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# $E_{11}$ and Supersymmetry

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We introduce fermions into the $E_{11}$ non-linear realisation. We show, at low levels, that the commutators of the Cartan involution invariant subalgebra of $E_{11}$ with the known supersymmetry transformations of eleven dimensional supergravity lead to symmetries of the theory indicating the consistency of supersymmetry and $E_{11}$.

## 1. Introduction

The first paper which conjectured $E_{11}$ symmetry [1] only considered the bosonic sectors of the theories under consideration and the same is true for all subsequent papers. However, there have been a number of results which follow from $E_{11}$ symmetry which have traditionally been, or have subsequently been, shown to follow from supersymmetry. Two such examples are the two and five form central charges in the $l_1$ representation of $E_{11}$, which is conjectured to contain all brane charges, [2] and the representations carried by form fields which imply the classification of gauged supergravities [3,4]. A brief account of some of the evidence for an underlying $E_{11}$ symmetry of strings and branes is summarised in the first seven pages of [5].

The dimensionally reduced maximal supergravities contain non-linear realisations that encode the scalar fields. In fact this statement is true for all supergravity theories which possess scalars in their supergravity multiplet. The prototype example is the maximal supergravity theory in four dimensions which possesses an $E_7$ non-linear realisation with local subgroup SU(8) [6]. The other fields in the supergravity multiplet transform as matter representations of the non-linear realisation, that is under the local subgroup. This includes the fermions.

The local subalgebra adopted in the non-linear realisation of $E_{11}$ is the Cartan involution invariant subalgebra denoted $K(E_{11})$. The commutation relations of this latter algebra were given at low levels in [2] and it was found that the generators could be represented at low levels by the eleven dimensional $\gamma$-matrices. As such $K(E_{11})$ possesses, at low levels, a 32 component spinor representation that might be used as the supersymmetry parameter [2].

It has also been proposed [7] that non-linearly realised $E_{10}$ is a symmetry of maximal eleven dimensional supergravity. This is a subalgebra of $E_{11}$, but it differs from the earlier proposal [1] in the way it incorporates space-time; the fields are taken to depend only on time and the spatial derivatives of the fields are proposed to occur at higher level in $E_{10}$. Fermions have been incorporated in the $E_{10}$ non-linear realisation [8-11]. Following the pattern found in supergravity theories in lower dimensions these authors took the fermions to belong to linear representations of $K(E_{10})$. They found that there exists at low levels an unfaithful representation which is a vector spinor of the ten dimensional Lorentz group which can be identified with the gravitino. The previously found [2] thirty two component unfaithful representation, which is a spinor of the ten dimensional Lorentz group, was used as the supersymmetry parameter.

In this paper we follow a similar path to incorporate the fermions into the $E_{11}$ non-linear realisation. The contents of this paper are as follows. In section two we summarise the algebra of $K(E_{11})$, in section three we compute the $K(E_{11})$ transformations of the fields at low levels, in section four we will find a unfaithful representation of $K(E_{11})$ which can be identified with the gravitino, in section five we compute the commutators of the low level fields between the known supersymmetry transformations and their previously found $K(E_{11})$ transformations and show that they are consistent in that they lead to known symmetries of the theory.

As very briefly indicated in reference [11] some calculations incorporating fermions, which are unpublished, have been carried out in the $E_{11}$ context by these authors.

