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On classical string solutions in $\mathrm{AdS}_5 \times \mathrm{S}^5$

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Abstract

We discuss some new simple closed bosonic string solutions in $\mathrm{AdS}_5 \times \mathrm{S}^5$ that may be of interest in the context of AdS/CFT duality. In the first part of this work we consider solutions with two spins (S_1, S_2) in AdS_5. Starting from the flat-space solutions and using perturbation theory in the curvature of AdS_5 space, we construct leading terms in the small two-spin solution. We find corrections to the leading Regge term in the classical string energy and uncover a discontinuity in the spectrum for certain type of solutions. We then analyze the connection between small-spin and large-spin limits of string solutions in AdS_5. We show that the $S_1 = S_2$ solution in AdS_5 found in earlier papers admits both limits only in the simplest cases of the folded and rigid circular strings. In the second part of the paper we construct a new class of chiral solutions in $R_t \times S^5$ for which embedding coordinates of S^5 satisfy the linear Laplace equations. They generalize the previously studied rigid string solutions. We study in detail a simple non-trivial example.

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1 Introduction

Semiclassical string solutions is a useful tool for probing AdS/CFT correspondence [1, 2, 3, 4]. In the closed string sector, AdS energy of a closed string expressed in terms of spins[1] and string tension $T = \frac{\sqrt{\lambda}}{2\pi}$, i.e. $E(S_i, J_m; \lambda)$, gives the strong coupling limit of the scaling dimension of the corresponding gauge-theory operator (see, e.g., [5, 6]). Also, in the open string sector, solutions ending at the boundary of AdS_5 describe the strong coupling limit of the associated Wilson loops [7, 8].

In this paper we present some new classical solutions for a closed bosonic string in the $AdS_5 \times S^5$.

We shall first consider strings with two spins in AdS_5 part of $AdS_5 \times S^5$. A natural ansatz for describing a rigid "rotating" string solution is [4] ($0 \le \sigma < 2\pi$):

$$Y_0 + iY_5 = y_0(\sigma)\, e^{i\kappa\tau} , \qquad Y_1 + iY_2 = y_1(\sigma)\, e^{i\omega_1\tau} , \qquad Y_3 + iY_4 = y_2(\sigma)\, e^{i\omega_2\tau} , \\ \kappa,\ \omega_1,\ \omega_2 = \text{const} . \tag{1.1}$$

Here Y_P are embedding coordinates of $R^{2,4}$ with the metric $\eta_{PQ} = (-1, +1, +1, +1, +1, -1)$; $Y_P Y^P = -1$. A general approach to finding such rigid string solutions was developed in [9]. Using the reduction of the conformal-gauge string sigma model to the 1-d Neumann integrable model, one finds that the equations for $y_0,\ y_1,\ y_2$ are those of a harmonic oscillator constrained to move on a 2d hyperboloid — an integrable system with two integrals of motion b_1, b_2 with $b_1 + b_2 = \kappa^2 + \omega_1^2 + \omega_2^2$. In general, the solutions are expressed in terms of hyperelliptic functions and thus are not easy to analyze. There are few special cases when they simplify — when the hyperelliptic surface degenerates into an elliptic one and $y_a(\sigma)$ can be expressed in terms of the standard elliptic functions. Two of such cases, $\omega_1 = \omega_2$, corresponding to $S_1 = S_2$ solution, and its boosted analog with $\kappa = \omega_2 \neq \omega_1$ were studied in [10]. The existence of simple but more general solution with two unequal spins is an open question. Recent study of $\mathcal{N} = 4$ SYM states dual to minimal energy spinning string configuration with two spins (S_1, S_2) with $\frac{S_1}{S_2}$ fixed using the asymptotic Bethe ansatz (ABA) [11] suggests that such simple solution might indeed exist. In the large-spin limit the energy of the $S_1 = S_2$ solution [10] matched the strong-coupling ABA result of [11].

Aiming at a better understanding of two-spin solutions in AdS_5, here we first study the case of small strings or small-spin limit. Starting from the flat-space case, in which the general two-spin solutions are [9]

$$\kappa^2 = n_1^2 a_1^2 + n_2^2 a_2^2 \\ y_1^{\text{flat}} = a_1 \sin(n_1\sigma) \qquad y_2^{\text{flat}} = a_2 \sin[n_2(\sigma + \sigma_0)] \\ \omega_1 = n_1, \qquad \omega_2 = n_2, \qquad n_i = \text{integer} \tag{1.2}$$

and performing perturbation with respect to the curvature of AdS_5, we find the corrections to the flat-space expression for the classical energy $E(S_1, S_2; \lambda)$. We uncover a discontinuity in the spectrum of classical strings with equal and unequal winding numbers in Y_1Y_2 and Y_3Y_4 planes (n_1 and n_2). It may indicate that there are deep differences between solutions with $n_1 \neq n_2$ and more symmetrical ones with $n_1 = n_2$. We then investigate the connection between small- (flat-space) and large-spin limits of two-spin string solutions in AdS_5. In the particular cases of $\omega_1 = \omega_2$ and $\kappa = \omega_2 \neq \omega_1$ the general solutions in AdS_5 were found in [10]. It was discussed there, for $\kappa = \omega_2 \neq \omega_1$ case, that strings which admit large-spin limit do not have the small-spin one and vice-versa. For $\omega_1 = \omega_2$, we find that apart from the trivial cases of folded and circular strings, the general rigid solution with $S_1 = S_2$

[1] The generic states of bosonic string in $AdS_5 \times S^5$ may be labeled by the values of three $SO(2,4)$ Cartan generators (E, S_1, S_2) and three $SO(6)$ Cartan generators (J_1, J_2, J_3). We will be interested in "spinning" string solutions that have non-zero value of these charges.

in AdS_5 admitting the large-spin limit does not have a small-spin limit. For more general two-spin solutions, having both limits might still be possible.

In the second part of the paper we consider another simple class of string solutions — chiral solutions in $R_t \times S^5$. Such solutions obey an additional constraint

$$\partial_+ X_M \partial_- X_M = 0, \tag{1.3}$$

where X_M are embedding coordinates of R^6 with the Euclidean metric δ_{MN}; $X_M X_M = 1$ and $\partial_\pm = \frac{\partial}{\partial \sigma_\pm} = \frac{1}{2}\left(\frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \sigma}\right)$, $\sigma_\pm = \tau \pm \sigma$. Then the classical string equations in conformal gauge become

$$\partial_+ \partial_- X_M = 0. \tag{1.4}$$

The simplest solution of this kind is [12],

$$Y_0 + iY_5 = e^{i\kappa\tau}\ , \quad X_1 + iX_2 = a_1 e^{im_1\sigma_\pm}\ , \quad X_3 + iX_4 = a_2 e^{im_2\sigma_\pm}\ , \quad X_5 + iX_6 = a_3 e^{im_3\sigma_\mp}\ , \tag{1.5}$$

where $\sum_{i=1}^{3} a_i = 1$ and m_i are integers. It was recently used in [13] as a model of a quantum string state with "small" quantum numbers. We expect that more general chiral solutions may also find useful applications.

Here we consider the following ansatz

$$X_1 + iX_2 = a_1 e^{iF_1(\sigma_+)}, \quad X_3 + iX_4 = a_2 e^{iF_2(\sigma_+)}, \quad X_5 + iX_6 = a_3 e^{iF_3(\sigma_-)}, \tag{1.6}$$

and obtain the general solution for the functions $F_i(\sigma_\pm)$. A particular simple non-trivial case

$$F_1(\sigma_+) = \alpha \cos n\sigma_+\ , \qquad F_2(\sigma_+) = \alpha \sin n\sigma_+\ , \qquad F_3(\sigma_-) = m\sigma_- \tag{1.7}$$

we analyze in detail. It reduces to (1.6) in the limit $n \to 0$. Note that chiral solutions treat τ and σ on an equal footing, i.e. non-trivial dependence on τ implies that the shape of the string is not rigid, in general, so such solutions are similar to "pulsating" ones.

