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Stokes phenomena and non-perturbative completion in the multi-cut two-matrix models

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Abstract

The Stokes multipliers in the matrix models are invariants in the string-theory moduli space and related to the D-instanton chemical potentials. They not only represent non-perturbative information but also play an important role to connect various perturbative string theories in the moduli space. They are a key concept to the non-perturbative completion of string theory and also expected to imply some remnant of strong coupling dynamics from M theory. In this paper, we investigate the non-perturbative completion problem consisted of two constraints on Stokes multipliers. As the first constraint, Stokes phenomena which realize the multi-cut geometry are studied in the $\mathbb{Z}_k$ symmetric critical points of the multi-cut two-matrix models. Sequence of solutions to the constraints are obtained in general k-cut critical points. A discrete set of solutions and a continuum set of solutions are explicitly shown, and they can be classified by several constrained configurations of Young diagram. As the second constraint, we discuss non-perturbative stability of backgrounds in terms of the Riemann-Hilbert problem. In particular, our procedure in the 2-cut $(1, 2)$ case (pure-supergravity case) completely fixes the D-instanton chemical potentials and results in the Hastings-McLeod solution to the Painlevé II equation. It is also stressed that the Riemann-Hilbert approach realizes an off-shell background independent formulation of non-critical string theory.

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1 Introduction and summary

Non-critical string theory [1] has provided interesting theoretical laboratories which uncover various intriguing features about string theory. This string theory is known as solvable system not only in the perturbative world-sheet formulation, Liouville theory [2–10], but also in the non-perturbative matrix-model formulation [11–31]. Recently, among various kinds of matrix models, the multi-cut matrix models [32] have turned out to be a fruitful system. The first discovery was on the two-cut matrix models [33–38], which were found to describe type 0 superstring theory [39–41]. Furthermore, the multi-cut two-matrix models were generally found to have a correspondence with the so-called fractional superstring theory [42] and also with non-critical M theory as its strong-coupling dual theory [43], which realizes the philosophy proposed in the Hořava-Keeler non-critical M theory [44].

Quantitative analyses of critical points and perturbative amplitudes in the multi-cut two-matrix models have been carried out in [43, 45]. The main observables used there are macroscopic loop amplitudes (or resolvent) [12–17, 28, 29, 46–50] which provide the information of spectral curves, the classical spacetime of this string theory [31, 51, 52]. The concrete expression for spectral curve is important because they provide relevant information for reproducing all order perturbative amplitudes in the multi-cut two-matrix models by the method of topological recursions [53].

The main theme in this paper is, on the other hand, about *non-perturbative aspects of the multi-cut two-matrix models*. Non-perturbative aspects in matrix models have also been studied extensively [23, 26, 27, 30, 31, 50–52, 54–77]. The main concern is about non-perturbative contributions of the matrix-model free energy $\mathcal{F}(\mathcal{C}; g_{\rm str})$ on the large N spectral curve $\mathcal{C}$:[1]

$$\mathcal{F}(\mathcal{C}; g_{\rm str}) \underset{asym}{\simeq} \sum_{n=0}^{\infty} g_{\rm str}^{2n-2}\, \mathcal{F}_n(\mathcal{C}) + \mathcal{F}_{\text{non-perturb.}}(\mathcal{C}; g_{\rm str}), \qquad g_{\rm str} \to 0. \tag{1.1}$$

Here $\mathcal{F}_n(\mathcal{C})$ is the genus-n perturbative free energy on the spectral curve $\mathcal{C}$, and the information of the matrix-model potentials (so-called KP flows $\{t_n\}_{n\in\mathbb{Z}}$) is implicitly included in the spectral curve:

$$\mathcal{C} = \mathcal{C}\big(\{t_n\}_{n\in\mathbb{Z}}\big), \qquad \{t_n\}_{n\in\mathbb{Z}} \in \mathcal{M}_{\rm string}^{(\rm non-norm.)} \subset \mathbb{C}^{\mathbb{Z}}, \tag{1.2}$$

which is known as the non-normalizable string-theory moduli space $\mathcal{M}_{\rm string}^{(\rm non-norm.)}$ [78].[2] The first quantitative implication was given in the early 90's and is about the strength of string non-perturbative corrections which are $\mathcal{O}(e^{-1/g_{str}})$ order quantities [79], i.e. open-string (D-brane) degree of freedom [80]:

$$\mathcal{F}_{\text{non-perturb.}}(\mathcal{C}; g_{\rm str}) = \sum_I \theta_I \, \exp\Big[-\frac{1}{g_{\rm str}} S_{\rm inst}^{(I)}(\mathcal{C}; g_{\rm str})\Big]. \tag{1.3}$$

[1]We carefully put "asym" below the equation in order to emphasize that they are equal only in the asymptotic sense.

[2]The normalizable string-theory moduli space $\mathcal{M}_{\rm string}^{(\rm norm.)}$ is known as the space of filling fraction [74] which parametrizes the on-shell string backgrounds. The off-shell backgrounds are defined in Section 5.

Here I is a set of indices which labels multi-instanton sectors, $I = \{i_1, i_2, \cdots\}$,

$$S^{(I)}_{\rm inst}(\mathcal{C}; g_{\rm str}) = \sum_{i\in I=\{i_1,i_2,\cdots\}} S^{(i)}_{\rm inst}(\mathcal{C}; g_{\rm str}) + \mathcal{O}(g_{\rm str}), \qquad S^{(I)}_{\rm inst}(\mathcal{C}; g_{\rm str}) \sim \mathcal{O}(g^0_{\rm str}), \tag{1.4}$$

and each primitive instanton $S^{(i)}_{\rm inst}(\mathcal{C}; g_{\rm str})$ ($i = 1, 2, \cdots, N_{\rm inst}$), is shown to have the precise correspondence with a singular point of the spectral curve $\mathcal{C}$ [41, 55–59, 61, 68] and the leading disk amplitudes are identified with the ZZ-brane amplitudes in Liouville theory [7,9,10]. It is worth mentioning that these instanton corrections with further higher order $g_{\rm str}$ corrections $S^{(I)}_{\rm inst}(\mathcal{C}; g_{\rm str})$ are important in order to make the free energy $\mathcal{F}(\mathcal{C}; g_{\rm str})$ to be modular invariant under modular transformations of the spectral curve $\mathcal{C}$ and also to be background independent in the normalizable string-theory moduli space $\mathcal{M}^{(\rm norm.)}_{\rm string}$ (i.e. the filling fractions) [74]. The constant θ_I is called D-instanton chemical potential (or fugacity). These constants are understood as integration constants of corresponding string equations [27], that is,

$$\frac{\partial \theta_I}{\partial t_m} = 0, \qquad m \in \mathbb{Z}, \qquad \{t_n\}_{n\in\mathbb{Z}} \in \mathcal{M}^{(\rm non-norm.)}_{\rm string}, \tag{1.5}$$

for the flows in the non-normalizable moduli space $\mathcal{M}^{(\rm non-norm.)}_{\rm string}$. It was shown [31] that the only $N_{\rm inst}$ (i.e. the number of primitive instantons) chemical potentials θ_i ($i = 1, 2, \cdots, N_{\rm inst}$) are independent among all the chemical potentials θ_I.

Although various aspects of matrix models have been understood well so far, there still remains an important issue, which is about the D-instanton chemical potentials: *What is the physical requirement to determine the D-instanton chemical potentials?* Although the actual matrix models should employ some particular universal values [60], they seem to be totally free parameters at least within continuum formulations based on string (or loop) equations. This point has been studied in the bosonic minimal/2D string theories [60, 64, 65, 67], in the type 0 $(1, 2)$ superstring theory [62], in the collective string field theory [63] and in the free-fermion formulation [50, 66]. In this paper, we address this issue by solving *non-perturbative completion problem* within a continuous formulation for the critical points of the multi-cut two-matrix models. In practice, we pick up physically acceptable D-instanton chemical potentials which realize physically reasonable behaviors in the non-perturbative region $g_{\rm str} \to \infty$. Our solutions are based on two physical requirements: One is *multi-cut boundary condition* (in Section 4) and the other is *non-perturbative stability of perturbative backgrounds* (in Section 5).

The first requirement, the multi-cut boundary condition, is a non-perturbative constraint on the Baker-Akhiezer function system in these multi-cut critical points:

$$g_{\rm str}\frac{\partial}{\partial \zeta}\Psi(t;\zeta) = \mathcal{Q}(t;\zeta)\,\Psi(t;\zeta), \qquad g_{\rm str}\frac{\partial}{\partial t}\Psi(t;\zeta) = \mathcal{P}(t;\zeta)\,\Psi(t;\zeta), \tag{1.6}$$

where the equation system here is expressed as an ordinary differential equation in ζ and its isomonodromy deformation system in t.[3] Note that the Lax operators in Eq. (1.6) in the k-cut critical points are $k \times k$ matrix-valued operators [68]. The idea of the first constraint is motivated by the non-perturbative relationship between the Baker-Akhiezer

[3] The parameter t is one of the parameters in the non-normalizable moduli space $\mathcal{M}^{(\rm non-norm.)}_{\rm string}$, which is usually a coupling of the most relevant operator or the world-sheet cosmological constant.

functions and cuts in the resolvent curves. This kind of relationship is discussed in terms of Airy function [51]. Specifically, the asymptotic expansion of the Airy function around the cut ($\zeta \to -\infty$) is expressed as[4]

$$\mathrm{Ai}(t;\zeta) \underset{asym}{\simeq} \Big(\frac{g_{\mathrm{str}}\pi}{(\zeta+t)^{1/2}}\Big)^{1/2} \Big[e^{-\frac{2}{3g_{\mathrm{str}}}(\zeta+t)^{3/2}} + i e^{\frac{2}{3g_{\mathrm{str}}}(\zeta+t)^{3/2}}\Big] + \cdots, \tag{1.7}$$

where the relation to the resolvent (or macroscopic loop) operator $\mathcal{R}(\zeta)$ [19] is roughly expressed as

$$\mathrm{Ai}(t;\zeta) \sim \exp\Big[N\int^{\zeta} d\zeta'\,\mathcal{R}(\zeta')\Big], \qquad \mathcal{R}(\zeta) \equiv \frac{1}{N}\Big\langle \mathrm{tr}\,\frac{1}{\zeta - M}\Big\rangle \sim \sqrt{\zeta+t}, \tag{1.8}$$

with the expectation value $\langle\cdots\rangle$ taken with respect to the Hermitian one-matrix model of a matrix M. From this expression, one observes that *the cut in the negative axes* ($\zeta < -t$) *appears as a line where a competition between the exponents* $e^{\pm\frac{2}{3g_{\mathrm{str}}}(\zeta+t)^{3/2}}$ *(i.e. along the Stokes lines) happens*. Therefore, we interpret this as *a non-perturbative definition of the resolvent cuts*. This consideration turns out to be important in the fractional-superstring critical points of the multi-cut two-matrix models [43], since most of the cuts in these critical points are created by this procedure and cannot be read from the algebraic equations of the resolvent spectral curve. What is more, as we will see in Section 4, this procedure do not necessarily create the necessary and sufficient k cuts on the resolvent curve, even though the k-cut Baker-Akhiezer function Eq. (1.6) is obtained from the assumption that the critical points have k cuts around $\zeta \to \infty$. Therefore, we need to impose a physical constraint so that the resolvent curves in the k-cut critical points should have k cuts around $\zeta \to \infty$. This constraint is expressed in terms of *Stokes multipliers* for the *possible Stokes phenomena* in this system.

The second requirement, the non-perturbative stability of perturbative backgrounds, is imposed in the other formulation which is closely related to the Baker-Akhierzer function system: the so-called the Riemann-Hilbert (or inverse monodromy) approach [81–83] [22]. A brief flowchart of this approach is shown in Fig. 1. Details are given in Section 5, but in order to show how the Riemann-Hilbert approach works in resolving the issue, we here show the leading expression of the free energy (more precisely the two-point function of cosmological constant t) in the two-cut $(1,2)$ case:

$$\frac{\partial^2 \mathcal{F}(t;g_{\mathrm{str}})}{\partial t^2} = \big[f(t)\big]^2, \qquad f(t) = \sum_n s_{n,2,1}\int_{\mathcal{K}_n}\frac{d\lambda}{2\pi i}\, e^{g^{(2)}(t;\lambda) - g^{(1)}(t;\lambda)} + \cdots. \tag{1.9}$$

The parameter $s_{n,2,1}$ is *a Stokes multiplier* of the Baker-Akhierzer function system of the corresponding integrable system and the contour $\mathcal{K}_n$ is *an anti-Stokes line* corresponding to the Stokes multiplier $s_{n,2,1}$. As one can suspect from the expression, the Riemann-Hilbert approach is directly related to the study of *Stokes phenomena* at $\zeta \to \infty$ in the ordinary differential equation of the Baker-Akhierzer system.

In this expression, the function $g^{(j)}(t;\zeta)$ is *an arbitrary function* but should be properly chosen so that the integrals other than the "leading" expression shown in Eq. (1.9) are negligible [83]. From the matrix-model viewpoints (discussed in Section 5), this function can be interpreted as *an off-shell string background geometry of string theory.* Therefore,

[4] The asymptotic expansion of Airy function is reviewed in Appendix A.

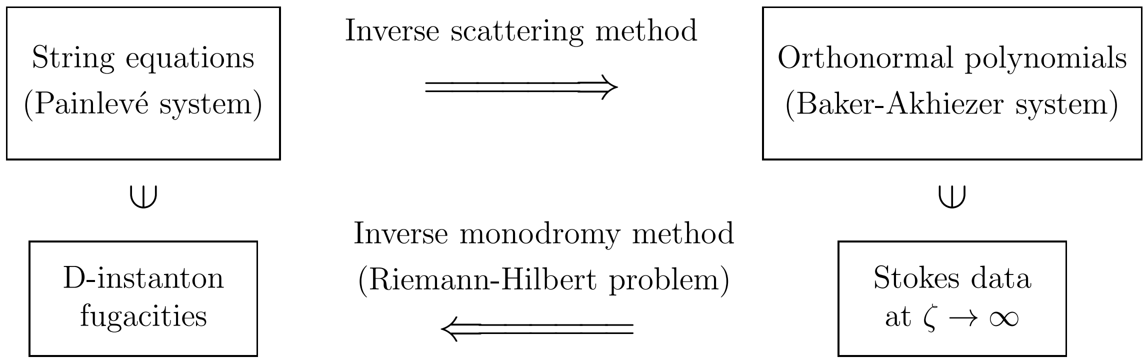


Figure 1: The Riemann Hilbert approach and the D-instanton chemical potentials (or fugacities)

if one *chooses* $g^{(j)}(t;\zeta)$ as a *macroscopic loop amplitude* realized in the large N limit of the matrix models, then the leading integral (1.9) becomes a similar expression to the mean field expression for a single eigenvalue of the matrix integral which appears in various studies in literature [27, 30, 31, 59, 60, 62].[5] Therefore, the Stokes multipliers $s_{n,2,1}$ in Eq. (1.9) are directly identified as *the D-instanton chemical potentials* in the semi-classical saddle-point analysis. That is, *the first constraint is directly related to the constraint on the D-instanton chemical potentials.* Furthermore, since the Riemann-Hilbert integral, Eq. (1.9), provides the complete integration representation based on the reference string background $g^{(j)}(t;\zeta)$, we can discuss *non-perturbative stability of the background* $g^{(j)}(t;\zeta)$, especially for the background which is obtained as large N limit of the matrix models. This consideration for the stability is also expressed as a constraint on the Stokes multipliers and therefore the D-instanton chemical potentials. Originally, the mean field analyses include ambiguity of *choice of contour* and *weight of these contours* [27] and this fact becomes a cause of the ambiguity about the D-instanton chemical potentials in continuum loop-equation systems. In the Riemann-Hilbert approach, however, these degrees of freedom are identified as anti-Stokes lines $\mathcal{K}_n$ and Stokes multipliers $s_{n,2,1}$, and they are tightly related to each other. As a consequence, the physical section of the D-instanton chemical potentials is obtained in the name of *non-perturbative completion.* This viewpoint is important in non-critical string theory because non-critical strings are sometimes defined as the large N (i.e. perturbative) expansion of unstable matrix-model critical points (e.g. $(2,3)$ bosonic minimal string theory) and therefore the matrix-model description does not necessary guarantee non-perturbative completion of string theory.

As we will see in the coming sections, the above procedures completely determine the D-instanton chemical potentials in the two-cut $(1,2)$ critical points and results in the Hastings-McLeod solution [84] to the Painlevé II equation (in Section 5.1). Actually it is known that this is the unique solution which realizes the two phases of the two-cut $(1,2)$ critical point of the two-cut matrix model,[6] and therefore the Hastings-McLeod solution is suitable for this critical point. An advantage of our work is the discovery of

[5] It is interesting that the Riemann-Hilbert expression gives a similar expression to the D-instanton operators obtained in the free-fermion formulation [30, 31].

[6] It was shown by Hastings-McLeod [84] that their solution is a unique solution to the Painlevé II equation, Eq. (2.43), which realizes the following asymptotic behaviors of $f(t)$ on the two sides of infinity

the actual physical requirements to obtain the correct solutions to the non-perturbative completion which are also applicable to the critical points with an arbitrary number of cuts. Furthermore, we carry out the non-perturbative completion procedures generally in higher number-cut (which even reaches to ∞-cut) critical points and obtain the concrete form of solutions (in Section 4.3). Interestingly, we found that the solutions are labeled by constrained Young diagrams. This result implies that there is a quite rich world beyond this non-perturbative horizon, and that the multi-cut matrix models provide fruitful fields for a quantitative study of these issues.

Organization of this paper is as follows: In Section 2, after summarizing the asymptotic expansion of the ODE system in the multi-cut critical points, the general facts about Stokes phenomenon in ordinary differential equations are reviewed. As a warming up, the case of the two-cut $(1,2)$ critical point is also shown. In Section 3, Stokes phenomena in the multi-cut critical points are studied. In particular, the practical way of reading the Stokes multipliers in the general cases is developed. In Section 4, the multi-cut boundary condition is proposed. In Section 4.3, the discrete and continuum solutions are obtained with imposing several ansatz. In Section 5, the non-perturbative stability condition is studied in terms of the Riemann-Hilbert problem. Section 6 is devoted to conclusion and discussion.

Context of Appendices is: Appendix A is about the Stokes phenomenon of Airy function (a review of [51]). Appendix B is about calculation of Lax operators. Appendix C is about calculation of the multi-cut boundary-condition recursive equations. Appendix D is about calculation of the 3 and 4-cut $(1,1)$ critical points.

2 Stokes phenomena in the ODE systems

Before we devote ourselves into the multi-cut systems, we here first review some general facts about Stokes phenomenon in ordinary differential equation systems, then we summarize the well-studied two-cut $(1,2)$ case. This two-cut system has been extensively studied not only in physical context [33–38, 41, 52, 68] but also in mathematical context [82–84, 86–89], since it is related to the Hastings-McLeod solution [84] of the Painlevé II system. For more comprehensive and rigorous reviews and references on the isomonodromy deformations, Stokes phenomenon and inverse monodromy problems, see [85]. We also note that the idea of ismonodromy deformation was introduced in non-critical string theory by [22].

2.1 The ODE system and asymptotic expansions

It was first proposed in [68] that the multi-cut matrix models are controlled by multi-component KP hierarchy [90] and therefore by the following Baker-Akhiezer function

$t \to \pm\infty$:

$$\frac{1}{2}\ddot{f}(t) - f^3(t) + 2tf(t) = 0 : \qquad f(t\to\infty) \sim 0, \qquad f(t\to-\infty) \sim \sqrt{-t}, \tag{1.10}$$

which is the same behavior as the two-cut $(1,2)$ critical point of the two-cut matrix model discussed in [41].

system:

$$\zeta\Psi(t;\zeta) = \boldsymbol{P}(t;\partial)\,\Psi(t;\zeta), \tag{2.1}$$

$$g_{\rm str}\frac{\partial}{\partial\zeta}\Psi(t;\zeta) = \boldsymbol{Q}(t;\partial)\,\Psi(t;\zeta). \tag{2.2}$$

Here the operator $\boldsymbol{P}(t;\partial)$ and $\boldsymbol{Q}(t;\partial)$ are $\hat{p}$-th and $\hat{q}$-th order differential operators in $\partial \equiv g_{\rm str}\partial_t$, respectively, which satisfy the Douglas equation [20]:

$$\big[\boldsymbol{P}(t;\partial),\boldsymbol{Q}(t;\partial)\big] = g_{\rm str} I_k. \tag{2.3}$$

Critical points in the multi-cut two-matrix models are characterized by these Lax operators and explicitly obtained in [45] with their critical potentials. There are two kinds of interesting critical points: the $\mathbb{Z}_k$-symmetric critical points and fractional-superstring critical points.

1. The $\mathbb{Z}_k$-symmetric critical points are characterized by the following $k\times k$ Lax operators [45]:

$$\boldsymbol{P}(t;\partial) = \Gamma\,\partial^{\hat{p}} + \sum_{n=0}^{\hat{p}-1} U_n^{(Z_kP)}(t)\,\partial^n, \qquad \boldsymbol{Q}(t;\partial) = \Gamma^{-1}\,\partial^{\hat{q}} + \sum_{n=0}^{\hat{q}-1} U_n^{(Z_kQ)}(t)\,\partial^n, \tag{2.4}$$

with the shift matrix Γ,

$$\Gamma = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \tag{2.5}$$

and the $k\times k$ matrix-valued real coefficients $U_n^{(Z_kP)}(t)$ and $U_n^{(Z_kQ)}(t)$ which satisfy

$$U_n^{(Z_kP)}(t) = \begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & \ddots & \\ & & & 0 & * \\ * & & & & 0 \end{pmatrix}, \qquad U_n^{(Z_kQ)}(t) = \begin{pmatrix} 0 & & & & * \\ * & 0 & & & \\ & \ddots & \ddots & & \\ & & * & 0 & \\ & & & * & 0 \end{pmatrix}, \tag{2.6}$$

as a result of the $\mathbb{Z}_k$ symmetry of the critical points. Macroscopic loop amplitudes (i.e. off-critical resolvent amplitudes with $t\neq 0$) in this kind of critical points are also obtained in [45] with the Daul-Kazakov-Kostov prescription [29] and written with the Jacobi polynomials or the third and fourth Chebyshev polynomials. In particular, the amplitudes in the the k-cut $(1,1)$ critical points are given as the eigenvalues of the Lax operators Eq. (2.4) in the weak coupling limit $g_{\rm str}\to 0$:[7]

$$\begin{aligned} \boldsymbol{P}(t;\partial) &\simeq \mathrm{diag}_{j=1}^k\Big(P^{(j)}_{\rm classical}(t;z)\Big) = \mathrm{diag}_{j=1}^k\Big(\omega^{j-1}\,x(z)\Big), \\ \boldsymbol{Q}(t;\partial) &\simeq \mathrm{diag}_{j=1}^k\Big(Q^{(j)}_{\rm classical}(t;z)\Big) = \mathrm{diag}_{j=1}^k\Big(\omega^{-(j-1)}\,y(z)\Big), \end{aligned} \tag{2.7}$$

[7]In this paper, the equality $\simeq$ means that they are equal up to some similarity transformation.

with

$$x(z) = t\sqrt[k]{\big(z-c\big)^l\big(z-b\big)^{k-l}}, \qquad y(z) = t\sqrt[k]{\big(z-c\big)^{k-l}\big(z-b\big)^l} \tag{2.8}$$

and $0 = c\,l + b\,(k-l)$ and the dimensionless variable $z \equiv g_{\mathrm{str}} t^{-1}\partial_t$.

2. The fractional-superstring critical points [42] are characterized by the following two kinds of Lax operators [45]: The first kind is given as

$$\boldsymbol{P}(t;\partial) = \Gamma\,\partial^{\hat{p}} + \sum_{n=0}^{\hat{p}-1} U_n^{(F_kP)}(t)\,\partial^n, \qquad \boldsymbol{Q}(t;\partial) = \Gamma\,\partial^{\hat{q}} + \sum_{n=0}^{\hat{q}-1} U_n^{(F_kQ)}(t)\,\partial^n. \tag{2.9}$$

These Lax operators are derived from the $\omega^{1/2}$-rotated critical potentials. The second kind is given as

$$\boldsymbol{P}(t;\partial) = \Gamma^{(\mathrm{real})}\,\partial^{\hat{p}} + \sum_{n=0}^{\hat{p}-1} U_n^{(R_kP)}(t)\,\partial^n, \qquad \boldsymbol{Q}(t;\partial) = \Gamma^{(\mathrm{real})}\,\partial^{\hat{q}} + \sum_{n=0}^{\hat{q}-1} U_n^{(R_kQ)}(t)\,\partial^n, \tag{2.10}$$

with the matrix $\Gamma^{(\mathrm{real})}$,

$$\Gamma^{(\mathrm{real})} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -1 & & & & 0 \end{pmatrix}. \tag{2.11}$$

These Lax operators are derived from the real critical potentials. In both cases, all the $k\times k$ matrix-valued coefficients $U_n^{(F_kP)}(t)$ and $U_n^{(F_kQ)}(t)$ (and $U_n^{(R_kP)}(t)$ and $U_n^{(R_kQ)}(t)$) are real functions. The macroscopic loop amplitudes in each case are obtained and written by the deformed Chebyshev functions [43].

In this paper, for the sake of simplicity, we concentrate on the $\hat{p}=1$ cases of the $\mathbb{Z}_k$-symmetric critical points. With this choice of critical points, the Lax operator $\boldsymbol{P}(t;\partial)$ becomes

$$\boldsymbol{P}(t;\partial) = \Gamma\partial + H(t), \tag{2.12}$$

and the Baker-Akhiezer function for the eigenvalue problem of the operator $\boldsymbol{P}(t;\partial)$, Eq. (2.1), is rewritten as

$$g_{\mathrm{str}}\frac{\partial}{\partial t}\Psi(t;\zeta) = \mathcal{P}(t;\zeta)\,\Psi(t;\zeta) \equiv \Gamma^{-1}\big[\zeta - H(t)\big]\,\Psi(t;\zeta), \tag{2.13}$$

and therefore Eq. (2.2) is also rewritten as a $k\times k$ matrix polynomial operator in ζ:

$$g_{\mathrm{str}}\frac{\partial\Psi(t;\zeta)}{\partial\zeta} = \mathcal{Q}(t;\zeta)\,\Psi(t;\zeta) \equiv \boldsymbol{Q}(t;\partial)\,\Psi(t;\zeta), \qquad \mathcal{Q}(t;\zeta) = \sum_{n=-1}^{-r}\frac{\mathcal{Q}_n(t)}{\zeta^{n+1}}. \tag{2.14}$$

Here we define r as

$$r \equiv \hat{q} + 1 > 0, \tag{2.15}$$

which is referred to as the Poincaré index in literature. The advantage of this formulation is that the pair of Lax operators $(\boldsymbol{P}(t;\partial), \boldsymbol{Q}(t;\partial))$ becomes a pair of the polynomial operators $(\mathcal{P}(t;\zeta), \mathcal{Q}(t;\zeta))$, and the system can be expressed as an $k \times k$ first order ordinary differential equation (ODE) system. These systems are called the Zakharov-Shabat eigenvalue problem [91] or AKNS hierarchy [92] in literature. Note that the Douglas equation becomes

$$\big[\boldsymbol{P}(t;\partial), \boldsymbol{Q}(t;\partial)\big] = g_{\mathrm{str}} I_k \quad \Leftrightarrow \quad \big[g_{\mathrm{str}}\partial_\zeta - \mathcal{Q}(t;\zeta), \partial - \mathcal{P}(t;\zeta)\big] = 0, \tag{2.16}$$

in terms of these Lax operators.

This ODE system Eq. (2.14) has the k independent order k column vector solutions $\Psi^{(j)}(t;\zeta)$, $(j = 1, 2, \cdots, k)$, and we here use the following matrix solution notation:

$$\Psi(t;\zeta) \equiv \Big(\Psi^{(1)}(t;\zeta), \cdots, \Psi^{(k)}(t;\zeta)\Big). \tag{2.17}$$

As in the usual ODE, we consider formal expansion around $\zeta \to \infty$. However the point $\zeta \to \infty$ is an irregular singularity and the formal series expansion around this irregular point in general does not converge absolutely. Up to proper redefinition of the k independent solutions, the formal series expansion of the solutions around $\zeta \to \infty$ is given as

$$\Psi_{asym}(t;\zeta) \equiv Y(t;\zeta)\, e^{\varphi(t;\zeta)} \equiv \Big[I_k + \sum_{n=1}^{\infty} \frac{Y_n(t)}{\zeta^n}\Big] \times \exp\Big[\varphi_0 \ln\zeta - \sum_{m=-r,\neq 0}^{\infty} \frac{\varphi_m(t)}{m\,\zeta^m}\Big]. \tag{2.18}$$

The coefficient matrices are obtained from the recursive equations,

$$0 = -nY_n(t) + \sum_{m=0}^{n+r} \Big(Y_m(t)\,\varphi_{n-m}(t) - \mathcal{Q}_{n-m}(t)\,Y_m(t)\Big), \quad \big(n = -r, -r+1, \cdots\big). \tag{2.19}$$

For convenience, we extend the indices of the coefficients:

$$Y_0(t) = I_k, \quad Y_n(t) = 0 \quad (n < 0), \qquad \varphi_m(t) = \mathcal{Q}_m(t) = 0 \quad (m < -r), \tag{2.20}$$

and imposing the following constraints on $Y_n(t)$ and $\varphi_n(t)$:

$$\big[\Gamma^l, \varphi_n(t)\big] = 0, \qquad \sum_{i=1}^{k} \big[Y_n(t)\big]_{i,i+l} = 0, \qquad (l = 0, 1, \cdots, k-1). \tag{2.21}$$

This recursive relation then can be solved uniquely and all the expansion coefficient are written with the coefficient matrix-valued function $H(t)$ in Eq. (2.12).

On the other hand, it is also convenient to use a diagonal basis, $\widetilde{\Psi}_{asym}(t;\zeta)$, which is defined by

$$\begin{aligned}\widetilde{\Psi}_{asym}(t;\zeta) &\equiv \widetilde{Y}(t;\zeta)\, e^{\widetilde{\varphi}(t;\zeta)} \equiv \Big[I_k + \sum_{n=1}^{\infty} \frac{\widetilde{Y}_n(t)}{\zeta^n}\Big] \times \exp\Big[\widetilde{\varphi}_0 \ln\zeta - \sum_{m=-r,\neq 0}^{\infty} \frac{\widetilde{\varphi}_m(t)}{m\,\zeta^m}\Big] \\ &\equiv U^\dagger\, \Psi_{asym}(t;\zeta)\, U,\end{aligned} \tag{2.22}$$

where the matrix U is given as

$$U_{jl} = \frac{1}{\sqrt{k}}\omega^{(j-1)(l-1)}, \qquad \Gamma U = U\,\Omega, \tag{2.23}$$

with $\Omega = \mathrm{diag}(1, \omega, \omega^2, \cdots, \omega^{k-1})$ and $\omega = e^{2\pi i/k}$. Since this is a similarity transformation, the coefficients also satisfy the same recursive relation (2.19). In this basis, the function $\widetilde{\varphi}(t;\zeta)$ is a diagonalized matrix and we write its eigenvalues as

$$\widetilde{\varphi}(t;\zeta) = \mathrm{diag}\big(\varphi^{(1)}(t;\zeta), \cdots, \varphi^{(k)}(t;\zeta)\big). \tag{2.24}$$

The vector components of the formal series, $\widetilde{\Psi}_{asym} = \big(\widetilde{\Psi}^{(1)}_{asym}, \cdots, \widetilde{\Psi}^{(k)}_{asym}\big)$, is given as

$$\widetilde{\Psi}^{(j)}_{asym}(t;\zeta) = \widetilde{Y}^{(j)}(t;\zeta)\, e^{\varphi^{(j)}(t;\zeta)}, \qquad (j = 1, 2, \cdots, k), \tag{2.25}$$

with $\widetilde{Y}(t;\zeta) = \big(\widetilde{Y}^{(1)}, \cdots, \widetilde{Y}^{(k)}\big)$.

Although the above formal solutions are formal series around the irregular singularity, they are related to the exact analytic solutions to the ODE system, $\widetilde{\Psi}(t;\zeta)$, in the sense of *asymptotic expansion*:

$$\widetilde{\Psi}(t;\zeta) \underset{asym}{\simeq} \widetilde{\Psi}_{asym}(t;\zeta)\, C, \tag{2.26}$$

in some specific angular domain [93]:

$$\zeta \to \infty \in D(a;b) \equiv \big\{\zeta \in \mathbb{C};\ a < \arg(\zeta) < b\big\}. \tag{2.27}$$

An example of the angular domain is shown in Fig. 2-a. Here C is a proper coefficient matrix, and the meaning of asymptotic expansion is following:

Definition 1 [asymptotic expansion] *For a holomorphic function $f(\zeta)$,* an asymptotic expansion of $f(\zeta)$ in a domain $D(a;b)$ *is defined as a formal series $\sum_n f_n \zeta^{-n}$ such that there exists a constant $B^{(N)}_{R;a,b} \in \mathbb{R}$ which satisfies*

$$\Big| f(\zeta) - \sum_{n=-r}^{N} \frac{f_n}{\zeta^n} \Big| < \frac{B^{(N)}_{R;a,b}}{|\zeta|^N}, \qquad \zeta \in D(a;b) \cap \big\{\zeta \in \mathbb{C};\ \big|\zeta\big| > R\big\} \tag{2.28}$$

for each integer $N = -r, -r+1, \cdots$ and sufficiently large $R \in \mathbb{R}$. This is written as

$$f(\zeta) \underset{asym}{\simeq} \sum_{n=-r}^{\infty} \frac{f_n}{\zeta^n}, \qquad \zeta \to \infty \in D(a;b). \tag{2.29}$$

The maximal angular domains are called *Stokes sectors.*

2.2 General facts on Stokes phenomena in the ODE system

In this subsection, in order to understand the asymptotic expansion Eqs. (2.22) and (2.26), we review some general theorem about the asympototic expansions and Stokes phenomena in the general $k \times k$ ODE systems,

$$\begin{aligned} g_{\mathrm{str}} \frac{\partial}{\partial \zeta} \widetilde{\Psi}(t;\zeta) &= \Big[\widetilde{\mathcal{Q}}_{-r}\, \zeta^{r-1} + \widetilde{\mathcal{Q}}_{-r+1}(t)\, \zeta^{r-2} + \cdots \widetilde{\mathcal{Q}}_{-1}(t)\Big] \widetilde{\Psi}(t;\zeta) \\ &\equiv \widetilde{\mathcal{Q}}(t;\zeta)\, \widetilde{\Psi}(t;\zeta). \end{aligned} \tag{2.30}$$

Note that proof of the theorems appearing in this subsection can be found in [85] and references therein. For sake of simplicity, we assume

$$\widetilde{\mathcal{Q}}_{-r} = \mathrm{diag}\big(A_1, A_2, \cdots, A_k\big), \qquad A_i - A_j \neq 0, \qquad A_i \neq 0, \qquad (i, j = 1, 2, \cdots, k). \tag{2.31}$$

Therefore, the exponents Eq. (2.24) are expressed as

$$\widetilde{\varphi}(t;\zeta) = \widetilde{\varphi}_0(t)\ln\zeta - \sum_{n=-r, n\neq 0}^{\infty} \frac{\widetilde{\varphi}_n(t)}{n\zeta^n} = \frac{1}{r}\widetilde{\mathcal{Q}}_{-r}\,\zeta^r + \cdots, \tag{2.32}$$

and $\varphi^{(i)}_{-r} = A_i$ $(i = 1, 2, \cdots, k)$ also satisfies (2.31).

The meaning of the asymptotic expansion Eq. (2.22) is that basically we ignore relatively small exponents. One takes some (small enough) anglular domain $D(a; e^{i\epsilon}a)$ then compares the relative magnitudes around $\zeta \to \infty$, for example,

$$\big|e^{\varphi^{(j_1)}(t;\zeta)}\big| < \big|e^{\varphi^{(j_2)}(t;\zeta)}\big| < \cdots < \big|e^{\varphi^{(j_k)}(t;\zeta)}\big|, \qquad \zeta \to \infty \in D(a; e^{i\epsilon}a). \tag{2.33}$$

Then one can obtain the following equality under the asymptotic expansion:

$$e^{\varphi^{(j_2)}(t;\zeta)} + \theta e^{\varphi^{(j_1)}(t;\zeta)} \underset{asym}{\simeq} e^{\varphi^{(j_2)}(t;\zeta)}, \qquad \zeta \to \infty \in D(a; e^{i\epsilon}a). \tag{2.34}$$

That is, the smaller exponents become practically invisible in view of the asymptotic expansion. Therefore, it is important to consider the angles of ζ where the the exponents, $\exp\big(\varphi^{(j)}(\zeta)\big)$ $(i = 1, 2, \cdots, k)$, change the relative magnitudes around $\zeta \to \infty$. This leads to the concept of *Stokes lines*:

Definition 2 [Stokes lines] *With the assumption (2.31), Stokes lines* $\mathrm{SL}_{j,l}$ *in this ODE system are defined for each pair of* (j, l) *as*

$$\mathrm{SL}_{j,l} \equiv \Big\{\zeta \in \mathbb{C}; \mathrm{Re}\big[\big(\varphi^{(j)}_{-r} - \varphi^{(l)}_{-r}\big)\zeta^r\big] = 0\Big\} = \bigcup_{n=0}^{2r-1} \mathrm{SL}^{(n)}_{j,l}, \tag{2.35}$$

which consists of $2r$ *semi-infinite lines,* $\mathrm{SL}^{(n)}_{j,l}$ $(n = 0, 1, \cdots, 2r-1)$*. The set of lines,* SL, *denotes a set of whole Stokes lines,* $\mathrm{SL} \equiv \bigcup_{j,l} \mathrm{SL}_{j,l}$.