## 2. The Cartan Involution invariant subgroup of $E_{11}$

In this section we summarise the commutation relations of $E_{11}$ generators and those of the Cartan involution invariant subgroup of $E_{11}$, denoted $K(E_{11})$ . The Dynkin diagram of $E_{11}$ is given by

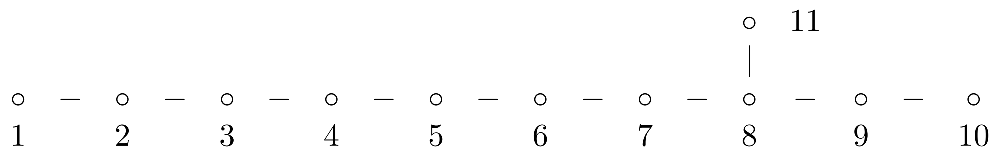


Figure 1. The Dynkin diagram of $E_{11}$

Deleting node eleven we find the algebra GL(11), which corresponds in the non-linear realisation to eleven dimensional gravity. As such it is natural to decompose the adjoint representation of $E_{11}$ in terms of GL(11) which consists of SL(11) and the remaining generator of the Cartan subalgebra. We denoted these generators by $K^a{}_b,\ a,b=1,\ldots,11$ and they obey the commutators

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \tag{2.1}$$

All generators of a Kac-Moody algebra are formed from multiple commutators of the Chevalley generators. The level of a generator is defined to be the number of times the Chevalley generator $E_{11}$ (not to be mistaken with the symbol for the algebra itself) occurs for positive root generators, or minus the number of times $F_{11}$ appears for negative root generators. The results of the decomposition can be classified by this level [12,7]. The positive root generators at level one and two respectively are given by[1]

$$R^{a_1\ldots a_3}, R^{a_1\ldots a_6} \tag{2.2}$$

while

$$R_{a_1\ldots a_3}, R_{a_1\ldots a_6} \tag{2.3}$$

are the negative root generators at levels -1 and -2 respectively.

The commutation relations of the positive root generators with GL(11) are

$$[K^b{}_c, R^{a_1\ldots a_3}] = 3\delta^{[a_1}_c R^{a_2a_3]b},\ \ [K^b{}_c, R^{a_1\ldots a_6}] = 6\delta^{[a_1}_c R^{a_2\ldots a_6]b} \tag{2.4}$$

While the commutators of the negative root generator with those of GL(11) are given by

$$[K^b{}_c, R_{a_1a_2a_3}] = -3\delta^b_{[a_1} R_{a_2a_3c}, \quad [K^b{}_c, R_{a_1\ldots a_6}] = -6\delta^b_{[a_1} R_{a_2\ldots a_6c} \tag{2.5}$$

Generators at level two, or minus two, can be found as the commutator of two level one, or minus one, generators.

$$[R^{a_1\ldots a_3}, R^{a_4\ldots a_6}] = 2R^{a_1\ldots a_6}, \quad [R_{a_1\ldots a_3}, R_{a_4\ldots a_6}] = 2R_{a_1\ldots a_6} \tag{2.6}$$

In these equations we have chosen the normalisation of these generators.

Finally, the commutators between the positive and negative root generators at levels one and two are [1]

$$
\begin{array}{rcl}
[R^{a_1\ldots a_3}, R_{b_1\ldots b_3}] &=& 18\delta^{[a_1a_2}_{[b_1b_2} K^{a_3]}{}_{b_3]} - 2\delta^{a_1\ldots a_3}_{b_1\ldots b_3}\left(\sum_b K^b{}_b\right) \\
[R^{a_1\ldots a_3}, R_{b_1\ldots b_6}] &=& \frac{5!}{2}\delta^{a_1\ldots a_3}_{[b_1\ldots b_3}R_{b_4\ldots b_6]} \\
[R^{a_1\ldots a_6}, R_{b_1\ldots b_6}] &=& -5!\left(9\delta^{[a_1\ldots a_5}_{[b_1\ldots b_5}K^{a_6]}{}_{b_6]} - \delta^{[a_1\ldots a_6]}_{[b_1\ldots b_6]}\sum_c K^c{}_c\right) \\
[R_{b_1\ldots b_3}, R^{a_1\ldots a_6}] &=& \frac{5!}{2}\delta^{[a_1\ldots a_3}_{b_1\ldots b_3}R^{a_4\ldots a_6]}
\end{array}
\tag{2.6}
$$