The rest of the paper is organized as follows. In section 2 we discuss basics of bosonic string solutions in $\mathrm{AdS}_5 \times \mathrm{S}^5$: action in the conformal gauge, equations of motion, etc. Section 3 is dedicated to small-string solutions in AdS_5. In section 4 we consider the relation between string solutions admitting small- and large-spin limits in AdS_5. In particular, we discuss small-spin limit of exact solutions with two equal spins. Section 5 is devoted to the chiral solutions in $R_t \times S^5$. In Appendix A we give an overview of circular and folded string solutions in AdS_5. In Appendix B curvature corrections to the folded string solution displaced from the center of AdS_5 are discussed. In Appendixes C and D we present technical details of calculation of spins for chiral solutions corresponding to (1.7).

2 Closed bosonic string in $\mathrm{AdS}_5 \times \mathrm{S}^5$

We will be interested in the classical bosonic solutions for closed string in $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$I_B = \frac{1}{2} T \int d\tau \int_0^{2\pi} d\sigma (L_{AdS} + L_S), \qquad T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \tag{2.1}$$

where

$$L_{AdS} = -\partial_a Y_P \partial^a Y^P - \tilde{\Lambda}(Y_P Y^P + 1), \qquad L_S = -\partial_a X_M \partial^a X_M + \Lambda(X_M X_M - 1). \tag{2.2}$$

Here $X_M,\ M = 1, ..., 6$ and $Y_P,\ P = 0, ..., 5$ are embedding coordinates of R^6 with the Euclidean metric δ_{MN} in L_S and of $R^{2,4}$ with $\eta_{PQ} = (-1, +1, +1, +1, +1, -1)$ in L_{AdS}, respectively ($Y_P = \eta_{PQ} Y^Q$). Λ and $\tilde{\Lambda}$ are the Lagrange multipliers imposing the two hypersurface conditions $Y_P Y^P = -1$ and $X_M X_M = 1$. The action (2.1) is to be supplemented with the conformal gauge constraints

$$\dot{Y}_P \dot{Y}^P + Y'_P Y'^P + \dot{X}_M \dot{X}_M + X'_M X'_M = 0, \qquad \dot{Y}_P Y'^P + \dot{X}_M X'_M = 0 \tag{2.3}$$

and the closed string periodicity conditions

$$Y_P(\tau, \sigma + 2\pi) = Y_P(\tau, \sigma), \qquad X_M(\tau, \sigma + 2\pi) = X_M(\tau, \sigma). \tag{2.4}$$

The classical equations of motion following from (2.1) are

$$\begin{aligned} &\partial^a \partial_a Y_P - \tilde{\Lambda} Y_P = 0, \quad \tilde{\Lambda} = \partial^a Y_P \partial_a Y^P, \quad Y_P Y^P = -1, \\ &\partial^a \partial_a X_M + \tilde{\Lambda} X_M = 0, \quad \Lambda = \partial^a X_M \partial_a X_M, \quad X_M X_M = 1. \end{aligned} \tag{2.5}$$

The action is invariant under the $SO(2,4)$ and $SO(6)$ rotations with correspondent conserved (on-shell) charges

$$S_{PQ} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (Y_P \dot{Y_Q} - Y_Q \dot{Y_P}), \qquad J_{MN} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (X_M \dot{X_N} - X_N \dot{X_M}) \ . \tag{2.6}$$

We are interested in finding "spinning" string solutions that have non-zero values of these charges.

It is useful to solve the constraints

$$Y_P Y^P = -1 \qquad X_M X_M = 1 \tag{2.7}$$

by choosing an explicit parametrization of the embedding coordinates Y_P and X_M, for example

$$\begin{aligned} &Y_{05} = Y_0 + iY_5 = \cosh\rho e^{it} \ , \\ &Y_{12} = Y_1 + iY_2 = \sinh\rho \cos\theta e^{i\phi_1} \ , \qquad Y_{34} = Y_3 + iY_4 = \sinh\rho \sin\theta e^{i\phi_2} \ ; \end{aligned} \tag{2.8}$$

$$\begin{aligned} &X_{12} = X_1 + iX_2 = \sin\gamma \cos\psi e^{i\varphi_1} \ , \qquad X_{34} = X_3 + iX_4 = \sin\gamma \sin\psi e^{i\varphi_2} \ , \\ &X_{56} = X_5 + iX_6 = \cos\gamma e^{i\varphi_3} \ . \end{aligned} \tag{2.9}$$

Then the corresponding metrics take the form

$$ds^2_{AdS_5} = -\cosh^2\rho \ dt^2 + d\rho^2 + \sinh^2\rho \ (d\theta^2 + \cos^2\theta \ d\phi_1^2 + \sin^2\theta \ d\phi_2^2) \tag{2.10}$$

$$ds^2_{S^5} = \cos^2\gamma \ d\varphi_1^2 + d\gamma^2 + \sin^2\gamma \ (d\psi^2 + \cos^2\psi \ d\varphi_1^2 + \sin^2\psi \ d\varphi_2^2). \tag{2.11}$$

The Cartan generators of $SO(2,4)$ corresponding to the three linear isometries of the AdS$_5$ metric are the translations in the AdS time t and two angles ϕ_a :

$$S_0 \equiv S_{05} \equiv E = \sqrt{\lambda}\mathcal{E}, \quad S_1 \equiv S_{12} = \sqrt{\lambda}\mathcal{S}_1, \quad S_2 \equiv S_{34} = \sqrt{\lambda}\mathcal{S}_2. \tag{2.12}$$

The Cartan generators of $SO(6)$ corresponding to the three linear isometries of the S^5 metric are the translations in the three angles φ_a :

$$J_1 \equiv J_{12} = \sqrt{\lambda}\mathcal{J}_1, \quad J_2 \equiv J_{34} = \sqrt{\lambda}\mathcal{J}_2, \quad J_3 \equiv J_{56} = \sqrt{\lambda}\mathcal{J}_3. \tag{2.13}$$

In this paper we also use the other type of embedding coordinates in AdS_5:

$$Y_{05} = y_0 \ e^{it} \ , \qquad Y_{12} = y_1 \ e^{i\phi_1} \ , \qquad Y_{34} = y_2 \ e^{i\phi_2} \ , \tag{2.14}$$

where

$$y_1 = \sinh\rho \ \cos\theta \ , \quad y_2 = \sinh\rho \ \sin\theta \quad \text{and} \quad y_0 = \sqrt{1 + y_1^2 + y_2^2} = \cosh\rho \ . \tag{2.15}$$

The corresponding AdS_5 metric takes the form

$$ds^2_{AdS_5} = -(1 + y_1^2 + y_2^2)dt^2 - \frac{(y_1 dy_1 + y_2 dy_2)^2}{1 + y_1^2 + y_2^2} + dy_1^2 + dy_2^2 + y_1^2 d\phi_1^2 + y_2^2 d\phi_2^2 \ . \tag{2.16}$$

Coordinates (2.8) we call "circular", coordinates (2.14) — "cartesian".

3 Small rigid strings in AdS_5

Aiming at a better understanding of two-spin solutions in AdS_5, in this section we study the case of small strings.

3.1 Rigid string ansatz

Our aim here is to study closed strings with two spins, i.e. rotating in $\phi_{1,2}$. A natural ansatz for describing such solutions is the "rigid" string ansatz [4] ($0 \le \sigma < 2\pi$):

$$\begin{array}{c} t = \kappa\tau \ , \quad \phi_1 = \omega_1 \tau \ , \quad \phi_2 = \omega_2 \tau \quad \kappa, \ \omega_1, \ \omega_2 = \text{const} \\ y_1 = y_1(\sigma) \ , \quad y_2 = y_2(\sigma) \qquad \text{or} \qquad \rho = \rho(\sigma) \ , \quad \theta = \theta(\sigma) \ . \end{array} \tag{3.1}$$

In the "circular" coordinates, the string equations of motion and the conformal constraint for this ansatz read

$$(\theta' \sinh^2\rho)' = (\omega_1^2 - \omega_2^2)\sin\theta\cos\theta \ \sinh^2\rho \tag{3.2}$$

$$\rho'' - \cosh\rho \ \sinh\rho \ (\kappa^2 + \theta'^2 - \omega_1^2\cos^2\theta - \omega_2^2\sin^2\theta) = 0 \tag{3.3}$$

$$\rho'^2 - \kappa^2\cosh^2\rho + \sinh^2\rho \ (\theta'^2 + \omega_1^2 \ \cos^2\theta + \omega_2^2 \ \sin^2\theta) = 0. \tag{3.4}$$

Note, that these equations are not independent, for example, (3.3) is a linear combination of (3.2) and (3.4)'s first derivative.