An example of Stokes lines $\mathrm{SL}_{j,l}$ is shown in Fig. 2-b. In particular, if the angular domain $D(a; b)$ of the asymptotic expansion includes a Stokes line, one cannot neglect the exponents as it happens in Eq. (2.34). This leads to the following definition of *Stokes sectors*:

Definition 3 [Stokes sectors] *A Stokes sector* D *in the ODE system is an angular domain,* $D = D(a; b)$*, with angles* (a, b) *such that for each pair of* (j, l) *there exist a unique Stokes line* $\mathrm{SL}^{(n_{j,l})}_{j,l}$ *which satisfies,*

$$\mathrm{SL}^{(n_{j,l})}_{j,l} \subset D = D(a; b), \tag{2.36}$$

that is, except for this line $\mathrm{SL}^{(n_{j,l})}_{j,l}$ *there is no other line* $\mathrm{SL}^{(n'_{j,l})}_{j,l}$ $(\neq \mathrm{SL}^{(n_{j,l})}_{j,l})$ *which runs inside the domain,* D.

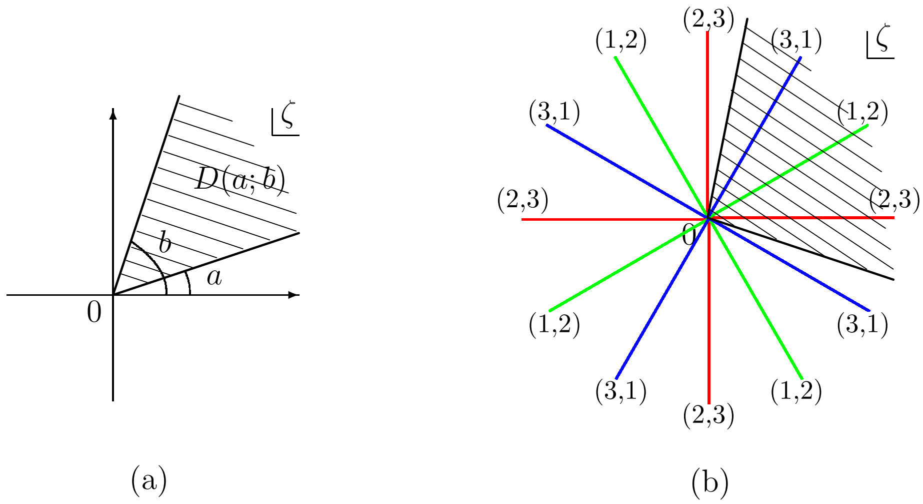


Figure 2: a) An angular domain of $D(a;b)$. b) Stokes lines and Stokes sectors. This is the 3-cut $(1,1)$ critical points. An example of Stokes sectors is also shown. In this critical point, there are three kinds of the Stokes lines $\mathrm{SL}_{i,j}$, $(i,j) = (1,2),(2,3),(3,1)$. Stokes sectors includes one and only one Stokes line of each kind.

An example of the Stokes sectors (the 3-cut $(1,1)$ critical point) is shown in Fig. 2-b.

Actually the definition of the Stokes sectors results in the following theorem:

Theorem 1 [93] *For a given Stokes sector D, any solutions to the ODE system $\widetilde{\Psi}(t;\zeta)$ has the following asymptotic expansion:*

$$\widetilde{\Psi}(t;\zeta) \underset{asym}{\simeq} \widetilde{\Psi}_{asym}(t;\zeta)\, C, \qquad \zeta \to \infty \in D, \tag{2.37}$$

with a matrix C. Furthermore, the coefficient matrix C (i.e. asymptotic expansion) is unique in the Stokes sector D.

This uniqueness enables us to define the following unique solution in a Stokes sector D:

Definition 4 [Canonical solution] *If the solution to the ODE system, $\widetilde{\Psi}_{can}(t;\zeta)$, has the asymptotic expansion with $C = I_k$ in a Stokes sector D,*

$$\widetilde{\Psi}_{can}(t;\zeta) \underset{asym}{\simeq} \widetilde{\Psi}_{asym}(t;\zeta), \qquad \zeta \to \infty \in D, \tag{2.38}$$

this solution is called the canonical solution in the Stokes sector D.

This theorem on the other hand means that the asymptotic expansion is not unique if one chooses some angular domain D' narrower than Stokes sectors. In particular, as is shown in Fig. 3, the intersection of two different Stokes sectors D_1 and D_2 is generally narrower than Stokes sectors, and therefore there appears difference between the canonical solutions $\widetilde{\Psi}_i(t;\zeta)$ of each sector $D_i (i=1,2)$:

$$\widetilde{\Psi}_2(t;\zeta) = \widetilde{\Psi}_1(t;\zeta) S, \qquad D_1 \cap D_2 \neq \emptyset. \tag{2.39}$$

This $k \times k$ matrix S which expresses the difference between $\widetilde{\Psi}_1(t;\zeta)$ and $\widetilde{\Psi}_2(t;\zeta)$ is called a *Stokes matrix* in the intersection $D_1 \cap D_2$. This indicates that solutions in the ODE

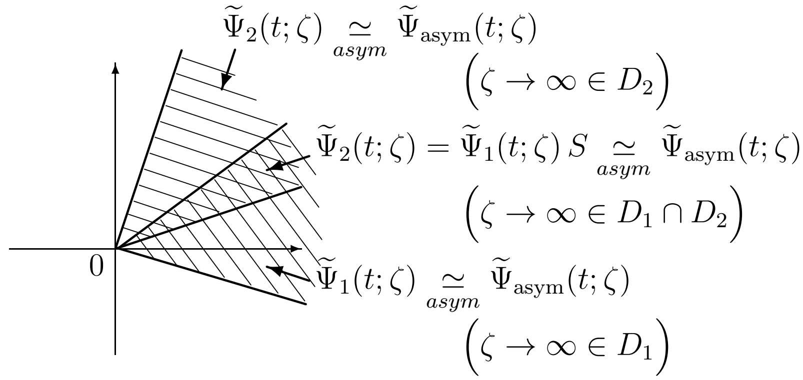


Figure 3: Explanation of Stokes phenomenon in ODE systems. For given two Stokes sectors, their canonical solutions are generally different by a Stokes matrix, S in the intersection $D_1 \cap D_2$. This behavior of analytic functions is called Stokes phenomenon.

system generally have different asymptotic expansion in different Stokes sectors. This analytic behavior of the solutions is referred to as the *Stokes phenomenon* in the ODE system.

A direct calculation shows that the Stokes matrices do not depend on ζ, and furthermore, they do not depend on the deformation parameter t either (as in (2.13)):

$$\frac{dS}{d\zeta} = \frac{dS}{dt} = 0. \tag{2.40}$$

This means that the Stokes matrices are understood as integration constants for the evolution system in the t space. Therefore, these integrable deformations in the original multi-component KP hierarchy are also called *isomonodromy deformation system* [81]. This also leads us to the concept of *inverse monodromy approach* [81, 82], which is also briefly reviewed in Section 5.

Components of Stokes matrices satisfy the following theorem (See [85], for example):

Theorem 2 [Stokes multipliers] *For given two Stokes sectors D_1 and D_2 ($D_1 \cap D_2 \neq \emptyset$), components of their Stokes matrices, i.e. Stokes multipliers, $S = (s_{ij})$, satisfy the following triangular condition:*

$$s_{ij} = 0 \qquad \textit{if} \qquad \mathrm{Re}\big[(\varphi^{(i)}_{-r} - \varphi^{(j)}_{-r})\zeta^r\big] \not< 0 \qquad \zeta \to \infty \in D_1 \cap D_2 \neq \emptyset, \tag{2.41}$$

with $s_{jj} = 1$.

Here note that $A - B \not< 0$ includes the cases, in which the ordering of (A, B) cannot be defined. Therefore, equivalently, this means that the Stokes multiplier $s_{i,j}$ can take non-zero values only when the exponents satisfy

$$\mathrm{Re}\big[(\varphi^{(i)}_{-r} - \varphi^{(j)}_{-r})\zeta^r\big] < 0 \text{ for all angular range of } \zeta \to \infty \in D_1 \cap D_2 \neq \emptyset. \tag{2.42}$$

In this paper, we often refer to these facts about Stokes phenomena in ODE systems.

2.3 Stokes phenomena in the two-cut case

In this subsection, we consider the above general consideration in the two-cut $(1,2)$ critical point.

2.3.1 The ODE system and asymptotic expansions in the two-cut case

The string equation in this system is known as the Painlevé II equation [34, 35],

$$\frac{g_{\rm str}^2}{2}\ddot{f} - f^3 + 2tf = 0, \tag{2.43}$$

which is equivalent to the following ODE system in ζ (Eq. (2.45)) with its isomonodromy deformations in t (Eq. (2.46)):[8]

$$g_{\rm str}\frac{\partial}{\partial\zeta}\widetilde{\Psi}(t;\zeta) = \Bigl[\sigma_3\zeta^2 - \bigl(\sigma_1 f\bigr)\zeta + \Bigl(-\frac{1}{2}f^2+\mu\Bigr)\sigma_3 - g_{\rm str}\frac{i}{2}\sigma_2\dot{f}\Bigr]\widetilde{\Psi}(t;\zeta), \tag{2.45}$$

$$g_{\rm str}\frac{\partial}{\partial t}\widetilde{\Psi}(t;\zeta) = \Bigl[\sigma_3\zeta - \sigma_1\, f(t)\Bigr]\,\widetilde{\Psi}(t;\zeta). \tag{2.46}$$

Since this 2×2 first-order ODE system has two independent column vector solutions $\widetilde{\Psi}^{(1)}(t;\zeta)$ and $\widetilde{\Psi}^{(2)}(t;\zeta)$, we use the matrix notation for the solutions:

$$\widetilde{\Psi}(t;\zeta) = \Bigl(\widetilde{\Psi}^{(1)}(t;\zeta),\widetilde{\Psi}^{(2)}(t;\zeta)\Bigr). \tag{2.47}$$

At the point $\zeta\to\infty$, the ODE has an irregular singularity (of the Poincaré order 3) and the formal expansion of the solutions (2.22) is given as

$$\begin{aligned}\widetilde{\Psi}_{asym}(\zeta;t) &= \Bigl[I_2 + \frac{i}{2\zeta}\sigma_2 f(t) + \mathcal{O}(1/\zeta^2)\Bigr]\exp\Bigl[\frac{1}{g_{\rm str}}\Bigl(\frac{1}{3}\sigma_3\zeta^3 + \mu\sigma_3\zeta + \mathcal{O}(1/\zeta)\Bigr)\Bigr] \\ &\equiv \widetilde{Y}(t;\zeta)\,e^{\widetilde{\varphi}(t;\zeta)}.\end{aligned} \tag{2.48}$$

This can be obtained with the recursion relation 2.19 (see also in Appendix B.2). Note that the exponent $\widetilde{\varphi}(t;\zeta)$ is a diagonal matrix which satisfies $\widetilde{\varphi}(t;\zeta)\propto\sigma_3$, and then each vector solution $\widetilde{\Psi}^{(i)}_{asym}(t;\zeta)$ $(i=1,2)$ has different exponents:

$$\widetilde{\Psi}^{(i)}_{asym}(t;\zeta) = \widetilde{Y}^{(i)}(t;\zeta)\,e^{\varphi^{(i)}(t;\zeta)}, \tag{2.49}$$

with

$$\widetilde{Y}(t;\zeta) = \bigl(\widetilde{Y}^{(1)}(t;\zeta),\widetilde{Y}^{(2)}(t;\zeta)\bigr), \qquad \widetilde{\varphi}(t;\zeta) = {\rm diag}\bigl(\varphi^{(1)}(t;\zeta),\varphi^{(2)}(t;\zeta)\bigr). \tag{2.50}$$

[8] In the later discussion (from Section 3), we also define the different basis: $\Psi(t;\zeta)\equiv U^\dagger\widetilde{\Psi}(t;\zeta)U$, with

$$U\sigma_3U^\dagger = \sigma_1, \qquad U\sigma_1U^\dagger = -\sigma_3, \qquad U\sigma_2U^\dagger = \sigma_2. \tag{2.44}$$

This basis naturally appears in the matrix-model calculations and is more suitable to read the Hermiticity of the multi-cut matrix models [45].

2.3.2 Stokes sectors and Stokes matrices

In this case, there is only one kind of the Stokes lines $\mathrm{SL}_{1,2}$ which is given by (2.35) as

$$\zeta = |\zeta| e^{i\theta}: \qquad \theta = \frac{\pi}{6} + \frac{n\pi}{3} \qquad (n = 0, 1, \cdots, 5). \tag{2.51}$$

Therefore, Stokes sectors D_n are given as

$$D_n = e^{ni\frac{\pi}{3}} D_0, \qquad (n = 0, 1, \cdots, 5), \qquad D_0 = \Big\{ \zeta \in \mathbb{C}\,;\, -\frac{\pi}{2} < \arg(\zeta) < \frac{\pi}{6} \Big\}. \tag{2.52}$$

This is shown in Fig. 4. The canonical solution on the Stokes sector D_n is denoted by $\widetilde{\Psi}_n(t;\zeta)$. The Stokes matrices S_n are now defined as

$$S_n \equiv \widetilde{\Psi}_n^{-1}(t;\zeta)\, \widetilde{\Psi}_{n+1}(t;\zeta), \qquad (n = 0, 1, \cdots, 5), \tag{2.53}$$

and therefore components of the Stokes matrices are read as

$$\begin{aligned} D_{2n} \cap D_{2n+1}: \quad & S_{2n} = \begin{pmatrix} 1 & 0 \\ s_{2n} & 1 \end{pmatrix}; \qquad \Big(\big| e^{\varphi^{(1)}(t;\zeta)} \big| > \big| e^{\varphi^{(2)}(t;\zeta)} \big|, \quad \zeta \to \infty \Big), \\ D_{2n+1} \cap D_{2n+2}: \quad & S_{2n+1} = \begin{pmatrix} 1 & s_{2n+1} \\ 0 & 1 \end{pmatrix}; \qquad \Big(\big| e^{\varphi^{(1)}(t;\zeta)} \big| < \big| e^{\varphi^{(2)}(t;\zeta)} \big|, \quad \zeta \to \infty \Big). \end{aligned} \tag{2.54}$$

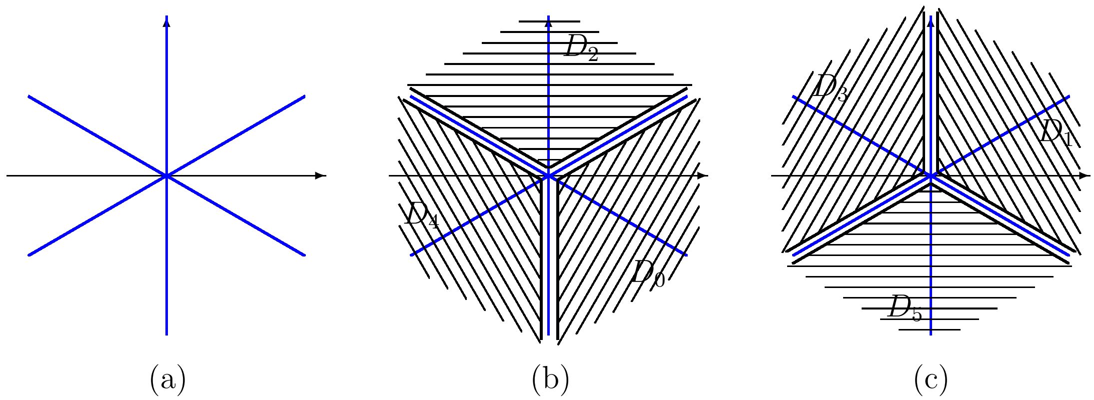


Figure 4: a) Stokes lines in the two-cut $(1,2)$ case. b) Stokes sectors of D_0, D_2 and D_4. c) Stokes sectors of D_1, D_3 and D_5.

2.3.3 Three basic constraints on the Stokes multipliers

The Stokes multipliers satisfy three constraints from the symmetry of the original ODE system.

$\mathbb{Z}_2$ **symmetry constraint** This symmetry originates from the $\mathbb{Z}_2$ symmetry of the matrix model. That is, this is the reflection symmetry $M \to -M$ of the one-matrix models:

$$\mathcal{Z} = \int dM e^{-N \,\mathrm{tr}\, V(M)}, \qquad V(-M) = V(M). \tag{2.55}$$

In terms of the ODE system, this symmetry is expressed by the reflection of $\zeta \to -\zeta$:

$$g_{\rm str}\frac{d\widetilde{\Psi}(t;-\zeta)}{d\zeta} = \big[-\widetilde{\mathcal{Q}}(t;-\zeta)\big]\,\widetilde{\Psi}(t;-\zeta) = \big[\sigma_1\widetilde{\mathcal{Q}}(t;\zeta)\sigma_1\big]\,\widetilde{\Psi}(t;-\zeta),$$

$$g_{\rm str}\frac{d\widetilde{\Psi}(t;-\zeta)}{dt} = \big[\widetilde{\mathcal{P}}(t;-\zeta)\big]\,\widetilde{\Psi}(t;-\zeta) = \big[\sigma_1\widetilde{\mathcal{P}}(t;\zeta)\sigma_1\big]\,\widetilde{\Psi}(t;-\zeta). \tag{2.56}$$

Therefore, each canonical solution is mapped to other canonical solution as:

$$\sigma_1\widetilde{\Psi}_n(t;-\zeta)\sigma_1 = \widetilde{\Psi}_{n+3}(t;\zeta), \qquad (n=0,2,\cdots,5), \tag{2.57}$$

and the Stokes matrices are mapped as

$$S_{n+3} = \sigma_1 S_n \sigma_1, \qquad s_{n+3} = s_n, \qquad (n=0,1,\cdots,5). \tag{2.58}$$

This means that there are only three independent Stokes multipliers,

$$s_0 = s_3 \equiv \alpha, \qquad s_1 = s_4 \equiv \beta, \qquad s_2 = s_5 = \gamma. \tag{2.59}$$

Hermiticity constraint This originates from Hermiticity of the matrix models. In the two-cut cases, they are studied in [38, 68]. This symmetry is expressed as[9]

$$\widetilde{\mathcal{Q}}^*(t;\zeta^*) = \widetilde{\mathcal{Q}}(t;\zeta^*), \qquad \widetilde{\mathcal{P}}^*(t;\zeta^*) = \widetilde{\mathcal{P}}(t;\zeta^*). \tag{2.60}$$

Therefore, each canonical solution is mapped to other canonical solution as:

$$\widetilde{\Psi}_n^*(t;\zeta^*) = \widetilde{\Psi}_{7-n}(t;\zeta), \qquad (n=0,1,\cdots,5), \tag{2.61}$$

and the Stokes matrices are mapped as

$$S_n^* = S_{6-n}^{-1}, \qquad s_n^* + s_{6-n} = 0, \qquad (n=0,1,\cdots,5). \tag{2.62}$$

This reduces three independent Stokes multipliers α, β and γ to be two real parameters:

$$\alpha^* + \alpha = 0, \qquad \beta^* + \gamma = 0. \tag{2.63}$$

Monodromy free constraint The last constraint is the requirement that the solutions to the ODE system are single-valued functions. Note that one can also introduce this degree of freedom in the context of matrix models. This corresponds to introducing background RR flux and/or D0-branes in 0A string background, that is, the system becomes the complex matrix models [41, 94]. This constraint for the single-valued solutions is expressed as

$$\widetilde{\Psi}_n(t;\zeta) = \widetilde{\Psi}_n(t;e^{2\pi i}\zeta) = \widetilde{\Psi}_{n+6}(t;\zeta), \tag{2.64}$$

therefore

$$S_0\,S_1\,S_2\,S_3\,S_4\,S_5 = I_2, \tag{2.65}$$

[9] Note that we use the following convention of complex conjugation in this paper: $[f(\zeta)]^* = f^*(\zeta^*) = \sum_n f_n^* \zeta^*$, with a function $f(\zeta) \equiv \sum_n f_n \zeta^n$.

which results in

$$s_0 + s_1 + s_2 + s_0 s_1 s_3 = \alpha(1 - |\beta|^2) + \beta - \beta^* = 0, \tag{2.66}$$

with the other constraints. Consequently, the Stokes multipliers have one real degree of freedom, say α. On the other hand, the Lax operators in this case also have one real degree of freedom, $f(t)$. In this sense, the system is completely fixed with one integration constant.

Here it is also worth mentioning that among the Stokes multipliers satisfying the algebraic relation (2.66), it is the following choice

$$\alpha = 0, \qquad \beta = \pm 1, \tag{2.67}$$

which realize the actual perturbative behavior (in $t \to \pm\infty$) of the matrix models argued in [41]. This choice of the Stokes multipliers is known as the Hastings-McLeod solution in the Painlevé II equation [84].[10] This means that there is a unique (or at least discrete) choice of the physical Stokes multipliers (therefore D-instantion chemical potentials). As mentioned in Introduction, this naturally rises the following question: *what is the physical requirements which specify the above multipliers?* This is also related to the issue cited by [60, 62]: *What is the boundary condition in continuum formulations which can fix the D-instanton chemical potentials in the matrix models?* Our procedure (discussed in Section 4 and Section 5) gives an answer to the question. In Section 4.2.1 and then Section 5.1, we will see that our physical requirements correctly choose this particular parametrization Eq. (2.67) of the Stokes multipliers.

3 Stokes phenomena in the multi-cut cases

In this section, we develop general framework for Stokes phenomena in the general multi-cut critical points, and show explicitly *how the actual systems can be controlled*. Key information is provided by *profile of dominant exponents* introduced in Section 3.2.

3.1 Stokes lines and Stokes sectors

First we focus on the Stokes lines,

$$\mathrm{SL}_{j,l}: \qquad \mathrm{Re}\big[\big(\varphi_{-r}^{(j)} - \varphi_{-r}^{(l)}\big)\zeta^r\big] = 0, \tag{3.1}$$

and the resulting Stokes sectors (2.36). In this paper, we are interested in the $\mathbb{Z}_k$-symmetric critical points, and as one can see in Appendix B, the leading coefficient of the exponents, $\varphi_{-r}^{(j)}$, is given as

$$\varphi_{-r}^{(j)} = \omega^{-r(j-1)}. \tag{3.2}$$

Consequently, the conditions on the Stokes lines (in terms of angle, $\zeta = |\zeta| e^{i\theta}$) are expressed as

$$\mathrm{Re}\big[\big(\varphi_{-r}^{(j)} - \varphi_{-r}^{(l)}\big)e^{ri\theta}\big] = -2\sin\big(r\theta - \pi\frac{r(j+l-2)}{k}\big)\sin\big(\pi\frac{r(j-l)}{k}\big). \tag{3.3}$$

[10] See [85] for reviews of this solution.

First of all, if there is a pair of (j,l) such that

$$r(j-l)\in k\mathbb{Z}, \tag{3.4}$$

then the condition (2.31) does not satisfy. This means that the highest exponents degenerate $(\varphi^{(j)}_{-r}-\varphi^{(l)}_{-r})\,\zeta^r=0$. In this case, we consider the next leading Stokes lines,

$$\mathrm{Re}\big[\big(\varphi^{(j)}_{-r+1}-\varphi^{(l)}_{-r+1}\big)\zeta^{r-1}\big]=0, \tag{3.5}$$

or more generally we consider the following Stokes lines:

Definition 5 [(General) Stokes lines] *The general Stokes lines* $\mathrm{SL}^g_{j,l}$ *in this ODE system are defined for each pair of* (j,l) *as*

$$\mathrm{SL}^g_{j,l}\equiv\Big\{\zeta\in\mathbb{C};\mathrm{Re}\big[\varphi^{(j)}(t;\zeta)-\varphi^{(l)}(t;\zeta)\big]=0\Big\}=\bigcup_{n=0}^{2r-1}\mathrm{SL}^{g;(n)}_{j,l}, \tag{3.6}$$

which consists of $2r$ *semi-infinite lines,* $\mathrm{SL}^{g;(n)}_{j,l}$ $(n=0,1,\cdots,2r-1)$*. The set of lines,* SL^g*, denotes a set of whole (general) Stokes lines,* $\mathrm{SL}^g\equiv\bigcup_{j,l}\mathrm{SL}^g_{j,l}$*.*

The physical interpretation of these general Stokes lines is discussed in Section 4.2. The situations (3.4) are also interesting critical points in the multi-cut matrix models, however here for sake of simplicity, we concentrate on the following cases,

$$(k,r):\qquad \text{a coprime}, \tag{3.7}$$

because Eq. (3.4) becomes trivial in this case:

$$r(j-l)\in k\mathbb{Z}\qquad\Leftrightarrow\qquad j-l\in k\mathbb{Z}. \tag{3.8}$$

Therefore Eq. (3.3) gives the angle $\theta^{(n)}_{j,l}$ for the Stokes lines $\mathrm{SL}_{j,l}$ as[11]

$$\theta=\theta^{(n)}_{j,l}=\frac{kn+r(j+l-2)}{rk}\pi,\qquad n\in\mathbb{Z}, \tag{3.10}$$

in the $\mathbb{Z}_k$-symmetric cases. From this formula, one can read several basic information about the Stokes lines. An example of Stokes lines (3-cut $(1,1)$ case) is shown in Fig. 2-b. For later convenience, we introduce the following concept:

Definition 6 [Segments] *Angular domains in between two Stokes lines which do not include any Stokes lines are called segments.*

[11] Note that if one considers the fractional-superstring cases the formula is expressed as

$$\theta=\theta^{(n)}_{j,l}=\frac{kn+(r-2)(j+l-2)}{rk}\pi,\qquad n\in\mathbb{Z} \tag{3.9}$$

with $\varphi^{(j)}_{-r}=\omega^{-(r-2)(j-1)}$ and the coprime condition is imposed on the pair $(k,r-2)$.

In our present cases with a coprime (k, r) and with $k \geq 3$, there are $2rk$ distinct segments δD_n $(n = 0, 1, \cdots, 2rk-1)$ given as

$$\delta D_n \equiv D\big(n\delta\theta - \delta\theta; n\delta\theta\big), \qquad \Big(n = 0, 1, \cdots, 2rk-1;\ \delta\theta = \frac{\pi}{rk}\Big), \tag{3.11}$$

which can fill the complex plane $\mathbb{C}$,

$$\bigcup_{n=0}^{2rk-1} \overline{\delta D_n} = \mathbb{C}, \qquad \delta D_m \cap \delta D_{m'} = \emptyset \qquad \big(m \neq m'\big). \tag{3.12}$$

According to the definition of Stokes sectors, Eq. (2.36), we define the following most basic Stokes sectors, D_n:

Definition 7 [Fine Stokes sectors/matrices] *The following angular domains D_n*

$$D_n = e^{ni\delta\theta} D_0, \qquad D_0 = D\big(-\delta\theta; k\delta\theta\big), \qquad (n = 0, 1, \cdots, 2rk-1), \tag{3.13}$$

are Stokes sectors of a coprime (k, r) system with $k \geq 3$, which are referred to as fine Stokes sectors. The canonical solution of the fine Stokes sector D_n is denoted as $\widetilde{\Psi}_n(t;\zeta)$ and the corresponding Stokes matrices S_n are given as

$$\widetilde{\Psi}_{n+1}(t;\zeta) = \widetilde{\Psi}_n(t;\zeta)\, S_n, \tag{3.14}$$

which is referred to as (fine) Stokes matrices.

Here we also define the other two kinds of Stokes sectors/matrices: First we define Stokes sectors/matrices which respect to the $\mathbb{Z}_k$ symmetry of the multi-cut matrix models:

Definition 8 [Symmetric Stokes sectors/matrices] *The following subset of the fine Stokes sectors,*

$$D_{2nr}, \qquad (n = 0, 1, \cdots, k-1), \tag{3.15}$$

are referred to as symmetric Stokes sectors,[12] *and the corresponding Stokes matrices $S_{2rn}^{(\mathrm{sym})}$*

$$S_{2rn}^{(\mathrm{sym})} \equiv \widetilde{\Psi}_{2rn}^{-1}(t;\zeta)\, \widetilde{\Psi}_{2r(n+1)}(t;\zeta) = S_{2rn} \cdot S_{2rn+1} \cdots S_{2r(n+1)-1}. \tag{3.16}$$

are referred to as symmetric Stokes matrices.

Next we define the following economical Stokes sectors/matrices:

Definition 9 [Coarse Stokes sectors/matrices] *The following subset of the fine Stokes sectors,*

$$D_{nk}, \qquad (n = 0, 1, \cdots, 2r-1), \tag{3.17}$$

are referred to as coarse Stokes sectors, and the corresponding Stokes matrices $S_{nk}^{(\mathrm{c})}$ are written as

$$S_{nk}^{(\mathrm{coa})} \equiv \widetilde{\Psi}_{nk}^{-1}(t;\zeta)\, \widetilde{\Psi}_{(n+1)k}(t;\zeta) = S_{nk} \cdot S_{nk+1} \cdots S_{(n+1)k-1}. \tag{3.18}$$

are referred to as coarse Stokes matrices.

Coarse Stokes sectors are most often used in literature. However, in the following discussions, one will see that the fine Stokes matrices are more convenient for our calculations.

[12]Note that this definition is not enough for the $k = 3, r = 2$ case. In these cases, we employ a modified version of the Stokes sectors, for example, D_{nr}.

3.2 Profile of dominant exponents

The non-zero components of Stokes matrices (Stokes multipliers) are determined by the triangular condition of Theorem 2. However, this definition is inconvenient for practical evaluations. Therefore we here develop a way to read the information of these non-zero Stokes multipliers by evaluating *the profile of dominant exponents*. The definition of this concept is following:

Since there is no Stokes line in the segments defined in Eq. (3.11), one can define the following ordered set J_l of indices $j_{l,i}$:

$$J_l = \left[\ j_{l,1} \,\middle|\, j_{l,2} \,\middle|\, \cdots \,\middle|\, j_{l,k}\ \right] \in \mathbb{N}^k, \tag{3.19}$$

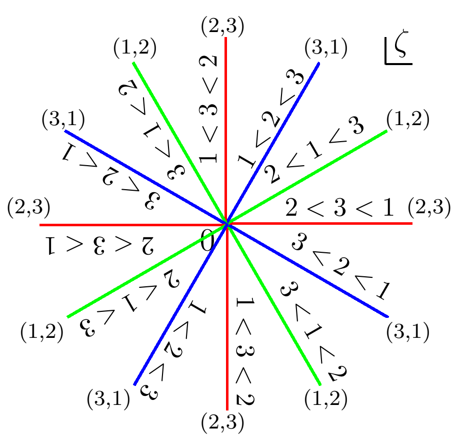
which describes the profile of dominant exponents in the segment D_l

$$\mathrm{Re}\big[\varphi_{-r}^{(j_{l,1})}\zeta^r\big] < \mathrm{Re}\big[\varphi_{-r}^{(j_{l,2})}\zeta^r\big] < \cdots < \mathrm{Re}\big[\varphi_{-r}^{(j_{l,k})}\zeta^r\big], \qquad \zeta \in \delta D_l. \tag{3.20}$$

This sequence of numbers, $\mathcal{J} = \{J_l\}_{l=0}^{2rk}$, is referred to as *profile of dominant exponents*. Here we express the profile $\mathcal{J}$ as follows:

$$\mathcal{J} = \left[\begin{array}{c|c|c|c} j_{2rk-1,1} & j_{2rk-1,2} & \cdots & j_{2rk-1,k} \\ \hline \vdots & \vdots & & \vdots \\ \hline j_{1,1} & j_{1,2} & \cdots & j_{1,k} \\ \hline j_{0,1} & j_{0,2} & \cdots & j_{0,k} \end{array}\right] \tag{3.21}$$

Note that the ordering of indices in the vertical direction is different from the usual matrix, and that elements are periodic in the index l, $J_l = J_{l+2rk}$. An example (3-cut $(1,1)$ critical point) and the relation to the ζ plane are shown in Fig. 5.



(a)

$$\mathcal{J} = \left[\begin{array}{c|c|c} 3 & 1 & 2 \\ \hline 1 & 3 & 2 \\ \hline 1 & 2 & 3 \\ \hline 2 & 1 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 2 & 1 \\ \hline 3 & 1 & 2 \\ \hline 1 & 3 & 2 \\ \hline 1 & 2 & 3 \\ \hline 2 & 1 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 2 & 1 \end{array}\right]$$

(b)

Figure 5: The two expressions for the profile of dominant exponents in the 3-cut $(1,1)$ critical point. $\mathrm{Re}[\varphi_{-2}^{(j_{l,1})}] < \mathrm{Re}[\varphi_{-2}^{(j_{l,2})}] < \mathrm{Re}[\varphi_{-2}^{(j_{l,3})}]$ is expressed as $j_{l,1} < j_{l,2} < j_{l,3}$. a) The profile in the ζ plane. b) The profile $\mathcal{J}$ in the table. In the same way, the dominance is expressed as $[j_{l,1}|j_{l,2}|j_{l,3}]$

Next we address how to fill the non-zero elements in the profiles. The basic equation for reading this information is the definition of Stokes lines (3.10),

$$\mathrm{SL}_{i,j}: \quad \theta = \theta_{i,j}^{(n)} = \frac{kn + r(i+j-2)}{rk}\pi, \qquad n \in \mathbb{Z}. \tag{3.22}$$

Since (r,k) is a coprime pair of integers, the Stokes lines appear at the following angles:

$$\theta^{(n)}_{i,j} = \frac{l}{rk}\pi = l\delta\theta, \qquad l \in \mathbb{Z}. \tag{3.23}$$

Therefore, we re-interpret this in the following way: the pair (i,j) will change their dominance at angle $\theta = l\delta\theta$ when there exists an integer n which satisfies

$$r(i+j-2) + kn = l, \qquad n, l \in \mathbb{Z}. \tag{3.24}$$

To study this relation, we introduce the Euclid reminder (m_l, n_l) of

$$rm_l + kn_l = l. \tag{3.25}$$

This relation uniquely determines pair of (m_l, n_l) up to the following shift:

$$(m_l, n_l) \quad \to \quad (m_l + ks, n_l - rs), \qquad s \in \mathbb{Z}. \tag{3.26}$$

Note that the redundancy (3.26) is understood as redundancy of the indices j, $j \to j + sk$ in Eq. (3.24). Therefore without loss of generality, one can choose one representative as $0 \le m_l < k$, and

$$m_l \equiv l m_1 \mod k. \tag{3.27}$$

Consequently, we obtain the following rule:

Proposition 1 [Sum rule of indices] *For a given J_l, a pair of indices (i,j) in the profile J_l change their relative dominance at angle $\theta = l\delta\theta$ when they satisfy the following sum rule:*

$$i + j - 2 = m_l, \tag{3.28}$$

with the Euclidean reminder (n_l, m_l) of Eq. (3.25).