The Cartan involution of a Lie algebra is defined on the Chevalley generators as

$$E_a \to -F_a, F_a \to -E_a, H_a \to -H_a \tag{2.7}$$

The effect on the generators used above is

$$K^a{}_b \to -K^b{}_a, R^{a_1\ldots a_3} \to -R_{a_1\ldots a_3}, R^{a_1\ldots a_6} \to R_{a_1\ldots a_6} \tag{2.8}$$

The Cartan involution invariant subalgebra $K(E_{11})$ is generated by the invariant combination of the Chevalley generators given by

$$S_a = E_a - F_a \tag{2.9}$$

A basis for the Cartan involution invariant subalgebra K($E_{11}$) is given, up to and including level 2, by [2]

$$
\begin{array}{rcl}
J^{ab} &=& K^a{}_c\eta^{cb} - K^b{}_c\eta^{ca} \\
S_{a_1\ldots a_3} &=& R^{b_1\ldots b_3}\eta_{b_1a_1}\ldots\eta_{b_3a_3} - R_{a_1\ldots a_3} \\
S_{a_1\ldots a_6} &=& R^{b_1\ldots b_6}\eta_{b_1a_1}\ldots\eta_{b_6a_6} + R_{a_1\ldots a_6}
\end{array}
\tag{2.10}
$$

The $J^{ab}$s generate the Lorentz algebra. Rather than introduce signs using the Minkowski metric $\eta_{ab}$, which ensures that we have the Lorentz group SO(1,10) rather than the group SO(11). We could also adopt the modified Cartan involution given by $E_1 \to F_1,\ F_1 \to E_1$ and $H_1 \to -H_1$ for which the above generators are the involution invariant algebra.

The commutators between the generators of K($E_{11}$) are given by[2]

$$
\begin{array}{c}
[J^{ab}, J^{cd}] = \eta^{bd}J^{ac} + \eta^{ac}J^{bd} - \eta^{bc}J^{ad} - \eta^{ad}J^{bc} \\
[S^{a_1\ldots a_3}, S_{b_1\ldots b_3}] = 2S^{a_1\ldots a_3}{}_{a_4\ldots a_6} - 18\delta^{[a_1a_2}_{[b_1b_2}J^{a_3]}{}_{b_3]} \\
[S_{a_1\ldots a_3}, S^{b_1\ldots b_6}] = -3S^{b_1\ldots b_6}{}_{[a_1a_2,a_3]} - \frac{5!}{2}\delta^{[b_1\ldots b_3}_{[a_1\ldots a_3]}S^{b_4\ldots b_6]}
\end{array}
\tag{2.11}
$$

### 3. The action of K($E_{11}$) on the bosonic fields

In this section we calculate the transformations of the bosonic fields under rigid K($E_{11}$). By definition, the group element from which the nonlinear realisation is constructed transforms under rigid transformations as

$$g \to g_0 g, \qquad g \to gh \tag{3.1}$$

where $g_0 \in E_{11}$ is a rigid, i.e. constant, transformation, but $h \in K(E_{11})$ is a local transformation.

Using the analogue of the Iwasawa decomposition we may write the general group element of $E_{11}$ as

$$g = e^{h^a H_a} e^{\sum_\alpha A^\alpha E_\alpha} e^{\sum_\beta B^\beta S_\beta} \tag{3.2}$$

Where the sums over $\alpha$ and $\beta$ run over all positive roots, and $S_\beta$ denotes an element of $K(E_{11})$. This group element is of the form of an element of the Borel subalgebra multiplied by a Cartan involution invariant group element. Using the local symmetry we can choose the group element to be of the form

$$g = e^{h_a{}^b K^a{}_b} e^{A_{a_1\ldots a_3} R^{a_1\ldots a_3}} e^{A_{a_1\ldots a_6} R^{a_1\ldots a_6}} \ldots \tag{3.3}$$

Thus we choose our coset representatives.We note that we did not use the local Lorentz group part of $K(E_{11})$ to choose the $h_a{}^b$ to be symmetric.