In the "cartesian" coordinates, the string equations of motion and the conformal constraint read

$$\frac{y_1 y_1'' + y_2 y_2''}{1 + y_1^2 + y_2^2} \ y_1 + \frac{(y_1 y_2' - y_2 y_1')^2 + y_1'^2 + y_2'^2}{(1 + y_1^2 + y_2^2)^2} y_1 - y_1'' + (1 + \omega_1^2) y_1 = 0 \tag{3.5}$$

$$\frac{y_1 y_1'' + y_2 y_2''}{1 + y_1^2 + y_2^2} \ y_2 + \frac{(y_1 y_2' - y_2 y_1')^2 + y_1'^2 + y_2'^2}{(1 + y_1^2 + y_2^2)^2} y_1 - y_2'' + (1 + \omega_2^2) y_2 = 0 \tag{3.6}$$

$$(y_2' \ y_1 - y_1' \ y_2)^2 + (y_1')^2 + (y_2')^2 = (1 + y_1^2 + y_2^2) \ (\kappa^2 \ (1 + y_1^2 + y_2^2) - \omega_1^2 \ y_1^2 - \omega_2^2 \ y_2^2). \tag{3.7}$$

We may rewrite this system in a more compact form (with only independent equations present):

$$(y_2' \ y_1 - y_1' \ y_2)' = (\omega_1^2 - \omega_2^2) \ y_1 \ y_2 \tag{3.8}$$

$$(y_2' \ y_1 - y_1' \ y_2)^2 + (y_1')^2 + (y_2')^2 = (1 + y_1^2 + y_2^2) \ (\kappa^2 \ (1 + y_1^2 + y_2^2) - \omega_1^2 \ y_1^2 - \omega_2^2 \ y_2^2), \tag{3.9}$$

where (3.8) is the difference between (3.5) and (3.6).

A general approach to finding such rigid string solutions in AdS_5 (and S^5) was developed in [9] using the reduction of the conformal-gauge string sigma model to the 1-d Neumann integrable model.[2] Starting with the $R^{2,4}$ embedding coordinates (2.14) one finds that the equations for $y_0, \ y_1, \ y_2$ are those of a harmonic oscillator constrained to move on a 2d hyperboloid — an integrable system with two integrals of motion b_1, b_2 with $b_1 + b_2 = \kappa^2 + \omega_1^2 + \omega_2^2$. In general, the solutions are expressed in terms of hyperelliptic functions and thus are not easy to analyze. There are few special cases when they simplify — when the hyperelliptic surface degenerates into an elliptic one and $y_a(\sigma)$ can be expressed in terms of the standard elliptic functions. Two of such cases, $\omega_1 = \omega_2$, corresponding to $S_1 = S_2$ solution, and its boosted analog with $\kappa = \omega_2 \neq \omega_1$ were studied in [10]. The existence of simple but more general solution with two unequal spins is an open question. Recent study of $\mathcal{N} = 4$ SYM states dual to minimal energy spinning string configuration with two spins (S_1, S_2) with $\frac{S_1}{S_2}$ fixed using ABA [11] suggests that such simple solution might indeed exist.

Here we study small strings with two-spin solutions in AdS_5, starting from the flat-space solutions and using perturbation theory in the curvature of AdS_5.

3.2 Flat-space limit

In this section, we review the flat-space limit for closed strings in AdS_5. Let us start from the expression for the metric in "circular" coordinates

$$ds^2_{AdS_5} = -\cosh^2\left(\frac{\rho}{R}\right) \ dt^2 + d\rho^2 + R^2 \ \sinh^2\left(\frac{\rho}{R}\right) \ (d\theta^2 + \cos^2\theta \ d\phi_1^2 + \sin^2\theta \ d\phi_2^2) \ . \tag{3.10}$$

Here R is the radius of curvature of AdS_5.

If the size of the string is small $\rho = \epsilon\tilde{\rho} \ll R, \ \epsilon \ll 1$, one can perform an expansion ($R = 1$):

$$ds^2_{AdS_5} = \epsilon^2(- \ d\tilde{t}^2 + d\tilde{\rho}^2 + \tilde{\rho}^2 \ d\Omega_3) + \epsilon^4\tilde{\rho}^2 \ \left(-d\tilde{t}^2 + \frac{1}{3}\tilde{\rho}^2 d\Omega_3\right) + O\left(\epsilon^6\right), \tag{3.11}$$

where $t = \epsilon\tilde{t}$, $d\Omega_3 = d\theta^2 + \cos^2\theta \ d\phi_1^2 + \sin^2\theta \ d\phi_2^2$. The leading term represents the metric of flat $R^{1,4}$ Minkowski space.

A similar expansion can be performed in terms of the "cartesian" coordinates. In the limit of small strings

$$y_1 = \epsilon\tilde{y}_1, \qquad y_2 = \epsilon\tilde{y}_2, \qquad \epsilon \ll 1, \tag{3.12}$$

where ϵ defines the size of the string with respect to the radius of curvature, we have

$$ds^2_{AdS_5} = \epsilon^2\left(-d\tilde{t}^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2 + \tilde{y}_1^2 d\phi_1^2 + \tilde{y}_2^2 d\phi_2^2\right) - \epsilon^4\left((\tilde{y}_1^2 + \tilde{y}_2^2)d\tilde{t}^2 + \tilde{y}_1 d\tilde{y}_1 + \tilde{y}_2 d\tilde{y}_2\right) + O(\epsilon^4) \ , \tag{3.13}$$

[2] A more general rigid string ansatz, where in addition to $\rho = \rho(\sigma), \ \theta = \theta(\sigma)$ one has $\phi_1 = \omega_1\tau + \alpha_1(\sigma), \ \phi_2 = \omega_2\tau + \alpha_2(\sigma)$ and where the corresponding 1-d system is the Neumann-Rosochatius one, was considered in [12].

where $dt = \epsilon d\tilde{t}$. Again, the leading term is the metric of flat $R^{1,4}$ Minkowski space.

In this paper we will mainly work with the expansion (3.13).

In the flat-space limit the string equations of motion and conformal constraint for the ansatz (3.1) become

$$\begin{array}{ll} \tilde{y}_1'' + \omega_1^2 \tilde{y}_1 = 0 & \tilde{y}_2'' + \omega_1^2 \tilde{y}_2 = 0 \\ (\tilde{y}_1')^2 + \omega_1^2 \tilde{y}_1^2 - \tilde{\kappa}_1^2 = 0 & (\tilde{y}_2')^2 + \omega_2^2 \tilde{y}_2^2 - \tilde{\kappa}_2^2 = 0 \\ \tilde{t} = \tilde{\kappa}\tau, & \tilde{\kappa}^2 = \tilde{\kappa}_1^2 + \tilde{\kappa}_2^2 \end{array} \tag{3.14}$$

The solutions of these equations are [9]

$$\begin{array}{ll} \tilde{t} = \tilde{\kappa}\tau, & \tilde{\kappa}^2 = n_1^2 a_1^2 + n_2^2 a_2^2 \\ \tilde{y}_1 = y_1^{\rm flat} = \ a_1 \sin(n_1 \sigma) & \tilde{y}_2 = y_2^{\rm flat} = \ a_2 \sin[n_2(\sigma + \sigma_0)] \\ \omega_1 = n_1, & \omega_2 = n_2 \end{array} \tag{3.15}$$

where n_i are integers and σ_0 is a constant phase shift. The energy and spins are given by

$$\mathcal{E} = \kappa, \quad \mathcal{S}_i = \frac{n_i a_i^2}{2}, \qquad \text{i.e.} \qquad \mathcal{E} = \sqrt{2(\mathrm{n}_1 \mathcal{S}_1 + \mathrm{n}_2 \mathcal{S}_2)}$$

or, restoring λ,

$$E = \sqrt{\lambda}\kappa, \quad S_i = \sqrt{\lambda}\frac{n_i a_i^2}{2}, \qquad \text{i.e.} \qquad \mathrm{E} = \sqrt{2\sqrt{\lambda}(\mathrm{n}_1 \mathrm{S}_1 + \mathrm{n}_2 \mathrm{S}_2)}.$$

To get the states on the leading Regge trajectory (having minimal energy for given values of the spins) one is to choose $n_1 = n_2 = 1$.