From this proposition, one can read all the pairs of indices which change the relative dominance at the angle $\theta = l\delta\theta$:

$$\begin{aligned} (i,j) &\equiv (a, m_l + 2 - a) \mod k, \quad (k - b + 1, m_l + 2 + b - 1) \mod k, \\ &\text{with} \quad a = 1, 2, \cdots, \left\lfloor \frac{m_l + 1}{2} \right\rfloor; \, b = 1, 2, \cdots, \left\lfloor \frac{k - m_l - 1}{2} \right\rfloor, \end{aligned} \tag{3.29}$$

the number of which is

$$\#_{(i,j)\,\text{pairs}} = \begin{cases} \frac{k-1}{2} & (k: \text{odd}) \\ \frac{k}{2} & (k: \text{even}, \, m_l: \text{odd}) \\ \frac{k}{2} - 1 & (k: \text{even}, \, m_l: \text{even}) \end{cases}. \tag{3.30}$$

By taking into account these results, one can show the following proposition:

Proposition 2 [Trajectory of the indices] *If the profile of dominance J_l in a segment δD_l is given by the following sequence:*

$$J_l = \left[\, j_{l,1} \,\middle|\, j_{l,2} \,\middle|\, \cdots \,\middle|\, j_{l,k} \,\right] = \left[\, a_1 \,\middle|\, a_2 \,\middle|\, \cdots \,\middle|\, a_k \,\right], \tag{3.31}$$

then the next profile $J_{l+1} = \left[\ j_{l+1,1}\ \middle|\ j_{l+1,2}\ \middle|\ \cdots\ \middle|\ j_{l+1,k}\ \right]$ *is given as follows:*

<u>k is odd</u>

$$l \in 2\mathbb{Z}: \qquad \begin{cases} j_{l+1,2m} = j_{l,2m-1} = a_{2m-1}, & (1 < 2m < k) \\ j_{l+1,2m-1} = j_{l,2m} = a_{2m}, & (1 \le 2m-1 < k) \\ j_{l+1,k} = j_{l,k} = a_k \end{cases}$$

$$l \in 2\mathbb{Z}+1: \qquad \begin{cases} j_{l+1,2m} = j_{l,2m+1} = a_{2m+1}, & (1 < 2m < k) \\ j_{l+1,2m+1} = j_{l,2m} = a_{2m}, & (1 < 2m+1 \le k) \\ j_{l+1,1} = j_{l,1} = a_1 \end{cases} ,$$

<u>k is even</u>

$$l \in 2\mathbb{Z}: \qquad \begin{cases} j_{l+1,2m} = j_{l,2m+1} = a_{2m+1}, & (1 < 2m < k) \\ j_{l+1,2m+1} = j_{l,2m} = a_{2m}, & (1 \le 2m+1 < k) \\ j_{l+1,1} = j_{l,1} = a_1 \\ j_{l+1,k} = j_{l,k} = a_k \end{cases}$$

$$l \in 2\mathbb{Z}+1: \qquad \begin{cases} j_{l+1,2m} = j_{l,2m-1} = a_{2m-1}, & (1 < 2m \le k) \\ j_{l+1,2m-1} = j_{l,2m} = a_{2m}, & (1 \le 2m-1 \le k) \end{cases} , \tag{3.32}$$

In terms of table, they are expressed as

<u>k is odd</u>

$$\mathcal{J} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline a_2 & a_1 & a_4 & a_3 & a_6 & \cdots & a_{k-1} & a_{k-2} & a_k \\ \hline a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{k-2} & a_{k-1} & a_k \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline b_1 & b_3 & b_2 & b_5 & b_4 & \cdots & b_{k-3} & b_k & b_{k-1} \\ \hline b_1 & b_2 & b_3 & b_4 & b_5 & \cdots & b_{k-2} & b_{k-1} & b_k \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \end{array}\right] \begin{array}{l} \\ \\ \leftarrow J_l,\ (l \in 2\mathbb{Z}) \\ \\ \\ \leftarrow J_{l'},\ (l' \in 2\mathbb{Z}+1) \\ \\ \end{array} ,$$

<u>k is even</u>

$$\mathcal{J} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline a_1 & a_3 & a_2 & a_5 & a_4 & \cdots & a_{k-1} & a_{k-2} & a_k \\ \hline a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{k-2} & a_{k-1} & a_k \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline b_2 & b_1 & b_4 & b_3 & b_6 & \cdots & b_{k-3} & b_k & b_{k-1} \\ \hline b_1 & b_2 & b_3 & b_4 & b_5 & \cdots & b_{k-2} & b_{k-1} & b_k \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \end{array}\right] \begin{array}{l} \\ \\ \leftarrow J_l,\ (l \in 2\mathbb{Z}) \\ \\ \\ \leftarrow J_{l'},\ (l' \in 2\mathbb{Z}+1) \\ \\ \end{array} , \tag{3.33}$$

with $J_{l'} = \left[\ j_{l',1}\ \middle|\ j_{l',2}\ \middle|\ \cdots\ \middle|\ j_{l',k}\ \right] = \left[\ b_1\ \middle|\ b_2\ \middle|\ \cdots\ \middle|\ b_k\ \right]$.

This proposition means that trajectories of the indices, for instance a and b, are given as follows:

$$
\underline{k \text{ is odd}} \qquad\qquad\qquad\qquad \underline{k \text{ is even}}
$$

$$
\mathcal{J} = \left[\begin{array}{c|c|c|c|c|c|c}
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline
 & a & & \cdots & b & & \\ \hline
a & & & \cdots & & b & \\ \hline
a & & & \cdots & & & b \\ \hline
 & a & & \cdots & & & b \\ \hline
 & & a & \cdots & & b & \\ \hline
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots
\end{array}\right], \qquad
\mathcal{J} = \left[\begin{array}{c|c|c|c|c|c|c}
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \hline
 & a & & \cdots & & b & \\ \hline
a & & & \cdots & & & b \\ \hline
a & & & \cdots & & & b \\ \hline
 & a & & \cdots & & b & \\ \hline
 & & a & \cdots & b & & \\ \hline
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots
\end{array}\right] \tag{3.34}
$$

Furthermore, the non-zero elements in the profile are recursively fixed by using the sum rule (3.28). The direct comparison of exponents $\{\varphi_{-r}^{(j)}\}_{j=1}^{k}$ results in the following initial numbers for the recursions:

Proposition 3 *The most dominant exponent in the segments δD_0 and δD_1 is given as*

$$
j_{0,k} = j_{1,k} = 1. \tag{3.35}
$$

with the assumption of coprime (k, r) and $k \geq 3$.

First we fix the numbers in J_0, which are now denoted as

$$
J_0 \equiv \left[\ A_k \,\middle|\, \cdots \,\middle|\, A_3 \,\middle|\, A_2 \,\middle|\, A_1\ \right] = \left[\ A_k \,\middle|\, \cdots \,\middle|\, A_3 \,\middle|\, A_2 \,\middle|\, 1\ \right]. \tag{3.36}
$$

From Proposition 2, one can see all the profile $\mathcal{J}$:

$$
\mathcal{J} = \left[\begin{array}{c|c|c|c|c|c|c}
\cdots & A_4 & (A_7 & A_2) & (A_5 & 1) & A_3 \\ \hline
\cdots & (A_7 & A_4) & (A_5 & A_2) & (A_3 & 1) \\ \hline
\cdots & \mathbf{A_6} & (\mathbf{A_5} & \mathbf{A_4}) & (\mathbf{A_3} & \mathbf{A}_2) & \mathbf{1} \\ \hline
\cdots & (A_5 & A_6) & (A_3 & A_4) & (1 & A_2)
\end{array}\right] \leftarrow J_0. \tag{3.37}
$$

Here we put parentheses, $(*|*)$, around two indices to indicate the pair of dominance changing of (3.29). The point is that we use two recursive relations with respect to the pair of $(A_{l+1}|A_l)$,

$$
\begin{aligned}
A_{2m} \to A_{2m+1}: &\quad A_{2m} + A_{2m+1} = m_0 + 2 = 2 \\
A_{2m+1} \to A_{2m+2}: &\quad A_{2m+1} + A_{2m+2} = m_{-1} + 2 = 2 - m_1,
\end{aligned} \tag{3.38}
$$

to determine the components. The result is given as

Lemma 1 [Components of J_0] *The components of the profile $J_0 \equiv \left[\ A_k \,\middle|\, \cdots \,\middle|\, A_3 \,\middle|\, A_2 \,\middle|\, A_1\ \right]$ is give as*

$$
A_n = 1 - (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \times m_1, \tag{3.39}
$$

especially the $2r$ shift of the index n gives $A_{n+2r} = A_n + (-1)^{n+1} \mod k$.

The last equation is useful to fill the profile. In particular, one can fill the numbers from 1 to k by using

$$A_{1+2rs} = 1 + s, \tag{3.40}$$

with reflection relations $A_n = A_{n+2k}$ and $A_n = A_{1-n}$ of Eq. (3.39). The vertical sequence in $\mathcal{J}$ is also read from this result:

Corollary 1 [Vertical components] *The sequence of* $\{j_{l,k}\}_{l=0}^{2rk}$ *is given as*

$$j_{l,k} = A_{2\lfloor \frac{l}{2} \rfloor + 1} = 1 + \left\lfloor \frac{l}{2} \right\rfloor \times m_1, \tag{3.41}$$

especially the $2r$ *shift of the index* l *gives* $j_{l+2r,k} = j_{l,k} + 1 \mod k$.

Therefore, repeating the same procedure, one obtains the following general formula:

Theorem 3 [General components] *The general components of the profile* $\mathcal{J}$ *are given as*

$$\begin{array}{lll} \underline{l : \textit{even}} & j_{l,n} = \left\{ \begin{array}{ll} 1 + \left(\left\lfloor \frac{k-n+1}{2} \right\rfloor + \left\lfloor \frac{l}{2} \right\rfloor \right) m_1 & (n : \textit{odd}) \\ 1 + \left(l - \left\lfloor \frac{k-n+1}{2} \right\rfloor - \left\lfloor \frac{l}{2} \right\rfloor \right) m_1 & (n : \textit{even}) \end{array} \right. \\ \underline{l : \textit{odd}} & j_{l,n} = \left\{ \begin{array}{ll} 1 - \left(\left\lfloor \frac{k-n}{2} \right\rfloor - \left\lfloor \frac{l}{2} \right\rfloor \right) m_1 & (n : \textit{odd}) \\ 1 + \left(l + \left\lfloor \frac{k-n}{2} \right\rfloor - \left\lfloor \frac{l}{2} \right\rfloor \right) m_1 & (n : \textit{even}) \end{array} \right. . \end{array} \tag{3.42}$$

These are modulo k, *and for* $n = 1, 2, \cdots, k$ *and* $l = 0, 1, \cdots, 2rk - 1$.

Finally, we show two examples of the profiles $\mathcal{J}_{k,r}$ for the case of $(k, r) = (3, 2)$ and $(5, 2)$:

$$\mathcal{J}_{3,2} = \left[\begin{array}{c|c|c} 3 & (1 & 2) \\ \hline (1 & 3) & 2 \\ \hline 1 & (2 & 3) \\ \hline (2 & 1) & 3 \\ \hline 2 & (3 & 1) \\ \hline (3 & 2) & 1 \\ \hline 3 & (1 & 2) \\ \hline (1 & 3) & 2 \\ \hline 1 & (2 & 3) \\ \hline (2 & 1) & 3 \\ \hline 2 & (3 & 1) \\ \hline (3 & 2) & 1 \end{array} \right] \begin{array}{l} \leftarrow J_{11} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow J_0 \end{array}, \qquad \mathcal{J}_{5,2} = \left[\begin{array}{c|c|c|c|c} 2 & (4 & 5) & (1 & 3) \\ \hline (4 & 2) & (1 & 5) & 3 \\ \hline 4 & (1 & 2) & (3 & 5) \\ \hline (1 & 4) & (3 & 2) & 5 \\ \hline 1 & (3 & 4) & (5 & 2) \\ \hline (3 & 1) & (5 & 4) & 2 \\ \hline 3 & (5 & 1) & (2 & 4) \\ \hline (5 & 3) & (2 & 1) & 4 \\ \hline 5 & (2 & 3) & (4 & 1) \\ \hline (2 & 5) & (4 & 3) & 1 \\ \hline 2 & (4 & 5) & (1 & 3) \\ \hline (4 & 2) & (1 & 5) & 3 \\ \hline 4 & (1 & 2) & (3 & 5) \\ \hline (1 & 4) & (3 & 2) & 5 \\ \hline 1 & (3 & 4) & (5 & 2) \\ \hline (3 & 1) & (5 & 4) & 2 \\ \hline 3 & (5 & 1) & (2 & 4) \\ \hline (5 & 3) & (2 & 1) & 4 \\ \hline 5 & (2 & 3) & (4 & 1) \\ \hline (2 & 5) & (4 & 3) & 1 \end{array} \right] \begin{array}{l} \leftarrow J_{19} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow J_0 \end{array}. \tag{3.43}$$

One can observe that there is a $2k$ periodicity, $j_{n+2k,i} = j_{n,i}$, or more precisely, a reflection by step k, $j_{n,i} = j_{n+k,k-i+1}$.

3.3 Components of Stokes matrices and the profiles

From the profile we have considered above, one can see the non-zero multipliers by referring to Theorem 2. For example, if one wants to calculate the symmetric Stokes matrix $S_0^{(\mathrm{sym})}$ in the $(r,k) = (2,5)$ case, one first sees the dominance profile in the domain $D_0 \cap D_4$,

$$D_0 \cap D_4 \supset \left[\begin{array}{c|c|c|c|c} 1 & 3 & 4 & 5 & 2 \\ \hline 3 & 1 & 5 & 4 & 2 \end{array}\right], \begin{array}{l} \leftarrow J_5 \\ \leftarrow J_4 \end{array} \tag{3.44}$$

and reads the ordering of magnitude:

$$(2) > (5), (4), (3), (1), \qquad (5) > (3), (1), \qquad (4) > (3), (1). \tag{3.45}$$

This results in the necessary and sufficient symmetric Stokes multipliers:

$$S_0^{(\mathrm{sym})} = \begin{pmatrix} 1 & s_{0,1,2}^{(\mathrm{sym})} & 0 & s_{0,1,4}^{(\mathrm{sym})} & s_{0,1,5}^{(\mathrm{sym})} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{0,3,2}^{(\mathrm{sym})} & 1 & s_{0,3,4}^{(\mathrm{sym})} & s_{0,3,5}^{(\mathrm{sym})} \\ 0 & s_{0,4,2}^{(\mathrm{sym})} & 0 & 1 & 0 \\ 0 & s_{0,5,2}^{(\mathrm{sym})} & 0 & 0 & 1 \end{pmatrix}. \tag{3.46}$$

In the same way, if one wants to calculate the fine Stokes matrix S_0, one first sees the dominance profile in the domain $D_0 \cap D_1$,

$$D_0 \cap D_1 \supset \left[\begin{array}{c|c|c|c|c} 1 & \mathbf{3} & \mathbf{4} & \underline{\mathit{5}} & \underline{\mathit{2}} \\ \hline \mathbf{3} & 1 & \underline{\mathit{5}} & \mathbf{4} & \underline{\mathit{2}} \\ \hline \mathbf{3} & \underline{\mathit{5}} & 1 & \underline{\mathit{2}} & \mathbf{4} \\ \hline \underline{\mathit{5}} & \mathbf{3} & \underline{\mathit{2}} & 1 & \mathbf{4} \\ \hline \underline{\mathit{5}} & \underline{\mathit{2}} & \mathbf{3} & \mathbf{4} & 1 \end{array}\right], \begin{array}{l} \leftarrow J_5 \\ \\ \\ \\ \leftarrow J_1 \end{array} \tag{3.47}$$

and reads the ordering of magnitude:

$$(4) > (3), \qquad (2) > (5). \tag{3.48}$$

This results in the necessary and sufficient Stokes multipliers:

$$S_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & s_{0,3,4} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & s_{0,5,2} & 0 & 0 & 1 \end{pmatrix}. \tag{3.49}$$

With this original way of reading the multipliers, it is quite difficult to see the general structure. From the above result, however, one may notice that there is a relation between indices of non-zero Stokes multipliers $s_{0,i,j}$ in the Stokes matrix S_0 and the dominance changing pairs (j,i) in the profile J_0:

$$s_{0,3,4}, \quad s_{0,5,2} \qquad \leftrightarrow \qquad (2|5), \quad (4|3) \quad \in \quad J_0 = \big[\ (2 \,|\, 5) \,\big|\, (4 \,|\, 3) \,\big|\, 1\ \big]. \tag{3.50}$$

In view of this, one can generally show the following statement:

Theorem 4 *The non-zero Stokes multipliers in the fine Stokes matrix S_l can be read from the profile J_l as*

$$s_{l,i,j} = 0 \quad (i \neq j) \qquad \textit{if} \qquad (i,j) \neq (j_{l,m}, j_{l,m-1}) \quad \textit{with} \quad k+l+m \in 2\mathbb{Z}+1. \tag{3.51}$$

That is, if there is a dominance changing pair $(i|j)$ in the profile J_l, then the Stokes multiplier $s_{l,j,i}$ can take non-zero value.[13]

The other Stokes matrices, say $S_n^{(\mathrm{sym})}$ and $S_n^{(\mathrm{coa})}$, are written as a product of the fine Stokes matrices S_n (as in (3.16) and (3.18)). For instance,

$$S_0^{(\mathrm{sym})} = \begin{pmatrix} 1 & s_{0,1,2}^{(\mathrm{sym})} & 0 & s_{0,1,4}^{(\mathrm{sym})} & s_{0,1,5}^{(\mathrm{sym})} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{0,3,2}^{(\mathrm{sym})} & 1 & s_{0,3,4}^{(\mathrm{sym})} & s_{0,3,5}^{(\mathrm{sym})} \\ 0 & s_{0,4,2}^{(\mathrm{sym})} & 0 & 1 & 0 \\ 0 & s_{0,5,2}^{(\mathrm{sym})} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s_{2,1,2}+s_{1,1,4}s_{3,4,2} & 0 & s_{1,1,4} & s_{3,1,5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{1,3,2}+s_{0,3,4}s_{3,4,2} & 1 & s_{0,3,4} & s_{2,3,5} \\ 0 & s_{3,4,2} & 0 & 1 & 0 \\ 0 & s_{0,5,2} & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} S_0^{(\mathrm{coa})} &= \begin{pmatrix} 1 & s_{0,1,2}^{(\mathrm{coa})} & s_{0,1,3}^{(\mathrm{coa})} & s_{0,1,4}^{(\mathrm{coa})} & s_{0,1,5}^{(\mathrm{coa})} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{0,3,2}^{(\mathrm{coa})} & 1 & s_{0,3,4}^{(\mathrm{coa})} & s_{0,3,5}^{(\mathrm{coa})} \\ 0 & s_{0,4,2}^{(\mathrm{coa})} & 0 & 1 & s_{0,4,5}^{(\mathrm{coa})} \\ 0 & s_{0,5,2}^{(\mathrm{coa})} & 0 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & s_{2,1,2}+s_{1,1,4}s_{3,4,2} & s_{4,1,3} & s_{1,1,4} & s_{3,1,5}+s_{1,1,4}s_{4,4,5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{1,3,2}+s_{0,3,4}s_{3,4,2} & 1 & s_{0,3,4} & s_{2,3,5}+s_{0,3,4}s_{4,4,5} \\ 0 & s_{3,4,2} & 0 & 1 & s_{4,4,5} \\ 0 & s_{0,5,2} & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{3.52}$$

As one can see from these special examples, the Stokes multipliers are always related as[14]

$$s_{0,i,j}^{(\mathrm{xxx})} = s_{*,i,j} + \cdots . \tag{3.53}$$

This phenomenon can be generally shown by using the following expression for the fine Stokes matrix S_l:

$$S_l = I_k + \Lambda_l \equiv I_k + \sum_{i,j} s_{l,i,j} E_{i,j}, \tag{3.54}$$

with the matrix unit $E_{i,j}$, then the multiplication rule is given as

$$\begin{aligned} S_l^{(\mathrm{xxx})} &\equiv S_l S_{l+1} \cdots S_{l+n_{\mathrm{xxx}}} = I_k + \sum_{i=0}^{n_{\mathrm{xxx}}} \Lambda_{l+i} + \sum_{0 \le i_1 < i_2 \le n_{\mathrm{xxx}}} \Lambda_{l+i_1}\Lambda_{l+i_2} + \cdots \\ &\equiv I_k + \Lambda_l^{(\mathrm{xxx})} \equiv I_k + \sum_{i,j} s_{l,i,j}^{(\mathrm{xxx})} E_{i,j}. \end{aligned} \tag{3.55}$$

One can then see that the number of independent Stokes multipliers in each Stokes matrix $S_0^{(\mathrm{sym})}$ and $S_0^{(\mathrm{coa})}$ is supplied by the fine Stokes matrices. Consequently, the same statement also holds for these cases:

[13] Note that the orderings of indices $(i|j)$ and $s_{l,j,i}$ are different: $i \leftrightarrow j$.

[14] More generally the cases of $(k,r) = (k,2)$ are shown in Eq. (C.7).

Corollary 2 *The non-zero Stokes multipliers in other kinds of Stokes matrices are also read from the profile $\mathcal{J}$ as follows:*

$$\begin{aligned} s^{(\mathrm{sym})}_{2rn,i,j} = 0 \quad (i \neq j) \qquad & \textit{if} \qquad (i,j) \neq (j_{l,m}, j_{l,m-1}) \\ & \qquad\quad \textit{with} \quad k+l+m \in 2\mathbb{Z}+1, \quad 2rn \leq l < 2r(n+1), \\ s^{(\mathrm{coa})}_{kn,i,j} = 0 \quad (i \neq j) \qquad & \textit{if} \qquad (i,j) \neq (j_{l,m}, j_{l,m-1}) \\ & \qquad\quad \textit{with} \quad k+l+m \in 2\mathbb{Z}+1, \quad kn \leq l < k(n+1). \end{aligned} \tag{3.56}$$

Here $s^{(\mathrm{sym})}_{2rn,i,j}$ is the multiplier of $S^{(\mathrm{sym})}_{2rn}$ and $s^{(\mathrm{coa})}_{kn,i,j}$ is the multiplier of $S^{(\mathrm{coa})}_{kn}$.

3.4 Three basic constraints on the Stokes matrices

Finally we show the three basic constraints as the extension of the two-cut cases (See Section 2.3 for the two-cut cases).

$\mathbb{Z}_k$ **symmetry condition** This condition generally results in

$$S_{n+2r} = \Gamma^{-1} S_n \Gamma, \tag{3.57}$$

for the fine Stokes matrices S_n, $(n = 0, 1, \cdots, 2rk)$, which is obtained from the $\mathbb{Z}_k$ symmetry (see [45]) of the ODE system as follows: First we replace ζ as $\zeta \to \omega^{-1}\zeta$,

$$\begin{aligned} & g\frac{\partial \Psi(t;\zeta)}{\partial \zeta} = \mathcal{Q}(t;\zeta)\, \Psi(t;\zeta) \qquad \text{in} \qquad \zeta \in D \\ & \quad \to \qquad g\frac{\partial \Psi(t;\omega^{-1}\zeta)}{\partial \zeta} = \omega^{-1}\mathcal{Q}(t;\omega^{-1}\zeta)\, \Psi(t;\omega^{-1}\zeta) \qquad \text{in} \qquad \zeta \in \omega D. \end{aligned} \tag{3.58}$$

Since the symmetry of the ODE is expressed as[15]

$$\omega^{-1}\mathcal{Q}(t;\omega^{-1}\zeta) = \Omega^{-1}\, \mathcal{Q}(t;\zeta)\, \Omega, \qquad \text{with} \qquad \Omega^{-1} E_{i,i+1} \Omega = \omega E_{i,i+1}, \tag{3.59}$$

the ODE is expressed as

$$g\frac{\partial \big[\Omega \Psi(t;\omega^{-1}\zeta)\Omega^{-1}\big]}{\partial \zeta} = \mathcal{Q}(t;\zeta)\, \big[\Omega \Psi(t;\omega^{-1}\zeta)\Omega^{-1}\big] \qquad \text{in} \qquad \zeta \in \omega D. \tag{3.60}$$

This defines a map from $\Psi_n(t;\zeta)$ to $\Psi_{n+2r}(t;\zeta)$:

$$\begin{aligned} & \Psi_n(t;\zeta) \qquad \text{in} \qquad \zeta \in D_n \\ & \quad \to \qquad \Psi_{n+2r}(t;\zeta) = \big[\Omega \Psi(t;\omega^{-1}\zeta)\Omega^{-1}\big] \qquad \text{in} \qquad \zeta \in D_{n+2r} = \omega D_n. \end{aligned} \tag{3.61}$$

The above translation for the canonical solutions is given as

$$\Psi_{n+2r}(t;\zeta) = \Omega \Psi_n(t;\omega^{-1}\zeta)\Omega^{-1} \quad \Leftrightarrow \quad \widetilde{\Psi}_{n+2r}(t;\zeta) = \Gamma^{-1}\widetilde{\Psi}_n(t;\omega^{-1}\zeta)\Gamma, \tag{3.62}$$

with $U^{-1}\Omega U = \Gamma^{-1}$. Therefore, the translation of the Stokes matrices is given as

$$\begin{aligned} S_{n+2r} = \widetilde{\Psi}^{-1}_{n+2r}(t;\zeta)\widetilde{\Psi}_{n+2r+1}(t;\zeta) &= \Gamma^{-1}\widetilde{\Psi}^{-1}_n(t;\zeta)\widetilde{\Psi}_{n+1}(t;\zeta)\Gamma \\ &= \Gamma^{-1} S_n \Gamma, \end{aligned} \tag{3.63}$$

[15] Note Eq. (2.6).

and this proves the above relation. This means that only the first $2r$ Stokes matrices $S_n\,(n=0,1,\cdots,2r-1)$ are independent. In this sense, we use the first $2r$ dominance profiles to read the components of these Stokes matrices

$$\mathcal{J}^{(\mathrm{sym})}_{k,r} \equiv \left[\begin{array}{c} J_{2r-1} \\ \hline \vdots \\ \hline J_1 \\ \hline J_0 \end{array}\right] \qquad \Leftrightarrow \qquad S_n \qquad (n=0,1,\cdots,2r-1). \tag{3.64}$$

Here we show several examples of $r=2$:

$$\mathcal{J}^{(\mathrm{sym})}_{5,2} = \left[\begin{array}{c|c|c|c|c} 3 & (5 & 1) & (2 & 4) \\ \hline (5 & 3) & (2 & 1) & 4 \\ \hline 5 & (2 & 3) & (4 & 1) \\ \hline (2 & 5) & (4 & 3) & 1 \end{array}\right]\begin{array}{l} : J_3 \\ : J_2 \\ : J_1 \\ : J_0 \end{array}, \quad \mathcal{J}^{(\mathrm{sym})}_{7,2} = \left[\begin{array}{c|c|c|c|c|c|c} 7 & (3 & 4) & (6 & 1) & (2 & 5) \\ \hline (3 & 7) & (6 & 4) & (2 & 1) & 5 \\ \hline 3 & (6 & 7) & (2 & 4) & (5 & 1) \\ \hline (6 & 3) & (2 & 7) & (5 & 4) & 1 \end{array}\right]\begin{array}{l} : J_3 \\ : J_2 \\ : J_1 \\ : J_0 \end{array},$$

$$\mathcal{J}^{(\mathrm{sym})}_{9,2} = \left[\begin{array}{c|c|c|c|c|c|c|c|c} 4 & (8 & 9) & (3 & 5) & (7 & 1) & (2 & 6) \\ \hline (8 & 4) & (3 & 9) & (7 & 5) & (2 & 1) & 6 \\ \hline 8 & (3 & 4) & (7 & 9) & (2 & 5) & (6 & 1) \\ \hline (3 & 8) & (7 & 4) & (2 & 9) & (6 & 5) & 1 \end{array}\right]\begin{array}{l} : J_3 \\ : J_2 \\ : J_1 \\ : J_0 \end{array},$$

$$\mathcal{J}^{(\mathrm{sym})}_{11,2} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c} 10 & (4 & 5) & (9 & 11) & (3 & 6) & (8 & 1) & (2 & 7) \\ \hline (4 & 10) & (9 & 5) & (3 & 11) & (8 & 6) & (2 & 1) & 7 \\ \hline 4 & (9 & 10) & (3 & 5) & (8 & 11) & (2 & 6) & (7 & 1) \\ \hline (9 & 4) & (3 & 10) & (8 & 5) & (2 & 11) & (7 & 6) & 1 \end{array}\right]\begin{array}{l} : J_3 \\ : J_2 \\ : J_1 \\ : J_0 \end{array},$$

$$\mathcal{J}^{(\mathrm{sym})}_{13,2} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} 5 & (11 & 12) & (4 & 6) & (10 & 13) & (3 & 7) & (9 & 1) & (2 & 8) \\ \hline (11 & 5) & (4 & 12) & (10 & 6) & (3 & 13) & (9 & 7) & (2 & 1) & 8 \\ \hline 11 & (4 & 5) & (10 & 12) & (3 & 6) & (9 & 13) & (2 & 7) & (8 & 1) \\ \hline (4 & 11) & (10 & 5) & (3 & 12) & (9 & 6) & (2 & 13) & (8 & 7) & 1 \end{array}\right]\begin{array}{l} : J_3 \\ : J_2 \\ : J_1 \\ : J_0 \end{array}, \tag{3.65}$$

and the general k-cut cases $\mathcal{J}^{(\mathrm{sym})}_{k,2}$ are shown as Eq. (C.3) and (C.4) in Appendix C.

Hermiticity condition This condition generally results in

$$S_n^* = \Delta\Gamma\, S^{-1}_{(2r-1)k-n}\,\Gamma^{-1}\Delta, \tag{3.66}$$

for the fine Stokes matrices S_n, $(n=0,1,\cdots,2rk)$, which is obtained from the Hermiticity condition of the ODE system as follows. First we consider the complex conjugation, $\zeta\to\zeta^*$:

$$\begin{aligned} & g\frac{\partial\Psi(t;\zeta)}{\partial\zeta} = \mathcal{Q}(t;\zeta)\,\Psi(t;\zeta) \qquad \text{in} \qquad \zeta\in D \\ & \qquad \to \qquad g\frac{\partial\Psi^*(t;\zeta^*)}{\partial\zeta^*} = \mathcal{Q}^*(t;\zeta^*)\,\Psi^*(t;\zeta^*) \qquad \text{in} \qquad \zeta^*\in D^*. \end{aligned} \tag{3.67}$$

Since then the Hermiticity condition [45] is expressed as

$$\mathcal{Q}^*(t;\zeta^*) = \mathcal{Q}(t;\zeta^*), \tag{3.68}$$

the ODE is expressed as

$$g\frac{\partial\Psi^*(t;\zeta)}{\partial\zeta}=\mathcal{Q}(t;\zeta)\,\Psi^*(t;\zeta). \tag{3.69}$$

This defines a map from $\Psi_n(t;\zeta)$ to $\Psi_{(2r-1)k+1-n}(t;\zeta)$:

$$\begin{aligned}&\Psi_n(t;\zeta)\qquad \text{in}\qquad \zeta\in D_n\\ &\qquad\to\qquad \Psi_{(2r-1)k+1-n}(t;\zeta)=\Psi^*_n(t;\zeta)\qquad\text{in}\qquad \zeta\in D_{(2r-1)k+1-n}=D_n^*.\end{aligned} \tag{3.70}$$

The above transformation for the canonical solutions is given as

$$\begin{aligned}\Psi_n^*(t;\zeta)=\Psi_{(2r-1)k+1-n}(t;\zeta)\quad\Leftrightarrow\quad \widetilde{\Psi}^*_n&=U^2\,\widetilde{\Psi}_{(2r-1)k+1-n}(t;\zeta)\,U^{-2}\\ &=\Delta\Gamma\,\widetilde{\Psi}_{(2r-1)k+1-n}(t;\zeta)\,\Gamma^{-1}\Delta.\end{aligned} \tag{3.71}$$

Note that $U^*=U^{-1}$, $U^2=\Delta\Gamma$ and $\Delta_{i,j}=\delta_{i,k-i+1}$. Therefore, the translation of the Stokes matrices is

$$\begin{aligned}S_n^*=\big[\widetilde{\Psi}_n^{-1}(t;\zeta)\widetilde{\Psi}_{n+1}(t;\zeta)\big]^*&=\Delta\Gamma\big[\widetilde{\Psi}^{-1}_{(2r-1)k+1-n}(t;\zeta)\widetilde{\Psi}_{(2r-1)k-n}(t;\zeta)\big]\Gamma^{-1}\Delta\\ &=\Delta\Gamma\big[\widetilde{\Psi}^{-1}_{(2r-1)k-n}(t;\zeta)\widetilde{\Psi}_{(2r-1)k+1-n}(t;\zeta)\big]^{-1}\Gamma^{-1}\Delta\\ &=\Delta\Gamma\,S^{-1}_{(2r-1)k-n}\,\Gamma^{-1}\Delta.\end{aligned} \tag{3.72}$$

This proves the above formula Eq. (3.66).

Monodromy free condition If the formal expansion satisfies $\varphi_0=0$ (discussed in Appendix B), then the canonical solutions are the single valued functions:

$$\widetilde{\Psi}_n(t;\zeta)=\widetilde{\Psi}_n(t;e^{2\pi i}\zeta)=\widetilde{\Psi}_{n+2kr}(t;\zeta), \tag{3.73}$$

therefore the Stokes matrices satisfy

$$\begin{aligned}S_0\cdot S_1\cdots S_{2rk-1}&=S_0^{(\mathrm{coa})}\cdot S_k^{(\mathrm{coa})}\cdots S_{k(2r-1)}^{(\mathrm{coa})}\\ &=S_0^{(\mathrm{sym})}\cdot S_{2r}^{(\mathrm{sym})}\cdots S_{2r(k-1)}^{(\mathrm{sym})}=I_k.\end{aligned} \tag{3.74}$$

Note that, with the $\mathbb{Z}_k$-symmetry constraints, $2rk$ Stokes matrices are reduced to fundamental $2r$ Stokes matrices, $\{S_n\}_{n=0}^{2r-1}$, and also that the monodromy free condition is written as

$$\big(S_0^{(\mathrm{sym})}\,\Gamma^{-1}\big)^k=I_k. \tag{3.75}$$

4 The multi-cut boundary condition and solutions

So far we have considered the Stokes phenomena in the ODE systems which appear in the multi-cut matrix models. In Section 3.4, we discussed three basic constraints required by the symmetries. As is mentioned in Introduction, however, not all the solutions to these constraints can realize the critical points in the multi-cut matrix models. In this section, we propose the first *physical constraints* which we refer to as *multi-cut boundary conditions*. The second physical condition is proposed in Section 5.

4.1 Non-perturbative definition of cuts in spectral curves

In this subsection, we first recall the set up of the multi-cut two-matrix models and the relationship between the Baker-Akhiezer function system (i.e. the ODE system) and cuts in the spectral curves. The definition of the multi-cut two-matrix models is given by the following matrix integral:

$$Z = \int_{\mathcal{C}_N^{(k)}\times\mathcal{C}_N^{(k)}} dX\,dY\, e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}, \tag{4.1}$$

with the matrix contour $\mathcal{C}_N^{(k)}$ of the following $N\times N$ k-cut normal matrix,

$$\mathcal{C}_N^{(k)} \equiv \Big\{X = U\,\mathrm{diag}(x_1,x_2,\cdots,x_N)\,U^\dagger;\ U\in U(N),\quad x_j\in\bigcup_{n=0}^{k-1} e^{2\pi i\frac{n}{k}}\,\mathbb{R}\Big\}. \tag{4.2}$$

The system of two-matrix models has the corresponding orthonormal polynomial system [95]:

$$\alpha_n(x) = \frac{1}{\sqrt{h_n}}\Big(x^n+\cdots\Big),\qquad \beta_n(y) = \frac{1}{\sqrt{h_n}}\Big(y^n+\cdots\Big), \tag{4.3}$$

with

$$\delta_{n,m} = \int_{\mathcal{C}^{(k)}\times\mathcal{C}^{(k)}} dx\,dy\,e^{-N[V_1(x)+V_2(y)-xy]}\,\alpha_n(x)\,\beta_m(y). \tag{4.4}$$

Here the contour $\mathcal{C}^{(k)}$ is given as

$$\mathcal{C}^{(k)} = \Big\{x\in\bigcup_{n=0}^{k-1} e^{2\pi i\frac{n}{k}}\,\mathbb{R}\Big\}, \tag{4.5}$$

an example of which is shown in Fig.6.

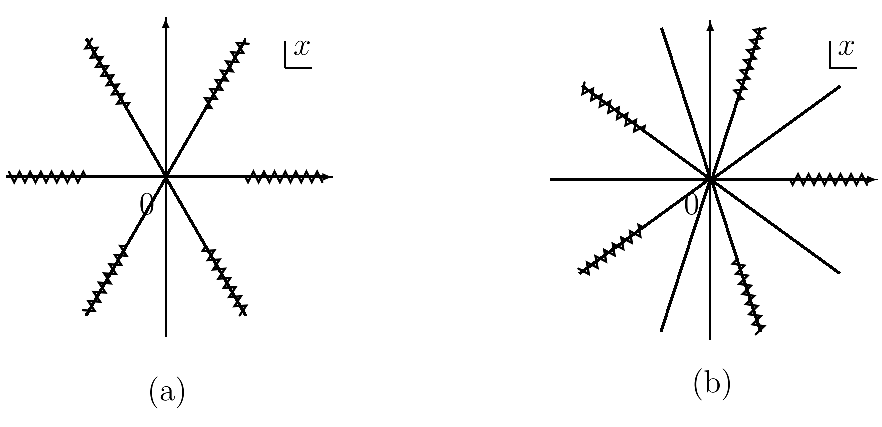


Figure 6: Examples of contours $\mathcal{C}^{(k)}$. (a) is 6-cut contour $\mathcal{C}^{(6)}$ and (b) is the 5-cut contour $\mathcal{C}^{(5)}$ which is equal to the 10-cut contour $\mathcal{C}^{(10)}$. For reference, the position of cuts (zig-zag lines) around $\zeta\to\infty$ is also denoted.