Carrying out a rigid Borel transformation takes us from one coset representative to another, so we can immediately read off the transformation of the fields. However, this is not the case for a general $E_{11}$ transformation and one has to perform an additional compensating local $K(E_{11})$ transformation to bring the group element back to being one of the coset representatives.

We must also include in the group element a part associated with space-time, that is a factor $e^{x^a P_a}$. In principal we should add further generators associated to the generalised space-time introduced in [2] corresponding to the non-linear realisation of $E_{11} \otimes_s l_1$, but these are likely to lead to higher order effects than those being considered in this paper. As such we take all $E_{11}$ generators except those of GL(11) to commute with $P_a$.

We now consider the rigid transformation

$$g_0 = e^{c^{a_1\ldots a_3} S_{a_1\ldots a_3}} \tag{3.4}$$

All other $K(E_{11})$ transformations can be found from this one by taking commutators. As a result we find at lowest order in the transformation parameter

$$\begin{aligned}
\delta_{c^3} e_\mu{}^a &= 2c^{\nu\rho\lambda}\left(9A_{\mu\nu\rho}e_\lambda{}^a - A_{\nu\rho\lambda}e_\mu{}^a\right)\\
\delta_{c^3} A_{a_1\ldots a_3} &= -3A_{a_2 a_3 b}\left(9c^{\mu\nu\rho}e_\mu{}^c e_\nu{}^d e_\rho{}^b A_{cda_1} - c^{\mu\nu\rho}e_\mu{}^c e_\nu{}^d e_\rho{}^e A_{cde}\delta^b_{a_1}\right)\\
&\quad +60A_{a_1\ldots a_3 cde}c^{\mu\nu\rho}e_\mu{}^c e_\nu{}^d e_\rho{}^e + c_{\mu\nu\rho}e_{a_1}{}^\mu e_{a_2}{}^\nu e_{a_3}{}^\rho - c^{\mu\nu\rho}e_{\mu a_1}e_{\nu a_2}e_{\rho a_3}
\end{aligned} \tag{3.5}$$

The quantity $c^3 \equiv c^{a_1 a_2 a_3}$ is the same constant regardless of whether it carries flat or curved indices. In other words we do not use the vielbein to convert the flat indices to the curved indices on $c^3$, but rather show explicitly the vielbein factors that are present. In other words, $c^{\mu\nu\rho} = \delta^\mu_a \delta^\nu_b \delta^\rho_c c^{abc}$ is also a constant.

To find the above result one must first move the $g_0$ of equation (3.4) past the $e^{h\cdot k}$ factor in the group element $g$ of equation (3.3) using the equation

$$g_0 e^{h\cdot k} = e^{h\cdot k} e^{-h\cdot k} e^{c^{a_1 a_2 a_3} S_{a_1 a_2 a_3}} e^{h\cdot k}$$

$$= e^{h\cdot k} e^{c_{\mu_1\mu_2\mu_3} e_{a_1}{}^{\mu_1} e_{a_2}{}^{\mu_2} e_{a_3}{}^{\mu_3} R^{a_1 a_2 a_3}} e^{c^{\mu_1\mu_2\mu_3} e_{\mu_1}{}^{a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3} R_{a_1 a_2 a_3}} \tag{3.6}$$