Note, that in the case $n_1 = n_2$ and $2\sigma_0 \neq \pi n$ there are also non-Cartan components of the spin present. We have not mentioned them above, as such solutions can always be rotated to[3]

$$\begin{array}{ll} y_1^{\rm flat} = a \sin(n\sigma), & y_2^{\rm flat} = b \cos(n\sigma) \\ \omega_1 = \omega_2 = n, & \kappa^2 = n^2(a^2 + b^2) \end{array} \tag{3.19}$$

i.e. ones without non-Cartan components.

3.3 Curvature corrections to the flat-space solutions in AdS$_5$

Expansions (3.11) and (3.13) suggest the possibility that solutions in full AdS$_5$ may be constructed as

$$\begin{array}{l} y_1(\sigma) = \epsilon\, y_1^{flat} + \epsilon^3 z_1(\sigma) + \epsilon^5 z_3(\sigma) + ... \\ y_2(\sigma) = \epsilon\, y_2^{flat} + \epsilon^3 z_2(\sigma) + \epsilon^5 z_4(\sigma) + ... \end{array} \tag{3.20}$$

[3] Let us set, for simplicity, $n_1 = n_2 = n = 1$ and rotate (3.15) by an angle β ($\beta \neq \frac{\pi}{2}m,\ m \in Z$) in $Y_1 Y_3$ and $Y_2 Y_4$ planes

$$\begin{pmatrix} a_1 \sin(\sigma) \\ a_2 \sin(\sigma + \sigma_0) \end{pmatrix} \to \begin{pmatrix} \sqrt{(a\cos\beta - b\sin\beta\cos\sigma_0)^2 + (b\sin\beta\sin\sigma_0)^2}\,\sin(\sigma - \varphi_1) \\ \sqrt{(a\sin\beta + b\cos\beta\cos\sigma_0)^2 + (b\cos\beta\sin\sigma_0)^2}\,\cos(\sigma - \varphi_2) \end{pmatrix}, \tag{3.16}$$

where

$$\sin\varphi_1 = \frac{b\sin\beta\sin\sigma_0}{\sqrt{(a\cos\beta - b\sin\beta\cos\sigma_0)^2 + (b\sin\beta\sin\sigma_0)^2}} \tag{3.17}$$

$$\sin\varphi_2 = \frac{a\sin\beta + b\cos\beta\cos\sigma_0}{\sqrt{(a\sin\beta + b\cos\beta\cos\sigma_0)^2 + (b\cos\beta\sin\sigma_0)^2}} \tag{3.18}$$

When $\varphi_1 = \varphi_2 = \varphi_0$ or equivalently $\tan 2\beta = \frac{a}{b\cos\sigma_0}$, the rotated solution is indeed of the form (3.19), with $\sigma \to \sigma - \varphi_0$.

where the first term corresponds to the flat-space solution (3.15), while the others may be found using perturbation theory in the curvature of AdS_5.

Here we will be interested in the first subleading corrections only.

Let us look for a solution of (3.8), (3.9) in the form:

$$\begin{aligned} y_1(\sigma) &= \epsilon\ a\sin(n_1\sigma) + \epsilon^3 z_1(\sigma) \\ y_2(\sigma) &= \epsilon\ b\sin(n_2(\sigma+\sigma_0)) + \epsilon^3 z_2(\sigma), \end{aligned} \tag{3.21}$$

where $n_{1,2} \in Z$ and

$$\begin{aligned} \omega_1 &= n_1(1+\epsilon^2\tilde{\omega}_1), \qquad & \omega_2 &= n_2(1+\epsilon^2\tilde{\omega}_2) \\ \kappa_0 &= \epsilon\ \kappa_0 + \epsilon^3\ \kappa_1, \qquad & \kappa_0^2 &= a^2n_1^2 + b^2n_2^2 \end{aligned} \tag{3.22}$$

with $\tilde{\omega}_i,\ \kappa_1$ are curvature corrections to $\omega_i,\ \kappa$.

From (3.8), (3.9) one obtains the following system of equations

$$\begin{aligned} -b\ \sin((\sigma+\sigma_0)n_2)\ (n_1^2z_1 + z_1'') + a\sin(\sigma n_1)\left(n_2^2z_2 + z_2''\right) \\ = 2ab\ \sin((\sigma+\sigma_0)n_2)\sin(\sigma n_1)(n_1^2\tilde{\omega}_1 - n_2^2\tilde{\omega}_2) \end{aligned} \tag{3.23}$$

$$\begin{aligned} & 2a\ (\cos(n_1\sigma)n_1z_1' + \sin(n_1\sigma)n_1^2z_1) + 2b\ (\cos(n_2(\sigma+\sigma_0))n_2z_2' + \sin(n_2(\sigma+\sigma_0))n_2^2z_2) \\ & \quad = 2\chi + \frac{1}{4}(a^2n_1\sin(2n_1\sigma) + b^2n_2\sin(2n_2(\sigma+\sigma_0)))^2 \\ & +a^2\ \sin^2(n_1\sigma)\left[a^2n_1^2 + b^2n_2^2 - 2a^2n_1^2\tilde{\omega}_1\right] + b^2\ \sin^2(n_1(\sigma+\sigma_0))\left[a^2n_1^2 + b^2n_2^2 - 2b^2n_2^2\tilde{\omega}_2\right]. \end{aligned} \tag{3.24}$$

Here $\chi^2 = \kappa_1^2\kappa_0^2$. The equations for z_1 and z_2 may be separated in the following way. Differentiate both sides of (3.24). The left-hand side reads

$$\begin{aligned} & \left(a\ (\cos(n_1\sigma)n_1z_1' + \sin(n_1\sigma)n_1^2z_1) + b\ (\cos(n_2(\sigma+\sigma_0))n_2z_2' + \sin(n_2(\sigma+\sigma_0))n_2^2z_2)\right)' \\ & \quad = a\cos(n_1\sigma)n_1\left(n_1^2z_1 + z_1''\right) + b\cos(n_2(\sigma+\sigma_0))n_2\left(n_2^2z_2[s] + (z_2)''[s]\right). \end{aligned} \tag{3.25}$$

Then, compare (3.25) with the left-hand side of (3.23). After some rearrangements we obtain

$$z_1'' + n_1^2z_1 = 2a\sin(n_1\sigma)\left[a^2n_1^2\cos^2(n_1\sigma) + b^2n_2\cos^2(n_2(\sigma+\sigma_0)) - n_1^2\tilde{\omega}_1\right] \tag{3.26}$$

$$z_2'' + n_2^2z_2 = 2b\sin(n_2(\sigma+\sigma_0))\left[a^2n_1^2\cos^2(n_1\sigma) + b^2n_2\cos^2(n_2(\sigma+\sigma_0)) - n_2^2\tilde{\omega}_2\right]. \tag{3.27}$$

These equations can be readily solved:

- If $n_1 = n_2 = n$ one finds

$$\begin{aligned} z_1 &= C_1\sin(n\sigma) + C_2\cos(n\sigma) \\ & \quad -\frac{1}{4}a\ n\sigma\left[(a^2 + 2b^2 - 4\tilde{\omega}_1)\cos(n\sigma) - b^2\cos(n\sigma + 2n\sigma_0)\right] \\ & \quad -\frac{a}{16}\left[a^2\sin(3n\sigma) + b^2\sin(3n\sigma + 2n\sigma_0) + 2b^2\sin(n\sigma + 2n\sigma_0)\right. \\ & \qquad\qquad \left. -2\sin(n\sigma)\left(a^2 + 2b^2 - 4\tilde{\omega}_1\right)\right] \\ z_2 &= C_3\cos(n\sigma) + C_4\sin(n\sigma) \\ & \quad -\frac{1}{4}b\ n\sigma\left[(b^2 + 2a^2 - 4\tilde{\omega}_2)\cos(n\sigma + n\sigma_0) - a^2\cos(n\sigma - n\sigma_0)\right] \\ & \quad -\frac{b}{16}\left[b^2\sin(3n\sigma + 3n\sigma_0) + a^2\sin(3n\sigma + n\sigma_0) + 2a^2\sin(n\sigma - n\sigma_0)\right. \\ & \qquad\qquad \left. -2\sin(n\sigma + n\sigma_0)\left(b^2 + 2a^2 - 4\tilde{\omega}_2\right)\right]. \end{aligned} \tag{3.28}$$

Here C_i $(i = 1, 2, 3, 4)$ are integration constants.