In the double scaling limit, the orthonormal polynomials $\alpha_n(x)$ (or their dual polynomials $\beta_n(y)$) turn out to be a continuum Baker-Akhiezer function, $\Psi_{\rm orth}(t;\zeta)$:

$$\alpha_n(x) = a^{-\hat{p}/2}\,\Psi_{\rm orth}(\zeta;t), \tag{4.6}$$

with the following scaling relations of $a \to 0$:

$$\begin{aligned} x = \omega^{-1/2}a^{\hat{p}/2}\zeta \to 0, \qquad & \frac{n}{N} = \exp\bigl(-ta^{\frac{\hat{p}+\hat{q}-1}{2}}\bigr) \to 1, \\ N^{-1} = g_{\rm str}\,a^{\frac{\hat{p}+\hat{q}}{2}} \to 0, \qquad & \partial_n = -a^{1/2}g_{\rm str}\partial_t \equiv -a^{1/2}\partial \to 0, \end{aligned} \tag{4.7}$$

which satisfies the differential equations (4.8) and (4.9):

$$\zeta\Psi_{\rm orth}(t;\zeta) = \boldsymbol{P}(t;\partial)\,\Psi_{\rm orth}(t;\zeta), \tag{4.8}$$

$$g_{\rm str}\frac{\partial}{\partial\zeta}\Psi_{\rm orth}(t;\zeta) = \boldsymbol{Q}(t;\partial)\,\Psi_{\rm orth}(t;\zeta). \tag{4.9}$$

This means that the orthonormal polynomial system is one of the solutions to the differential equations (4.8) and (4.9), and eventually the ODE systems (2.13) and (2.14). Consequently, its asymptotic expansion in the Stokes sector, $\zeta \in D_n$, is given by the canonical solutions $\Psi_n(t;\zeta)$ with some proper vector $X^{(n)}$ as

$$\Psi_{\rm orth}(t;\zeta) \underset{asym}{\simeq} \widetilde{\Psi}_n(t;\zeta)\,X^{(n)}, \qquad \zeta \to \infty \in D_n. \tag{4.10}$$

By taking into account the Stokes phenomena (3.14), these vectors of various Stokes sectors are related as follows:

$$X^{(n)} = S_n\,X^{(n+1)}, \qquad X^{(n+2rk)} = X^{(n)}. \tag{4.11}$$

Note that the scaled orthonormal polynomials $\Psi_{\rm orth}(t;\zeta)$ are entire functions in $\zeta \in \mathbb{C}$ because the original orthonormal polynomials are also entire functions.

On the other hand, another important approach to solving the multi-cut matrix models is the semi-classical approach with the resolvent operator $\mathcal{R}(x)$ of the matrix models,

$$\mathcal{R}(x) = \Bigl\langle \frac{1}{N}\,{\rm tr}\,\frac{1}{x-X}\Bigr\rangle = \int_{\mathcal{C}^{(k)}} dz\,\frac{\rho(z)}{x-z}, \tag{4.12}$$

where $\rho(z)$ is the density function of eigenvalues of the matrix X and it produces the cuts along the matrix-model contour $\mathcal{C}^{(k)}$ on the ζ space. This resolvent operator is also related to the orthonormal polynomial solution $\Psi_{\rm orth}(t;\zeta)$ in the following way [19]:[16]

$$\Psi_{\rm orth}(t;\zeta) \sim \bigl\langle \det\bigl(x-X\bigr)\bigr\rangle \sim \exp\Bigl[N\int^x dx'\mathcal{R}(x')\Bigr], \tag{4.13}$$

with the scaling relation, $x = \omega^{-1/2}a^{\hat{p}/2}\zeta$, of Eq. (4.7). This relation is quite precise, and therefore one can consider the following equivalent expression:[17]

$$\mathcal{R}(x) \sim \lim_{g_{\rm str}\to 0} g_{\rm str}\frac{\partial}{\partial\zeta}\ln\Psi_{\rm orth}(t;\zeta). \tag{4.14}$$

[16]For the precise relations, see Appendix A in [45], for example.

[17] Although $\Psi_{\rm orth}(t;\zeta)$ is a vector valued function, the behaviors of exponents are the same among the vector components. Therefore, it is understood by taking one particular element of the function $\Psi_{\rm orth}(t;\zeta)$.

4.2 The multi-cut boundary conditions

In the previous subsection, we introduced two types of definitions for the resolvent operator $\mathcal{R}(x)$. One is from the condensation of eigenvalues in the semi-classical limit (4.12) and the other is from the orthonormal polynomials (4.14). The difference between these two actually provides another type of *physical constraints* on the Stokes multipliers, since the position of cuts in the orthonormal polynomials (4.14) is quite non-trivial. The position of cuts in the semi-classical approach (4.12) is clear and given by the original matrix-model contour $\mathcal{C}^{(k)}$ (as shown in Fig. 6). It is because the eigenvalues of the matrix models can have condensation only along the matrix-model contour:[18]

$$\mathrm{Cuts} \equiv \bigcup_{j=1}^{k} \mathrm{Cut}^{(j)} \quad \subset \quad \omega^{1/2} \times \mathcal{C}_x \equiv \bigcup_{n=0}^{k-1} \omega^{n+1/2}\, \mathbb{R}. \tag{4.15}$$

On the other hand, the position of cuts in the orthonormal polynomials (4.14) can be read in the following way:

First of all, the resolvent operators $R(x)$ are related to the exponents $\varphi^{(j)}(t;\zeta)$ in the asymptotic expansions (2.22). In particular, when $\Psi_{\rm orth}(t;\zeta)$ is expressed by a superposition of several exponents $e^{\varphi^{(j_1)}(t;\zeta)}, \cdots, e^{\varphi^{(j_L)}(t;\zeta)}$ in some angular domain D:

$$\Psi_{\rm orth}(t;\zeta) \sim \sum_{i=1}^{L} \theta_i\, e^{\varphi^{(j_i)}(t;\zeta)}, \qquad \zeta \to \infty \in {}^{\exists} D, \tag{4.16}$$

the definition (4.14) extracts only the most dominant exponents, say

$$\mathcal{R}(x) \sim g_{\rm str}\, \partial_\zeta \varphi^{(j_M)}(t;\zeta) + \mathcal{O}\big((g_{\rm str})^0\big), \tag{4.17}$$

with $\mathrm{Re}\big[\varphi^{(j_M)}(t;\zeta) - \varphi^{(j)}(t;\zeta)\big] > 0$ for $j\,(\neq j_M) = j_1, \cdots, j_L$ and $\theta_j \neq 0$. Therefore, when ζ crosses the Stokes lines (2.35), the resolvent operator (4.17) realizes discontinuities in the ζ plane (at least around $\zeta \to \infty$). This is understood as a tail of cuts. However, for instance, if one generally chooses the Stokes multipliers and the vector $X^{(n)}$ in the orthonormal polynomials (4.10) and (4.11), the cuts (4.17) appear almost along all the Stokes lines SL in (2.35) and do not generally realize the cuts in the original definition of the resolvent operator (4.12). Therefore, we impose some conditions on the multipliers so that the resolvent (4.17) realizes the multi-cut geometry (4.15) expected from the definition of the multi-cut matrix models (4.1). This constraint on the Stokes multipliers is referred to as *multi-cut boundary condition*.

Quantitatively, the multi-cut constraints are imposed as follows: The total number of cuts around $\zeta \to \infty$ is k and the angles of the k cuts are given as

$$\mathrm{Cut}^{(j)} \quad \subset \quad \omega^{j-1/2}\, \mathbb{R}_+, \qquad (j = 1, 2, \cdots, k). \tag{4.18}$$

Therefore, we obtain:

Definition 10 [Multi-cut boundary condition] *The following requirement is called the multi-cut boundary condition:*

1. *There is an ordered set of k distinct indices, $(a_1, a_2, \cdots, a_k)$.*

[18] See also [45] for the reason why we have the $\omega^{1/2}$ rotation.

2. *There are the discontinuities along* $\mathrm{Cut}^{(j)}$ $(j=1,2,\cdots,k)$*:*

$$\Psi_{\rm orth}(t;e^{i\epsilon}\zeta)\underset{asym}{\simeq} c_{j+1}\times\widetilde{\Psi}^{(a_{j+1})}_{\rm asym}(t;\zeta),\qquad \Psi_{\rm orth}(t;e^{-i\epsilon}\zeta)\underset{asym}{\simeq} c_{j}\times\widetilde{\Psi}^{(a_{j})}_{\rm asym}(t;\zeta),$$
$$\textit{along}\quad \zeta\to\infty\in \mathit{Cut}^{(j)}\subset\omega^{j/2}\,\mathbb{R}_+;\ \epsilon\to 0_+,\tag{4.19}$$

with some constant c_j $(j=1,2,\cdots,k)$.

3. *The orthonormal polynomial* $\Psi_{\rm orth}(t;\zeta)$ *in the angular domain between* $\mathit{Cut}^{(j-1)}$ *and* $\mathit{Cut}^{(j)}$ *has the following leading behavior:*

$$\Psi_{\rm orth}(t;\zeta)\underset{asym}{\simeq} c_j\times\widetilde{\Psi}^{(a_j)}_{\rm asym}(t;\zeta),\qquad \zeta\to\infty\in D\Big(\frac{2\pi(j-3/2)}{k};\frac{2\pi(j-1/2)}{k}\Big).\tag{4.20}$$

In the case of the real-potential critical points (2.10), the position of cuts is given by $\mathcal{C}^{(k)}$.

An example of the boundary condition in the ζ plane is shown in Fig. 7.

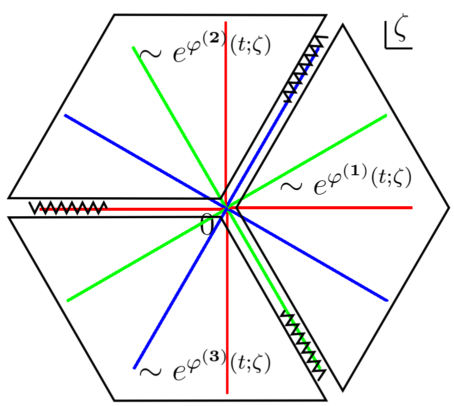


Figure 7: The multi-cut boundary condition in the 3-cut $(1,1)$ critical point. Although the general solutions to the Baker-Akhiezer function system generally have 12 cuts, there are three cuts in the orthonormal polynomial.

Below we make several comments:

- From the orthonormal polynomial approach (4.17), the position of cuts has the mathematical meaning as Stokes lines. Since our definition of Stokes line (2.35) only cares the position at $\zeta\to\infty$, in order to extract the information of the position of cuts in the ζ plane (especially in the weak coupling semi-classical limit), we have to use the following definition of Stokes lines:

$$\mathrm{Re}\big[\varphi^{(j)}(t;\zeta)-\varphi^{(l)}(t;\zeta)\big]=0,\tag{4.21}$$

  instead of (2.35). Although these lines are generally curved, we understand it as an (analytic) deformation of matrix contour. This consideration is important if one considers some matrix models with complex potentials.[19] See for example the semi-classical solution in fractional-superstring critical points [43]. We will come back to this consideration in Section 5.

[19]In this sense, perturbations of potentials with complex coefficients in matrix models are interpreted as deformations of the matrix-model contour.

- In the $\hat{p} > 1$ cases, the exponents $\varphi^{(j)}(t;\zeta)$ have non-trivial cuts in the ζ plane, say $\varphi^{(j)}(t;\zeta) \sim \zeta^{(\hat{q}+1)/\hat{p}}$. This $\hat{p}$-th root cut should be smeared by a proper supplement of exponents [51]. This is also reviewed in Appendix A. Since we concentrate on the $\hat{p} = 1$ cases in this paper, we do not encounter this phenomenon and this point in the general k-cut cases remains to be studied for future investigations.

Before devoting into general cases, we first consider the multi-cut boundary condition in the two-cut case, as a warming up for the general systems.

4.2.1 The multi-cut boundary condition in the two-cut case

Here we show how to solve the multi-cut boundary conditions in the two-cut $(1,2)$ case. The orthonormal polynomial $\Psi_{\rm orth}(t;\zeta)$ in a Stokes sector D_n is generally given as a superposition of independent solutions, $\widetilde{\Psi}_n^{(j)}(t;\zeta)$:

$$\Psi_{\rm orth}(t;\zeta) = \widetilde{\Psi}_n(t;\zeta) X^{(n)} = x_1^{(n)} \widetilde{\Psi}_n^{(1)}(t;\zeta) + x_2^{(n)} \widetilde{\Psi}_n^{(2)}(t;\zeta), \qquad \zeta \to \infty \in D_n. \tag{4.22}$$

However this assumption results in the 6-cut geometry of resolvent as shown in Fig. 8-a, even though this system is called "two-cut". Therefore, one has to choose proper Stokes multipliers in order to satisfy the multi-cut boundary condition and therefore to obtain the geometry which only includes two cuts as shown in Fig. 8-b.

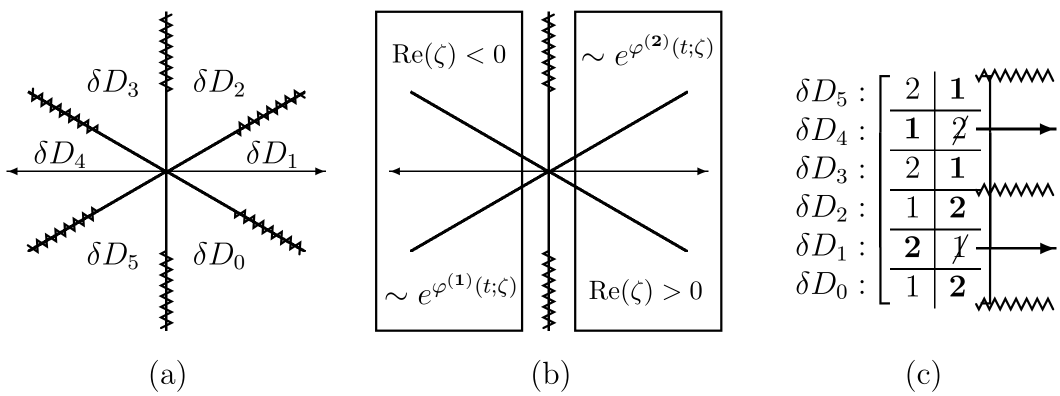


Figure 8: The positions of cuts in the two-cut $(1,2)$ ODE system. a) A general configuration of cuts for the general Stokes multipliers. There are 6 cuts. b) A configuration of cuts for the $(1,2)$ critical point in the two-cut matrix models. The boxes indicate the regions $\mathrm{Re}(\zeta) > 0$ and $\mathrm{Re}(\zeta) < 0$, in which the asymptotic expansion is given by $\sim e^{\varphi^{(i)}(\zeta)}$ $(i = 1,2)$. c) The profile of dominance depicted with the position of cuts and the weak coupling infinity $\zeta \to \pm\infty \in \mathbb{R}$.

The multi-cut boundary condition is then given as follows: Since we wish to erase the cuts of orthonormal polynomial (4.10) along the Stokes lines of

$$\theta = \pm\frac{\pi}{3}, \quad \pm\frac{5\pi}{3}, \tag{4.23}$$

we impose the following boundary condition:

$$X^{(0)} = \begin{pmatrix} 0 \\ x_2^{(0)} \end{pmatrix}, \qquad X^{(1)} = \begin{pmatrix} 0 \\ x_2^{(1)} \end{pmatrix}, \qquad X^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix},$$
$$X^{(3)} = \begin{pmatrix} x_1^{(3)} \\ 0 \end{pmatrix}, \qquad X^{(4)} = \begin{pmatrix} x_1^{(4)} \\ 0 \end{pmatrix}, \qquad X^{(5)} = \begin{pmatrix} x_1^{(5)} \\ x_2^{(5)} \end{pmatrix}, \tag{4.24}$$

where all the $x_i^{(n)}$ appearing here are non-zero. This can be also expressed in the dominance profile as in Fig. 8-c. That is, if the Stokes sector D_n includes the following profile,

$$\left[\ m_1 \,\middle|\, \cdots \,\middle|\, m_{I-1} \,\middle|\, \mathbf{m_I} \,\middle|\, \not{m}_{I+1} \,\middle|\, \cdots \,\middle|\, \not{m}_{k-1} \,\middle|\, \not{m}_k\ \right] \in D_n, \tag{4.25}$$

then the boundary condition can be read as

$$\Psi_{\mathrm{orth}}(t;\zeta) = \sum_{j=1}^{I} x_{m_j}^{(n)}\, \widetilde{\Psi}_n^{(m_j)}(t;\zeta), \qquad x_{m_I}^{(n)} \neq 0. \tag{4.26}$$

By imposing this boundary condition

$$X^{(n)} = S_n\, X^{(n+1)}, \qquad X^{(n+6)} = X^{(n)}, \tag{4.27}$$

with the Stokes matrix (2.54) of $\mathbb{Z}_2$ symmetry condition (2.59), one obtains

$$\begin{aligned} &\begin{pmatrix} 0 \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ x_2^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ x_2^{(1)} \end{pmatrix} = \begin{pmatrix} x_1^{(2)} + \beta x_2^{(2)} \\ x_2^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = \begin{pmatrix} x_1^{(3)} \\ \gamma x_1^{(3)} \end{pmatrix}, \\ &\begin{pmatrix} x_1^{(3)} \\ 0 \end{pmatrix} = \begin{pmatrix} x_1^{(4)} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x_1^{(4)} \\ 0 \end{pmatrix} = \begin{pmatrix} x_1^{(5)} \\ \beta x_1^{(5)} + x_2^{(5)} \end{pmatrix}, \qquad \begin{pmatrix} x_1^{(5)} \\ x_2^{(5)} \end{pmatrix} = \begin{pmatrix} \gamma x_2^{(0)} \\ x_2^{(0)} \end{pmatrix}, \end{aligned} \tag{4.28}$$

which results in

$$\begin{aligned} &\beta^2 = 1, \qquad \gamma^2 = 1, \qquad 1 + \beta\gamma = 0, \\ &x_1^{(2)} = x_1^{(3)} = x_1^{(4)} = x_1^{(5)} = \gamma x_2^{(0)} \neq 0, \qquad x_2^{(5)} = x_2^{(0)} = x_2^{(1)} = x_2^{(2)} = \gamma x_1^{(3)} \neq 0. \end{aligned} \tag{4.29}$$

Therefore, the solutions consistent with the Hermiticity condition (2.63) and monodromy free condition (2.66) are given as

$$\alpha \in i\mathbb{R}, \qquad \beta = -\gamma = \pm 1. \tag{4.30}$$

Consequently, the solution to the multi-cut boundary condition in the two-cut case has a real continuum parameter. However, another additional constraint for non-perturbative stability of semi-classical background completely fixes the Stokes multipliers. This is based on the Riemann-Hilbert approach and is discussed in Section 5.

4.2.2 The multi-cut boundary-condition recursions ($r = 2$)

From here, we solve the multi-cut boundary condition for an arbitrary number of cuts, k. In order to solve the constraints, we use the symmetric Stokes sectors (See Def. 8),

$$\Psi_{\mathrm{orth}}(t;\zeta) \underset{asym}{\simeq} \widetilde{\Psi}_{2rl}(t;\zeta)\, X^{(2rl)}, \qquad \zeta \to \infty \in D_{2rl}, \quad (l = 0,1,2,\cdots,k-1), \tag{4.31}$$

and its Stokes matrices, $S_{2rl}^{(\mathrm{sym})} = \Gamma^{-l}\, S_0^{(\mathrm{sym})}\, \Gamma^l$. For sake of simplicity, however, we here focus on the $r = 2$ cases, and therefore $k = 5, 7, 9, \cdots$.[20] Some of the results can be generalized to the general r cases.

[20] Here $k = 3$ is special because $k < 2r = 4$. This case is also separately calculated in Appendix D.

We first read the boundary condition in terms of the dominance profile. We first see the 5-cut cases:

<u>*The 5-cut case*</u>

$$D_0:\left[\begin{array}{c|c|c}\cdots & 5 & \mathbf{2}\\ \hline \cdots & 4 & \mathbf{2}\\ \hline \cdots & \mathbf{2} & \cancel{4}\\ \hline \cdots & \mathbf{1} & \cancel{4}\\ \hline \cdots & 4 & \mathbf{1}\\ \hline \cdots & 3 & \mathbf{1}\end{array}\right],\qquad D_4:\left[\begin{array}{c|c|c}\cdots & 1 & \mathbf{3}\\ \hline \cdots & 5 & \mathbf{3}\\ \hline \cdots & \mathbf{3} & \cancel{5}\\ \hline \cdots & \mathbf{2} & \cancel{5}\\ \hline \cdots & 5 & \mathbf{2}\\ \hline \cdots & 4 & \mathbf{2}\end{array}\right],\qquad D_8:\left[\begin{array}{c|c|c}\cdots & 2 & \mathbf{4}\\ \hline \cdots & 1 & \mathbf{4}\\ \hline \cdots & \mathbf{4} & \cancel{1}\\ \hline \cdots & \mathbf{3} & \cancel{1}\\ \hline \cdots & 1 & \mathbf{3}\\ \hline \cdots & 5 & \mathbf{3}\end{array}\right],$$

$$D_{12}:\left[\begin{array}{c|c|c}\cdots & 3 & \mathbf{5}\\ \hline \cdots & 2 & \mathbf{5}\\ \hline \cdots & \mathbf{5} & \cancel{2}\\ \hline \cdots & \mathbf{4} & \cancel{2}\\ \hline \cdots & 2 & \mathbf{4}\\ \hline \cdots & 1 & \mathbf{4}\end{array}\right],\qquad D_{16}:\left[\begin{array}{c|c|c}\cdots & 4 & \mathbf{1}\\ \hline \cdots & 3 & \mathbf{1}\\ \hline \cdots & \mathbf{1} & \cancel{3}\\ \hline \cdots & \mathbf{5} & \cancel{3}\\ \hline \cdots & 3 & \mathbf{5}\\ \hline \cdots & 2 & \mathbf{5}\end{array}\right]. \tag{4.32}$$

Therefore, the general k-cut cases are given as

<u>*The general k-cut cases*</u>

<u>$k = 4k_0 + 1$</u> $\quad D_{2rn}$: <u>$k = 4k_0 + 3$</u> $\quad D_{2rn}$:

$$\left[\begin{array}{c|c|c}
\cdots & n+\frac{k+5}{2}+\lfloor\frac{k-3}{4}\rfloor & \mathbf{n}+\lfloor\frac{\mathbf{k+3}}{\mathbf{4}}\rfloor\\ \hline
\cdots & n+\frac{k+3}{2}+\lfloor\frac{k-3}{4}\rfloor & \mathbf{n}+\lfloor\frac{\mathbf{k+3}}{\mathbf{4}}\rfloor\\ \hline
\cdots & \mathbf{n}+\lfloor\frac{\mathbf{k+3}}{\mathbf{4}}\rfloor & \cancel{n}+\frac{k+3}{2}+\lfloor\frac{k-3}{4}\rfloor\\ \hline
\cdots & \mathbf{n}+\lfloor\frac{\mathbf{k-1}}{\mathbf{4}}\rfloor & \cancel{n}+\frac{k+3}{2}+\lfloor\frac{k-3}{4}\rfloor\\ \hline
 & \vdots & \vdots\\ \hline
\cdots & n+\frac{k+5}{2} & \mathbf{n+2}\\ \hline
\cdots & n+\frac{k+3}{2} & \mathbf{n+2}\\ \hline
\cdots & \mathbf{n+2} & \cancel{n}+\frac{k+3}{2}\\ \hline
\cdots & \mathbf{n+1} & \cancel{n}+\frac{k+3}{2}\\ \hline
\cdots & n+\frac{k+3}{2} & \mathbf{n+1}\\ \hline
\cdots & n+\frac{k+1}{2} & \mathbf{n+1}
\end{array}\right],\quad
\left[\begin{array}{c|c|c}
\cdots & \mathbf{n}+\lfloor\frac{\mathbf{k+7}}{\mathbf{4}}\rfloor & \cancel{n}+\frac{k+3}{2}+\lfloor\frac{k-3}{4}\rfloor\\ \hline
\cdots & \mathbf{n}+\lfloor\frac{\mathbf{k+3}}{\mathbf{4}}\rfloor & \cancel{n}+\frac{k+3}{2}+\lfloor\frac{k-3}{4}\rfloor\\ \hline
\cdots & n+\frac{k+3}{2}+\lfloor\frac{k-3}{4}\rfloor & \mathbf{n}+\lfloor\frac{\mathbf{k+3}}{\mathbf{4}}\rfloor\\ \hline
\cdots & n+\frac{k+1}{2}+\lfloor\frac{k-3}{4}\rfloor & \mathbf{n}+\lfloor\frac{\mathbf{k+3}}{\mathbf{4}}\rfloor\\ \hline
 & \vdots & \vdots\\ \hline
\cdots & n+\frac{k+5}{2} & \mathbf{n+2}\\ \hline
\cdots & n+\frac{k+3}{2} & \mathbf{n+2}\\ \hline
\cdots & \mathbf{n+2} & \cancel{n}+\frac{k+3}{2}\\ \hline
\cdots & \mathbf{n+1} & \cancel{n}+\frac{k+3}{2}\\ \hline
\cdots & n+\frac{k+3}{2} & \mathbf{n+1}\\ \hline
\cdots & n+\frac{k+1}{2} & \mathbf{n+1}
\end{array}\right]. \tag{4.33}$$

Equivalently, the components of $X^{(4n)}$ ($r = 2$ and $k \geq 5$) is given as

$$x^{(4n)}_{n+i} \neq 0 \qquad (i = 1, 2, \cdots, \Big\lfloor\frac{k+3}{4}\Big\rfloor),$$
$$x^{(4n)}_{n+\frac{k+1}{2}+i} = 0 \qquad (i = 1, 2, \cdots, \Big\lfloor\frac{k+1}{4}\Big\rfloor). \tag{4.34}$$

It is convenient to introduce another vector $Y^{(4n)} = \left(y_{n,j}\right)_{j=1}^{k} \equiv \Gamma^n X^{(4n)}$, and in the vector form, they are given as

$$X^{(4n)} = \begin{pmatrix} \vdots \\ x^{(4n)}_{n+1} \neq 0 \\ \vdots \\ x^{(4n)}_{n+\lfloor \frac{k+3}{4} \rfloor} \neq 0 \\ \vdots \\ x^{(4n)}_{n+\frac{k+3}{2}} = 0 \\ \vdots \\ x^{(4n)}_{n+\frac{k+3}{2}+\lfloor \frac{k-3}{4} \rfloor} = 0 \\ \vdots \end{pmatrix}, \qquad Y^{(4n)} = \begin{pmatrix} y_{n,1} \neq 0 \\ \vdots \\ y_{n,\lfloor \frac{k+3}{4} \rfloor} \neq 0 \\ \vdots \\ y_{n,\frac{k+3}{2}} = 0 \\ \vdots \\ y_{n,\frac{k+3}{2}+\lfloor \frac{k-3}{4} \rfloor} = 0 \\ \vdots \end{pmatrix}. \tag{4.35}$$

Therefore, in terms of the vector $Y^{(4n)}$, the boundary condition can be expressed in the uniform way. Also note that

$$X^{(4n)} = X^{(4(n+k))}, \qquad Y^{(4n)} = Y^{(4(n+k))}, \tag{4.36}$$

and therefore the periodicity of index n follows: $y_{n+k,j} = y_{n,j}$. Here we also mention the boundary condition for general $r\,(=2,3,\cdots)$ in terms of the dominance profile:

$$D_{2r(n-1)} \supset \left[\begin{array}{c|c|c|c|c|c|c}
 & & & & & \vdots & \vdots \\ \hline
 & & & & & A_2 & \mathbf{n+1} \\ \hline
 & & & & & A_3 & \mathbf{n+1} \\ \hline
 & & & & & \mathbf{n+1} & \cancel{A}^{(n)}_{2r-1} \\ \hline
 & & & & \mathbf{n+1} & \cancel{A}^{(n)}_{2r-3} & \cancel{A}^{(n)}_{2r-1} \\ \hline
 & & & \mathbf{n+1} & \cancel{A}^{(n)}_{2r-5} & \cancel{A}^{(n)}_{2r-1} & \cancel{A}^{(n)}_{2r-3} \\ \hline
 & & \iddots & \iddots & \iddots & & \vdots \\ \hline
 & \mathbf{n+1} & \cancel{A}^{(n)}_{5} & \cancel{A}^{(n)}_{2r-1} & & & \vdots \\ \hline
\mathbf{n+1} & \cancel{A}^{(n)}_{3} & \cancel{A}^{(n)}_{2r-1} & & & & \vdots \\ \hline
\mathbf{n} & \cancel{A}^{(n)}_{2r-1} & \cancel{A}^{(n)}_{3} & & & & \vdots \\ \hline
 & \mathbf{n} & \cancel{A}^{(n)}_{2r-3} & \cancel{A}^{(n)}_{3} & & & \vdots \\ \hline
 & & \ddots & \ddots & \ddots & & \vdots \\ \hline
 & & & \mathbf{n} & \cancel{A}^{(n)}_{7} & \cancel{A}^{(n)}_{3} & \cancel{A}^{(n)}_{5} \\ \hline
 & & & & \mathbf{n} & \cancel{A}^{(n)}_{5} & \cancel{A}^{(n)}_{3} \\ \hline
 & & & & & \mathbf{n} & \cancel{A}^{(n)}_{3} \\ \hline
 & & & & & A^{(n)}_{3} & \mathbf{n} \\ \hline
 & & & & & A_2 & \mathbf{n}
\end{array}\right], \tag{4.37}$$

where $A_i^{(n)} \equiv A_{2r(n-1)+i}$ of Eq. (3.39), and $A_1^{(n)} = n$ and $A_{2r+1}^{(n)} = n+1$ (See Cor. 1). Basically if the Stokes sector D_n includes the following indices

$$A_i^{(n)} \qquad (i = 3, 5, \cdots, 2r-1), \tag{4.38}$$

in the profiles, then we impose

$$x^{(n)}_{A^{(n)}_i} = 0 \quad (i = 3, 5, \cdots) \qquad \text{and} \qquad x_{n+i} \neq 0 \quad (i = 0, 1, \cdots). \tag{4.39}$$

The ending points of these series depend on how D_n includes these profiles. This general classification is a little tedious, and therefore kept remained for the future study. In addition to this, the set of indices $(a_1, a_2, \cdots, a_k)$ in Def. 10 for the $\mathbb{Z}_k$ symmetric $(\hat{p}, \hat{q}) = (1, 1)$ critical points is given as

$$(a_1, a_2, \cdots, a_k) = (1, 2, \cdots, k). \tag{4.40}$$

Note that this ordering is generally different in other critical points.

Let us come back to the $r = 2$ cases. With the multi-cut boundary condition (4.35), the constraints on the Stokes multipliers are expressed with Eq. (4.11) as

$$X^{(4n)} = S^{(\mathrm{sym})}_{4n}\, X^{(4(n+1))} \qquad \Leftrightarrow \qquad Y^{(4n)} = \big(S^{(\mathrm{sym})}_0 \Gamma^{-1}\big)\, Y^{(4(n+1))}, \tag{4.41}$$

and, in terms of components, we obtain the following recursive relations for $y_{n,i}$:

$$y_{n,i} = y_{n+1,i-1} + \sum_{j=1}^{k} s^{(\mathrm{sym})}_{0,i,j}\, y_{n+1,j-1}, \qquad y_{n+k,j} = y_{n,j}. \tag{4.42}$$

This is the central equations for the multi-cut boundary condition. After imposing the boundary condition, all the components $\{y_{n,i}\}$ and then the vectors $Y^{(n)}$ are expressed only by $\{y_{n,1}\}$. For instance, the $k = 5, 7, 9$ and 11 cases are expressed as

$$\underline{k=5}: \quad Y^{(n)} = \left(\begin{array}{c} y_{n,1} \\ y_{n+1,1} \\ \hline -s^{(\mathrm{sym})}_{0,4,2}\, y_{n,1} \\ \hline 0 \\ \hline s^{(\mathrm{sym})}_{0,5,2}\, y_{n+1,1} \end{array}\right), \qquad \underline{k=7}: \quad Y^{(n)} = \left(\begin{array}{c} y_{n,1} \\ y_{n+1,1} \\ \hline y_{n+2,1} \\ -s^{(\mathrm{sym})}_{0,5,2}\, y_{n,1} \\ \hline 0 \\ 0 \\ \hline s^{(\mathrm{sym})}_{0,7,2}\, y_{n+1,1} + s^{(\mathrm{sym})}_{0,7,3}\, y_{n+2,1} \end{array}\right),$$

$$\underline{k=9}: \qquad Y^{(n)} = \left(\begin{array}{c} y_{n,1} \\ y_{n+1,1} \\ y_{n+2,1} \\ \hline y_{n+3,1} + s^{(\mathrm{sym})}_{0,4,3}\, y_{n+2,1} \\ -s^{(\mathrm{sym})}_{0,6,2}\, y_{n,1} \\ \hline 0 \\ 0 \\ \hline s^{(\mathrm{sym})}_{0,8,3}\, y_{n+2,1} \\ s^{(\mathrm{sym})}_{0,8,3}\, y_{n+3,1} + s^{(\mathrm{sym})}_{0,9,3}\, y_{n+2,1} + s^{(\mathrm{sym})}_{0,9,2}\, y_{n+1,1} \end{array}\right),$$

$$\underline{k=11}: \qquad Y^{(n)} = \left(\begin{array}{c} y_{n,1} \\ y_{n+1,1} \\ y_{n+2,1} \\ \hline y_{n+3,1} \\ y_{n+4,1} + s^{(\mathrm{sym})}_{0,5,3}\, y_{n+2,1} + s^{(\mathrm{sym})}_{0,5,4}\, y_{n+3,1} \\ -s^{(\mathrm{sym})}_{0,7,2}\, y_{n,1} \\ \hline 0 \\ 0 \\ 0 \\ \hline s^{(\mathrm{sym})}_{0,10,3}\, y_{n+2,1} + s^{(\mathrm{sym})}_{0,10,4}\, y_{n+3,1} \\ s^{(\mathrm{sym})}_{0,10,3}\, y_{n+3,1} + s^{(\mathrm{sym})}_{0,10,4}\, y_{n+4,1} + s^{(\mathrm{sym})}_{0,11,2}\, y_{n+1,1} + s^{(\mathrm{sym})}_{0,11,3}\, y_{n+2,1} \end{array}\right), \tag{4.43}$$

where we divided the components of the vectors into four categories:

$$\begin{aligned} &(\mathrm{I}) \quad 1 \le i \le \left\lfloor \frac{k+3}{4} \right\rfloor, && (\mathrm{II}) \quad \left\lfloor \frac{k+3}{4} \right\rfloor + 1 \le i \le \frac{k+1}{2}, \\ &(\mathrm{III}) \quad \frac{k+1}{2} + 1 \le i \le \left\lfloor \frac{3k+3}{4} \right\rfloor, && (\mathrm{IV}) \quad \left\lfloor \frac{3k+3}{4} \right\rfloor + 1 \le i \le k. \end{aligned} \tag{4.44}$$

The general cases are also similarly expressed and denoted as

$$y_{n,i} = y_{n,i}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big), \qquad Y^{(n)}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) \equiv \Big(y_{n,i}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big)\Big)_{i=1}^{k}, \tag{4.45}$$

which are shown in Appendix C. As is shown in Appendix C, however, the following components,

$$y_{n,1}, \qquad y_{n,\frac{k+1}{2}} \tag{4.46}$$

have two different expressions with $\{y_{n,1}\}$. This fact results in two recursive equations for $\{y_{n,1}\}$. Several examples are shown below:

$$\begin{aligned} &\underline{k=5} \\ &\quad \mathcal{F}_5[y_{n,1}] \equiv y_{n+2,1} + s_{1,3,2}\, y_{n+1,1} + s_{3,4,2}\, y_{n,1} = 0, \\ &\quad \mathcal{G}_5[y_{n,1}] \equiv s_{0,5,2}\, y_{n+2,1} + s_{2,1,2}\, y_{n+1,1} - y_{n,1} = 0, \\ &\underline{k=7} \\ &\quad \mathcal{F}_7[y_{n,1}] \equiv y_{n+3,1} + s_{3,4,3}\, y_{n+2,1} + s_{1,4,2}\, y_{n+1,1} + s_{3,5,2}\, y_{n,1} = 0, \\ &\quad \mathcal{G}_7[y_{n,1}] \equiv s_{2,7,3}\, y_{n+3,1} + s_{0,7,2}\, y_{n+2,1} + s_{2,1,2}\, y_{n+1,1} - y_{n,1} = 0, \\ &\underline{k=9} \\ &\quad \mathcal{F}_9[y_{n,1}] \equiv y_{n+4,1} + s_{1,4,3}\, y_{n+3,1} + s_{3,5,3}\, y_{n+2,1} + s_{1,5,2}\, y_{n+1,1} + s_{3,6,2}\, y_{n,1} = 0, \\ &\quad \mathcal{G}_9[y_{n,1}] \equiv s_{0,8,3}\, y_{n+4,1} + s_{2,9,3}\, y_{n+3,1} + s_{0,9,2}\, y_{n+2,1} + s_{2,1,2}\, y_{n+1,1} - y_{n,1} = 0, \\ &\underline{k=11} \\ &\quad \mathcal{F}_{11}[y_{n,1}] \equiv y_{n+5,1} + s_{3,5,4}\, y_{n+4,1} + s_{1,5,3}\, y_{n+3,1} + s_{3,6,3}\, y_{n+2,1} + s_{1,6,2}\, y_{n+1,1} + s_{3,7,2}\, y_{n,1} = 0, \\ &\quad \mathcal{G}_{11}[y_{n,1}] \equiv s_{2,10,4}\, y_{n+5,1} + s_{0,10,3}\, y_{n+4,1} + s_{2,11,3}\, y_{n+3,1} + s_{0,11,2}\, y_{n+2,1} + s_{2,1,2}\, y_{n+1,1} - y_{n,1} = 0. \end{aligned} \tag{4.47}$$

Generally one can show the following recursion equations for $y_{n,1}$:

$$\mathcal{F}_k[y_{n,1}] = y_{n+\frac{k-1}{2},1} + \sum_{j=1}^{\lfloor\frac{k-1}{4}\rfloor} s_{1,\frac{k-1}{2}+2-j,1+j} \times y_{n+2j-1,1} + \sum_{j=1}^{\lfloor\frac{k+1}{4}\rfloor} s_{3,\frac{k-1}{2}+3-j,1+j} \times y_{n+2j-2,1} = 0,$$

$$\mathcal{G}_k[y_{n,1}] = -y_{n,1} + \sum_{j=1}^{\lfloor\frac{k-1}{4}\rfloor} s_{0,k+1-j,1+j} \times y_{n+2j,1} + \sum_{j=1}^{\lfloor\frac{k+1}{4}\rfloor} s_{2,k+2-j,1+j} \times y_{n+2j-1,1} = 0. \qquad (4.48)$$

Note that the indices are understood as modulo k, say $s_{2,i,j} = s_{2,i+k,j}$. These two equations provide a non-trivial constraints on $\{y_{n,1}\}$. A brief proof for these equations is also shown in Appendix C. Note that all the Stokes multipliers in this expression are *fine Stokes multipliers*.