The presence of the vielbeins in equation (3.6) is explained in the appendix. Moving the expression in equation (3.6) after the $e^{h\cdot k}$ factor past the next factor in the group element $g$ , namely $e^{A_{a_1a_2a_3}R^{a_1a_2a_3}}$ the $e^{c^{\mu_1\ldots\mu_3}e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2}e_{\mu_3}{}^{a_3}R_{a_1\ldots a_3}}$ term creates a GL(11) transformation that must be reordered in the group element. Similar considerations apply to the passage of $e^{c^{\mu_1\ldots\mu_3}e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2}e_{\mu_3}{}^{a_3}R_{a_1\ldots a_3}}$ past the factor containing the six form field. Finally, one can recognise $e^{c^{\mu_1\ldots\mu_3}e_{\mu_1}{}^{a_1}e_{\mu_2}{}^{a_2}e_{\mu_3}{}^{a_3}R_{a_1\ldots a_3}}$ as part of the compensating local transformation

$$h = e^{c^{\mu\nu\rho}e_\mu{}^a e_\nu{}^b e_\rho{}^c S_{abc}} \tag{3.7}$$

We note that this contains a term $e^{c^{\mu\nu\rho}e_{\mu a}e_{\nu b}e_{\rho c}R^{abc}}$ which must be reabsorbed into the change in the three form field together with the similar term that arises from the passage of the factor $e^{c_{\mu_1\mu_2\mu_3}e_{a_1}{}^{\mu_1}e_{a_2}{}^{\mu_2}e_{a_3}{}^{\mu_3}R^{a_1a_2a_3}}$ in equation (3.6).

To calculate the variation of the vielbein under $S_6$ we repeat this procedure with $g_0 = e^{c_{a_1\ldots a_6}S^{a_1\ldots a_6}}$ and a suitably chosen compensating local transformation. We find the result

$$\delta_{c^6} e_\mu{}^a = 5! c^{\nu_1\ldots\nu_6} A_{\nu_1\ldots\nu_6} e_\mu{}^a - 5!9 c^{\nu_1\ldots\nu_6} A_{\nu_1\ldots\nu_5\mu} e_{\nu_6}{}^a - \frac{5!9}{2} c^{\nu_1\ldots\nu_6} A_{\nu_1\ldots\nu_3} A_{\nu_4\nu_5\mu} e_{\nu_6}{}^a \tag{3.8}$$

Finally we write down the effect of a rigid Lorentz transformation on the vielbein in this formalism so as to fix the normalisation. That is we take $g_0 = e^{c_{ab}J^{ab}}$ and process it as in equation (3.6) to find a local transformation. The result is

$$\delta_{c^2} e_\mu{}^a = 2 c_\mu{}^\nu e_\nu{}^a \tag{3.9}$$

where the index is raised with a constant GL(11,R) metric.

## 4. Spinorial representations of K($E_{11}$)

In this paper we wish to include fermions in the $E_{11}$ non-linear realisation. As we have already mentioned the prototypical example is the maximal supergravity in four dimensions which has an $E_7$ symmetry [6]. In this theory, and indeed all supergravity theories in which the scalars are part of the supergravity multiplet, the spinors appear in the nonlinear realisation as matter representations. The matter representations transform as a linear representation of the chosen local subalgebra, which is SU(8) in the example just considered. This isthe Cartan involution invariant subalgebra and so the maximal compact subgroup of $E_7$. We note that once one has chosen a coset representative, one must in general carry out compensating local transformations, which act on matter representations.

Spinors have already been introduced in the $E_{10}$ approach [8-10] where they also took the spinors to transform under the Cartan involution invariant subalgebra. To construct the representation of K($E_{10}$) appropriate to the gravitino, these authors started with the vector spinor representation of SO(10) and introduced a transformation for $S_3$, up to level three, that satisfied the known commutation relations for the $K(E_{10})$. It turned out that it was enough at low levels to introduce only the gravitino field and so the representation found was highly unfaithful.