The closed string periodicity condition (2.4) requires z_1, z_2 being periodic in σ, i.e. the linear terms must vanish:

$$\begin{aligned} &(a^2 + 2b^2 - 4\tilde{\omega}_1)\cos(n\sigma) - b^2\cos(n\sigma + 2n\sigma_0) = 0 \\ &(b^2 + 2a^2 - 4\tilde{\omega}_2)\cos(n\sigma + n\sigma_0) - a^2\cos(n\sigma - n\sigma_0) = 0. \end{aligned} \tag{3.29}$$

These equations can be solved for constant values of $\tilde{\omega}_1, \tilde{\omega}_2$ only

– in the elliptic string case, when $2\sigma_0 n = \pi + 2\pi m$, $m \in Z$,

$$\tilde{\omega}_1 = \frac{1}{4}(a^2 + 3b^2), \qquad \tilde{\omega}_2 = \frac{1}{4}(3a^2 + b^2). \tag{3.30}$$

This case is considered in section 3.4.

– in the folded string case, when $2\sigma_0 n = 2\pi m$, $m \in Z$

$$\tilde{\omega}_1 = \tilde{\omega}_2 = \frac{1}{4}(a^2 + b^2). \tag{3.31}$$

This case is considered in section 3.5.

The restriction on σ_0 might first look surprising. One can always rotate (3.15) with arbitrary σ_0 to (3.19) (see section 3.2) and, using the method given above, find the curvature corrections to any flat-space solution with $n_1 = n_2$. However, rotating back, we would not remain in the framework of the rigid string ansatz as the frequencies ω_1 and ω_2 are now different (see (3.30)).

- If $n_1 \neq n_2$ one finds

$$\begin{aligned} z_1 = {} & C_1\sin(n_1\sigma) + C_2\cos(n_1\sigma) \\ & + \frac{ab^2}{4\left(n_1^2 - n_2^2\right)}\left[-\cos(n_1\sigma)\sin(2n_2(\sigma+\sigma_0))n_1n_2 + \cos(2n_2(\sigma+\sigma_0))\sin(n_1\sigma)n_2^2\right] \\ & - \frac{a^3}{16}\Bigg[\sin(3n_1\sigma) - 2\sin(n_1\sigma)\frac{a^2n_1^2 + 2b^2n_2^2 - 4n_1^2\tilde{\omega}_1}{n_1^2} \\ & \qquad + 4n_1\sigma\cos(n_1\sigma)\frac{a^2n_1^2 + 2b^2n_2^2 - 4n_1^2\tilde{\omega}_1}{n_1^2}\Bigg] \\ z_2 = {} & C_3\cos(n_2(\sigma+\sigma_0)) + C_4\sin(n_2(\sigma+\sigma_0)) \\ & - \frac{ba^2}{4\left(n_1^2 - n_2^2\right)}\left(-\cos(n_2(\sigma+\sigma_0))\sin(2n_1\sigma)n_1n_2 + \cos(2n_1\sigma)\sin(n_2(\sigma+\sigma_0))n_1^2\right) \\ & - \frac{b^3}{16}\Bigg[\sin(3n_2(\sigma+\sigma_0)) - 2\sin(n_2(\sigma+\sigma_0))\frac{b^2n_2^2 + 2a^2n_1^2 - 4n_2^2\tilde{\omega}_2}{n_2^2} \\ & \qquad - 4n_2\sigma\cos(n_2(\sigma+\sigma_0))\frac{b^2n_2^2 + 2a^2n_1^2 - 4n_2^2\tilde{\omega}_2}{n_2^2}\Bigg]. \end{aligned} \tag{3.32}$$

Here C_i $(i = 1, 2, 3, 4)$ are integration constants.

The closed string periodicity condition (2.4) requires the linear terms vanish:

$$a^2n_1^2 + 2b^2n_2^2 - 4n_1^2\tilde{\omega}_1 = 0, \qquad b^2n_2^2 + 2a^2n_1^2 - 4n_2^2\tilde{\omega}_2 = 0. \tag{3.33}$$

Then (for any value of σ_0) we have

$$\tilde{\omega}_1 = \frac{a^2 n_1^2 + 2b^2 n_2^2}{4n_1^2} \,, \qquad \tilde{\omega}_2 = \frac{2a^2 n_1^2 + b^2 n_2^2}{4n_2^2} \,. \tag{3.34}$$

This case is considered in section 3.6.

In fact, the restriction on σ_0 in the $n_1 = n_2$ case singles out the solutions with zero non-Cartan components of spin. Indeed, for $n_1 \neq n_2$ there are no such components for any value of σ_0, while for $n_1 = n_2$ they vanish only if $2\sigma_0 = \pi m$.

Only flat-space solutions with zero non-Cartan components of spin receive curvature corrections in the framework of the rigid string ansatz. An attempt to find the corrections to the solutions with non-vanishing non-Cartan components leads out of the rigid string ansatz.

3.4 The elliptic string solution ($n_1 = n_2$)

Curvature corrections to the string solution with $n_1 = n_2 = n$ and $2\sigma_0 n = \pi + 2\pi m$, $m \in Z$ are (see (3.28), (3.30))

$$\begin{aligned} z_1 &= C_1 \sin(n\sigma) + C_2 \cos(n\sigma) - \frac{1}{16} a \, (a^2 - b^2) \sin(3n\sigma) \\ z_2 &= C_3 \cos(n\sigma) + C_4 \sin(n\sigma) + \frac{1}{16} b \, (b^2 - a^2) \cos(3n\sigma). \end{aligned} \tag{3.35}$$

Here vanishing of non-Cartan components of spin requires $aC_4 = -bC_2$.

Recall that in order to get from the system (3.23), (3.24) to (3.26), (3.27), we take a derivative from (3.24), thus we must check if it is satisfied. Substituting (3.35) into (3.24), one finds

$$-16\chi - 3n^2 \left(a^2 - b^2\right)^2 + 16n^2 \left(aC_1 + bC_3\right) = 0 \,. \tag{3.36}$$

Then the classical energy of the string reads

$$\mathcal{E}_{n_1=n_2} = \sqrt{2n(\mathcal{S}_1 + \mathcal{S}_2)} \left[1 + \frac{3}{8n} \, (\mathcal{S}_1 + \mathcal{S}_2) + \frac{1}{2n} \frac{\mathcal{S}_1 \mathcal{S}_2}{\mathcal{S}_1 + \mathcal{S}_2} + O(\mathcal{S}_i \mathcal{S}_j)\right] \tag{3.37}$$

or, restoring λ,

$$E_{n_1=n_2} = \sqrt{2n\sqrt{\lambda}(S_1 + S_2)} \left[1 + \frac{3}{8n\sqrt{\lambda}} \, (S_1 + S_2) + \frac{1}{2n\sqrt{\lambda}} \frac{S_1 S_2}{S_1 + S_2} + O(\lambda^{-1})\right] \,. \tag{3.38}$$

This expression is a generalization of circular and folded string cases (for a review see Appendix A and references therein). In the limit $\mathcal{S}_1 = \mathcal{S}, \ \mathcal{S}_2 = 0$, it gives the small-spin expansion of the classical energy of the folded string (see (A.14))

$$\mathcal{E} = \sqrt{2n\mathcal{S}} \left(1 + \frac{3\mathcal{S}}{8n} + O(\mathcal{S}^2)\right) ; \tag{3.39}$$

in the limit $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$ — the small-spin expansion of the classical energy of the circular string (see (A.5))

$$\mathcal{E} = 2\sqrt{n\mathcal{S}} \left(1 + \frac{\mathcal{S}}{n} + O(\mathcal{S}^2)\right) . \tag{3.40}$$

3.5 The folded string solution ($n_1 = n_2$)

Curvature corrections to the string solution with $n_1 = n_2 = n$ and $2\sigma_0 n = \pi + 2\pi m$, $m \in Z$ are (see (3.28), (3.31))

$$\begin{aligned} z_1 &= C_1 \sin(n\sigma) + C_2 \cos(n\sigma) - \frac{1}{16} a \ (a^2 + b^2) \sin(3n\sigma) \\ z_2 &= C_3 \cos(n\sigma) + C_4 \sin(n\sigma) - \frac{1}{16} b \ (b^2 + a^2) \cos(3n\sigma). \end{aligned} \tag{3.41}$$

Here vanishing of non-Cartan components of spin requires $aC_4 = -bC_2$.