Therefore, solving the multi-cut boundary condition means *finding out the solutions $\{y_{n,1}\}$ to the recursive relations (4.48) with properly choosing the Stokes multipliers which satisfy the basic three constraints discussed in Section 3.4.* This is the non-perturbative completion problem in this non-critical string theory.

4.3 Ansatz and solutions in the general k-cut cases

Before solving the boundary conditions, we here summarize the equations we solve: After imposing the $\mathbb{Z}_k$ symmetry condition (3.57),

$$\underline{\mathbb{Z}_k \text{ symmetry:}} \qquad S_{2rl}^{(\mathrm{sym})} = \Gamma^{-l}\, S_0^{(\mathrm{sym})}\, \Gamma^{l} \qquad (4.49)$$

the system becomes

$$\underline{\text{Multi-cut BC recursion:}} \qquad Y^{(4n)} = \big(S_0^{(\mathrm{sym})}\Gamma^{-1}\big)\, Y^{(4(n+1))} \qquad (4.50)$$

$$\underline{\text{Monodromy free condition:}} \qquad \big(S_0^{(\mathrm{sym})}\,\Gamma^{-1}\big)^k = I_k, \qquad (4.51)$$

$$\underline{\text{Hermiticity condition:}} \qquad S_n^* = \Delta\Gamma\, S_{(2r-1)k-n}^{-1}\,\Gamma^{-1}\Delta. \qquad (4.52)$$

First of all, we focus on the monodromy free condition (4.51). Let us assume that one can diagonalize the matrix $S_0^{(\mathrm{sym})}\,\Gamma^{-1}$, then the eigenvalues of the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$, $\lambda_j\,(j = 1, 2, \cdots, k)$, satisfy

$$(\lambda_j)^k = 1, \qquad S_0^{(\mathrm{sym})}\,\Gamma^{-1} \cong \mathrm{diag}_j(\lambda_j). \qquad (4.53)$$

Here $\cong$ indicates equality up to a similarity transformation. This means that the eigenvalues of this matrix are the k-th roots of unity. However, conversely if one cannot diagonalize the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$, then the matrix is generally given as the Jordan normal form,

$$S_0^{(\mathrm{sym})}\Gamma^{-1} \cong \Lambda + T, \qquad \Lambda = \mathrm{diag}_j(\lambda_j), \qquad (4.54)$$

with a proper triangular matrix T. Therefore, the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$ cannot satisfy the monodromy free condition (4.51). Hence, the monodromy free condition is rephrased as

$$\underline{\text{Monodromy free condition (4.51)}}$$

$$\Leftrightarrow \quad S_0^{(\mathrm{sym})}\Gamma^{-1} \textit{ is diagonalizable with eigenvalues } \lambda_j \textit{ of } \lambda_j^k = 1. \qquad (4.55)$$

For later convenience, we introduce the characteristic equation of the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$:

$$H(x) \equiv \det\Big(xI_k - S_0^{(\mathrm{sym})}\,\Gamma^{-1}\Big) = 0. \tag{4.56}$$

According to the basic theorem of linear algebra, if the minimal polynomial of a matrix A,

$$H_{\mathrm{minimal}}(A) = 0, \tag{4.57}$$

has no degenerate root, then the matrix A is diagonalizable. In particular, if all the eigenvalues of A are different, then the matrix A is diagonalizable and the minimal polynomial is the characteristic equation.

If one tries to solve the multi-cut boundary conditions with some direct analysis in several cases, one observes that this system has several solutions. Here we derive special solutions which can be generalized to the cases with an arbitrary number of cuts. In order to simplify the system, we impose a few ansatz for this system.

First we assume the following ansatz for the vector $Y^{(4n)}$:

$$\textbf{Ansatz 1:} \qquad Y^{(4(n+1))} = y \times Y^{(4n)} \qquad \Leftrightarrow \qquad y_{n,i} = y^n \times y_{0,i}. \tag{4.58}$$

This immediately results in $y^k = 1$ because the boundary condition recursion (4.50) with this ansatz becomes an eigenvalue equation for the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$:

$$\big(S_0^{(\mathrm{sym})}\Gamma^{-1}\big)\,Y^{(4n)} = y^{-1}\,Y^{(4n)} \qquad \Leftrightarrow \qquad H(y^{-1}) = 0, \tag{4.59}$$

and the monodromy free condition (4.55) concludes that y^{-1} is a k-th root of unity. As a result, the recursion equations (4.48) for $y_{n,1}$ turn out to be the following two algebraic equations in y:

$$\mathcal{F}_k(y) \equiv \frac{\mathcal{F}_k[y_{n,1} \to y^n \times y_{0,1}]}{y^n \times y_{0,1}} = 0, \qquad \mathcal{G}_k(y) \equiv \frac{\mathcal{G}_k[y_{n,1} \to y^n \times y_{0,1}]}{y^n \times y_{0,1}} = 0, \tag{4.60}$$

and actually they are related to each other by complex conjugation:

$$\big[\mathcal{F}_k(y)\big]^* = -y^{-\lfloor\frac{k}{2}\rfloor}\mathcal{G}_k(y), \tag{4.61}$$

with taking into account the Hermiticity condition (4.52) and $y^k = 1$. Therefore, the boundary conditions are given as the following algebraic equation system:

$$\mathcal{F}_k(y) = H(y^{-1}) = 0, \qquad y = \omega^n, \qquad {}^{\exists}n \in \mathbb{Z}. \tag{4.62}$$

Secondly, we assume that all the solutions to the boundary-condition algebraic equation $\mathcal{F}_k(y) = 0$ are k-th roots of unity. That is,

$$\textbf{Ansatz 2:} \qquad \mathcal{F}_k(y) = H(y^{-1}) = 0, \qquad y = \omega^{n_j}, \qquad (j = 1, 2, \cdots, \Big\lfloor\frac{k}{2}\Big\rfloor). \tag{4.63}$$

This ansatz fixes a half of not-yet-determined Stokes multipliers, with keeping the consistency with the boundary conditions and the Hermiticity conditions. By using the relation between roots and coefficients in (4.48), one obtains

$$s_{l,i,j} = (-1)^{l-1}\sigma_{L_{l,i,j}}(\{-\omega^{(-1)^{l-1}n_j}\}_{j=1}^{\lfloor\frac{k}{2}\rfloor}), \tag{4.64}$$

Here $\sigma_n(\{x_i\})$ is the symmetric polynomials among $\{x_i\}_{i=1}^N$ of degree n:[21]

$$\sigma_n(\{x_i\}_{i=1}^N) = \sum_{1\le i_1<i_2<\cdots<i_n\le N} x_{i_1}x_{i_2}\cdots x_{i_n}, \tag{4.65}$$

and the integer $L_{l,i,j}$ is defined as

$$0 \le L_{l,i,j} < k, \qquad L_{l,i,j} \equiv (-1)^{l-1}(i-j) \mod k. \tag{4.66}$$

The above formula is applied for the indices (i,j) which satisfy

$$\underline{k = 4k_0+1:}\quad L_{l,i,j} + \left\lfloor \frac{l-1}{2}\right\rfloor \in 2\mathbb{Z}+1; \qquad \underline{k=4k_0+3:}\quad L_{l,i,j} + \left\lfloor\frac{l}{2}\right\rfloor \in 2\mathbb{Z}. \tag{4.67}$$

In terms of the dominance profile, it is easy to see that they are a half of the Stokes multipliers. We show the right hand side of $\mathcal{J}_{k,2}^{(\mathrm{sym})}$:

$$\left.\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\cdots & k-1) & (\mathbf{4} & \frac{\mathbf{k-1}}{\mathbf{2}}) & (\frac{k+7}{2} & k) & (\mathbf{3} & \frac{\mathbf{k+1}}{\mathbf{2}}) & (\frac{k+5}{2} & 1) & (\mathbf{2} & \frac{\mathbf{k+3}}{\mathbf{2}}) \\ \hline
\cdots & (\mathbf{4} & \mathbf{k-1}) & (\frac{k+7}{2} & \frac{k-1}{2}) & (\mathbf{3} & \mathbf{k}) & (\frac{k+5}{2} & \frac{k+1}{2}) & (\mathbf{2} & \mathbf{1}) & \frac{k+3}{2} \\ \hline
\cdots & \frac{\mathbf{k-3}}{\mathbf{2}}) & (\frac{k+7}{2} & k-1) & (\mathbf{3} & \frac{\mathbf{k-1}}{\mathbf{2}}) & (\frac{k+5}{2} & k) & (\mathbf{2} & \frac{\mathbf{k+1}}{\mathbf{2}}) & (\frac{k+3}{2} & 1) \\ \hline
\cdots & (\frac{k+7}{2} & \frac{k-3}{2}) & (\mathbf{3} & \mathbf{k-1}) & (\frac{k+5}{2} & \frac{k-1}{2}) & (\mathbf{2} & \mathbf{k}) & (\frac{k+3}{2} & \frac{k+1}{2}) & 1
\end{array}\right]\begin{array}{l} :3 \\ :2 \\ :1 \\ :0 \end{array}. \tag{4.68}$$

The numbers in the bold type correspond to the Stokes multipliers satisfying the constraints (4.67). It is worth mentioning the plot of the number $L_{l,i,j}$ on this profile:

$$\left.\begin{array}{ccccccccccccc}
\cdots & \frac{k-11}{2} &)(& \frac{\mathbf{k-9}}{\mathbf{2}} &)(& \frac{k-7}{2} &)(& \frac{\mathbf{k-5}}{\mathbf{2}} &)(& \frac{k-3}{2} &)(& \frac{\mathbf{k-1}}{\mathbf{2}} &) \\ \hline
\cdots &)(& \mathbf{5} &)(& 4 &)(& \mathbf{3} &)(& 2 &)(& \mathbf{1} &) & \\ \hline
\cdots & \frac{\mathbf{k-11}}{\mathbf{2}} &)(& \frac{k-9}{2} &)(& \frac{\mathbf{k-7}}{\mathbf{2}} &)(& \frac{k-5}{2} &)(& \frac{\mathbf{k-3}}{\mathbf{2}} &)(& \frac{k-1}{2} &) \\ \hline
\cdots &)(& 5 &)(& \mathbf{4} &)(& 3 &)(& \mathbf{2} &)(& 1 &) &
\end{array}\right]\begin{array}{l} :3 \\ :2 \\ :1 \\ :0 \end{array}. \tag{4.69}$$

Therefore, the numbers, $L_{l,i,j} = 1,2,\cdots,\lfloor\frac{k}{2}\rfloor (=\frac{k-1}{2})$, are arranged in the increasing (or decreasing) order. It is also convenient to introduce the following notation for the fine Stokes multipliers:

$$(j|i) \in \mathcal{J}_{k,2}^{(\mathrm{sym})} \text{ satisfying Eq. (4.67)}: \qquad s_{l,i,j} = \left\{ \begin{array}{ll} \theta_{L_{l,i,j}} & (l=1,3) \\ -\theta^*_{L_{l,i,j}} & (l=0,2) \end{array}\right. , \tag{4.70}$$

and then for the complementary cases:

$$(j|i) \in \mathcal{J}_{k,2}^{(\mathrm{sym})} \text{ \underline{not} satisfying Eq. (4.67)}: \qquad s_{l,i,j} = \left\{ \begin{array}{ll} \widetilde{\theta}_{L_{l,i,j}} & (l=1,3) \\ -\widetilde{\theta}^*_{L_{l,i,j}} & (l=0,2) \end{array}\right. . \tag{4.71}$$

Therefore, under our assumption, we need to fix a half of Stokes multipliers which can be written by $\widetilde{\theta}_n$. Below we provide two kinds of solutions to the multi-cut boundary condition.

[21] Here σ_n stands for the symmetric polynomials. Do not be confused with the Pauli matrices.

4.3.1 Discrete solutions and configurations of avalanches

As we have seen in Eqs. (4.68), our ansatz fixes a half of the Stokes multipliers with the formula (4.64). That is, we have assumed that there are $\lfloor \frac{k}{2} \rfloor$ eigenvectors of the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$ which satisfy the boundary condition (4.35). The concrete expression for the eigenvectors is given by (4.45) as

$$\mathcal{Y}_j \equiv Y^{(0)}\big(\{y_{m,1} \to \omega^{m\,n_j} y_{0,1}\}\big), \qquad S_0^{(\mathrm{sym})}\Gamma^{-1}\mathcal{Y}_j = \omega^{-n_j}\mathcal{Y}_j. \tag{4.72}$$

The $\lfloor \frac{k}{2} \rfloor$ vectors $\{\mathcal{Y}_j\}_{j=1}^{\lfloor \frac{k}{2} \rfloor}$ turn out to be distinct if the following set of integers:

$$\mathbf{N} \equiv \big(n_1, n_2, \cdots, n_{\lfloor \frac{k}{2} \rfloor}\big), \tag{4.73}$$

are distinct up to modulo k.

Actually there is a similar way to fix another half of the Stokes multipliers, which is similar to the multi-cut boundary condition (4.35). Therefore, we consider another type of "boundary condition":

$$\begin{aligned} y_{n,i} &= 0 \qquad (i = 1, 2, \cdots, \Big\lfloor \frac{k+3}{4} \Big\rfloor), \\ y_{n,\frac{k+1}{2}+i} &\neq 0 \qquad (i = 1, 2, \cdots, \Big\lfloor \frac{k+1}{4} \Big\rfloor), \end{aligned} \tag{4.74}$$

which is referred to as *complementary boundary condition*. Of course, this condition has nothing to do with the boundary condition (4.35), however it is useful to obtain the following solutions to the non-perturbative completion. As is discussed in Appendix C.3, with this boundary condition, one can obtain similar algebraic equations to Eqs. (4.60):

$$\widetilde{\mathcal{F}}_k(y) \equiv \frac{\widetilde{\mathcal{F}}_k[y_{n,\frac{k+3}{2}} \to y^n \times y_{0,\frac{k+3}{2}}]}{y^n \times y_{0,\frac{k+3}{2}}} = 0, \qquad \widetilde{\mathcal{G}}_k(y) \equiv \frac{\widetilde{\mathcal{G}}_k[y_{n,\frac{k+3}{2}} \to y^n \times y_{0,\frac{k+3}{2}}]}{y^n \times y_{0,\frac{k+3}{2}}} = 0 \tag{4.75}$$

which are also related to each other by complex conjugation:

$$\big[\widetilde{\mathcal{F}}_k(y)\big]^* = -y^{-\lfloor \frac{k}{2} \rfloor}\widetilde{\mathcal{G}}_k(y), \tag{4.76}$$

with the Hermiticity condition (4.52) and $y^k = 1$. Therefore, by imposing another ansatz for the solution:

$$\textbf{Ansatz 3:} \qquad \widetilde{\mathcal{F}}_k(y) = H(y^{-1}) = 0, \qquad y = \omega^{\widetilde{n}_j}, \qquad (j = 1, 2, \cdots, \Big\lfloor \frac{k}{2} \Big\rfloor), \tag{4.77}$$

one obtains the following expression for the other half of the Stokes multipliers:

$$s_{l,i,j} = (-1)^{l-1}\sigma_{L_{l,i,j}}\big(\{-\omega^{(-1)^{l-1}\widetilde{n}_j}\}_{j=1}^{\lfloor \frac{k}{2} \rfloor}\big), \tag{4.78}$$

with the indices $(j|i) \in J_l$ satisfying

$$\underline{k = 4k_0 + 1:} \quad L_{l,i,j} + \Big\lfloor \frac{l-1}{2} \Big\rfloor \in 2\mathbb{Z}; \qquad \underline{k = 4k_0 + 3:} \quad L_{l,i,j} + \Big\lfloor \frac{l}{2} \Big\rfloor \in 2\mathbb{Z} + 1. \tag{4.79}$$

This formula also guarantees existence of another $\lfloor\frac{k}{2}\rfloor$ eigenvectors which is expressed with (C.29) as

$$\widetilde{\mathcal{Y}}_j \equiv \widetilde{Y}^{(0)}\big(\{\widetilde{y}_{m,\frac{k+3}{2}} \to \omega^{m\widetilde{n}_j} y_{0,\frac{k+3}{2}}\}\big), \qquad S_0^{(\mathrm{sym})}\Gamma^{-1}\widetilde{\mathcal{Y}}_j = \omega^{-\widetilde{n}_j}\,\widetilde{\mathcal{Y}}_j. \tag{4.80}$$

The $\lfloor\frac{k}{2}\rfloor$ vectors $\{\widetilde{\mathcal{Y}}_j\}_{j=1}^{\lfloor\frac{k}{2}\rfloor}$ turn out to be distinct if the following set of integers:

$$\widetilde{\mathbf{N}} \equiv \big(\widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big), \tag{4.81}$$

are distinct up to modulo k. By construction, these vectors $\{\widetilde{\mathcal{Y}}_j\}_{j=1}^{\lfloor\frac{k}{2}\rfloor}$ are independent vectors from the other vectors $\{\mathcal{Y}_j\}_{j=1}^{\lfloor\frac{k}{2}\rfloor}$. Therefore, these expressions for the Stokes multipliers are characterized by the following $k-1$ integers

$$(\mathbf{N};\widetilde{\mathbf{N}}) \equiv \big(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}; \widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big) \tag{4.82}$$

which satisfy the following condition:

$$0 < n_1 < n_2 < \cdots < n_{\lfloor\frac{k}{2}\rfloor} \le k; \qquad 0 < \widetilde{n}_1 < \widetilde{n}_2 < \cdots < \widetilde{n}_{\lfloor\frac{k}{2}\rfloor} \le k, \tag{4.83}$$

up to some re-ordering of the integers.

From the above construction, we fixed $k-1$ eigenvalues of the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$. The last eigenvalue can be read with taking into account the following fact:

$$\det\Big(S_0^{(\mathrm{sym})}\Gamma^{-1}\Big) = 1, \tag{4.84}$$

therefore the remaining eigenvector is given as the following value:

$$\omega^{-n_0}, \qquad n_0 \equiv -\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}(n_j + \widetilde{n}_j), \tag{4.85}$$

and then the characteristic equation (4.56) is expressed as

$$H(x) = \Big(x - \omega^{\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}(n_j+\widetilde{n}_j)}\Big)\prod_{i=1}^{\lfloor\frac{k}{2}\rfloor}\big(x-\omega^{-n_i}\big)\big(x-\omega^{-\widetilde{n}_i}\big) = 0. \tag{4.86}$$

Consequently, in order for the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$ to be diagonalizable, the following condition is sufficient:

$$(n;\widetilde{n}): \qquad -\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}\big(n_j + \widetilde{n}_j\big) = n_0 \not\equiv n_i,\ \widetilde{n}_i \mod k \qquad (i = 1, 2, \cdots, \Big\lfloor\frac{k}{2}\Big\rfloor), \tag{4.87}$$

since the last eigenvalue is distinct from the others. Actually, as is shown in Appendix C.4, this condition is also the necessary and sufficient condition. Essence of the proof is follows: The non-trivial situation is when the condition (4.87) is not satisfied. However, if we assume the situation and also assume that there exists an eigenvector of the last eigenvalue, then such an eigenvector is shown to be given by a linear combination of the vectors $\mathcal{Y}_j$ and/or $\widetilde{\mathcal{Y}}_j$. Therefore, the eigenvectors are not sufficient to diagonalize the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$, and this case is not a solution. This proves the statement. Below, we consider the meaning of this condition.

First of all, the following transformation does not break the condition (4.87) and maps a solution to a different solution:

$$\begin{aligned}&\big(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}; \widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big)\\ &\qquad\to\qquad \big(n_1+1, n_2+1, \cdots, n_{\lfloor\frac{k}{2}\rfloor}+1; \widetilde{n}_1+1, \widetilde{n}_2+1, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}+1\big),\end{aligned} \tag{4.88}$$

since this also maps n_0 as

$$n_0 \to n_0 + 1, \tag{4.89}$$

and therefore forms the $\mathbb{Z}_k$ group. Therefore, we choose the following representative of this $\mathbb{Z}_k$ transformation:

$$n_0 = -\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}\big(n_j + \widetilde{n}_j\big) \equiv 0 \qquad \mathrm{mod}\ k, \tag{4.90}$$

for the solutions. Under this condition, the condition (4.87) (with (4.90)) is rephrased as the following equivalent expression of the conditions:

$$(\mathbf{N}; \widetilde{\mathbf{N}}): \qquad \sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}\big(n_j + \widetilde{n}_j\big) \equiv 0, \qquad \begin{cases} 1 \le n_1 < n_2 < \cdots < n_{\lfloor\frac{k}{2}\rfloor} \le k-1, \\ 1 \le \widetilde{n}_1 < \widetilde{n}_2 < \cdots < \widetilde{n}_{\lfloor\frac{k}{2}\rfloor} \le k-1 \end{cases}. \tag{4.91}$$

We also introduce the following transformation which is called *dual*:

$$\begin{aligned}&\big(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}; \widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big)\\ &\qquad\to\qquad \big(k-n_1, k-n_2, \cdots, k-n_{\lfloor\frac{k}{2}\rfloor}; k-\widetilde{n}_1, k-\widetilde{n}_2, \cdots, k-\widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big),\end{aligned} \tag{4.92}$$

which also maps a solution to a solution with fixing the condition (4.91). Also the following is called *reflection*:

$$\big(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}; \widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big) \quad\to\quad \big(\widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}; n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}\big). \tag{4.93}$$

Below we show several examples. Since $\mathbf{N}$ and $\widetilde{\mathbf{N}}$ basically share the same sets of indices, we only show the $\mathbf{N}$ side with the following notation:

$$\mathbf{N}_{|\mathbf{N}|} \equiv \big(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}\big)_{n_1+n_2+\cdots+n_{\lfloor\frac{k}{2}\rfloor}}. \tag{4.94}$$

Therefore, a solution is given as a pair of these indices,

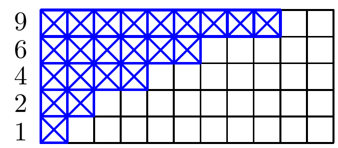
$$\big(\mathbf{N}_{|\mathbf{N}|}; \mathbf{N}'_{|\mathbf{N}'|}\big), \tag{4.95}$$

which satisfies $|\mathbf{N}|+|\mathbf{N}'| \equiv 0 \mod k$. For instance, the following indices are the solutions in the 5 and 7-cut cases:

$$
\begin{aligned}
&\underline{k=5}: \\
&\quad (1,4)_0, \quad (2,3)_0; \quad (2,4)_1; \quad (3,4)_2; \quad (1,2)_3; \quad (1,3)_4, \\
&\underline{k=7}: \\
&\quad (1,2,4)_0, \quad (3,5,6)_0; \\
&\quad (1,2,5)_1, \quad (1,3,4)_1, \quad (4,5,6)_1; \quad (1,2,3)_6, \quad (2,5,6)_6, \quad (3,4,6)_6; \\
&\quad (1,2,6)_2, \quad (1,3,5)_2, \quad (2,3,4)_2; \quad (1,5,6)_5, \quad (2,4,6)_5, \quad (3,4,5)_5; \\
&\quad (1,3,6)_3, \quad (1,4,5)_3, \quad (2,3,5)_3; \quad (1,4,6)_4, \quad (2,3,6)_4, \quad (2,4,5)_4.
\end{aligned} \tag{4.96}
$$

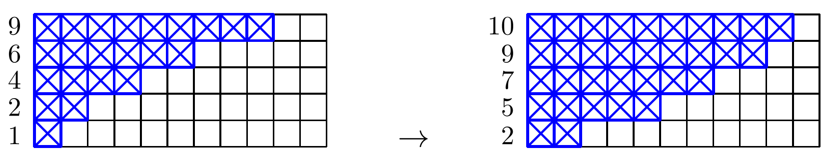
It is convenient to introduce a Young-diagram notation for expressing these indices. We here show it by examples (solutions in the 11-cut case). The indices $\mathbf{N}$ is denoted as

$$\mathbf{N} = (n_1, n_2, n_3, n_4, n_5) = (1,2,4,6,9) \qquad \Leftrightarrow$$



(4.97)

Therefore, the i-th row from the bottom has n_i boxes in the diagram. We also draw $\lfloor \frac{k}{2} \rfloor \times k$ total boxes for later convenience. In particular, the upper-left Young diagram (written with $\boxtimes$) is referred to as *sky* and the lower-right Young diagram (written with $\square$) is as *snow*. In this terminology, the dual transformation (4.92) exchanges the sky and snow in a Young digram:

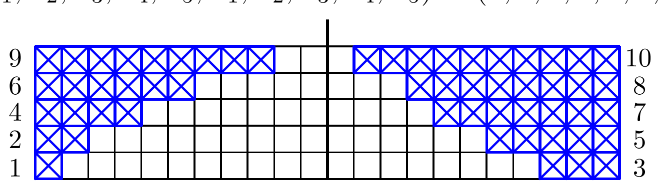


(4.98)

The pair $(\mathbf{N}; \widetilde{\mathbf{N}})$ is denoted as

$$(\mathbf{N}; \widetilde{\mathbf{N}}) = (n_1, n_2, n_3, n_4, n_5; \widetilde{n}_1, \widetilde{n}_2, \widetilde{n}_3, \widetilde{n}_4, \widetilde{n}_5) = (1,2,4,6,9; 3,5,7,8,10)$$

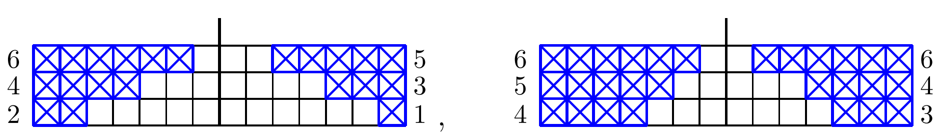
$\Leftrightarrow$



(4.99)

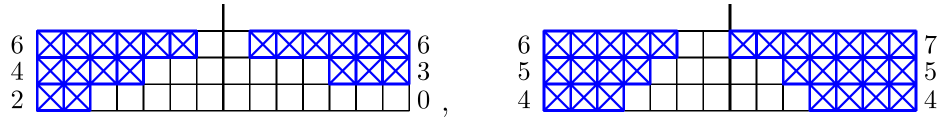
The rules to draw the Young diagrams are as follows:

1. The number of the boxes $\square$ (amount of snow) is always multiplied by k, and the following configurations are allowed solutions in the 7-cut case:



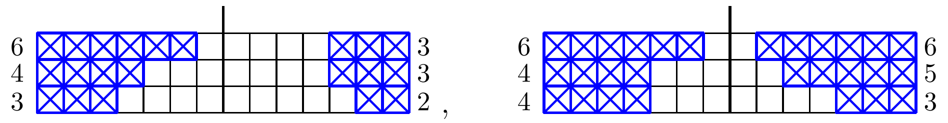
(4.100)

2. n_i and $\widetilde{n}_i$ ($i = 1, 2, \cdots, \lfloor \frac{k}{2} \rfloor$) cannot take 0 and k, therefore the following configurations are not allowed:

forbidden: 

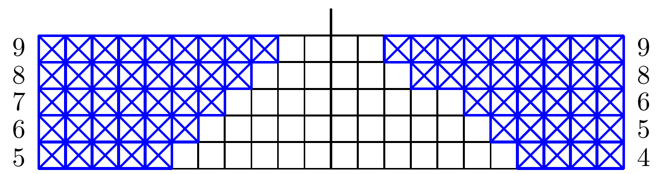
(4.101)

3. The solutions cannot have vertical cliffs, therefore the following configurations are not allowed:

forbidden: 

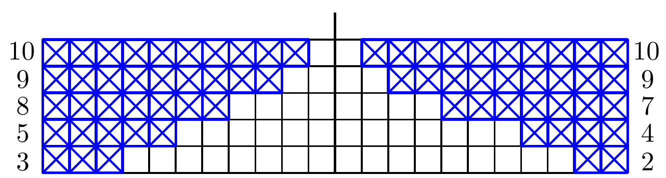
(4.102)

One of the ways to exhaust the solutions is first to take the most steepest configurations:



(4.103)

and then to consider possible ways for snow to slide on the surface with satisfying the condition (4.91), for example,



(4.104)

Therefore, the discrete solutions to the non-perturbative completion are labeled by configurations of *avalanches* in terms of Young diagram. Note that one can also move some snow on the one side to the other side.

4.3.2 Continuum solutions

Next we consider another solutions replacing the third ansatz (4.77). A simple way to solve the monodromy free condition is to require that the matrix $S_0^{(\mathrm{sym})}\,\Gamma^{-1}$ has k different eigenvectors. Therefore, we assume that the characteristic equations are given as

$$\textbf{Ansatz 4:} \qquad H(x) = \det\Big(xI_k - S_0^{(\mathrm{sym})}\,\Gamma^{-1}\Big) = x^k - 1. \tag{4.105}$$

Since the characteristic equation is written with the Stokes multipliers, one can obtain several constraint equations. Here are several examples:

The 5-cut case:

$$
\begin{aligned}
H(x) = &-1 + x^5 + x\left(-s_{1,3,2} + s_{0,5,2}s_{2,3,5} - s_{3,1,5}\right) + \\
&+ x^2\left(s_{0,3,4}s_{0,5,2} - s_{1,1,4} + s_{2,1,2}s_{2,3,5} - s_{1,3,2}s_{3,1,5} - s_{3,4,2}\right) + \\
&+ x^3\left(-s_{0,5,2} - s_{1,1,4}s_{1,3,2} + s_{0,3,4}s_{2,1,2} - s_{2,3,5} - s_{3,1,5}s_{3,4,2}\right) + \\
&+ x^4\left(-s_{0,3,4} - s_{2,1,2} - s_{1,1,4}s_{3,4,2}\right),
\end{aligned}
$$

The 7-cut case:

$$
\begin{aligned}
H(x) = &-1 + x^7 + x\left(-s_{1,7,6} + s_{0,3,6}s_{2,7,3} - s_{3,4,3}\right) + \\
&+ x^2\left(s_{0,3,6}s_{0,7,2} - s_{1,4,2} + s_{2,4,6}s_{2,7,3} - s_{3,1,6} - s_{1,7,6}s_{3,4,3}\right) + \\
&+ x^3\left(-s_{1,1,5} - s_{1,4,2}s_{1,7,6} + s_{0,3,6}s_{2,1,2} + s_{0,7,2}s_{2,4,6} + s_{0,4,5}s_{2,7,3} - s_{3,1,6}s_{3,4,3} - s_{3,5,2}\right) + \\
&+ x^4\left(-s_{0,3,6} + s_{0,4,5}s_{0,7,2} + s_{2,1,2}s_{2,4,6} - s_{2,7,3} - s_{1,4,2}s_{3,1,6} - s_{1,1,5}s_{3,4,3} - s_{1,7,6}s_{3,5,2}\right) + \\
&+ x^5\left(-s_{0,7,2} - s_{1,1,5}s_{1,4,2} + s_{0,4,5}s_{2,1,2} - s_{2,4,6} - s_{3,1,6}s_{3,5,2}\right) + \\
&+ x^6\left(-s_{0,4,5} - s_{2,1,2} - s_{1,1,5}s_{3,5,2}\right).
\end{aligned} \tag{4.106}
$$

These equations become simpler if one uses the notation given in Eqs. (4.70) and (4.71). One can read the general formula:

$$
\begin{aligned}
H(x) = x^k - 1 + &\sum_{n=1}^{\lfloor\frac{k}{2}\rfloor} x^n \Big[\sum_{i=1}^{n} \theta^*_{\lfloor\frac{k}{2}\rfloor+1-i}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-n+i} - \sum_{i=0}^{n}\theta_i\widetilde{\theta}_{n-i}\Big] + \\
&+ \sum_{n=1}^{\lfloor\frac{k}{2}\rfloor} x^{k-n}\Big[\sum_{i=0}^{n}\theta^*_i\widetilde{\theta}^*_{n-i} - \sum_{i=1}^{n}\theta_{\lfloor\frac{k}{2}\rfloor+1-i}\widetilde{\theta}_{\lfloor\frac{k}{2}\rfloor-n+i}\Big],
\end{aligned} \tag{4.107}
$$

where we have introduced $\theta_0 \equiv 1$. Therefore, the constraints are expressed as

$$
\begin{aligned}
0 &= \theta^*_{\lfloor\frac{k}{2}\rfloor}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor} - \theta_1 - \widetilde{\theta}_1, \\
0 &= \theta^*_{\lfloor\frac{k}{2}\rfloor}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-1} + \theta^*_{\lfloor\frac{k}{2}\rfloor-1}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor} - \theta_2 - \theta_1\widetilde{\theta}_1 - \widetilde{\theta}_2, \\
0 &= \theta^*_{\lfloor\frac{k}{2}\rfloor}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-2} + \theta^*_{\lfloor\frac{k}{2}\rfloor-1}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-1} + \theta^*_{\lfloor\frac{k}{2}\rfloor-2}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor} - \theta_3 - \theta_2\widetilde{\theta}_1 - \theta_1\widetilde{\theta}_2 - \widetilde{\theta}_3, \\
0 &= \theta^*_{\lfloor\frac{k}{2}\rfloor}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-3} + \theta^*_{\lfloor\frac{k}{2}\rfloor-1}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-2} + \theta^*_{\lfloor\frac{k}{2}\rfloor-2}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-1} + \theta^*_{\lfloor\frac{k}{2}\rfloor-3}\widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor} - \theta_4 - \theta_3\widetilde{\theta}_1 - \theta_2\widetilde{\theta}_2 - \theta_1\widetilde{\theta}_3 - \widetilde{\theta}_4, \\
&\cdots.
\end{aligned} \tag{4.108}
$$

Here we used Mathematica and checked these formulae up to $k = 23$. Since we assume the second ansatz, the half of the Stokes multipliers are given as (4.63), that is,

$$
\begin{aligned}
s_{l,i,j} = \theta_{L_{l,i,j}} \quad &\text{for} \quad (j|i) \in \mathcal{J}^{(\mathrm{sym})}_{k,2} \quad \text{and} \quad \text{Eq. (4.67)} \\
&\text{with} \quad \theta_m = \sigma_m(\{-\omega^{n_j}\}_{j=1}^{\lfloor\frac{k}{2}\rfloor}),
\end{aligned} \tag{4.109}
$$

therefore this solution is also labeled by $\lfloor\frac{k}{2}\rfloor$ distinct integers:

$$
(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}): \qquad 1 \le n_1 < n_2 < \cdots < n_{\lfloor\frac{k}{2}\rfloor} \le k. \tag{4.110}
$$

With noting the following relation:[22]

$$\sigma^*_{\lfloor\frac{k}{2}\rfloor}\sigma_n = \sigma^*_{\lfloor\frac{k}{2}\rfloor-n} \qquad \Leftrightarrow \qquad \theta^*_{\lfloor\frac{k}{2}\rfloor}\theta_n = \theta^*_{\lfloor\frac{k}{2}\rfloor-n}, \tag{4.111}$$

the forth ansatz (4.105) results in the following solution:

$$\begin{aligned} &s_{l,i,j} = \widetilde{\theta}_{L_{l,i,j}} \quad \text{for} \quad (j|i) \in \mathcal{J}^{(\mathrm{sym})}_{k,2} \quad \text{and} \quad \text{Eq. (4.79)} \\ &\qquad \text{with} \quad \widetilde{\theta}_n \equiv \mathcal{S}_n\big(\{\sigma_m\}_{m\in\mathbb{Z}}\big) + \widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-n+1}\sigma^*_{\lfloor\frac{k}{2}\rfloor}, \qquad \Big(n = 1,2,\cdots,\Big\lfloor\frac{k}{2}\Big\rfloor\Big). \end{aligned} \tag{4.112}$$

Here the polynomials $\mathcal{S}_n\big(\{x_m\}_{m\in\mathbb{Z}}\big)$ are defined by the following recursion relation:

$$\mathcal{S}_n\big(\{x_m\}_{m\in\mathbb{Z}}\big) = -\sum_{i=1}^{n} x_i\,\mathcal{S}_{n-i}\big(\{x_m\}_{m\in\mathbb{Z}}\big), \qquad S_0\big(\{x_m\}_{m\in\mathbb{Z}}\big) = 1. \tag{4.113}$$

The concrete expression is given as

$$\begin{aligned} \widetilde{\theta}_1 &= \big(-\sigma_1\big) + \widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor}\sigma^*_{\lfloor\frac{k}{2}\rfloor}, \\ \widetilde{\theta}_2 &= \big(-\sigma_2 + \sigma_1^2\big) + \widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-1}\sigma^*_{\lfloor\frac{k}{2}\rfloor}, \\ \widetilde{\theta}_3 &= \big(-\sigma_3 + 2\sigma_1\sigma_2 - \sigma_1^3\big) + \widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-2}\sigma^*_{\lfloor\frac{k}{2}\rfloor}, \\ \widetilde{\theta}_4 &= \big(-\sigma_4 + \sigma_2^2 - 3\sigma_1^2\sigma_2 + 2\sigma_1\sigma_3 + \sigma_1^4\big) + \widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-3}\sigma^*_{\lfloor\frac{k}{2}\rfloor}, \\ \widetilde{\theta}_5 &= \big(-\sigma_5 + \sigma_4\sigma_1 - 3\sigma_1^2\sigma_3 + \sigma_2\sigma_3 + 4\sigma_1^3\sigma_2 - 3\sigma_1\sigma_2^2 - \sigma_1^5\big) + \widetilde{\theta}^*_{\lfloor\frac{k}{2}\rfloor-4}\sigma^*_{\lfloor\frac{k}{2}\rfloor}. \end{aligned} \tag{4.114}$$

It is worth mentioning the relation to the Schur polynomials $P_n\big(\{x_m\}_{m\in\mathbb{Z}}\big)$:

$$\mathcal{S}_n\big(\{x_m\}_{m\in\mathbb{Z}}\big) = P_n\big(\{y_m\}_{m\in\mathbb{Z}}\big), \qquad x_n = P_n\big(\{-y_m\}_{m\in\mathbb{Z}}\big), \tag{4.115}$$

where the Schur polynomials $P_n\big(\{x_m\}_{m\in\mathbb{Z}}\big)$ are defined as

$$\sum_{n=0}^{\infty} z^n P_n\big(\{x_m\}_{m\in\mathbb{Z}}\big) = \exp\Big[\sum_{n=1}^{\infty} z^n x_n\Big]. \tag{4.116}$$

Note that these solutions includes $\lfloor\frac{k}{2}\rfloor$ real parameters. Sometimes, eigenvalues of the matrix $S_0^{(\mathrm{sym})}\,\Gamma^{-1}$ of the discrete solutions are distinct. In this case, such a discrete solution is a special case of the continuum solution. However generally these solutions do not include the discrete solutions in Section 4.3.1, since the discrete solutions generally include degeneracy of eigenvalues which cannot be resolved by these continuum parameters.