These techniques also apply to $E_{11}$ and we also take the gravitino to be a matter representation. We start with the standard Lorentz transformation of the gravitino SO(10,1) with a tangent space vector index;

$$J_{ab}\psi_c = -\frac{1}{2}\gamma_{ab}\psi_c - 2\eta_{c[a}\psi_{b]} \tag{4.1}$$

To find a suitable transformation of the vector spinor under $S_3$ we write down all possible terms with the correct SO(1,10) character and demand that it obey the algebra given in equation (2.11) involving the $S_3$ generator. In particular from the commutator between two $S^3$ generators in equation (2.11), one derives the following two relations

$$[S^{abc}, S_{ade}] = 0, \quad [S^{abc}, S_{abd}] = -J^c{}_d \tag{4.2}$$

Where $a, b, c, d, e$ are distinct indices.The second relation relates the $S_3$ transformation back to the known SO(1,10) transformation of equation (4.1). Given the $S_3$ transformation we can find all higher level $K(E_{11})$ transformations by taking repeated commutators. We find that the transformations of the vector spinor, that is the gravitino, up to level two, are given by

$$\begin{aligned} S_{abc}\psi_d &= \tfrac{1}{2}\gamma_{abc}\psi_d - \gamma_{d[ab}\psi_{c]} + 4\eta_{d[a}\gamma_b\psi_{c]} \\ S_{abcdef}\psi_g &= -\tfrac{1}{4}\gamma_{abcdef}\psi_g - 2\gamma_{g[abcde}\psi_{f]} + 5\eta_{g[a}\gamma_{bcde}\psi_{f]} \end{aligned} \tag{4.3}$$

One can repeat this procedure, starting with the spin 1/2 representation of SO(10,1) and recover the result [2]

$$J_{ab}\psi = -\frac{1}{2}\gamma_{ab}\psi \quad S_{abc}\psi = \frac{1}{2}\gamma_{abc}\psi \quad S_{abcdef}\psi = -\frac{1}{4}\gamma_{abcdef}\psi \tag{4.4}$$

### 5. Commutator of K($E_{11}$) and Supersymmetry

In this section we will calculate the commutator of the supersymmetry variations and the $K(E_{11})$ transformations on the vielbein and the three form. For our supersymmetry variations we take the well known transformations from eleven dimensional supergravity. We will find that the commutators result in symmetries of the theory and so demonstrate the consistency of $E_{11}$ with supersymmetry at least at low levels. This is far from guaranteed as $E_{11}$ has so far been based entirely on the bosonic fields. We take the supersymmetry transformations of the vielbein, the three form, and its dual, the six form with the Grassmann parameter $\epsilon_\alpha$ to be [13]

$$\begin{aligned} \delta_\epsilon e_\mu{}^a &= \bar{\epsilon}\gamma^a\psi_\mu \\ \delta_\epsilon A_{\mu\nu\rho} &= \tfrac{1}{2}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} \\ \delta_\epsilon A_{\mu_1\ldots\mu_6} &= -\tfrac{1}{60}\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} + \tfrac{1}{2}\bar{\epsilon}\gamma_{[\mu_1\mu_2}\psi_{\mu_3}A_{\mu_4\ldots\mu_6]} \end{aligned} \tag{5.1}$$

We note that the normalisation of the fields was already determined by their appearance in the $E_{11}$ group element of equation (3.3) and those chosen in equation (5.1) are the ones compatible with this previous choice.

One finds that the commutator of the variation of $Q_\alpha$ and $S_3$ on the vielbein is given by

$$[\bar{\epsilon}Q, c^3 \cdot S_3]e_\mu{}^a = \frac{1}{2}c^{bcd}\bar{\epsilon}\gamma_{bcd}\gamma^a\psi_\mu - 4c_\mu{}^{ad}\bar{\epsilon}\psi_d - 4c_\mu{}^{cd}\bar{\epsilon}\gamma^a{}_c\psi_d + 4c^{acd}\bar{\epsilon}\gamma_{\mu c}\psi_d + c^{bcd}\bar{\epsilon}\gamma^a{}_{\mu bc}\psi_d \tag{5.2}$$

When carrying out this calculation it is important to remember that the K($E_{11}$) transformation of the gravitino discussed in section three was defined in the tangent frame, however the gravitino in the supersymmetry transformations has a curved index, so when considering the K($E_{11}$) variation of the gravitino, we must include the vielbein required to convert a flat to a curved index, that is $\psi_\mu = e_\mu{}^a\psi_a$. The same applies to the threeform which we must write as $A_{\mu\nu\rho} = e_\mu{}^a e_\nu{}^b e_\rho{}^c A_{abc}$.