From (3.24), one obtains the constraint on C_i :

$$-16\chi - 3n^2 \left(a^2 + b^2\right)^2 + 16n^2 \left(aC_1 + bC_3\right) = 0. \tag{3.42}$$

Then the classical energy of the string reads

$$\mathcal{E} = \sqrt{2nS}\left(1 + \frac{3}{8n}\mathcal{S} + O(\mathcal{S}^2)\right) \quad \text{or} \quad E = \sqrt{2n\sqrt{\lambda}S}\left(1 + \frac{3}{8n\sqrt{\lambda}}\ S + O(\lambda^{-1})\right), \tag{3.43}$$

i.e. coincides with the small-spin expansion of the classical energy of the folded string (A.14).

3.6 $n_1 \neq n_2$ solutions

Curvature corrections to the string solution with $n_1 \neq n_2$ and arbitrary phase shift σ_0 are (see (3.32), (3.34))

$$\begin{aligned} z_1 &= C_1 \sin(n_1\sigma) + C_2 \cos(n_1\sigma) \\ &\quad + \frac{ab^2}{4\left(n_1^2 - n_2^2\right)} \ \left[-\cos(n_1\sigma)\sin(2n_2(\sigma+\sigma_0))n_1 n_2 + \cos(2n_2(\sigma+\sigma_0))\sin(n_1\sigma)n_2^2\right] \\ &\qquad - \frac{a^3}{16}\sin(3n_1\sigma) \\ z_2 &= C_3 \sin(n_2(\sigma+\sigma_0)) + C_4 \cos(n_2(\sigma+\sigma_0)) \\ &\quad + \frac{ba^2}{4\left(n_2^2 - n_1^2\right)} \ \left(-\cos(n_2(\sigma+\sigma_0))\sin(2n_1\sigma)n_1 n_2 + \cos(2n_1\sigma)\sin(n_2(\sigma+\sigma_0))n_1^2\right) \\ &\qquad - \frac{b^3}{16}\sin(3n_2(\sigma+\sigma_0)). \end{aligned} \tag{3.44}$$

From (3.24), one obtains the constraint on C_i :

$$-16\chi - 3\left(a^4 n_1^2 + b^4 n_2^2\right) + 16\left(a\ n_1^2\ C_1 + b\ n_2^2\ C_3\right) = 0. \tag{3.45}$$

Then the classical energy of the string reads

$$\mathcal{E}_{n_1\neq n_2} = \sqrt{2n_1\mathcal{S}_1 + 2n_2\mathcal{S}_2}\left[1 + \frac{3}{8}\ \frac{(\mathcal{S}_1+\mathcal{S}_2)^2}{n_1\mathcal{S}_1 + n_2\mathcal{S}_2} + \frac{1}{2}\ \frac{\mathcal{S}_1\mathcal{S}_2}{n_1\mathcal{S}_1 + n_2\mathcal{S}_2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1} - \frac{3}{2}\right) + O(\mathcal{S}_i\mathcal{S}_j)\right] \tag{3.46}$$

or, restoring λ,

$$\begin{aligned} E_{n_1\neq n_2} = \sqrt{2\sqrt{\lambda}(n_1\mathcal{S}_1 + n_2\mathcal{S}_2)}\ \Bigg[1 &+ \frac{3}{8\sqrt{\lambda}}\ \frac{(\mathcal{S}_1+\mathcal{S}_2)^2}{n_1\mathcal{S}_1 + n_2\mathcal{S}_2} \\ &+ \frac{1}{2\sqrt{\lambda}}\ \frac{\mathcal{S}_1\mathcal{S}_2}{n_1\mathcal{S}_1 + n_2\mathcal{S}_2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1} - \frac{3}{2}\right) + O(\lambda^{-1})\Bigg]. \end{aligned} \tag{3.47}$$

Note, that in the limit $n_1 = n_2 = n$ expression (3.47) becomes

$$E_{n_1 \neq n_2} \stackrel{n_1=n_2}{\longrightarrow} \sqrt{2\sqrt{\lambda}n(\mathcal{S}_1+\mathcal{S}_2)}\left[1+\frac{3}{8n\sqrt{\lambda}}\,\frac{(\mathcal{S}_1+\mathcal{S}_2)^2}{\mathcal{S}_1+\mathcal{S}_2}+\frac{1}{2n\sqrt{\lambda}}\,\frac{\mathcal{S}_1\mathcal{S}_2}{\mathcal{S}_1+\mathcal{S}_2}\left(\frac{1}{2}\right)+O(\lambda^{-1})\right],$$

which differs from (3.38) by the factor of 1/2 in the third term in the brackets.

This discontinuity may indicate that there are deep differences between solutions with $n_1 \neq n_2$ and more symmetrical ones with $n_1 = n_2$.

4 Small-string limit of the exact string solutions in AdS_5

In this section we investigate the connection between small- (flat-space) and large-spin limits of two-spin string solutions in AdS_5. In the particular cases of $\omega_1 = \omega_2$ and $\kappa = \omega_2 \neq \omega_1$ the general solutions in AdS_5 were found in [10]. It was discussed there, for $\kappa = \omega_2 \neq \omega_1$ case, that strings which admit large-spin limit do not have the small-spin one and vice-versa. Thus we study only solutions with $\omega_1 = \omega_2$, corresponding to $S_1 = S_2$ case.

When $\omega_1 = \omega_2 = \omega$, string sigma model equations reduce to

$$\theta' = \frac{c}{\sinh^2\rho} \tag{4.1}$$

$$\rho'^2 = \kappa^2\cosh^2\rho - \frac{c^2}{\sinh^2\rho} - \omega^2\sinh^2\rho\ , \tag{4.2}$$

where c is an integration constant. The solution for ρ is [10]

$$\cosh\rho = \frac{\sqrt{a_-}}{\mathrm{dn}[\sqrt{a_+(\omega^2-\kappa^2)}\sigma, \frac{a_+-a_-}{a_+}]}\ . \tag{4.3}$$

Here

$$a_\pm = \frac{2\omega^2-\kappa^2\pm\sqrt{\kappa^4-4c^2(\omega^2-\kappa^2)}}{2(\omega^2-\kappa^2)} \tag{4.4}$$

define the size of the string: $\sqrt{a_-} \leq \cosh\rho \leq \sqrt{a_+}$. Parameters $\kappa,\ \omega,\ c$ are related to $a_\pm$ as

$$c^2 = (a_+-1)(a_--1)(\omega^2-\kappa^2), \qquad \kappa^2 = \omega^2\frac{a_++a_--2}{a_++a_--1}\ . \tag{4.5}$$

Solution (4.3) is valid for $\sqrt{a_-} \leq \cosh\rho \leq \sqrt{a_+}$ only.

Let us expand (4.3) in the small-string limit.

When the size of the string is small with respect to the curvature of AdS_5 space ($R = 1$), one has

$$\begin{array}{l} a_+ = \cosh\rho_{max} = 1+\epsilon^2\ a^2+\epsilon^4\ A+O(\epsilon^6) \\ a_- = \cosh\rho_{min} = 1+\epsilon^2\ b^2+\epsilon^4\ B+O(\epsilon^6) \end{array} \qquad \epsilon \ll 1\ . \tag{4.6}$$

In what follows we omit orders higher then ϵ^4.