5 Non-perturbative stability of backgrounds

In this section, we briefly review the Riemann-Hilbert approach and the Deift-Zhou method [81–83], and also discuss its physical interpretations in non-critical string theory. In particular, we argue that this procedure implies *an additional physical requirement*

[22] Below we use the following short notation: $\sigma_n \equiv \sigma_n\big(\{-\omega^{n_j}\}_j\big)$.

about non-perturbative stability of classical backgrounds. We will see that this constraint results in the proper Stokes multipliers expected in the two-cut $(1, 2)$ critical point. Classical background here means the spectral curves which appear as semi-classical (large N) solutions of matrix models.

The role of the Riemann-Hilbert approach is to obtain the t dependence of physical amplitudes (for example, asymptotic expansion in t) by using an integration expression which can be derived from the ODE system in ζ. For references of the Riemann-Hilbert approach, the authors recommend to read the review article [85].

Roughly speaking, in the Riemann-Hilbert approach, we first discard the analytic continuity of the canonical solutions (2.38) and keep the form of asymptotic expansion (2.25) in the complex plane $\mathbb{C}$. In practice, we introduce some Stokes sectors (here we consider fine Stokes sectors) D_n and canonical solutions on them, $\widetilde{\Psi}_n(t;\zeta)$. As it has been reviewed in Section 2, these canonical solutions have the same asymptotic expansion in each Stokes sector (2.38) and the difference of these canonical solutions is expressed by Stokes matrices (2.39). Therefore, we introduce a semi-infinite straight line from the origin, $\mathcal{K}_n$,

$$\mathcal{K}_n = \{\zeta = u\, e^{i\chi_n};\, u \in \mathbb{R}_+\} \quad \text{with} \quad {}^\exists\chi_n \in [0, 2\pi) \quad \text{s.t.} \quad \mathcal{K}_n \subset D_n \cap D_{n+1}, \tag{5.1}$$

and define the following new partially analytic function $\Psi_{\rm RH}(t;\zeta)$ in $\zeta \in \mathbb{C} \setminus \bigcup_n \Gamma_n$:

$$\Psi_{\rm RH}(t;\zeta) = \widetilde{\Psi}_n(t;\zeta) \qquad \zeta \in D(\chi_n;\chi_{n-1}), \qquad (n = 1, 2, \cdots), \tag{5.2}$$

which has the following uniform asymptotic expansion in $\zeta \in \mathbb{C} \setminus \bigcup_n \mathcal{K}_n$:

$$\Psi_{\rm RH}(t;\zeta) \underset{asym}{\simeq} \widetilde{\Psi}_{\rm asym}(t;\zeta) = \widetilde{Y}(t;\zeta)\, e^{\widetilde{\varphi}(t;\zeta)}, \qquad \zeta \to \infty \in \mathbb{C} \setminus \bigcup_n \mathcal{K}_n. \tag{5.3}$$

The lines $\mathcal{K}$ is referred to as discontinuity lines, and examples are shown in Fig. 9. Note that the function $\Psi_{\rm RH}(t;\zeta)$ has enough information to recover all the canonical solutions simply by analytically continuing the argument ζ.

An essence of the Deift-Zhou method for the Riemann-Hilbert problem [83] is introduction of the following $k \times k$ function $g(t;\zeta)$ which we shall call *(off-shell) string background*:

$$\begin{aligned} &g(t;\zeta) = \mathrm{diag}\big(g^{(1)}(t;\zeta), \cdots, g^{(k)}(t;\zeta)\big), \\ &\qquad \text{with} \quad g^{(i)}(t;\zeta) \equiv \sum_{n=1}^{r} t_n^{(i)} \zeta^n + t_0^{(i)} \ln\zeta + \sum_{n=0}^{\infty} \frac{1}{n}\, g_n^{(i)}\, \zeta^{-n}. \end{aligned} \tag{5.4}$$

If one focuses on the aspect of algebraic curves, the function $g(t;\zeta)$ is referred to as *(off-shell) background spectral curve*. We then obtain the following seting of the Riemann-Hilbert problem:

Lemma 2 [Setting of the Riemann-Hilbert problem] *There exists the set of parameters* $t_n^{(i)}$ $(i = 1, 2, \cdots, k;\ n = 1, 2, \cdots, r)$ *which satisfies*

$$Z(t;\zeta) \equiv \Psi_{\rm RH}(t;\zeta) e^{-g(t;\zeta)} \to I_k, \qquad \big(\zeta \to \infty \in \mathbb{C} \setminus \bigcup_n \mathcal{K}_n\big). \tag{5.5}$$

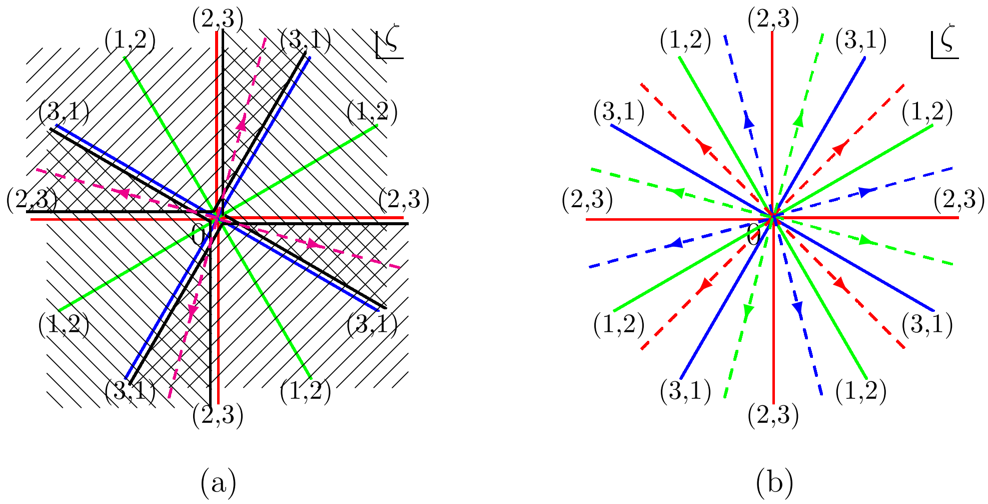


Figure 9: These are examples in the 3-cut $(1,1)$ critical point. a) The coarse Stokes sectors (shadowed domains) and the discontinuity lines $\mathcal{K}$ (dashed lines). Basically, any lines in the intersections $D_{3n} \cap D_{3(n+1)}$ are allowed. b) The discontinuity lines $\mathcal{K}$ (dashed lines) with respect to the fine Stokes sectors. They are related to the lines in (a) by continuous deformations which do not cross any divergence in the Riemann-Hilbert integral (5.10).

The $k \times k$ matrix function $Z(t;\zeta)$ then satisfies the following discontinuity relation:

$$Z_+(t;\zeta) = Z_-(t;\zeta)\, G(t;\zeta), \qquad G_n(t;\zeta) \equiv e^{g(t;\zeta)} S_n e^{-g(t;\zeta)}, \qquad \textit{along} \qquad \zeta \in \mathcal{K}_n, \quad (5.6)$$

where $n = 1, 2, \cdots$ and we define $Z_\pm(t;\zeta) \equiv \lim_{a\to 0} Z(t;\zeta \pm a\epsilon)$ with a vector ϵ which directs the left hand side of the line $\mathcal{K}_n$.

In general, the parameters $t_n^{(i)}$ ($i = 1, 2, \cdots, k$; $n = 1, 2, \cdots, r$) are the integrable deformations of the k-component KP hierarchy [90]. These are then given by the Lax equations:

$$g_{\rm str} \frac{\partial}{\partial t_n^{(i)}} \widetilde{\Psi}(t;\zeta) = \Big[\widetilde{\mathcal{P}}_{-n}^{(i)}\, \zeta^n + \widetilde{\mathcal{P}}_{-n+1}^{(i)}(t)\, \zeta^{n-1} + \cdots + \widetilde{\mathcal{P}}_0^{(i)}\Big] \widetilde{\Psi}(t;\zeta) \equiv \widetilde{\mathcal{P}}^{(i)}(t;\zeta)\, \widetilde{\Psi}(t;\zeta). \quad (5.7)$$

This information is understood as given information of the system and non-normalizable string moduli space which should not be minimized by the string dynamics [78]. Note that the Stokes matrices are invariants of these integrable deformation:

$$\frac{\partial S_m}{\partial t_n^{(i)}} = 0, \qquad \big(i = 1, \cdots, k;\ n = 1, 2, \cdots, r;\ m = 1, 2, \cdots\big), \quad (5.8)$$

and therefore the multipliers are integration constants (initial conditions) of these deformations. In this sense, they are also understood as non-normalizable string moduli space of the dynamics in the strong-coupling region of string theory.

Since the Stokes multipliers are integration constants of the system, we can uniquely obtain all the information by identifying the deformation parameters $t_n^{(i)}$ ($i = 1, 2, \cdots, k$; $n = 1, 2, \cdots, r$) and the Stokes multipliers. The fact is given in the form of the following theorem:

Theorem 5 [The Riemann-Hilbert problem (see [85])] *For a given analytic function $G(t;\zeta)$ on the discontinuity line $\zeta \in \mathcal{K} \equiv \bigcup_n \mathcal{K}_n$,*

$$G(t;\zeta) = G_n(t;\zeta) \equiv e^{g(t;\zeta)} S_n e^{-g(t;\zeta)} \qquad \zeta \in \mathcal{K}_n \qquad (n = 1, 2, \cdots), \tag{5.9}$$

there exists a unique holomorphic function $Z(t;\zeta)$ which satisfies Eqs. (5.5) and (5.6), and then $Z(t;\zeta)$ is given as

$$\begin{aligned} Z(t;\zeta) &= I_k + \int_{\mathcal{K}} \frac{d\lambda}{2\pi i} \frac{\rho(\lambda)(G(\lambda) - I_k)}{\lambda - \zeta} \\ &= I_k + \sum_{n,i,j} s_{n,i,j} \int_{\mathcal{K}_n} \frac{d\lambda}{2\pi i} \frac{\rho(\lambda) E_{i,j}}{\lambda - \zeta} e^{g^{(i)}(t;\lambda) - g^{(j)}(t;\lambda)}, \end{aligned} \tag{5.10}$$

with $\rho(\zeta) \equiv Z_-(\zeta)$ on $\zeta \in \mathcal{K} = \bigcup_n \mathcal{K}_n$.

By using the Riemann-Hilbert solution (5.10), one can obtain the canonical solutions to the ODE system (defined in (5.3)) as a function of t:

$$\Psi_{\mathrm{RH}}(t;\zeta) = Z(t;\zeta)\, e^{g(t;\zeta)} \underset{asym}{\simeq} \widetilde{\Psi}_{\mathrm{asym}}(t;\zeta) = \widetilde{Y}(t;\zeta)\, e^{\widetilde{\varphi}(t;\zeta)}, \qquad \zeta \to \infty \in \mathbb{C} \setminus \mathcal{K}. \tag{5.11}$$

Note that the "density function $\rho(\zeta)$" is given by $Z(t;\zeta)$ itself, and then the function $\rho(\zeta)$ satisfies the following integral equation:

$$\rho(\zeta) = I_k + \int_{\mathcal{K}} \frac{d\lambda}{2\pi i} \frac{\rho(\lambda)(G(\lambda) - I_k)}{\lambda - \zeta + \epsilon}, \qquad \zeta \in \mathcal{K}. \tag{5.12}$$

Therefore, one can recursively solve it and the solution is given as the following infinite sum of integrals:

$$Z(t;\zeta) = I_k + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \Big[\int_{\mathcal{K}} \frac{d\lambda_i}{2\pi i} \Big] \prod_{j=2}^{n} \Big[\frac{G(\lambda_j) - I_k}{\lambda_j - \lambda_{j-1} + \epsilon} \Big] \frac{G(\lambda_1) - I_k}{\lambda_1 - \zeta}, \tag{5.13}$$

with the assumption that

$$\int_{\mathcal{K}} \frac{d\lambda}{2\pi i} \big(G(\lambda) - I_k \big), \tag{5.14}$$

is sufficiently small. Note that we use the following multiplication rule of matrices: $\prod_{j=1}^{n} A_j \equiv A_n A_{n-1} \cdots A_1$. In terms of componets, this is expressed as

$$\begin{aligned} Z(t;\zeta) = I_k &+ \sum_{n,i,j} s_{n,i,j} E_{i,j} \int_{\mathcal{K}_n} \frac{d\lambda_1}{2\pi i} \frac{e^{g^{(i)}(t;\lambda_1) - g^{(j)}(t;\lambda_1)}}{\lambda_1 - \zeta} + \\ &+ \sum_{n_1,n_2,i,j,l} s_{n_2,i,l} s_{n_1,l,j} E_{i,j} \int_{\mathcal{K}_{n_1}} \frac{d\lambda_1}{2\pi i} \int_{\mathcal{K}_{n_2}} \frac{d\lambda_2}{2\pi i} \frac{e^{g^{(i)}(t;\lambda_2) - g^{(l)}(t;\lambda_2) + g^{(l)}(t;\lambda_1) - g^{(j)}(t;\lambda_1)}}{(\lambda_2 - \lambda_1 - \epsilon)(\lambda_1 - \zeta)} + \cdots . \end{aligned} \tag{5.15}$$

This expression is meaningful if the integral can be small enough to have convergence. In this case, one can evaluate the leading contribution by truncating higher terms (the so-called Born approximation). It is worth mentioning that this integral is quite similar to the D-instanton operator formalism in the free-fermion formulation [30,31] by interpreting $g^{(i)}(t;\zeta)$ as the free boson operator $\varphi_0^{(i)}(\zeta)$ in the system.

An important point here is that, in the Riemann-Hilbert approach, the string background $g(t;\zeta)$ is *arbitrary* except for the parameters $t_n^{(i)}$ $(i=1,2,\cdots,k;\ n=1,2,\cdots,r)$, and then generally is different from the semi-classical resolvent amplitudes $\widetilde{\varphi}(t;\zeta)$ of Eq. (2.32) which is obtained as a solution to the equation of motion (or loop equations) in the large N limit of the matrix models. As one can see in Theorem 5, the role of the string background $g(t;\zeta)$ is a reference background in the Riemann-Hilbert problem. Therefore, from the string-theory viewpoints, the string backgrounds $g(t;\zeta)$ are generally understood as *off-shell backgrounds of string theory* and in this sense the Riemann-Hilbert approach realizes *an off-shell background independent formulation of string theory.*

In order to understand $g(t;\zeta)$ as off-shell backgrounds of string theory, it is worth mentioning the interpretation of *the position of cuts.* Taking into account the consideration given around Eq. (4.21), we can define the cuts on the off-shell background as a combination of general Stokes lines:

$$\mathrm{Re}\Big(g^{(i)}(t;\zeta)-g^{(j)}(t;\zeta)\Big)=0, \tag{5.16}$$

which is obtained by an analytic deformation of the matrix contour $\omega^{1/2}\mathcal{C}^{(k)}$ (so that it realizes Eq. (4.15) in the leading of $\zeta\to\infty$). Note that this consideration is possible after imposing proper Stokes phenomena which solve the multi-cut boundary condition, as it is carried out in Section 4.

This viewpoint also provides the following consideration: If one chooses $g(t;\zeta)$ as a semi-classical resolvent function $\widetilde{\varphi}(t;\zeta)$, then the evaluation of Eqs. (5.11) and (5.10) in $g_{\mathrm{str}}\to 0$,

$$\Psi_{\mathrm{RH}}(t,\zeta)=Z(t;\zeta)\,e^{g(t;\zeta)}=\Big[I_k+\cdots\Big]\,e^{g(t;\zeta)}, \tag{5.17}$$

is calculation of quantum corrections from the background spectral curve $g(t;\zeta)$ which is given by the semi-classical resolvent. Therefore, *if the resolvent background is a stable vacuum of this system, the non-perturbative corrections should be exponentially small.* This is the additional constraint for the Stokes multipliers and is referred to as *small-instanton condition.*

5.1 The small-instanton condition for the 2-cut critical point

Here we consider the small-instanton condition in the 2-cut $(1,2)$ critical point. Mathematically, the Riemann-Hilbert problem in this case has been evaluated in [82,83,86–89] in the larger classes of Stokes multipliers (See the review [85]). In particular, according to the Deift-Zhou procedure [83], one first deforms the discontinuity lines $\mathcal{K}$ to *anti-Stokes lines.* The concept of anti-Stokes lines depends on saddle points of the string background $g(t;\zeta)$:

$$\text{saddle points } \zeta_*=\zeta_{i,j}^{(n)}:\qquad \frac{\partial}{\partial\zeta}\Big[g^{(i)}(t;\zeta_*)-g^{(j)}(t;\zeta_*)\Big]=0,\qquad (i,j=1,2,\cdots,k). \tag{5.18}$$

Definition 11 [Anti-Stokes lines] *Anti-Stokes lines* $\mathrm{ASL}_{i,j}^{(n)}$ *are defined for each pair of* (i,j) *as*

$$\mathrm{ASL}_{i,j}^{(n)} = \Big\{\zeta \in \mathbb{C};\ \mathrm{Im}\Big[g^{(i,j)}(t;\zeta)\Big] = \mathrm{Im}\Big[g^{(i,j)}(t;\zeta_{i,j}^{(n)})\Big]\Big\}, \tag{5.19}$$

where $\zeta_{i,j}^{(n)}$ *is a saddle point of the function* $g^{(i,j)}(t;\zeta) \equiv g^{(i)}(t;\zeta) - g^{(j)}(t;\zeta)$*.*

In the procedure of the Deift-Zhou method, one can choose the string background $g(t;\zeta)$, however, we know that the 2-cut $(1,2)$ critical point has two phases with respect to the sign of t cosmological constant [41]. Therefore, we *choose* the string background according to the actual phase appearing in the two-cut matrix model:[23]

$$g(t;\zeta) = \sigma_3\Big[\frac{1}{3}\zeta^3 + t\zeta + \cdots\Big] = \begin{cases} \sigma_3\Big[\frac{1}{3}\big(\zeta^2+2t\big)^{3/2}\Big] & : \text{two-cut phase } (t>0) \\ \sigma_3\Big[\frac{1}{3}\zeta^3 + t\zeta\Big] & : \text{one-cut phase } (t<0) \end{cases}. \tag{5.20}$$

Since we know that these curves are realized in the critical point as its stable vacua, these perturbative vacua should satisfy the small-instanton condition. Below we separately consider each case. We skip the calculation which is the same as [85].

The two-cut phase $(t>0)$ There are three saddle points of the function $g^{(1,2)}(t;\zeta) \equiv g^{(1)}(t;\zeta) - g^{(2)}(t;\zeta)$:

$$\zeta = \zeta_{1,2}^{(n)}: \qquad \zeta_{1,2}^{(0)} = 0, \qquad \zeta_{1,2}^{(\pm1)} = \pm i\sqrt{2t}, \tag{5.21}$$

and the values of the function at these saddle points are

$$g^{(1,2)}(t;\zeta_{1,2}^{(0)}) = \frac{2}{3}\big(2t\big)^{3/2}, \qquad g^{(1,2)}(t;\zeta_{1,2}^{(\pm1)}) = 0. \tag{5.22}$$

Note the saddle-point value of the function $g^{(2,1)}(t;\zeta) = -g^{(1,2)}(t;\zeta)$ is opposite of them. They are understood as instanton actions for the saddle points. The deformation of discrete lines $\mathcal{K}$ to the DZ curves is given in Fig. 10.

On the DZ curves, we then evaluate the integral (5.15) at saddle points [85]. The small-instanton condition becomes relevant when the saddle point $\zeta_{1,2}^{(0)} = 0$ of $g^{(1,2)}(t;\zeta)$ contributes in the Riemann-Hilbert integral (5.15). This happens in the integral on the curve $\mathcal{K}_3$. The relevant part is given as

$$Z(t;\zeta) = \alpha E_{1,2} \int_{\mathcal{K}_3} \frac{d\lambda}{2\pi i} \frac{e^{g^{(1,2)}(t;\zeta)}}{\lambda - \zeta} + \cdots. \tag{5.23}$$

[23] Note that we are here imposing *a physical requirement*, by taking into account the Deift-Zhou method [83]. In the Deift-Zhou procedure, one considers an arbitrary Stokes multipliers, and the function $g(t;\zeta)$ is a function which we choose so that there is no divergence in the RH calculation. In this way, we can obtain the asymptotic form in t for these arbitrary Stokes multipliers. In this section, on the other hand, we impose a physical constraint in which the physical background $g(t;\zeta)$ obtained from the matrix models is non-perturbatively stable. Therefore, this constraint picks up the special and physical Stokes multipliers.

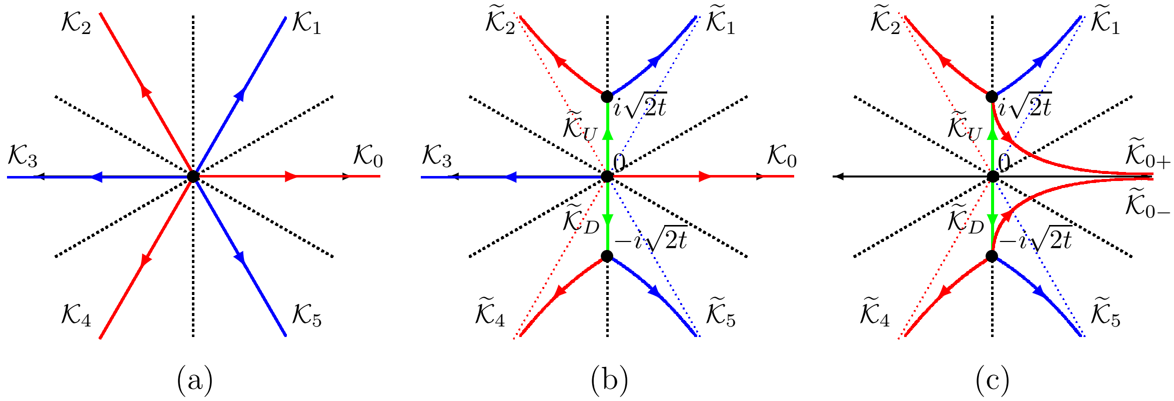


Figure 10: The discontinuity lines and the DZ curves in the two-cut $(1, 2)$ critical point of the two-cut phase. a) The discontinuity lines $\mathcal{K}$. There are two kinds of lines: the one kind is the lines $\mathcal{K}_{2n+1}$ on which the integral (5.15) only includes the contributions from the exponent $e^{g^{(1,2)}(\zeta)}$. The other kind is the lines $\mathcal{K}_{2n}$ on which the integral (5.15) only includes the contributions from the exponent $e^{g^{(2,1)}(\zeta)}$. b) The DZ curves which are obtained from analytic deformation of the original lines $\mathcal{K}$. A large D-instanton effect appears around the origin on the line $\mathcal{K}_3$. Therefore, we require $\alpha = 0$ so that this large instanton vanishes. c) The resulting DZ lines with $\alpha = 0$. Two lines along the real axes $\widetilde{\mathcal{K}}_{0\pm}$ come from the Stokes matrices on the lines $\widetilde{\mathcal{K}}_U$ and $\widetilde{\mathcal{K}}_D$. Saddle point approximation on each line gives ZZ branes in the Liouville theory, however contributions from these lines are the same and canceled by the $\mathbb{Z}_2$ symmetry.

The parameter α is the Stokes multipliers of this system (2.59). Therefore, in order to satisfy the small-instanton condition, the following condition is necessary and sufficient:

$$\alpha = s_0 = s_3 = 0, \tag{5.24}$$

otherwise this perturbative vacuum (5.20) breaks down (or decays into some stable vacuum) by the large non-perturbative effects. Consequently, the solutions to the non-perturbative completion are finally given as

$$\alpha = 0, \qquad \beta = \pm 1 = -\gamma, \tag{5.25}$$

which is known as the Hastings-McLeod solution in the Painlevé II equation [84]. As it has been calculated in [84], the final result is given as[24]

$$f(t) = -2\beta\sqrt{2t} + \cdots \qquad \text{with} \quad \beta = \pm 1, \tag{5.26}$$

especially, the instanton effect which comes from a single ZZ-brane at the origin $\zeta = 0$ vanishes in this phase.

[24]In this calculation, we use the local Riemann-Hilbert problems. Since the evaluation of the Riemann-Hilbert problem is not our purpose, we here skip the calculation. See the review [85]. An intuitive reason for vanishing the D-instanton effects (or physical interpretation of the mathematical result) is cancellation due to the $\mathbb{Z}_2$ symmetry of the system. For example, if one introduces the formal monodromy (as mentioned around (2.64), i.e. adding D0-brane charges in the background) then the instanton effect from the origin $\zeta = 0$ appears.

The one-cut phase ($t<0$) There are two saddle points of the function $g^{(1,2)}(t;\zeta)\equiv g^{(1)}(t;\zeta)-g^{(2)}(t;\zeta)$:

$$\zeta=\zeta_{1,2}^{(n)}: \qquad \zeta_{1,2}^{(\pm 1)}=\pm\sqrt{-t}, \tag{5.27}$$

and the values of the function at these saddle points are

$$g^{(1,2)}(t;\zeta_{1,2}^{(\pm 1)})=\mp\frac{4}{3}(-t)^{3/2}. \tag{5.28}$$

The deformation of discrete lines $\mathcal{K}$ to the DZ curves is given in Fig. 11. Note that existence of this phase also requires the same constraint $\alpha=0$. By taking into account the solution to the non-perturbative completion (5.25), the Riemann-Hilbert integral (5.15) becomes the following simple contour integrals:

$$\begin{aligned} Z(t;\zeta) &= I_k+\beta E_{1,2}\int_{\mathcal{K}_{1,2}}\frac{d\lambda}{2\pi i}\frac{e^{g^{(1,2)}(t;\lambda)}}{\lambda-\zeta}-\beta E_{2,1}\int_{\mathcal{K}_{2,1}}\frac{d\lambda}{2\pi i}\frac{e^{g^{(2,1)}(t;\lambda)}}{\lambda-\zeta}+\cdots, \\ &= I_k+\frac{\beta}{2\pi i}\Big[i\sqrt{\frac{\pi}{2\sqrt{-t}}}\frac{E_{1,2}}{\sqrt{-t}-\zeta}-i\sqrt{\frac{\pi}{2\sqrt{-t}}}\frac{E_{2,1}}{-\sqrt{-t}-\zeta}\Big]e^{-\frac{4}{3}(-t)^{3/2}}+\cdots, \end{aligned} \tag{5.29}$$

therefore the asymptotic expression of $f(t)$ is given as

$$f(t)=-\frac{\beta}{\sqrt{2\pi\sqrt{-t}}}e^{-\frac{4}{3}(-t)^{3/2}}+\cdots \qquad \text{with} \quad \beta=\pm 1. \tag{5.30}$$

See Eq. (2.48). It is worth mentioning that a similar expression was found in the 2-cut $(1,2)$ critical points [70] which comes from an explicit expression of fermion state within the free-fermion formulation [24, 30, 31, 68], although the expression there is given by an infinite sum of super-matrix integrals.

5.2 The small-instanton condition for the k-cut critical points

Here we consider the small-instanton constraint in the k-cut $(1,1)$ critical points. Since we here focus on the additional constraint, we only study the saddle point actions for the semi-classical string background and evaluation of the Riemann-Hilbert integrals is remained for future investigation. The classical backgrounds in these cases are calculated in [45] and given in terms of parameter z as

$$\begin{aligned} & g(t;\zeta)=\mathrm{diag}\big(g^{(1)}(t;\zeta),\cdots,g^{(k)}(t;\zeta)\big), \qquad g^{(j)}(t;\zeta)=\int^{\omega^{-(j-1)}\zeta}y(z)\,dx(z), \\ & \text{with} \qquad x(z)=t\sqrt[k]{\big(z-c\big)^{l}\big(z-b\big)^{k-l}}, \qquad y(z)=t\sqrt[k]{\big(z-c\big)^{k-l}\big(z-b\big)^{l}}, \end{aligned} \tag{5.31}$$

with $0=c\,l+b\,(k-l)$. The index $l\,(=0,1,2,\cdots,k-1)$ labels generally different solutions. The classical background $g(t;\zeta)$ is then expressed as

$$g^{(j)}(t;\zeta)=g^{(1)}\big(t;\omega^{-(j-1)}\zeta\big), \qquad g^{(1)}\big(t;x\big)=\frac{1}{2}\big(z(x)\big)^2-(c+b)\,z(x). \tag{5.32}$$

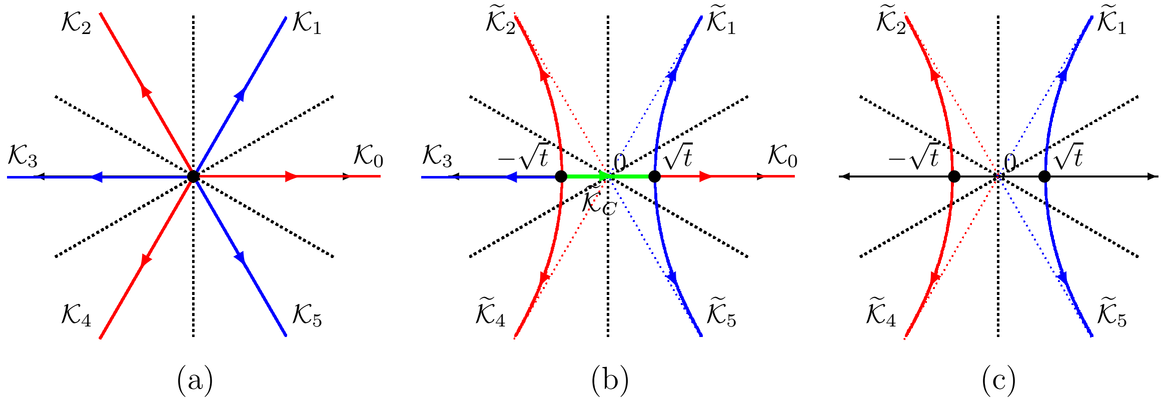


Figure 11: The discontinuity lines and the DZ curves in the two-cut $(1,2)$ critical point of the one-cut phase. a) The discontinuity lines $\mathcal{K}$ which is the same as two-cut phase. b) The DZ curves which are obtained from analytic deformation of the original lines $\mathcal{K}$. A large D-instanton effect appears around the saddle point $\zeta = +\sqrt{t}$ on the line $\mathcal{K}_0$, and around the saddle point $\zeta = -\sqrt{t}$ on the line $\mathcal{K}_3$. Therefore, we require $\alpha = 0$ so that these large instantons vanishes. c) The resulting DZ lines with $\alpha = 0$. By taking into account the sign of the Stokes multipliers, one observes that the integral (5.15) along connected lines $\widetilde{\mathcal{K}}_2$ and $\widetilde{\mathcal{K}}_4$ (and also $\widetilde{\mathcal{K}}_1$ and $\widetilde{\mathcal{K}}_5$ in the same way) can be considered as an integral on the single contour. Saddle point approximation on each line gives ZZ branes in the Liouville theory of the one-cut phase.

Here $z(x)$ is the inverse of the function $x(z)$ in Eq. (5.31). The saddle points for $g^{(i,j)}(t;\zeta) = g^{(i)}(t;\zeta) - g^{(j)}(t;\zeta)$ are given as

$$\frac{d}{d\zeta} g^{(i,j)}(t;\zeta) = 0 \qquad \Leftrightarrow \qquad \omega^{i-1} x(z) = \omega^{j-1} x(z'), \quad \omega^{-(i-1)} y(z) = \omega^{-(j-1)} y(z'), \tag{5.33}$$

and then this can be solved as

$$z' = z_{i,j}^{(n)} \equiv \Big(\frac{b\, e^{\frac{i}{2}\chi_{i,j}^{(n)}} + c\, e^{-\frac{i}{2}\chi_{i,j}^{(n)}}}{2\cos\big(\chi_{i,j}^{(n)}/2\big)} \Big), \qquad z = z_{j,i}^{(-n)} \equiv \Big(\frac{b\, e^{-\frac{i}{2}\chi_{i,j}^{(n)}} + c\, e^{\frac{i}{2}\chi_{i,j}^{(n)}}}{2\cos\big(\chi_{i,j}^{(n)}/2\big)} \Big), \tag{5.34}$$

with $\chi_{i,j}^{(n)} \equiv 2\pi \frac{(i-j)+nk}{k-2l}$, $(n = 1, 2, \cdots)$. Substituting these values in Eq. (5.32), we obtain the saddle point action:

$$\begin{aligned} g^{(i,j)}\big(t;\zeta_{i,j}^{(n)}\big) &= \frac{1}{2}\Big((z_{i,j}^{(n)})^2 - (z_{j,i}^{(-n)})^2 \Big) - (b+c)\Big(z_{i,j}^{(n)} - z_{j,i}^{(-n)} \Big) \\ &= i\frac{c^2 - b^2}{2} \tan\big(\frac{\chi_{i,j}^{(n)}}{2} \big) \in i\,\mathbb{R}. \end{aligned} \tag{5.35}$$

This means that the saddle point action always contributes order $\mathcal{O}(g_{\rm str}^0)$ and then identified as perturbative corrections (not as instantons). Therefore, there is no additional (small-instanton) constraints on the solutions obtained in Section 4.

6 Conclusion and discussions

In this paper, we give concrete solutions to the non-perturbative completion in the k-cut two-matrix models by a quantitative study of Stokes phenomena. The non-perturbative completion problem consists of the multi-cut boundary condition for the orthonormal polynomial systems and the non-perturbative stability condition for the semi-classical spectral curves in the large N limit. By carrying out these procedures, we demonstrated two classes of solutions, which are referred to as discrete and continuum solutions. Interestingly, the solutions possess kind of "charges" in terms of Young diagram representation.

We note that the continuum solutions to the non-perturbative completion still include continuous free parameters, although the two-cut cases have been completely fixed. It is conceivable that we might need to rely on further independent physical arguments to reduce these degrees of freedom, here we would like to interpret these free parameters as physical moduli parameters in the non-perturbative region of the string theory. Since the strong-coupling dual theory of the multi-cut matrix models seems to be non-critical M theory [43], these continuous parameters would correspond to the non-perturbative (non-normalizable) moduli space of M theory, $\mathcal{M}^{(\text{non}-\text{norm.})}_{\text{M-theory}}$ which is a distinct parameter space from the string-theory moduli space, $\mathcal{M}^{(\text{non}-\text{norm.})}_{\text{string}}$ and $\mathcal{M}^{(\text{norm.})}_{\text{string}}$. Below we provide a list of issues which deserve further exploration.

- In this paper, we have solved Stokes phenomena in $\mathbb{Z}_k$-symmetric critical points. It is also interesting to consider similar program in the fractional-superstring critical points [42]. In particular,we would like to see the emargence of the non-critical M theory from the $k \to \infty$ limit [43].

- Our procedure is directly related to Riemann-Hilbert calculus. It is useful to examine higher order instanton sectors and generalize the results in [76].