From equation (5.2) we extract the generic form of the commutator

$$[Q, S_{bcd}] = \frac{1}{2}\gamma_{abc}Q + \left(\frac{1}{2}\gamma^f{}_{ebc} + 2\eta_{eb}\delta^f_c - 2(\eta_{eb}\gamma^f{}_c - \delta^f_b\gamma_{ec})\right)\psi_d J_L{}^e{}_f \tag{5.3}$$

which we recognise as a supersymmetry transformation and a **local** Lorentz transformation denoted by the symbol $J_L{}^e{}_f$. On the metric, which is a Lorentz invariant object, the field dependent Lorentz transformations do not appear, and we are left with $[Q, S_{abc}] = \frac{i}{2}\gamma_{abc}Q$. This is expected, because the supercharge $Q$ is a spinor, which transforms as in equation (4.4).

We note that the commutator (5.3) is field dependent. This is a well known phenomenon that occurs in the commutator of supersymmetry and gauge transformations when some of the fields have been set to zero using the supermultiplet of gauge symmetries, the prototype example is to fix the Wess-Zumino gauge in supersymmetric Yang-Mills theory; for a review see [14]. It is to be expected here as we have used a local symmetry, that is the $K(E_{11})$, to gauge away the non-Borel part of the group element.

A similar calculation on the threeform field gives

$$[\bar{\epsilon}Q, c^3 S_3]A_{\mu\nu\rho} = \frac{1}{4}c^{abc}\bar{\epsilon}\gamma_{abc}\gamma_{\mu\nu}\psi_\rho \tag{5.4}$$

The spacetime threeform is a Lorentz invariant object, and one does not expect to see the field dependent terms of equation (5.3). Thus one finds that the generic commutator of a supersymmetry transformation with a rigid $S^{a_1a_2a_3}$ transformation is, up to level two, of the form

$$[Q, S_{abc}] = \frac{1}{2}\gamma_{abc}Q \tag{5.5}$$

plus local transformations.

The commutator of supersymmetry and $S_6$ on the vielbein is given by

$$\begin{aligned}[\bar{\epsilon}Q, c^6 S_6]e_\mu{}^a = \; & +\tfrac{1}{4}c^{\nu_1\ldots\nu_6}\bar{\epsilon}\gamma_{\nu_1\ldots\nu_6}\gamma^a\psi_\mu \\ & -\tfrac{5!}{2}c^{\nu_1\ldots\nu_6}\bar{\epsilon}\gamma_{\nu_1\nu_2}\psi_{\nu_3}(9A_{\mu\nu_4\nu_5}e_{\nu_6}{}^a - A_{\nu_4\ldots\nu_6}e_\mu{}^a) \\ & +2c^{\nu_1\ldots\nu_6}\bar{\epsilon}\gamma^a{}_{\mu\nu_1\ldots\nu_5}\psi_{\nu_6} - 20c_\mu{}^{a\nu_1\ldots\nu_4}\bar{\epsilon}\gamma_{\nu_1\ldots\nu_3}\psi_{\nu_4} \\ & -5\left(c_\mu{}^{\nu_1\ldots\nu_5}\bar{\epsilon}\gamma^a{}_{\nu_1\ldots\nu_4}\psi_{\nu_5} - c^{a\nu_1\ldots\nu_5}\bar{\epsilon}\gamma_{\mu\nu_1\ldots\nu_4}\psi_{\nu_5}\right)\end{aligned} \tag{5.6}$$