In that limit the elliptic modulus of dn in (4.3) is small

$$\frac{a_+-a_-}{a_+} = \frac{\epsilon^2\ (a^2-b^2)+\epsilon^4\ (A-B)}{1+\epsilon^2\ a^2+\epsilon^4\ A} \sim \epsilon^2 \ll 1\ ,$$

so we can perform an expansion

$$
\begin{aligned}
\cosh\rho = 1 + \epsilon^2 \ \frac{1}{2}(a^2 \sin^2(W\sigma) + b^2 \cos^2(W\sigma)) \\
+ \epsilon^4 \ \frac{1}{8}\big[\ W(a^4 - b^4) \ \sigma \ \sin(2W\sigma) - \frac{1}{4} \ (a^2 - b^2)^2 \ \sin^2(2W\sigma) \\
- \big(a^4 - 4A\big)\sin^2(W\sigma) - \big(b^4 - 4B\big)\cos^2(W\sigma)\big] + O(\epsilon^6) \ .
\end{aligned}
\tag{4.7}
$$

Here $W^2 = \omega^2 - \kappa^2$. To satisfy the closed string periodicity condition, the ϵ^2 and ϵ^4 terms must both be periodic. There are two options:

- W is an integer. Then the ϵ^2 term is periodic and the linearity in the ϵ^4 term cancels if $a = b$.
- W has the form $W = W_0 + \epsilon^2 \ W_1$. Then from (4.7) we have

$$
\begin{aligned}
\cosh\rho = 1 + \epsilon^2 \ \frac{1}{2}(a^2 \sin^2(W_0\sigma) + b^2 \cos^2(W_0\sigma)) \\
+ \epsilon^4 \ \frac{1}{8}\big[\ (a^2 - b^2)(4W_1 + m(a^2 + b^2)) \ \sigma \ \sin(2W_0\sigma) - \frac{1}{4} \ (a^2 - b^2)^2 \ \sin^2(2W_0\sigma) \\
- \big(a^4 - 4A\big)\sin^2(W_0\sigma) - \big(b^4 - 4B\big)\cos^2(W_0\sigma)\big].
\end{aligned}
\tag{4.8}
$$

The ϵ^2 term is periodic if W_0 is an integer and the linearity in the ϵ^4 term cancels if $a = b$ or

$$
W_1 = -\epsilon^2 \ \frac{1}{4} \ W_0 \ (a^2 + b^2) \ . \tag{4.9}
$$

The case $a = b$ brings us to the trivial limit of the circular string, so we will not discuss it here. Let us investigate the other option.

Assuming that W has the form (4.9), we get

$$
\begin{aligned}
\cosh\rho = 1 + \epsilon^2 \ \frac{1}{2}(a^2 \sin^2(W_0\sigma) + b^2 \cos^2(W_0\sigma)) \\
- \epsilon^4 \ \frac{1}{8} \left[\frac{1}{4} \ (a^2 - b^2)^2 \ \sin^2(2W_0\sigma) \right. \\
\left. + \big(a^4 - 4A\big)\sin^2(W_0\sigma) + \big(b^4 - 4B\big)\cos^2(W_0\sigma)\right] .
\end{aligned}
\tag{4.10}
$$

Making use of (4.1) and (4.10), one obtains the following equation for θ

$$
\begin{aligned}
\theta' = \frac{\tilde{c} \ \epsilon^2}{\sinh^2\rho} = \frac{\tilde{c}_0 + \epsilon^2 \ \tilde{c}_1}{a^2 \sin^2(W_0\sigma) + b^2 \cos^2(W_0\sigma)} \\
-\epsilon^2 \ \tilde{c}_0 \ \frac{2A\sin^2(W_0\sigma) + 2B\cos^2(W_0\sigma) - (a^2 - b^2)^2 \cos^2(W_0\sigma)\sin^2(W_0\sigma)}{2(a^2 \sin^2(W_0\sigma) + b^2 \cos^2(W_0\sigma))^2},
\end{aligned}
\tag{4.11}
$$

where

$$
c = \tilde{c} \ \epsilon^2 = \epsilon^2 \ \tilde{c}_0 + \epsilon^4 \ \tilde{c}_1. \tag{4.12}
$$

Its solution is

$$
\theta(\sigma) = \theta_0(\sigma) + \epsilon^2 \ \theta_1(\sigma), \tag{4.13}
$$

where

$$
\begin{aligned}
&\theta_0(\sigma) = \frac{\tilde{c}_0}{W_0\ ab} \arctan\left[\frac{a}{b}\tan(W_0\sigma)\right];\\
&\theta_1(\sigma) = \frac{\tilde{c}_0}{4W_0\ ab}\left(a^2+b^2-2\frac{A}{a^2}-2\frac{B}{b^2}+4\frac{\tilde{c}_1}{\tilde{c}_0}\right)\arctan\left[\frac{a}{b}\tan(W_0\sigma)\right]\\
&\quad -\frac{\tilde{c}_0}{4W_0}\left(a^2-b^2-2\frac{A}{a^2}+2\frac{B}{b^2}\right)\frac{\cos(W_0\sigma)\sin(W_0\sigma)}{a^2\sin^2(W_0\sigma)+b^2\cos^2(W_0\sigma)}-\frac{\tilde{c}_0}{2W_0}\arctan\left[\tan(W_0\sigma)\right].
\end{aligned}
\tag{4.14}
$$

This expression, as well as (4.3) and (4.8), is valid for $0 \le W_0\sigma \le \frac{\pi}{2}$ only. Within this interval θ may change only by a rational value of π : $0 \le \theta \le \frac{n}{k}\theta$ and may not gain any small corrections, otherwise the solution would not satisfy the closed string periodicity condition. We must have that $\theta_1(W_0\sigma = 0) = \theta_1(W_0\sigma = \frac{\pi}{2})$. The latter gives the following constraint on $\tilde{c}_i,\ A,\ B$

$$4\frac{\tilde{c}_1}{\tilde{c}_0} = 2\left(\frac{A}{a^2}+\frac{B}{b^2}\right)-(a-b)^2. \tag{4.15}$$

So far we have not used the relations given in (4.5). Substitution of (4.9) and (4.12) into (4.5) gives

$$4\frac{\tilde{c}_1}{\tilde{c}_0} = 2\left(\frac{A}{a^2}+\frac{B}{b^2}\right)-(a-b)^2+ab. \tag{4.16}$$

Comparing this to (4.15), one finds

$$ab = 0,$$

which implies $a = 0$ or $b = 0$ and brings us to the limit of folded string.

Apart from the trivial cases of folded and circular strings, we find that the general rigid solution with $\omega_1 = \omega_2$ ($S_1 = S_2$) in AdS_5 admitting the large-spin limit does not have a small-spin limit. For more general two-spin solutions it might still be possible to have both limits.

5 Chiral solutions for a bosonic string in $R_t \times S^5$

In this section we discuss chiral solutions in $R_t \times S^5$. Such solutions obey an additional constraint

$$\partial_+ X_M \partial_- X_M = 0, \tag{5.1}$$

where X_M are embedding coordinates of R^6 with the Euclidean metric δ_{MN}; $X_M X_M = 1$ and

$$\partial_\pm = \frac{\partial}{\partial\sigma_\pm} = \frac{1}{2}\left(\frac{\partial}{\partial\tau}\pm\frac{\partial}{\partial\sigma}\right), \qquad \sigma_\pm = \tau\pm\sigma.$$

We will discuss the string located at the center of AdS_5 and rotating in S^5, trivially embedded in AdS_5 as $Y_5 + iY_0 = e^{it}$, with the global AdS time being $t = \kappa\tau$ and $Y_1, ..., Y_4 = 0$ (see (2.5)).

The classical string equations in conformal gauge become[4]

$$\partial_-\partial_+ X_M = 0 \tag{5.4}$$

$$\partial_- X_M \partial_+ X_M = 0 \tag{5.5}$$

$$\kappa^2 = 4\partial_\pm X_M \partial_\pm X_M \tag{5.6}$$

[4] Chiral solutions may also be considered via Pohlmeyer reduction [14]. For example, let only four of X^M's are

The simplest solution of this kind is [12],

$$\kappa = 2a_3m_3 \ , \quad X_1 + iX_2 = a_1 e^{im_1\sigma_\pm} \ , \quad X_3 + iX_4 = a_2 e^{im_2\sigma_\pm} \ , \quad X_5 + iX_6 = a_3 e^{im_3\sigma_\mp} \ ,$$

where $\sum_{i=1}^{3} a_i = 1$ and m_i are integers. It was recently used in [13] as a model of a quantum string state with "small" quantum numbers. We expect that more general chiral solutions may also find useful applications.