- In this paper, we focus on the cases with $\hat{p} = 1$ and small $\hat{q}$. In order to extend this procedure to the general $\hat{q}$ cases, one should resolve several complexities as shown in Eq. (4.37). It is of great interest to obtain the Stokes multipliers in higher $(\hat{p}, \hat{q})$ critical points. In particular, evaluation in the bosonic cases would clarify the issue raised in [60]. Also we have to take into account the smoothing of the cuts as shown in [51] (also see Appendix A).

- It is interesting to investigate *whether the Riemann-Hilbert representation can be written in language of matrix models?* This resembles the supermatrix models [70] which appear by evaluating tau-function in terms of free fermions. Also it is interesting to compare it with Kontsevich type matrix models [96] and also with the non-perturbative topological string-theory block recently proposed in [97].

- The Riemann-Hilbert representation is a background independent formulation, which allows us to introduce general off-shell background in string theory. Therefore, it is interesting to study physics in off-shell backgrounds and general concept of background independence in matrix models/string theory.

- In the multi-cut matrix models, there are two kinds of perturbative string vacua [43]: One is perturbatively isolated sectors (perturbative superselection sectors) which are decoupled with other sectors in all-order perturbation theory. This phenomenon is an origin of the extra-dimension in M theory. The other is perturbative vacua in the string-theory moduli space. For survey for the second vacua, the Riemann-Hilbert representation is even more powerful, since the off-shell moduli space is understand as the space of off-shell string-theory backgrounds. Furthermore, the $\mathbb{Z}_k$ symmetric critical points in the multi-cut matrix models have several perturbative vacua which satisfy loop equations. Therefore, it is interesting to study non-perturbative string-theory landscape from the Riemann-Hilbert approach. In particular, it might be possible to identify which observables are suitable for a discription of *a potential picture in the moduli space*.

- We obtained several solutions to Stokes phenomena which are characterized by several charges carried by Young diagrams. What is the physical meaning of our solutions? Any relation to W-symmetry or WZNW?
- Our solutions are natural generalizations of the Hastings-McLeod solution in the Painlevé II equation. The Hastings-McLeod solution is known to have several special features, for instance analyticity of the solution (See also [85]). Therefore, it is mathematically interesting to understand the standing point of our solutions in general solutions of the string equations.
- As is well-known, the integrable deformations in the usual integrable system correspond to the moduli space of worldsheet conformal field theory. On the other hand, non-trivial deformations of our solutions can be interpreted as non-perturbative integrable deformations in *physical solutions of string equations*. Therefore, these deformations are related to the moduli space of the dynamical degree of freedom in the strong coupling region, i.e. degree of freedom in non-critical M theory. It is interesting if there is a comprehensive understanding of these non-perturbative integrable deformations.

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A Stokes phenomenon in the Airy function

The non-perturbative relations between the resolvent and the orthonormal polynomials are first studied in [51] in the $(2,1)$ critical point of bosonic minimal string. Since this study also uncovers another aspect of cuts in the resolvent curves for the cases of $\hat{p} \geq 2$, we here briefly review the results and summarize the key points.

In the bosonic $(2,1)$ critical point, the orthonormal polynomials satisfy the following differential equation:

$$\zeta\Psi_{\rm orth}(t;\zeta) = \big(\partial^2 + u(t)\big)\,\Psi_{\rm orth}(t;\zeta), \tag{A.1}$$

$$g_{\rm str}\frac{\partial}{\partial\zeta}\Psi_{\rm orth}(t;\zeta) = \partial\,\Psi_{\rm orth}(t;\zeta). \tag{A.2}$$

By taking into account the definition $\partial \equiv g_{\rm str}\partial_t$, one can show that the orthonormal polynomial is given as Airy function:

$$0 = \Big(g_{\rm str}^2\frac{\partial^2}{\partial\zeta^2} - \zeta - t\Big)\,\Psi_{\rm orth}(t;\zeta), \qquad \Psi_{\rm orth}(t;\zeta) = {\rm Ai}(\zeta + t). \tag{A.3}$$

Here we have concluded $u(t) = -t$ by imposing the integrability condition of (A.1) and (A.2), and also have chosen the damping solution (Airy function) as the physical solution [51]:

$$\Psi_{\rm orth}(t;\zeta) \to 0, \qquad \zeta \to \infty. \tag{A.4}$$

As it is well-known, the asymptotic behavior of the orthonormal polynomial $\Psi_{\rm orth}(t;\zeta)$ (i.e. the Airy function) around the real axes, $\zeta \to \pm\infty$, is given as

$$\Psi_{\rm orth}(t;\zeta) \underset{asym}{\simeq} \left(\frac{g_{\rm str}\pi}{(\zeta+t)^{1/2}}\right)^{1/2} e^{-\frac{2}{3g_{\rm str}}(\zeta+t)^{3/2}} + \cdots\,, \tag{A.5}$$

in $\zeta \to \infty$ with the angle, $|\arg(\zeta)| < \pi$, and

$$\Psi_{\rm orth}(t;\zeta) \underset{asym}{\simeq} \left(\frac{g_{\rm str}\pi}{(\zeta+t)^{1/2}}\right)^{1/2} \Big[e^{-\frac{2}{3g_{\rm str}}(\zeta+t)^{3/2}} + ie^{\frac{2}{3g_{\rm str}}(\zeta+t)^{3/2}}\Big] + \cdots\,, \tag{A.6}$$

in $\zeta \to e^{\pi i}\times\infty$ with the angle, $|\arg(-\zeta)| < 2\pi/3$. Note that both two expressions in the intersections, $\pi/3 < |\arg(\zeta)| < \pi$, have common asymptotic expansions, and therefore, the appearance/disappearance of different exponents in different asymptotic regions is understood as the Stokes phenomenon. As a consequence, the resolvent in the weak coupling limit $g_{\rm str} \to 0$ is smooth in ζ with $\arg(\zeta) < \pi$, and the discontinuity only appears along $\zeta \in (-\infty, -t)$, that is,

$$\lim_{\epsilon\to\pm 0}\Big[\lim_{g_{\rm str}\to 0}\Psi_{\rm orth}(t;\zeta + i\epsilon)\Big] \sim e^{\mp\frac{2}{3g_{\rm str}}(\zeta+t)^{3/2}}, \qquad \zeta \in (-\infty, -t). \tag{A.7}$$

An important point in [51] is that the resolvent curve itself has a cut around $\zeta \to \infty$. However the explicit cuts are smeared by the superposition of the exponents $e^{(\zeta+t)^{3/2}}$ and $e^{-(\zeta+t)^{3/2}}$.

B Lax operators in the multi-cut matrix models

Here we summarize the Lax operators used in this paper.

B.1 The $\mathbb{Z}_k$ symmetric $(1,1)$ critical points

This class of critical points are characterized by the following Lax operators:

$$\begin{aligned}\boldsymbol{P}(t;\partial) &= \Gamma\partial + H(t),\\ \boldsymbol{Q}(t;\partial) &= \big(\boldsymbol{\Gamma}^{-2}(t;\partial)\,\boldsymbol{P}(t;\partial)\big)_+ - \mu\big(\boldsymbol{\Gamma}^{-1}(t;\partial)\big)_+\\ &= \Gamma^{-1}\partial - \Gamma^{-1}H\Gamma^{-1} - \mu\Gamma^{-1}.\end{aligned} \tag{B.1}$$

Note that the $\mathbb{Z}_k$ symmetry requires

$$H(t) = \begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & \ddots & \\ & & & 0 & * \\ * & & & & 0 \end{pmatrix}, \tag{B.2}$$

and the Lax operator $\boldsymbol{\Gamma}(t;\partial)$ is defined as

$$\boldsymbol{\Gamma}(t;\partial) = \Gamma + \sum_{n=1}^{\infty} S_n(t)\,\partial^{-n}, \qquad \big(\boldsymbol{\Gamma}(t;\partial)\big)^k = I_k, \qquad \big[\boldsymbol{\Gamma}(t;\partial), \boldsymbol{P}(t;\partial)\big] = 0. \tag{B.3}$$

From these operators, one can calculate the operator $\mathcal{Q}(t;\zeta)$ (see Eq. (2.14)) which is given as

$$\mathcal{Q}(t;\zeta) = \Gamma^{-2}\zeta - \Gamma^{-1}\big(\{\Gamma^{-1}, H\} + \mu\big). \tag{B.4}$$

The coefficients of the asymptotic expansion (2.18) are then calculated as

$$\begin{aligned}\varphi(\zeta) &= \frac{(\Gamma^{-1}\zeta)^2}{2} - \mu\,\Gamma^{-1}\zeta + \mathcal{O}(1/\zeta),\\ Y(\zeta) &= I_k + \frac{1}{\zeta}\mathrm{adj}^{-1}(\Gamma^{-2})\big[\Gamma^{-1}\{\Gamma^{-1}, H(t)\}\big] + \mathcal{O}(1/\zeta^2),\end{aligned} \tag{B.5}$$

where adj^{-1} is the inverse operator of $\mathrm{adj}(A)[B] = AB - BA$. In the $k=3$ case, by using the formula, $\mathrm{adj}^{-1}(\Gamma^{-1})[X] = [\Gamma^{-1}, X]/3$, one can show

$$Y_1(t) = \frac{1}{3}\Big(H(t) - \Gamma^{-1}H(t)\Gamma\Big). \tag{B.6}$$

Here we have checked that $\varphi_0(t) = 0$ is true for first cases $k = 3, 4, 5$ and this is consistent with our solutions.

B.2 Fractional-superstring $(\hat{p}, \hat{q}) = (1, 2)$ critical points $(r = 3)$

In this case, we only study $k = 2$ case, but generally one can calculate as follows: The Lax operators in these cases are

$$\begin{aligned}\boldsymbol{P}(t;\partial) = \Gamma\partial + H(t), \qquad \boldsymbol{Q}(t;\partial) &= \big(\boldsymbol{\Gamma}^{-1}(t;\partial)\,\boldsymbol{P}^2(t;\partial)\big)_+ - \mu\big(\boldsymbol{\Gamma}^{-1}(t;\partial)\big)_+\\ &= \Gamma\,\partial^2 + H(t)\,\partial - S_2(t) - \mu\Gamma^{-1}\end{aligned} \tag{B.7}$$

Therefore, the operator $\mathcal{Q}(t;\zeta)$ is given as

$$\mathcal{Q}(t;\zeta) = \Gamma^{-1}\zeta^2 - \Gamma^{-1}H(t)\,\zeta - \partial H(t) - S_2(t) - \mu\Gamma^{-1}, \tag{B.8}$$

or

$$\mathcal{Q}_{-3}(t) = \Gamma^{-1}, \quad \mathcal{Q}_{-2}(t) = -\Gamma^{-1}H(t), \quad \mathcal{Q}_{-1}(t) = -\partial H(t) - S_2(t) - \mu\Gamma^{-1}. \tag{B.9}$$

Here $S_2(t)$ satisfies

$$\big[\Gamma, S_2(t)\big] + \Gamma\partial H = 0, \qquad \{\Gamma^{(k-1)}, S_2(t)\} + \{\Gamma^{(k-2)}, H(t), H(t)\} = 0. \tag{B.10}$$

In the $k=2$ case, $S_2(t)$ is given as

$$S_2(t) = \frac{1}{2}\Big(\sigma_1 f^2(t) - i\sigma_2\,\partial f(t)\Big), \qquad H(t) = i\sigma_2 f(t). \tag{B.11}$$

The coefficients of the asymptotic expansion are given as

$$\begin{aligned}\varphi(\zeta) &= \big(\frac{\zeta^3}{3} - \mu\zeta\big)\Gamma^{-1} + \mathcal{O}(1/\zeta),\\ Y(\zeta) &= I_k - \frac{1}{\zeta}\mathrm{adj}^{-1}(\Gamma^{-1})\big[H(t)\big] + \mathcal{O}(1/\zeta^2)\end{aligned} \tag{B.12}$$

Here we have checked that $\varphi_0(t) = 0$ is true for first cases $k = 3, 4, 5$ and this is consistent with our solutions.

C Calculation for the boundary-condition recursions

In this appendix, we prove the general recursive equations for the boundary conditions (4.48) from the original form (4.42). First we divide the indices i of $y_{n,i}$ into four categories:

$$\begin{aligned}&\text{(I)}\quad 1 \le i \le \Big\lfloor\frac{k+3}{4}\Big\rfloor =: A, && \text{(II)}\quad B := \Big\lfloor\frac{k+3}{4}\Big\rfloor + 1 \le i \le \frac{k+1}{2},\\ &\text{(III)}\quad \frac{k+1}{2} + 1 \le i \le \Big\lfloor\frac{3k+3}{4}\Big\rfloor =: C, && \text{(IV)}\quad D := \Big\lfloor\frac{3k+3}{4}\Big\rfloor + 1 \le i \le k,\end{aligned} \tag{C.1}$$

with respect to the boundary condition (4.35), which is expressed as

$$y_{n,i} \neq 0 \qquad i \in \text{(I)}, \qquad \text{and} \qquad y_{n,i} = 0 \qquad i \in \text{(III)}. \tag{C.2}$$

To read the Stokes multipliers, we also show the profile of dominance $\mathcal{J}^{(\mathrm{sym})}_{k,2}$ in the general k-cut cases. Details are shown in Section 3.3. The left-hand sides of the profile $\mathcal{J}^{(\mathrm{sym})}_{k,2}$

are given as

$\underline{k = 4k_0 + 1:}$

$\mathcal{J}_{k,2}^{(\mathrm{sym})} =$

$$
\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
B & (D & D+1) & (\mathbf{A} & B+1) & (\mathbf{C} & D+2) & (\mathbf{A-1} & B+2) & (\mathbf{C-1} & \cdots \\ \hline
(D & B) & (\mathbf{A} & D+1) & (\mathbf{C} & B+1) & (\mathbf{A-1} & D+2) & (\mathbf{C-1} & B+2) & \cdots \\ \hline
D & (\mathbf{A} & B) & (\mathbf{C} & D+1) & (\mathbf{A-1} & B+1) & (\mathbf{C-1} & D+2) & (\mathbf{A-2} & \cdots \\ \hline
(\mathbf{A} & D) & (\mathbf{C} & B) & (\mathbf{A-1} & D+1) & (\mathbf{C-1} & B+1) & (\mathbf{A-2} & D+2) & \cdots
\end{array}\right.
$$

$\underline{k = 4k_0 + 3:}$

$\mathcal{J}_{k,2}^{(\mathrm{sym})} =$

$$
\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
D & (B & B+1) & (\mathbf{C} & D+1) & (\mathbf{A} & B+2) & (\mathbf{C-1} & D+2) & (\mathbf{A-1} & \cdots \\ \hline
(B & D) & (\mathbf{C} & B+1) & (\mathbf{A} & D+1) & (\mathbf{C-1} & B+2) & (\mathbf{A-1} & D+2) & \cdots \\ \hline
B & (\mathbf{C} & D) & (\mathbf{A} & B+1) & (\mathbf{C-1} & D+1) & (\mathbf{A-1} & B+2) & (\mathbf{C-2} & \cdots \\ \hline
(\mathbf{C} & B) & (\mathbf{A} & D) & (\mathbf{C-1} & B+1) & (\mathbf{A-1} & D+1) & (\mathbf{C-2} & B+2) & \cdots
\end{array}\right. \tag{C.3}
$$

and the right-hand sides of $\mathcal{J}_{k,2}^{(\mathrm{sym})}$ are the same expression for both cases of k:

$$
\left.\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\cdots & k-1) & (\mathbf{4} & \frac{k-1}{2}) & (\mathbf{\frac{k+7}{2}} & k) & (\mathbf{3} & \frac{k+1}{2}) & (\mathbf{\frac{k+5}{2}} & \mathbf{1}) & (\mathbf{2} & \mathbf{\frac{k+3}{2}}) \\ \hline
\cdots & (\mathbf{4} & k-1) & (\mathbf{\frac{k+7}{2}} & \frac{k-1}{2}) & (\mathbf{3} & k) & (\mathbf{\frac{k+5}{2}} & \frac{k+1}{2}) & (\mathbf{2} & \mathbf{1}) & \mathbf{\frac{k+3}{2}} \\ \hline
\cdots & \frac{k-3}{2}) & (\mathbf{\frac{k+7}{2}} & k-1) & (\mathbf{3} & \frac{k-1}{2}) & (\mathbf{\frac{k+5}{2}} & k) & (\mathbf{2} & \frac{k+1}{2}) & (\mathbf{\frac{k+3}{2}} & \mathbf{1}) \\ \hline
\cdots & (\mathbf{\frac{k+7}{2}} & \frac{k-3}{2}) & (\mathbf{3} & k-1) & (\mathbf{\frac{k+5}{2}} & \frac{k-1}{2}) & (\mathbf{2} & k) & (\mathbf{\frac{k+3}{2}} & \frac{k+1}{2}) & \mathbf{1}
\end{array}\right]
\begin{array}{l} : J_3 \\ : J_2 \\ : J_1 \\ : J_0 \end{array}. \tag{C.4}
$$

The numbers in the bold type are the number relevant to the boundary condition, (I) and (III). From this dominance profile, one can see the following facts:

1. There are only four Stokes multipliers of the following type:

$$
s_{0,i,*}^{(\mathrm{sym})}, \qquad i \in (\mathrm{I}) \qquad \text{or} \qquad (\mathrm{III}), \tag{C.5}
$$

   which are given as

$$
s_{0,1,\frac{k+3}{2}}^{(\mathrm{sym})}, \qquad s_{0,1,2}^{(\mathrm{sym})}, \qquad s_{0,1,\frac{k+5}{2}}^{(\mathrm{sym})} \qquad \text{and} \qquad s_{0,\frac{k+3}{2},2}^{(\mathrm{sym})}. \tag{C.6}
$$

2. The difference between symmetric Stokes multipliers and fine Stokes multipliers happens only as

$$
s_{0,1,2}^{(\mathrm{sym})} = s_{2,1,2} + s_{1,1,\frac{k+3}{2}}\, s_{3,\frac{k+3}{2},2}, \qquad s_{0,\frac{k+1}{2},2}^{(\mathrm{sym})} = s_{1,\frac{k+1}{2},2} + s_{0,\frac{k+1}{2},\frac{k+3}{2}}\, s_{3,\frac{k+3}{2},2}, \tag{C.7}
$$

   or equivalently,

$$
s_{2,1,2} = s_{0,1,2}^{(\mathrm{sym})} - s_{0,1,\frac{k+3}{2}}^{(\mathrm{sym})}\, s_{0,\frac{k+3}{2},2}^{(\mathrm{sym})}, \qquad s_{1,\frac{k+1}{2},2} = s_{0,\frac{k+1}{2},2}^{(\mathrm{sym})} - s_{0,\frac{k+1}{2},\frac{k+3}{2}}^{(\mathrm{sym})}\, s_{0,\frac{k+3}{2},2}^{(\mathrm{sym})}. \tag{C.8}
$$

   The others are related by a simple linear relation: $s_{l,i,j} = s_{0,i,j}^{(\mathrm{sym})}$.

Taking into account these facts, we see how the coefficients $\{y_{n,i}\}$ are expressed by $\{y_{n,1}\}$ with using Eq. (4.42).

C.1 Expressions in terms of $y_{n,1}$

Region I: $\left(1 \leq i \leq \left\lfloor \frac{k+3}{4} \right\rfloor = A\right)$ The first equation ($i=1$) in (I) is special and given as

$$y_{n,1} = y_{n+1,k} + s^{(\mathrm{sym})}_{0,1,\frac{k+3}{2}} \times y_{n+1,\frac{k+1}{2}} + s^{(\mathrm{sym})}_{0,1,2} \times y_{n+1,1}. \tag{C.9}$$

The first term is in the region (IV) and the second term is in the region (II). The other equations are simply given as

$$y_{n,i} = y_{n+1,i-1} \neq 0, \qquad (i = 2, 3, \cdots, \left\lfloor \frac{k+3}{4} \right\rfloor = A), \tag{C.10}$$

which is equivalent to

$$y_{n,i}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) = y_{n+i-1,1}, \qquad i \in (\mathrm{I}). \tag{C.11}$$

We often use these equations in the following discussions without mentioning it.

Region II $\left(B = \left\lfloor \frac{k+3}{4} \right\rfloor + 1 \leq i \leq \frac{k+1}{2}\right)$ In this case, the equations depend on k of modulo 4. The first equation ($i = B = \lfloor \frac{k+7}{4} \rfloor$)[25] is given as

$$y_{n,B} = \begin{cases} y_{n+A,1} + s^{(\mathrm{sym})}_{0,B,A} \times y_{n+A-1,1} & (k \equiv 1 \mod 4), \\ y_{n+A,1} & (k \equiv 3 \mod 4). \end{cases} \tag{C.12}$$

This is written only with $y_{n,1}$. The other equations, except for the last one ($i = \frac{k+1}{2}$), are given as

$$y_{n,B+j} = \begin{cases} y_{n+1,B+j-1} + s^{(\mathrm{sym})}_{0,B+j,A-j} \times y_{n+A-j-1,1} + s^{(\mathrm{sym})}_{0,B+j,A-j+1} \times y_{n+A-j,1} \\ \hfill (k \equiv 1 \mod 4) \\ y_{n+1,B+j-1} + s^{(\mathrm{sym})}_{0,B+j,A-j+1} \times y_{n+A-j,1} + s^{(\mathrm{sym})}_{0,B+j,A-j+2} \times y_{n+A-j+1,1} \\ \hfill (k \equiv 3 \mod 4) \end{cases},$$
$$\text{for} \qquad j = 1, 2, \cdots, \left(\frac{k+1}{2} - B\right) - 1. \tag{C.13}$$

Note that this expression is written with $y_{n,1}$ except for the first term, $y_{n+1,B+j-1}$. The first term is resolved by recursively using Eqs. (C.13). We finally obtain the expression with $y_{n,1}$:

$$\begin{aligned} y_{n,B+j} &= y_{n,B+j}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) \\ &\equiv y_{n+A+j,1} + \sum_{a=0}^{j-\epsilon(k)} s^{(\mathrm{sym})}_{0,B+j-a,A-j+a+\epsilon(k)} \times y_{n+A-j-1+2a+\epsilon(k),1} + \\ &\qquad + \sum_{a=0}^{j-1} s^{(\mathrm{sym})}_{0,B+j-a,A-j+a+1+\epsilon(k)} \times y_{n+A-j+2a+\epsilon(k),1}. \end{aligned} \tag{C.14}$$

[25] Note that the $k=5$ case is special and we should only see Eq. (C.16) and skip $s_{0,3,3}$ which is not allowed.

Here we introduced $\epsilon(k)$ as

$$\epsilon(k) = \begin{cases} 0 & (k = 4k_0 + 1) \\ 1 & (k = 4k_0 + 3) \end{cases}, \tag{C.15}$$

and the last equation ($i = \frac{k+1}{2}$) is given as

$$y_{n,\frac{k+1}{2}} = y_{n+1,\frac{k-1}{2}} + s^{(\text{sym})}_{0,\frac{k+1}{2},\frac{k+3}{2}} \times y_{n+1,\frac{k+1}{2}} + s^{(\text{sym})}_{0,\frac{k+1}{2},2} \times y_{n+1,1} + s^{(\text{sym})}_{0,\frac{k+1}{2},3} \times y_{n+2,1}. \tag{C.16}$$

This is not the equation to express $y_{n,\frac{k+1}{2}}$ by $y_{n,1}$. Such an equation is provided by the first equation in the next region III.

Region III $\left(\dfrac{k+1}{2} + 1 \leq i \leq \left\lfloor \dfrac{3k+3}{4} \right\rfloor = C\right)$ The first equation ($i = \frac{k+3}{2}$) is special and is given as

$$0 = y_{n,\frac{k+3}{2}} = y_{n+1,\frac{k+1}{2}} + s^{(\text{sym})}_{0,\frac{k+3}{2},2} \times y_{n+1,1}. \tag{C.17}$$

This is the equation for expressing $y_{n+1,\frac{k+1}{2}}$ by $y_{n,1}$:

$$y_{n,\frac{k+1}{2}} = y_{n,\frac{k+1}{2}}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) \equiv -s^{(\text{sym})}_{0,\frac{k+3}{2},2} \times y_{n,1}. \tag{C.18}$$

The other equations are simply give as

$$y_{n,i} = y_{n+1,i-1} = 0 \qquad \left(i = \frac{k+3}{2} + 1, \cdots, \left\lfloor \frac{3k+3}{4} \right\rfloor = C\right). \tag{C.19}$$

Region IV $\left(D = \left\lfloor \dfrac{3k+3}{4} \right\rfloor + 1 \leq i \leq k\right)$ The first equation ($i = D = \lfloor \frac{3k+7}{4} \rfloor$) depends on k of modulo 4 and is given as

$$y_{n,D} = \begin{cases} s^{(\text{sym})}_{0,D,A} \times y_{n+A-1,1} & (k \equiv 1 \mod 4), \\ s^{(\text{sym})}_{0,D,A} \times y_{n+A-1,1} + s^{(\text{sym})}_{0,D,A+1} \times y_{n+A,1} & (k \equiv 3 \mod 4), \end{cases} \tag{C.20}$$

The others are given as

$$\begin{aligned} y_{n,D+j} = y_{n+D+j-1,1} + s^{(\text{sym})}_{0,D+j,A-j} \times y_{n+A-j-1,1} + s^{(\text{sym})}_{0,D+j,A-j+1} \times y_{n+A-j,1}, \\ \text{for} \qquad j = 1, 2, \cdots, (k-D) \end{aligned} \tag{C.21}$$

Therefore, in the same way that we performed in the region II, we obtain the following expression:

$$\begin{aligned} y_{n,D+j} &= y_{n,D+j}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) \\ &\equiv \sum_{a=0}^{j} s^{(\text{sym})}_{0,D+j-a,A-j+a} \times y_{n+A-j-1+2a,1} + \sum_{a=0}^{j-1+\epsilon(k)} s^{(\text{sym})}_{0,D+j-a,A-j+a+1} \times y_{n+A-j+2a,1}. \end{aligned} \tag{C.22}$$

C.2 The recursion equations for $y_{n,1}$

The recursion equation $\mathcal{F}_k[y_{n,1}] = 0$ The recursion equation $\mathcal{F}_k[y_{n,1}] = 0$ in (4.48) originates from Eq. (C.16). First of all, by using Eq. (C.18), one obtains

$$-s^{(\mathrm{sym})}_{0,\frac{k+3}{2},2} \times y_{n,1} = y_{n+1,\frac{k-1}{2}} + \left(s^{(\mathrm{sym})}_{0,\frac{k+1}{2},2} - s^{(\mathrm{sym})}_{0,\frac{k+1}{2},\frac{k+3}{2}}\, s^{(\mathrm{sym})}_{0,\frac{k+3}{2},2}\right) \times y_{n+1,1} + s^{(\mathrm{sym})}_{0,\frac{k+1}{2},3} \times y_{n+2,1}. \tag{C.23}$$

Note that the difference between symmetric Stokes multipliers and fine Stokes multipliers (shown in Eq. (C.8)) only appears in Eq. (C.23) and the equation becomes

$$-s_{3,\frac{k+3}{2},2} \times y_{n,1} = y_{n+1,\frac{k-1}{2}} + s_{1,\frac{k+1}{2},2} \times y_{n+1,1} + s_{3,\frac{k+1}{2},3} \times y_{n+2,1}. \tag{C.24}$$

By substituting (C.14), one obtains the recursive equation:

$$\mathcal{F}_k[y_{n,1}] = y_{n+\frac{k-1}{2},1} + \sum_{j=1}^{\lfloor\frac{k-1}{4}\rfloor} s_{1,\frac{k-1}{2}+2-j,1+j} \times y_{n+2j-1,1} + \sum_{j=1}^{\lfloor\frac{k+1}{4}\rfloor} s_{3,\frac{k-1}{2}+3-j,1+j} \times y_{n+2j-2,1} = 0. \tag{C.25}$$

The recursion equation $\mathcal{G}_k[y_{n,1}] = 0$ The recursion equation $\mathcal{G}_k[y_{n,1}] = 0$ in (4.48) originates from Eq. (C.26). Here we also by using Eq. (C.18) obtain:

$$\begin{aligned} y_{n,1} &= y_{n+1,k} + \left(s^{(\mathrm{sym})}_{0,1,2} - s^{(\mathrm{sym})}_{0,1,\frac{k+3}{2}}\, s^{(\mathrm{sym})}_{0,\frac{k+3}{2},2}\right) \times y_{n+1,1} \\ &= y_{n+1,k} + s_{2,1,2} \times y_{n+1,1}. \end{aligned} \tag{C.26}$$

Therefore, in the same way as below, by substituting Eq. (C.22) one obtains the recursive equation:

$$\mathcal{G}_k[y_{n,1}] = -y_{n,1} + \sum_{j=1}^{\lfloor\frac{k-1}{4}\rfloor} s_{0,k+1-j,1+j} \times y_{n+2j,1} + \sum_{j=1}^{\lfloor\frac{k+1}{4}\rfloor} s_{2,2-j,1+j} \times y_{n+2j-1,1} = 0. \tag{C.27}$$

C.3 The complementary boundary condition

Recursive equations in this Appendix C are based on the boundary condition (C.2) which has been discussed in Section 4. In this subsection, we consider a complementary "boundary condition" which is expressed by

$$y_{n,i} = 0 \qquad i \in (\mathrm{I}), \qquad \text{and} \qquad y_{n,i} \neq 0 \qquad \in (\mathrm{III}). \tag{C.28}$$

This complementary boundary condition uncovers several interesting structures of the recursive equation (4.42) for the Stokes multipliers in the $\mathbb{Z}_k$ symmetric critical points. By using this complementary condition, one can perform the same procedure shown in Section C.1 and C.2. Here we consider expressions with $y_{n,\frac{k+3}{2}}$, i.e.

$$\widetilde{Y}^{(n)}\big(\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}\big) = \Big(\widetilde{y}_{j,n}\big(\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}\big)\Big)_{j=1}^{k}, \tag{C.29}$$

in the following ordering: III → I → II → IV.

C.3.1 Expression in terms of $y_{n,\frac{k+3}{2}}$

Region III $\left(\frac{k+1}{2}+1 \leq i \leq \left\lfloor\frac{3k+3}{4}\right\rfloor = C\right)$ The first equation ($i=\frac{k+3}{2}$) is given as

$$y_{n,\frac{k+3}{2}} = y_{n+1,\frac{k+1}{2}}. \tag{C.30}$$

We interpret that this is an equation to express $y_{n+1,\frac{k+1}{2}}$ in terms of $y_{n,\frac{k+3}{2}}$:

$$y_{n,\frac{k+1}{2}} = \widetilde{y}_{n,\frac{k+1}{2}}\big(\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}\big) \equiv y_{n-1,\frac{k+3}{2}}. \tag{C.31}$$

The other equations are simply give as

$$y_{n,i} = \widetilde{y}_{n,i}\big(\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}\big) = y_{n+i-\frac{k+3}{2},\frac{k+3}{2}} \qquad \left(i=\frac{k+3}{2}+1,\cdots,\left\lfloor\frac{3k+3}{4}\right\rfloor = C\right). \tag{C.32}$$

Therefore, we use these relations to rewrite all the $y_{n,i}$ $i\in$ (III) in terms of $\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}$.

Region I: $\left(1 \leq i \leq \left\lfloor\frac{k+3}{4}\right\rfloor = A\right)$ The first equation ($i=1$) in (I) is given as

$$0 = y_{n,1} = y_{n+1,k} + s^{(\text{sym})}_{0,1,\frac{k+3}{2}} \times y_{n,\frac{k+3}{2}} + s^{(\text{sym})}_{0,1,\frac{k+5}{2}} \times y_{n+1,\frac{k+3}{2}}. \tag{C.33}$$

This is an equation for the recursive equation in $\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}$. The other equations are simply given as

$$y_{n,i} = y_{n+1,i-1} = 0, \qquad \left(i=2,3,\cdots,\left\lfloor\frac{k+3}{4}\right\rfloor = A\right). \tag{C.34}$$

Region II $\left(B = \left\lfloor\frac{k+3}{4}\right\rfloor + 1 \leq i \leq \frac{k+1}{2}\right)$ The first equation ($i=B=\lfloor\frac{k+7}{4}\rfloor$) depends on k of modulo 4 and is given as

$$y_{n,B} = \begin{cases} s^{(\text{sym})}_{0,B,C} \times y_{n+C-\frac{k+3}{2},\frac{k+3}{2}} + s^{(\text{sym})}_{0,B,C+1} \times y_{n+C+1-\frac{k+3}{2},\frac{k+3}{2}} & (k\equiv 1 \mod 4), \\ s^{(\text{sym})}_{0,B,C} \times y_{n+C-\frac{k+3}{2},\frac{k+3}{2}} & (k\equiv 3 \mod 4). \end{cases} \tag{C.35}$$

This is written only with $y_{n,1}$. The other equations, except for the last one ($i=\frac{k+1}{2}$), are given as

$$\begin{aligned} y_{n,B+j} &= y_{n+1,B+j-1} + s^{(\text{sym})}_{0,B+j,C-j} \times y_{n+C-j-\frac{k+3}{2},\frac{k+3}{2}} + s^{(\text{sym})}_{0,B+j,C-j+1} \times y_{n+1+C-j-\frac{k+3}{2},\frac{k+3}{2}}, \\ &\text{for} \qquad j=1,2,\cdots,\left(\frac{k+1}{2}-B\right)-1, \end{aligned} \tag{C.36}$$

Therefore, by recursively using Eqs. (C.36), we obtain the expression with $\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}$:

$$\begin{aligned} y_{n,B+j} &= \widetilde{y}_{n,B+j}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) \\ &\equiv \sum_{a=0}^{j} s^{(\text{sym})}_{0,B+j-a,C-j+a} \times y_{n+C-j-\frac{k+3}{2}+2a,\frac{k+3}{2}} + \\ &\qquad + \sum_{a=0}^{j-\epsilon(k)} s^{(\text{sym})}_{0,B+j-a,C-j+a+1} \times y_{n+C-j+2a+1-\frac{k+3}{2},\frac{k+3}{2}}. \end{aligned} \tag{C.37}$$

The last equation ($i = \frac{k+1}{2}$) is given as

$$y_{n,\frac{k+1}{2}} = y_{n+1,\frac{k-1}{2}} + s^{(\mathrm{sym})}_{0,\frac{k+1}{2},\frac{k+3}{2}} \times y_{n+1,\frac{k+1}{2}} + s^{(\mathrm{sym})}_{0,\frac{k+1}{2},\frac{k+5}{2}} \times y_{n+1,\frac{k+3}{2}}. \tag{C.38}$$

This is an equation for the recursive equation in $\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}$.

Region IV $\left(D = \left\lfloor \dfrac{3k+3}{4} \right\rfloor + 1 \le i \le k\right)$ The first equation ($i = D = \lfloor \frac{3k+7}{4} \rfloor$) depends on k of modulo 4 and is given as

$$y_{n,D} = \begin{cases} y_{n+D-\frac{k+3}{2},\frac{k+3}{2}} & (k \equiv 1 \mod 4), \\ y_{n+D-\frac{k+3}{2},\frac{k+3}{2}} + s^{(\mathrm{sym})}_{0,D,C} \times y_{n+C-\frac{k+3}{2},\frac{k+3}{2}} & (k \equiv 3 \mod 4), \end{cases} \tag{C.39}$$

The others are given as

$$y_{n,D+j} = \begin{cases} y_{n+D+j-1,1} + s^{(\mathrm{sym})}_{0,D+j,C-j+1} \times y_{n+C-j+1-\frac{k+3}{2},\frac{k+3}{2}} + \\ \qquad + s^{(\mathrm{sym})}_{0,D+j,C-j+2} \times y_{n+C-j+2-\frac{k+3}{2},\frac{k+3}{2}}, & (k \equiv 1 \mod 4) \\ y_{n+D+j-1,1} + s^{(\mathrm{sym})}_{0,D+j,C-j} \times y_{n+C-j-\frac{k+3}{2},\frac{k+3}{2}} + \\ \qquad + s^{(\mathrm{sym})}_{0,D+j,C-j+1} \times y_{n+C-j+1-\frac{k+3}{2},\frac{k+3}{2}}, & (k \equiv 3 \mod 4) \end{cases},$$
$$\text{for} \qquad j = 1, 2, \cdots, (k-D) \tag{C.40}$$

Therefore, we obtain the following expression:

$$\begin{aligned} y_{n,D+j} &= \widetilde{y}_{n,D+j}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) \\ &\equiv y_{n+D+j-\frac{k+3}{2},\frac{k+3}{2}} + \sum_{a=0}^{j-1} s^{(\mathrm{sym})}_{0,D+j-a,C-j+a+1-\epsilon(k)} \times y_{n+C-j+a+1-\epsilon(k)-\frac{k+3}{2},\frac{k+3}{2}} + \\ &\qquad + \sum_{a=0}^{j-1+\epsilon(k)} s^{(\mathrm{sym})}_{0,D+j-a,C-j+a+2-\epsilon(k)} \times y_{n+C-j+a+2-\epsilon(k)-\frac{k+3}{2},\frac{k+3}{2}}. \end{aligned} \tag{C.41}$$

C.3.2 The recursion equations for $y_{n,\frac{k+3}{2}}$

Here we extend the recursive equations in Section C.2 to this complementary boundary condition. If we can find solutions to these recursive equations, then this means that the original recursive equations (4.42) are consistent with the complementary boundary condition.