These variations lead to the commutator relation

$$[Q, S_{a\ldots f}] = \frac{1}{4}\gamma_{a\ldots f}Q - 30\gamma_{ab}\psi_c S_{def} + \left(\tfrac{5}{2}(\delta_{fj}\gamma^k{}_{a\ldots d}\psi_e - \delta^k_f\gamma_{ja\ldots d}\psi_e) + \gamma^k{}_{ja\ldots e}\psi_f - 10\delta_{ej}\delta^b_f\gamma_{a\ldots c}\psi_d\right)\psi_f J_L{}^j{}_k \tag{5.7}$$

where the right hand side is understood to be antisymmetrised over the indices $abcdef$. Thus we may write the commutator as

$$[Q, S_{abcdef}] = \frac{1}{4}\gamma_{abcdef}Q \tag{5.8}$$

plus local transformations. We note that equations (5.5) and (5.8) are compatible with regarding the supercharge as a spinor which we found to transform as in equation (4.4). The commutators of the Cartan involution subalgebra with the supersymmetry as anticipated in [2].

## Appendix A. Vielbeins in $E_{11}$

In the calculations given in this paper the vielbein plays an important role, and in this appendix we briefly discuss how the vielbein appears in the $E_{11}$ non-linear realisation. For this purpose we can take our group elementto contain just the part appropriate for gravity, namely

$$g = e^{x^a P_a} e^{h_a{}^b K^a{}_b} \ldots \tag{A.1}$$

where $\ldots$ indicates factors involving higher level fields. The most direct way to see the presence of the vielbein is to compute the Cartan form

$$\mathcal{V} = g^{-1}\partial_\mu g = dx^\mu e_\mu{}^a P_a + (e^{-1}\partial_\mu e)_a{}^b K^a{}_b + \ldots \tag{A.2}$$

where $e_\mu{}^a = (e^h)_\mu{}^a$. The Cartan forms transform under the local subalgebra $K(E_{11})$ as $\mathcal{V} \to h^{-1}\mathcal{V}h + h^{-1}\partial_\mu h$. At lowest level this is just the Lorentz group and so $e_\mu{}^a$ transforms on its upper $a$ index just like a vector under the Lorentz group while any reparameterisation of $x^\mu$ gives a corresponding change in the the lower $\mu$ index of $e_\mu{}^a$. Thus $e_\mu{}^a$ does transform as a vielbein should. Indeed constructing the theory of gravity from the non-linear realisation as was first done in [15], and again in a more vielbein orientated approach in [16], one finds that $e_\mu{}^a$ does indeed appear in the theory as the vielbein should.

Effectively, the above calculation of the vielbein evaluates $e^{h_a{}^b K^a{}_b}$ in the vector representation as this factor acts on $P_a$. In this representation

$$(K^a{}_b)_c{}^d = \delta^a_c \delta_b{}^d \tag{A.3}$$

where $c, d$ are the representation matrix indices clearly giving $(e^{h_a{}^b K^a{}_b})_c{}^d = e^{hd}_c$ in vector representation.

In the paper we encounter expressions where we move $e^{h_a{}^b K^a{}_b}$ past generators in representations of GL(11), for example equation (2.4). In particular we find that

$$e^{-h\cdot K} c^{a_1\ldots a_3} R_{a_1\ldots a_3} e^{h\cdot K} = c^{\mu\nu\rho} e_\mu{}^a e_\nu{}^b e_\rho{}^c R_{abc} \tag{A.4}$$

We recall that the parameter $c^3$ is the same constant no matter what indices it displays, but it is natural to write its indices so as to reflect what it is contracted with. We also give the analogous result for the positive root generators

$$e^{-h\cdot K} c_{a_1\ldots a_3} R^{a_1\ldots a_3} e^{h\cdot K} = c_{\mu\nu\rho} e_a{}^{\mu} e_b{}^{\nu} e_c{}^{\rho} R^{abc} \qquad (A.5)$$

which involves the inverse vielbeins $e_a{}^{\mu} = (e^{-h\cdot K})_a{}^{\mu}$.

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