tilde{G}_{ABCD} are typical of couplings to the R-R fields. Supersymmetric Yang-Mills theories with a spacetime dependent coupling were originally constructed in [36, 37] and are known as Janus field theories. An extension to include a spacetime dependent θ -angle for the 4-dimensional supersymmetric Yang-Mills was performed in [38] as an application to study spacetime singularities using holography. Similar field theory constructions also appear in the work [39].

In the open case, one needs to impose a boundary condition to decouple the \hat{X}^{10} field at the boundary. In particular we are interested in a supersymmetric boundary condition in this paper. This, together with the other boundary conditions that are needed to maintain supersymmetry of the system, will be discussed next.

4.2 Multiple D2-branes ending on a D4

We now derive the supersymmetric boundary conditions for the flux modified Lorentzian Bagger-Lambert theory. Since the field X_-^I has been integrated out, the boundary condition (2.30) cannot be applied immediately and one needs to derive the boundary condition from the reduced action (4.28) directly.

Since rotational invariance is explicitly broken by the σ_1 dependence of the coupling, it makes a difference where the boundary is. For example, the theory with a boundary at $\sigma_1 = 0$ is not equivalent to the theory with a boundary at $\sigma_2 = 0$. In particular, to decouple the field \hat{X}^{10} at the boundary, one needs to impose the boundary condition

$$\text{Tr}(2\hat{F}_{02}\hat{X}^{10} + mv(\hat{X}^{10})^2) = 0, \quad \text{boundary at } \sigma_1 = 0, \quad (4.32)$$

$$\text{Tr}(\hat{F}_{01}\hat{X}^{10}) = 0, \quad \text{boundary at } \sigma_2 = 0. \quad (4.33)$$

Let us first discuss the second case and assume that the condition (4.33) is satisfied for the moment and come back to discuss whether it is supersymmetric later.

The supersymmetric variation of the Lagrangian is given by

$$\delta\mathcal{L} = \frac{i}{2}\partial_\mu\text{Tr}(\hat{\Psi}, \Gamma^\mu\delta\hat{\Psi}) - \partial_\mu\text{Tr}(\delta\hat{X}^A, \hat{D}^\mu\hat{X}^A) + \text{bulk terms}. \quad (4.34)$$

Imposing the boundary condition $\sigma_2 = 0$ gives

$$\delta\int d^3\sigma\mathcal{L} = \frac{i}{2}\int d^2\sigma\left(\text{Tr}(\hat{\Psi}, \Gamma^2\delta\hat{\Psi}) - 2\text{Tr}(\delta\hat{X}^A, \hat{D}_2\hat{X}^A)\right), \quad (4.35)$$

so we obtain the boundary equation of motion

$$\begin{aligned} \hat{D}_\mu\hat{X}^A\hat{\Psi}\Gamma^2\Gamma^\mu\Gamma^A\epsilon - \frac{v}{2}[\hat{X}^A, \hat{X}^B]\hat{\Psi}\Gamma^2\Gamma^{(10)AB}\epsilon - \frac{1}{2v}\epsilon_{\mu\nu\lambda}\hat{F}^{\nu\lambda}\hat{\Psi}\Gamma^2\Gamma^{\mu(10)}\epsilon \\ + \frac{1}{4}\hat{\Psi}\Gamma^2\tilde{\mathcal{G}}\Gamma^A\epsilon\hat{X}^A - 2\hat{D}_2\hat{X}^A\hat{\Psi}\Gamma^A\epsilon = 0, \end{aligned} \quad (4.36)$$

where, $\mu = 0, 1, 2$.

Let us now consider a system of D2-branes ending on a D4-brane. In general a system of two intersecting D-branes is supersymmetric if the relative transverse space has dimension in multiples of 4. Therefore, with this in anticipation, let us look for a solution to the boundary condition (4.36) with

$$\hat{X}^{3,4,5,6} = 0. \quad (4.37)$$

This corresponds to a D4-brane with worldvolume in the 01789-directions. The R-symmetry is reduced from $SO(7)$ to $SO(3)$. In addition to (2.13), we also impose the condition

$$\Gamma^{01789(10)}\epsilon = \epsilon. \quad (4.38)$$

We remark that this is not the same as the D4-brane projector in the κ -symmetric formulation of D-branes, see for example [26, 40]. The effect of a background flux is already taken into account in terms of the M2-branes description and so the D4-brane supersymmetry is represented simply by the condition (4.38) in the M2-branes model. From the above conditions, we obtain

$$\Gamma^{(10)ij}\epsilon = -\varepsilon^{ijk}\Gamma^2\Gamma^k\epsilon, \quad (4.39)$$

$$\Gamma^2\tilde{Q}\epsilon = 4m\epsilon, \quad (4.40)$$

where we have used the indices $i, j, k = 7, 8, 9$. Let us also impose the condition

$$\Gamma^{01789(10)}\Psi = -\Psi. \quad (4.41)$$

It follows that $\delta\hat{A}_\alpha = 0$ for $\alpha = 0, 1$ and hence \hat{F}_{01} is supersymmetric invariant. Therefore one can impose the supersymmetric boundary condition

$$\hat{F}_{01} = 0, \quad (4.42)$$

which also implies (4.33).

The boundary condition (4.36) then simplifies to

$$\hat{D}_2\hat{X}^i = \frac{1}{2}v\varepsilon^{ijk}[\hat{X}^j, \hat{X}^k] + m\hat{X}^i. \quad (4.43)$$

The Nahm equation describes the profile of the D4-brane where the D2-branes end. The fuzzy funnel solution is obtained with the ansatz

$$\hat{X}^i(\sigma_2) = f(\sigma_2)T^i, \quad (4.44)$$

where T^i obey the $SU(2)$ algebra $[T^i, T^j] = \varepsilon^{ijk}T^k$ and f obeys

$$f' = vf^2 + mf. \quad (4.45)$$

This has solution

$$f = \frac{m}{ce^{-m\sigma_2} - v}, \quad (4.46)$$

where c is a constant. The solution behaves as $f = v_0^{-1}/(s_0 - \sigma_2)$ for small m , where s_0 is a constant. This is the expected profile in the absence of flux. In the presence of flux, the solution describes a fuzzy sphere

$$\sum_{i=7}^9(\hat{X}^i)^2 = R^2, \quad (4.47)$$

whose radius $R = Cf$ depends on the Casimir C of the representation as well as f . Since v actually depends on σ_1 , the fuzzy funnel has an S^2 cross section whose radius depends on both σ_1 and σ_2 . This is a new feature of the flux we consider.

As before, it is straightforward to check that the boundary condition (4.37), (4.41) and (4.43) are supersymmetric invariant. Finally we comment on the other possibility of having a boundary at $\sigma_1 = 0$. Our analysis above can be performed in exactly the same way, with only a straightforward change of the index 2 to 1 in the equations (4.35), (4.36), (4.38), (4.39), (4.40), (4.41). However, it is easy to convince oneself that there is no way to impose a supersymmetric boundary condition such that (4.32) holds. This is due to the fact that \hat{X}^{10} has a non-trivial supersymmetry variation. Therefore we conclude that with the solution $v = v_0 e^{m\sigma_1}$, the flux Lorentzian Bagger-Lambert theory is 1/2 BPS if there is a boundary at $\sigma_2 = 0$. On the other hand, if the boundary is at $\sigma_1 = 0$, then all supersymmetries are broken.

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