The recursion equation $\widetilde{\mathcal{F}}_k[y_{n,\frac{k+3}{2}}] = 0$ Here we consider the first recursion equation $\widetilde{\mathcal{F}}_k[y_{n,\frac{k+3}{2}}] = 0$ which is similar to the equation $\mathcal{F}_k[y_{n,\frac{k+3}{2}}] = 0$ in (4.48). This equation originates from Eq. (C.33), and is given as

$$\begin{aligned} \widetilde{\mathcal{F}}_k[y_{n,\frac{k+3}{2}}] \equiv y_{n+\frac{k-1}{2},\frac{k+3}{2}} &+ \sum_{j=1}^{\lfloor\frac{k-1}{4}\rfloor} s_{3,k+2-j,\frac{k+3}{2}+j} \times y_{n+2j-1,\frac{k+3}{2}} + \\ &+ \sum_{j=1}^{\lfloor\frac{k+1}{4}\rfloor} s_{1,k+2-j,\frac{k+1}{2}+j} \times y_{n+2j-2,\frac{k+3}{2}} = 0. \end{aligned} \tag{C.42}$$

The recursion equation $\widetilde{\mathcal{G}}_k[y_{n,\frac{k+3}{2}}]=0$ The second recursion equation $\widetilde{\mathcal{G}}_k[y_{n,\frac{k+3}{2}}]=0$ is similar to $\mathcal{G}_k[y_{n,\frac{k+3}{2}}]=0$ in (4.48). This equation originates from Eq. (C.38), and is given as

$$\widetilde{\mathcal{G}}_k[y_{n,\frac{k+3}{2}}] = -y_{n,\frac{k+3}{2}} + \sum_{j=1}^{\lfloor\frac{k-1}{4}\rfloor} s_{2,\frac{k+3}{2}-j,\frac{k+3}{2}+j} \times y_{n+2j,\frac{k+3}{2}} + \\ + \sum_{j=1}^{\lfloor\frac{k+1}{4}\rfloor} s_{0,\frac{k+3}{2}-j,\frac{k+1}{2}+j} \times y_{n+2j-1,\frac{k+3}{2}} = 0. \tag{C.43}$$

It is worth mentioning that the change of positions of the Stokes multipliers (which appear in these recursive equations, $\widetilde{\mathcal{F}}_k$ and $\widetilde{\mathcal{G}}_k$) in the dominance profile. We show the right hand side of $\mathcal{J}_{k,2}^{(\mathrm{sym})}$:

$$\left.\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \cdots & \mathbf{k-1}) & (4 & \frac{k-1}{2}) & (\mathbf{\frac{k+7}{2}} & \mathbf{k}) & (3 & \frac{k+1}{2}) & (\mathbf{\frac{k+5}{2}} & \mathbf{1}) & (2 & \frac{k+3}{2}) \\ \hline \cdots & (4 & k-1) & (\mathbf{\frac{k+7}{2}} & \mathbf{\frac{k-1}{2}}) & (3 & k) & (\mathbf{\frac{k+5}{2}} & \mathbf{\frac{k+1}{2}}) & (2 & 1) & \frac{k+3}{2} \\ \hline \cdots & \frac{k-3}{2}) & (\mathbf{\frac{k+7}{2}} & \mathbf{k-1}) & (3 & \frac{k-1}{2}) & (\mathbf{\frac{k+5}{2}} & \mathbf{k}) & (2 & \frac{k+1}{2}) & (\mathbf{\frac{k+3}{2}} & \mathbf{1}) \\ \hline \cdots & (\mathbf{\frac{k+7}{2}} & \mathbf{\frac{k-3}{2}}) & (3 & k-1) & (\mathbf{\frac{k+5}{2}} & \mathbf{\frac{k-1}{2}}) & (2 & k) & (\mathbf{\frac{k+3}{2}} & \mathbf{\frac{k+1}{2}}) & 1 \end{array}\right] \begin{array}{l} :3 \\ :2 \\ :1 \\ :0 \end{array}. \tag{C.44}$$

The numbers in the bold type correspond to the Stokes multipliers appearing in the recursive equations in the complementary boundary condition. As one can see from Eq. (4.68), the Stokes multipliers here are complement to the Stokes multipliers appearing in the usual boundary condition Eq. (4.67).

C.4 General recursion equations and the third eigenvector

As one may notice in the calculations in the previous section, the procedures with the boundary condition and with the complementary boundary condition are almost the same. Therefore, the vector $Y^{(n)}$ in the recursion equations (4.42) are generally solved as

$$Y^{(n)}_{\mathrm{general}}\big(\{y_{m,1}, y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}\big) \equiv Y^{(n)}\big(\{y_{m,1}\}_{m\in\mathbb{Z}}\big) + \widetilde{Y}^{(n)}\big(\{y_{m,\frac{k+3}{2}}\}_{m\in\mathbb{Z}}\big), \tag{C.45}$$

with the constraints:

$$\mathcal{F}_k[y_{n,1}] + \widetilde{\mathcal{G}}_k[y_{n,\frac{k+3}{2}}] = 0, \qquad \mathcal{G}_k[y_{n,1}] + \widetilde{\mathcal{F}}_k[y_{n,\frac{k+3}{2}}] = 0. \tag{C.46}$$

This formula also enables us to obtain the general solutions to the eigenvalue problem of the matrix $S_0^{(\mathrm{sym})}\Gamma^{-1}$. Here, however, we focus on the discrete solutions discussed in Section 4.3.1. If one assumes the discrete solution with the indices

$$\big(n_1, n_2, \cdots, n_{\lfloor\frac{k}{2}\rfloor}; \widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_{\lfloor\frac{k}{2}\rfloor}\big), \tag{C.47}$$

then generally eigenvectors with the eigenvalues ω^{-n_i} (or $\omega^{-\widetilde{n}_i}$) are given by

$$Y^{(0)}_{\mathrm{general}}\big(\{y_{m,1}\to\omega^n a, y_{m,\frac{k+3}{2}}\to\omega^n b\}_{m\in\mathbb{Z}}\big), \tag{C.48}$$

the dimension of the eigenspace is give as

$$\dim\Big[Y^{(0)}_{\rm general}\big(\{y_{m,1}\to\omega^{n(m-1)}a,y_{m,\frac{k+3}{2}}\to\omega^{n(m-1)}b\}_{m\in\mathbb{Z}}\big)\Big]=\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}\big(\delta_{n,n_j}+\delta_{n,\widetilde{n}_j}\big). \quad (C.49)$$

Therefore, if the indices do not satisfy Eq. (4.87), then the number of distinct eigenvalues is $k-1$, i.e. the matrix $S_0^{(\rm sym)}\Gamma^{-1}$ is not diagonalizable.

On the other hand, if the condition Eq. (4.87) is satisfied, then the last eigenvector is given by Eq. (C.48) with

$$\mathcal{F}_k(\eta)a+\widetilde{\mathcal{G}}_k(\eta)b=0,\qquad \eta\equiv\omega^{-\sum_{j=1}^{\lfloor\frac{k}{2}\rfloor}(n_j+\widetilde{n}_j)}. \quad (C.50)$$

Interestingly, if $n_j=\widetilde{n}_j$ $(j=1,2,\cdots,\lfloor\frac{k}{2}\rfloor)$ then the vector is given as

$$\begin{aligned}&Y^{(0)}_{\rm general}\big(\{y_{m,1}\to\eta^{m-1},y_{m,\frac{k+3}{2}}\to\eta^{m-1}(-1)^{\lfloor\frac{k}{2}\rfloor}\eta^{1/2}\}_{m\in\mathbb{Z}}\big)\\&={}^t\Big(1,\eta,\cdots,\eta^{\lfloor\frac{k-1}{4}\rfloor},0,\cdots,0,(-1)^{\lfloor\frac{k}{2}\rfloor}\eta^{1/2},\cdots,(-1)^{\lfloor\frac{k}{2}\rfloor}\eta^{1/2+\lfloor\frac{k-3}{4}\rfloor},0,\cdots,0\Big). \quad (C.51)\end{aligned}$$

D Calculation in the 3-cut $(1,1)$ critical point $(r=2)$

The specialty of the 3-cut $(1,1)$ critical point is that the symmetric Stokes sectors D_{4n} (see Eq. (3.15)) do not cover the whole plane $\mathbb{C}$. Therefore, we consider doubling of the sectors

$$D_{2n},\qquad S_{2n}^{(\rm sym)}\equiv S_{2n}S_{2n+1},\qquad (n=0,1,\cdots,5), \quad (D.1)$$

and express the boundary condition (4.35) as follows:

$$\begin{aligned}Y^{(4n)}&=\begin{pmatrix}y_{4n,1}\\y_{4n,2}\\y_{4n,3}\end{pmatrix}\equiv\Gamma^nX^{(4n)}=\begin{pmatrix}x^{(4n)}_{n+1}\neq0\\x^{(4n)}_{n+2}\\x^{(4n)}_{n+3}=0\end{pmatrix},\\Y^{(4n+2)}&=\begin{pmatrix}y_{4n+2,1}\\y_{4n+2,2}\\y_{4n+2,3}\end{pmatrix}\equiv\Gamma^nX^{(4n+2)}=\begin{pmatrix}x^{(4n+2)}_{n+1}\\x^{(4n+2)}_{n+2}\neq0\\x^{(4n+2)}_{n+3}=0\end{pmatrix},\end{aligned} \quad (D.2)$$

with

$$X^{(2n)}=S_{2n}^{(\rm sym)}X^{(2n+2)},\qquad (n=0,1,\cdots,5). \quad (D.3)$$

This is then written as

$$\begin{aligned}&Y^{(4n)}=S_0^{(\rm sym)}Y^{(4n+2)},\qquad Y^{(4n+2)}=\big(S_2^{(\rm sym)}\Gamma^{-1}\big)Y^{(4n+4)},\\\Leftrightarrow\qquad&\begin{cases}y_{4n,i}=y_{4n+2,i}+\sum_j\Big[s^{(\rm sym)}_{0,i,j}\times y_{4n+2,j}\Big],\\y_{4n+2,i}=y_{4n+4,i-1}+\sum_j\Big[s^{(\rm sym)}_{2,i,j}\times y_{4n+4,j-1}\Big].\end{cases}\end{aligned} \quad (D.4)$$

These recursion relations are expressed as

$$\begin{aligned} &y_{4n,3} = y_{4n+2,3} = 0, \qquad && y_{4n,1} = y_{4n+2,1} \neq 0, \\ &y_{4n,2} = y_{4n+2,2} \neq 0, \qquad && y_{4n+2,2} = y_{4n+4,1} \neq 0, \end{aligned} \tag{D.5}$$

and the following two recursion equation for $y_{4n,1}$

$$y_{4n,1} = s_{2,1,2} \times y_{4n+4,1}, \qquad y_{4n+4,1} = -s_{3,3,2} \times y_{4n,1}. \tag{D.6}$$

As one may notice, this equation itself is the same as Eq. (4.48). The solution to this boundary condition is easily solved as

$$y_{4n,1} = \omega^{nl}, \qquad s_{3,3,2} = -\omega^{l}, \qquad s_{2,1,2} = \omega^{-l}, \qquad (l = 0, 1, 2), \tag{D.7}$$

and the general solution is given as

$$s_{0,2,3} = -\omega^{-l} + \omega^{l} s_{1,1,3}, \tag{D.8}$$

with Eq. (D.7). This provides the first case of the continuum solution (4.112).

E Calculation in the 4-cut $(1, 1)$ critical point

Here we calculate the 4-cut $(1, 1)$ critical point as an example in which the coprime condition of Eq. (3.7) is violated:

$$(k, r) = (4, 2). \tag{E.1}$$

In this case, the leading exponents are degenerate:

$$\varphi^{(1)}(t;\zeta) \sim \varphi^{(3)}(t;\zeta), \qquad \varphi^{(2)}(t;\zeta) \sim \varphi^{(4)}(t;\zeta), \tag{E.2}$$

and consider the subleading Stokes lines:

$$\mathrm{Re}\Big[\big(\varphi^{(1)}_{-r+1} - \varphi^{(3)}_{-r+1}\big)\zeta^{r-1}\Big] = 0, \qquad \mathrm{Re}\Big[\big(\varphi^{(2)}_{-r+1} - \varphi^{(4)}_{-r+1}\big)\zeta^{r-1}\Big] = 0. \tag{E.3}$$

The dominance profile in the ζ plane is shown in Fig. 12.

Here we use the fine Stokes sectors D_n (calculated in the leading Stokes lines) which are defined as

$$D_n \equiv D\Big(\frac{(n-1)\pi}{4}; \frac{n\pi}{4}\Big), \qquad n = 0, 1, 2, 3. \tag{E.4}$$

All fine Stokes matrices can be expressed in terms of S_0 as

$$S_n = \Gamma^{-n} S_0 \Gamma^{n}, \qquad S_0 = \begin{pmatrix} 1 & & & \\ \alpha & 1 & \beta & \\ \epsilon & & 1 & \\ \gamma & & \delta & 1 \end{pmatrix}. \tag{E.5}$$

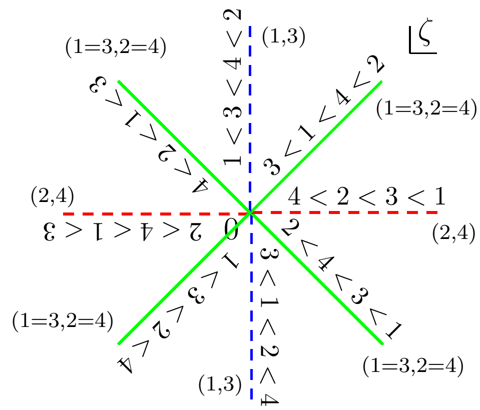

Figure 12: The dominance profile in the 4-cut $(1,1)$ case in terms of ζ. The bold lines express the leading Stokes lines with degeneracy $\varphi^{(1)} \sim \varphi^{(3)}$ and $\varphi^{(2)} \sim \varphi^{(4)}$. The dashed lines express the sub leading Stokes lines for $(1,3)$ and $(2,4)$.

Then the multi-cut boundary condition is given as

$$Y^{(n)} \equiv \Gamma^n X^{(n)} = \begin{pmatrix} y_{n,1} \neq 0 \\ y_{n,2} = 0 \\ y_{n,3} = 0 \\ y_{n,4} \neq 0 \end{pmatrix}. \tag{E.6}$$

The recursive equations are expressed as

$$y_{n,1} = y_{n+1,4}, \qquad 0 = \epsilon \times y_{n+1,4}, \qquad y_{n+1,1} + \alpha \times y_{n,1} = 0, \qquad \gamma \times y_{n+1,1} - y_{n,1} = 0 \tag{E.7}$$

and the solutions are given as

$$\alpha = -\omega^l, \qquad \gamma = \omega^{-l}, \qquad \epsilon = 0, \qquad y_{n,1} = \omega^{nl} \qquad (l = 0,1,2,3). \tag{E.8}$$

By directly solving the monodromy free condition, the other Stokes multipliers are also fixed and the solution is given as

$$S_0 = \begin{pmatrix} 1 & & & \\ -\omega^l & 1 & -\omega^{-l} & \\ 0 & & 1 & \\ \omega^{-l} & & \omega^l & 1 \end{pmatrix}, \qquad (l = 0,1,2,3). \tag{E.9}$$

References

[1] A. M. Polyakov, "Quantum geometry of bosonic strings," Phys. Lett. B **103** (1981) 207; "Quantum geometry of fermionic strings," Phys. Lett. B **103** (1981) 211.

[2] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, "Fractal structure of 2d-quantum gravity," Mod. Phys. Lett. A **3** (1988) 819.

[3] F. David, “Conformal field theories coupled to 2-D gravity in the conformal gauge,” Mod. Phys. Lett. A **3** (1988) 1651;
J. Distler and H. Kawai, “Conformal field theory and 2-D quantum gravity, or who’s afraid of Joseph Liouville?,” Nucl. Phys. B **321** (1989) 509.

[4] H. Dorn and H. J. Otto, “Two and three point functions in Liouville theory,” Nucl. Phys. B **429** (1994) 375 [arXiv:hep-th/9403141];
A. B. Zamolodchikov and Al. B. Zamolodchikov, “Structure constants and conformal bootstrap in Liouville field theory,” Nucl. Phys. B **477** (1996) 577 [arXiv:hep-th/9506136].

[5] J. Teschner, “On the Liouville three point function,” Phys. Lett. B **363** (1995) 65 [arXiv:hep-th/9507109].

[6] V. Fateev, A. B. Zamolodchikov and Al. B. Zamolodchikov, “Boundary Liouville field theory. I: Boundary state and boundary two-point function,” arXiv:hep-th/0001012;
J. Teschner, “Remarks on Liouville theory with boundary,” arXiv:hep-th/0009138.

[7] A. B. Zamolodchikov and Al. B. Zamolodchikov, “Liouville field theory on a pseudosphere,” arXiv:hep-th/0101152.

[8] R. H. Poghosian, “Structure constants in the $N = 1$ super-Liouville field theory,” Nucl. Phys. B **496** (1997) 451 [arXiv:hep-th/9607120];
R. C. Rashkov and M. Stanishkov, “Three-point correlation functions in $N = 1$ Super Liouville Theory,” Phys. Lett. B **380** (1996) 49 [arXiv:hep-th/9602148].

[9] T. Fukuda and K. Hosomichi, “Super Liouville theory with boundary,” Nucl. Phys. B **635** (2002) 215 [arXiv:hep-th/0202032];
C. Ahn, C. Rim and M. Stanishkov, “Exact one-point function of $N = 1$ super-Liouville theory with boundary,” Nucl. Phys. B **636** (2002) 497 [arXiv:hep-th/0202043].

[10] N. Seiberg and D. Shih, “Branes, rings and matrix models in minimal (super)string theory,” JHEP **0402** (2004) 021 [arXiv:hep-th/0312170].

[11] E. Brezin and V. A. Kazakov, “Exactly solvable field theories of closed strings,” Phys. Lett. B **236** (1990) 144;
M. R. Douglas and S. H. Shenker, Nucl. Phys. B **335** (1990) 635;
D. J. Gross and A. A. Migdal, “Nonperturbative Two-Dimensional Quantum Gravity,” Phys. Rev. Lett. **64** (1990) 127.

[12] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. Math. Phys. **59** (1978) 35.

[13] V. A. Kazakov, “The Appearance of Matter Fields from Quantum Fluctuations of 2D Gravity,” Mod. Phys. Lett. A **4** (1989) 2125.

[14] I. K. Kostov, “Strings embedded in Dynkin diagrams,” Cargese 1990, Proceedings, Random surfaces and quantum gravity, pp.135-149.

[15] I. K. Kostov, “Loop amplitudes for nonrational string theories,” Phys. Lett. B **266** (1991) 317.

[16] I. K. Kostov, “Strings with discrete target space,” Nucl. Phys. B **376** (1992) 539 [arXiv:hep-th/9112059].

[17] T. Banks, M. R. Douglas, N. Seiberg and S. H. Shenker, “Microscopic and macroscopic loops in nonperturbative two-dimensional gravity,” Phys. Lett. B **238** (1990) 279.

[18] E. Brezin, M. R. Douglas, V. Kazakov and S. H. Shenker, “The Ising model coupled to 2-d Gravity: A nonperturbative analysis,” Phys. Lett. B **237** (1990) 43;
D. J. Gross and A. A. Migdal, “Nonperturbative Solution of the Ising Model on a Random Surface,” Phys. Rev. Lett. **64** (1990) 717.

[19] D. J. Gross and A. A. Migdal, “A nonperturbative treatment of two-dimensional quantum gravity,” Nucl. Phys. B **340** (1990) 333.

[20] M. R. Douglas, “Strings in less than one-dimension and the generalized KdV hierarchies,” Phys. Lett. B **238** (1990) 176.

[21] T. Tada and M. Yamaguchi, “P and Q operator analysis for two matrix model,” Phys. Lett. B **250** (1990) 38;
M. R. Douglas, “The Two matrix model,” *In *Cargese 1990, Proceedings, Random surfaces and quantum gravity* 77-83. (see HIGH ENERGY PHYSICS INDEX 30 (1992) No. 17911)*;
T. Tada, “(Q, P) Critical Point From Two Matrix Models,” Phys. Lett. B **259** (1991) 442.

[22] G. W. Moore, “Geometry Of The String Equations,” Commun. Math. Phys. **133** (1990) 261; “Matrix Models Of 2-D Gravity And Isomonodromic Deformation,” Prog. Theor. Phys. Suppl. **102** (1990) 255.

[23] P. H. Ginsparg and J. Zinn-Justin, “Action principle and large order behavior of nonperturbative gravity,” in Proceedings, Random Surfaces and Quantum Gravity, Cargese, 1990, pp. 85-109.

[24] M. Fukuma, H. Kawai and R. Nakayama, “Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity,” Int. J. Mod. Phys. A **6** (1991) 1385; “Infinite dimensional Grassmannian structure of two-dimensional quantum gravity,” Commun. Math. Phys. **143** (1992) 371; “Explicit solution for p–q duality in two-dimensional quantum gravity,” Commun. Math. Phys. **148** (1992) 101.

[25] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, “Loop Equations And Virasoro Constraints In Nonperturbative 2-D Quantum Gravity,” Nucl. Phys. B **348** (1991) 435.

[26] B. Eynard and J. Zinn-Justin, “Large order behavior of 2-D gravity coupled to $d < 1$ matter,” Phys. Lett. B **302** (1993) 396 [arXiv:hep-th/9301004].

[27] F. David, “Phases of the large N matrix model and nonperturbative effects in 2-d gravity,” Nucl. Phys. B **348** (1991) 507; “Nonperturbative effects in matrix models and vacua of two-dimensional Phys. Lett. B **302** (1993) 403 [arXiv:hep-th/9212106].

[28] G. W. Moore, N. Seiberg and M. Staudacher, “From loops to states in 2-D quantum gravity,” Nucl. Phys. B **362** (1991) 665.

[29] J. M. Daul, V. A. Kazakov and I. K. Kostov, “Rational theories of 2-D gravity from the two matrix model,” Nucl. Phys. B **409** (1993) 311 [arXiv:hep-th/9303093].

[30] M. Fukuma and S. Yahikozawa, “Nonperturbative effects in noncritical strings with soliton backgrounds,” Phys. Lett. B **396** (1997) 97 [arXiv:hep-th/9609210]; “Combinatorics of solitons in noncritical string theory,” Phys. Lett. B **393** (1997) 316 [arXiv:hep-th/9610199];

[31] M. Fukuma and S. Yahikozawa, “Comments on D-instantons in c $<$ 1 strings,” Phys. Lett. B **460** (1999) 71 [arXiv:hep-th/9902169].

[32] C. Crnkovic and G. W. Moore, “Multicritical multicut matrix models,” Phys. Lett. B **257** (1991) 322.

[33] D. J. Gross and E. Witten, “Possible Third Order Phase Transition In The Large N Lattice Gauge Theory,” Phys. Rev. D **21** (1980) 446.

[34] V. Periwal and D. Shevitz, “Unitary matrix models as exactly solvable string theories,” Phys. Rev. Lett. **64** (1990) 1326; “Exactly solvable unitary matrix models: multicritical potentials and correlations,” Nucl. Phys. B **344** (1990) 731.

[35] M. R. Douglas, N. Seiberg and S. H. Shenker, “Flow and instability in quantum gravity,” Phys. Lett. B **244** (1990) 381.

[36] C. R. Nappi, “Painleve-II And Odd Polynomials,” Mod. Phys. Lett. A **5** (1990) 2773.

[37] C. Crnkovic, M. R. Douglas and G. W. Moore, “Loop equations and the topological phase of multi-cut matrix models,” Int. J. Mod. Phys. A **7** (1992) 7693 [arXiv:hep-th/9108014].

[38] T. J. Hollowood, L. Miramontes, A. Pasquinucci and C. Nappi, “Hermitian Versus Anti-Hermitian One Matrix Models And Their Hierarchies,” Nucl. Phys. B **373** (1992) 247 [arXiv:hep-th/9109046].

[39] T. Takayanagi and N. Toumbas, “A matrix model dual of type 0B string theory in two dimensions,” JHEP **0307** (2003) 064 [arXiv:hep-th/0307083].

[40] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. M. Maldacena, E. J. Martinec and N. Seiberg, “A new hat for the $c = 1$ matrix model,” arXiv:hep-th/0307195.

[41] I. R. Klebanov, J. M. Maldacena and N. Seiberg, “Unitary and complex matrix models as 1-d type 0 strings,” Commun. Math. Phys. **252** (2004) 275 [arXiv:hep-th/0309168].

[42] H. Irie, "Fractional supersymmetric Liouville theory and the multi-cut matrix models," Nucl. Phys. B **819** (2009) 351 [arXiv:0902.1676 [hep-th]].

[43] C. T. Chan, H. Irie and C. H. Yeh, "Fractional-superstring amplitudes, multi-cut matrix models and non-critical M theory," Nucl. Phys. B **838** (2010) 75 [arXiv:1003.1626 [hep-th]].

[44] P. Horava and C. A. Keeler, "Noncritical M-theory in 2+1 dimensions as a nonrelativistic Fermi liquid," JHEP **0707** (2007) 059 [arXiv:hep-th/0508024].

[45] C. T. Chan, H. Irie, S. Y. Shih and C. H. Yeh, "Macroscopic loop amplitudes in the multi-cut two-matrix models," Nucl. Phys. B **828** (2010) 536 [arXiv:0909.1197 [hep-th]].

[46] C. Kristjansen, "Multiloop correlators for rational theories of 2-d gravity from the generalized Kontsevich models," Nucl. Phys. B **436** (1995) 342 [arXiv:hep-th/9409066].

[47] M. Anazawa, A. Ishikawa and H. Itoyama, "Universal annulus amplitude from the two matrix model," Phys. Rev. D **52** (1995) 6016 [arXiv:hep-th/9410015].

[48] M. Anazawa, A. Ishikawa and H. Itoyama, "Macroscopic three loop amplitudes from the two matrix model," Phys. Lett. B **362** (1995) 59 [arXiv:hep-th/9508009].

[49] M. Anazawa and H. Itoyama, "Macroscopic n-Loop Amplitude for Minimal Models Coupled to Two-Dimensional Gravity: Fusion Rules and Interactions," Nucl. Phys. B **471** (1996) 334 [arXiv:hep-th/9511220].

[50] M. Fukuma, H. Irie and Y. Matsuo, "Notes on the algebraic curves in (p,q) minimal string theory," JHEP **0609** (2006) 075 [arXiv:hep-th/0602274].

[51] J. M. Maldacena, G. W. Moore, N. Seiberg and D. Shih, "Exact vs. semiclassical target space of the minimal string," JHEP **0410** (2004) 020 [arXiv:hep-th/0408039].

[52] N. Seiberg and D. Shih, "Flux vacua and branes of the minimal superstring," JHEP **0501** (2005) 055 [arXiv:hep-th/0412315].

[53] B. Eynard, "Topological expansion for the 1-hermitian matrix model correlation functions," JHEP **0411** (2004) 031 [arXiv:hep-th/0407261];
B. Eynard and N. Orantin, "Invariants of algebraic curves and topological expansion," arXiv:math-ph/0702045; "Geometrical interpretation of the topological recursion, and integrable string theories," arXiv:0911.5096 [math-ph].

[54] G. Bonnet, F. David and B. Eynard, "Breakdown of universality in multi-cut matrix models," J. Phys. A **33** (2000) 6739 [arXiv:cond-mat/0003324].

[55] J. McGreevy and H. L. Verlinde, "Strings from tachyons: The c = 1 matrix reloaded," JHEP **0312** (2003) 054 [arXiv:hep-th/0304224].

[56] E. J. Martinec, "The annular report on non-critical string theory," arXiv:hep-th/0305148.

[57] I. R. Klebanov, J. M. Maldacena and N. Seiberg, "D-brane decay in two-dimensional string theory," JHEP **0307** (2003) 045 [arXiv:hep-th/0305159].

[58] S. Y. Alexandrov, V. A. Kazakov and D. Kutasov, "Non-perturbative effects in matrix models and D-branes," JHEP **0309** (2003) 057 [arXiv:hep-th/0306177].

[59] V. A. Kazakov and I. K. Kostov, "Instantons in non-critical strings from the two-matrix model," arXiv:hep-th/0403152.

[60] M. Hanada, M. Hayakawa, N. Ishibashi, H. Kawai, T. Kuroki, Y. Matsuo and T. Tada, "Loops versus matrices: The nonperturbative aspects of noncritical string," Prog. Theor. Phys. **112** (2004) 131 [arXiv:hep-th/0405076].

[61] D. Kutasov, K. Okuyama, J. w. Park, N. Seiberg and D. Shih, "Annulus amplitudes and ZZ branes in minimal string theory," JHEP **0408** (2004) 026 [arXiv:hep-th/0406030].

[62] H. Kawai, T. Kuroki and Y. Matsuo, "Universality of nonperturbative effect in type 0 string theory," Nucl. Phys. B **711** (2005) 253 [arXiv:hep-th/0412004].

[63] R. de Mello Koch, A. Jevicki and J. P. Rodrigues, "Instantons in c = 0 CSFT," JHEP **0504** (2005) 011 [arXiv:hep-th/0412319].

[64] A. Sato and A. Tsuchiya, "ZZ brane amplitudes from matrix models," JHEP **0502** (2005) 032 [arXiv:hep-th/0412201].

[65] N. Ishibashi and A. Yamaguchi, "On the chemical potential of D-instantons in c = 0 noncritical string theory," JHEP **0506** (2005) 082 [arXiv:hep-th/0503199];
N. Ishibashi, T. Kuroki and A. Yamaguchi, "Universality of nonperturbative effects in c < 1 noncritical string theory," JHEP **0509** (2005) 043 [arXiv:hep-th/0507263].

[66] M. Fukuma, H. Irie and S. Seki, "Comments on the D-instanton calculus in $(p, p+1)$ minimal string theory," Nucl. Phys. B **728** (2005) 67 [arXiv:hep-th/0505253].

[67] Y. Matsuo, "Nonperturbative effect in c = 1 noncritical string theory and Penner model," Nucl. Phys. B **740** (2006) 222 [arXiv:hep-th/0512176].

[68] M. Fukuma and H. Irie, "A string field theoretical description of (p, q) minimal superstrings," JHEP **0701** (2007) 037 [arXiv:hep-th/0611045].

[69] T. Kuroki and F. Sugino, "T-duality of ZZ-brane," arXiv:hep-th/0612042; "T duality of the Zamolodchikov-Zamolodchikov brane," Phys. Rev. D **75** (2007) 044008.

[70] M. Fukuma and H. Irie, "Supermatrix models and multi ZZ-brane partition functions in minimal superstring theories," JHEP **0703** (2007) 101 [arXiv:hep-th/0701031].

[71] M. Marino, R. Schiappa and M. Weiss, "Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings," arXiv:0711.1954 [hep-th].

[72] M. Marino, "Nonperturbative effects and nonperturbative definitions in matrix models and topological strings," JHEP **0812** (2008) 114 [arXiv:0805.3033 [hep-th]].

[73] M. Marino, R. Schiappa and M. Weiss, "Multi-Instantons and Multi-Cuts," J. Math. Phys. **50** (2009) 052301 [arXiv:0809.2619 [hep-th]].

[74] B. Eynard and M. Marino, "A holomorphic and background independent partition function for matrix models and topological strings," arXiv:0810.4273 [hep-th].

[75] S. Pasquetti and R. Schiappa, "Borel and Stokes Nonperturbative Phenomena in Topological String Theory and c=1 Matrix Models," Annales Henri Poincare **11** (2010) 351 [arXiv:0907.4082 [hep-th]].

[76] S. Garoufalidis, A. Its, A. Kapaev and M. Marino, "Asymptotics of the instantons of Painleve I," arXiv:1002.3634 [math.CA].

[77] A. Klemm, M. Marino and M. Rauch, "Direct Integration and Non-Perturbative Effects in Matrix Models," JHEP **1010** (2010) 004 [arXiv:1002.3846 [hep-th]].

[78] N. Seiberg and S. H. Shenker, "A Note on background (in)dependence," Phys. Rev. D **45** (1992) 4581 [arXiv:hep-th/9201017].

[79] S. H. Shenker, "The Strength of nonperturbative effects in string theory," in Proceedings, Random Surfaces and Quantum Gravity, Cargese, 1990, pp. 191-200.

[80] J. Polchinski, "Combinatorics of boundaries in string theory," Phys. Rev. D **50** (1994) 6041 [arXiv:hep-th/9407031].

[81] H. Flaschka and A. C. Newell, "Monodromy And Spectrum Preserving Deformations. 1," Commun. Math. Phys. **76** (1980) 65;
M. Jimbo, T. Miwa and a. K. Ueno, "Monodromy Preserving Deformations Of Linear Differential Equations With Rational Coefficients. 1," Physica D **2** (1981) 407.

[82] A. S. Fokas and M. J. Ablowitz, "On the initial value problem of the second Painlevé transcendent," Comm. Math. Phys. **91** No. 3 (1983) 381;
A. S. Fokas and M. J. Ablowitz, "Linearization Of The Korteweg-De Vries And Painleve Ii Equations," Phys. Rev. Lett. **47** (1981) 1096;
A. S. Fokas and X. Zhou, "On the solvability of Painleve II and IV," Commun. Math. Phys. **144** (1992) 601.

[83] P. A. Deift and X. Zhou, "A Steepest Descent Method for Oscillatory Riemann–Hilbert Problems. Asymptotics for the MKdV Equation," Ann. of Math. **137** (1993) 295;
P. A. Deift and X. Zhou, "Asymptotics for the painleve II equation," Comm. Pure Appl. Math. **48** (1995) 277.

[84] S. P. Hastings and J. B. McLeod, "A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation," Arch. rational Mech. Anal. **73** (1980) 31-51.

[85] A. S. Fokas, A. R. Its, A. A. Kapaev and V. Y. Novokshenov, "Painlevé Transcendents: The Riemann-Hilbert Approach," Mathematical Surveys and Monographs, Vol. 128 (2006) 553 pp;
A. R. Its and V. Y. Novokshenov, "The Isomonodromic Deformation Method in the Theory of Painlevé Equations," Springer-Verlag (1986).

[86] A. A. Kapaev, “Global asymptotics of the second Painlevé transcendent,” Phys. Lett. A **167** (1992) 356; “The essential singularity of the Painlevé function of the second kind and nonlinear Stokes phenomenon,” Journal of Mathematical Science, Vol. 73 No. 4 (1995) 500.

[87] V. Yu. Novokshenov, “The Boutroux ansatz for the second Painlevé equation in the complex domain,” Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990) 1229.

[88] A. R. Its and A. A. Kapaev, “The nonlinear steepest descent approach to the asymptotics of the second Painlevé transcendent in the complex domain,” Math-Phys Odyssey 2001, 273-311, Prog. Math. Phys. 23 (2002) [nlin.SI/0108054].

[89] P. Bleher and A. Its, “Double scaling limit in the random matrix model: the Riemann-Hilbert approach,” arXiv:math-ph/0201003.

[90] M. Sato, RIMS Kokyuroku **439** (1981) 30;
E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations. 3. Operator approach to the Kadomtsev-Petviashvili equation,” RIMS-358;
M. Jimbo and T. Miwa, “Solitons and infinite dimensional Lie algebras,” Publ. Res. Inst. Math. Sci. Kyoto **19** (1983) 943;
V. G. Kac and J. W. van de Leur, “The n-component KP hierarchy and representation theory,” J. Math. Phys. **44** (2003) 3245 [arXiv:hep-th/9308137].

[91] V. E. Zakharov and A. B. Shabat, “Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media,” Sov. Phys. JETP **34** (1972) 62.

[92] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, “The inverse scattering transform-Fourier analysis for nonlinear problems,” Stud. Appl. Math. **53** (1974) 249.

[93] W. Wasow, “Asymptotic Expansions for Ordinary Differential Equations,” Interscience-Wiley, New York, 1985;
Y. Sibuya, “Linear differential equations in the complex domain: problems of analytic continuation,” Translations of mathematical monographs, **82**, Providence, 1990;
M. V. Fedoryuk, “Asymptotics. Integrals and series,” Moscow, Nauka, 1987.

[94] S. Dalley, C. V. Johnson and T. R. Morris, “Multicritical Complex Matrix Models And Nonperturbative 2-D Quantum Gravity,” Nucl. Phys. B **368** (1992) 625;
S. Dalley, C. V. Johnson, T. R. Morris and A. Watterstam, “Unitary matrix models and 2-D quantum gravity,” Mod. Phys. Lett. A **7** (1992) 2753 [arXiv:hep-th/9206060];
R. Lafrance and R. C. Myers, Mod. Phys. Lett. A **9** (1994) 101 [arXiv:hep-th/9308113];
P. Di Francesco, “Rectangular Matrix Models and Combinatorics of Colored Graphs,” Nucl. Phys. B **648** (2003) 461 [arXiv:cond-mat/0208037].

[95] M. L. Mehta, "A Method Of Integration Over Matrix Variables," Commun. Math. Phys. **79**, 327 (1981).

[96] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function," Commun. Math. Phys. **147** (1992) 1.

[97] M. C. N. Cheng, R. Dijkgraaf and C. Vafa, "Non-Perturbative Topological Strings And Conformal Blocks," arXiv:1010.4573 [hep-th].