Let us consider the ansatz

$$\begin{aligned} X_{12} &= X_1 + iX_2 = a_1 e^{iF_1(\sigma_+)} \\ X_{34} &= X_3 + iX_4 = a_2 e^{iF_2(\sigma_+)} \\ X_{56} &= X_5 + iX_6 = a_3 e^{iF_3(\sigma_-)} \end{aligned} \tag{5.7}$$

where $\sum_{i=1}^{3} a_i = 1$. To satisfy periodicity condition, F_1, F_2 must have the form

$$F_i(\sigma_+) = m_i\sigma_+ + \sum_n f_n^{(i)} \cos(n\sigma_+) + g_n^{(i)} \sin(n\sigma_+) \tag{5.8}$$

with f_n^i, g_n^i are real and m_i are integers.

From string equations (5.4), (5.5), (5.6) one finds

$$\kappa^2 = 4a_1^2 \left(\partial_+ F_1\right)^2 + 4a_2^2 \left(\partial_+ F_2\right)^2 , \qquad \kappa = 2m_3a_3, \tag{5.9}$$

$$F_3 = m_3\sigma_-. \tag{5.10}$$

Let us assume that F_1 is an arbitrary function of the form (5.8), then F_2 is expressed as

$$F_2(\sigma_+) = \pm \int \frac{1}{a_2} \sqrt{a_3^2 m_3^2 - a_1^2 \left(\partial_+ F_1\right)^2} d\sigma_+. \tag{5.11}$$

Being represented as an integral from the periodic function, F_2 possess periodic and linear terms only. So up to adjusting a_i, it has the form (5.8).

The general solution for the ansatz (5.7) is

$$\begin{aligned} &\kappa = 2m_3a_3 \\ &X_{12} = a_1\, e^{iF_1(\sigma_+)}, && F_1(\sigma_+) = m_1\sigma_+ + \sum_n f_n \cos(n\sigma_+) + g_n \sin(n\sigma_+) \\ &X_{34} = a_2\, e^{iF_2(\sigma_+)}, && F_2(\sigma_+) = \pm\frac{1}{a_2} \int d\sigma_+ \sqrt{a_3^2 m_3^2 - a_1^2 \left(\partial_+ F_1\right)^2} \\ &X_{56} = a_3 e^{im_3\sigma_-} \end{aligned} \tag{5.12}$$

non-zero. The reduced model corresponding to the string in $R_t \times S^3$ [15] is the complex sine-Gordon (CSG) model

$$\tilde{L} = \partial_+\alpha\partial_-\alpha + \tan^2\alpha\ \partial_+\theta\partial_-\theta + \frac{\kappa^2}{2}\cos 2\alpha\,. \tag{5.2}$$

The variables α and θ are expressed in terms of the $SO(4)$ invariant combinations of derivatives of the original variables X_m $(m = 1, 2, 3, 4)$

$$\kappa^2 \cos 2\alpha = \partial_+ X_M \partial_- X_M \ , \qquad \kappa^3 \sin^2\alpha\ \partial_\pm\theta = \mp\frac{1}{2}\epsilon^{MNKL} X_M \partial_+ X_N \partial_- X_K \partial_\pm^2 X_L\,. \tag{5.3}$$

Chiral solutions meet particular case of $\alpha = \frac{\pi}{4}$.

In general, the "phase function F_2" resulting from the integration (5.11) is expressed in elliptic functions. There are few cases when it simplify to elementary ones. Two of them $F_1(\sigma_+) = m_1\sigma_+$, $F_2(\sigma_+) = m_2\sigma_+$ and $F_1(\sigma_+) = \alpha\cos n\sigma_+$, $F_2(\sigma_+) = \alpha\sin n\sigma_+$ are discussed below.

Note that chiral solutions treat τ and σ on an equal footing, i.e. non-trivial dependence on τ implies that the shape of the string is not rigid, in general, so such solutions are similar to "pulsating" ones.

5.1 Rigid chiral solutions

The simplest chiral solution from ansatz (5.7) corresponds to

$$F_1(\sigma_+) = m_1\sigma_+, \qquad F_2(\sigma_+) = m_2\sigma_+ \ . \tag{5.13}$$

It reads [12]:

$$\kappa = 2a_3m_3 \ , \quad X_{12} = a_1 e^{im_1\sigma_+} \ , \quad X_{34} = a_2 e^{im_2\sigma_+} \ , \quad X_{56} = a_3 e^{im_3\sigma_-} \ , \tag{5.14}$$

where

$$a_1^2m_1^2 + a_2^2m_2^2 = a_3^2m_3^2 \tag{5.15}$$

with m_i are integers, $\sum_{i=1}^{3} a_i^2 = 1$.

Comparing that to (3.1), we see that it is a rigid string solution. In fact, it is the the only possible rigid chiral solution from ansatz (5.7).

For fixed m_i, the energy is given by the standard flat-space linear Regge relation

$$\mathcal{E} = \sqrt{2(m_1\mathcal{J}_1 + m_2\mathcal{J}_2 + m_3\mathcal{J}_3)}, \qquad m_3\mathcal{J}_3 = m_1\mathcal{J}_1 + m_2\mathcal{J}_2 \ , \tag{5.16}$$

where expressions for spins are

$$\mathcal{J}_1 = a_1^2m_1, \qquad \mathcal{J}_2 = a_2^2m_2, \qquad \mathcal{J}_3 = a_3^2m_3 \ . \tag{5.17}$$

Restoring λ, we get

$$E = \sqrt{2\sqrt{\lambda}(m_1J_1 + m_2J_2 + m_3J_3)}, \qquad m_3J_3 = m_1J_1 + m_2J_2 \ . \tag{5.18}$$

Note, that the non-Cartan components are zero only for $m_1 \neq m_2$. If $m_1 = m_2$ the solution can always be rotated to a two-spin one ($\mathcal{S}_2 = 0$).

5.2 Sine-Cosine solutions

A particularly simple non-trivial solution from ansatz (5.7) corresponds to

$$F_1(\sigma_+) = \alpha\cos n\sigma_+ \ , \qquad F_2(\sigma_+) = \alpha\sin n\sigma_+ \ . \tag{5.19}$$

It reads

$$
\begin{aligned}
&\kappa = 2m \sin\gamma \\
&X_1 = \frac{1}{\sqrt{2}}\cos\gamma \;\sin\left[\sqrt{2}\,\frac{m}{n}\tan\gamma\,\cos(n\sigma_+)\right] \\
&X_2 = \frac{1}{\sqrt{2}}\cos\gamma \;\sin\left[\sqrt{2}\,\frac{m}{n}\tan\gamma\,\sin(n\sigma_+)\right] \\
&X_3 = \frac{1}{\sqrt{2}}\cos\gamma \;\cos\left[\sqrt{2}\,\frac{m}{n}\tan\gamma\,\cos(n\sigma_+)\right] \\
&X_4 = \frac{1}{\sqrt{2}}\cos\gamma \;\cos\left[\sqrt{2}\,\frac{m}{n}\tan\gamma\,\sin(n\sigma_+)\right] \\
&X_5 = \sin\gamma\cos(m\sigma_-) \\
&X_6 = \sin\gamma\sin(m\sigma_-)
\end{aligned}
\tag{5.20}
$$

Here n, m are integers, $a_1 = a_2 = \frac{1}{\sqrt{2}}\cos\gamma \neq 0$, $a_3 = \sin\gamma \neq 0$. Snap shots of the string at $\tau = 0$ and $\tau = \frac{\pi}{4}$ are given on figure 1. One could see how it changes shape: a bended circle at $\tau = 0$ and folded (in projection on $X_1X_3X_5$) at $\tau = \frac{\pi}{2}$.

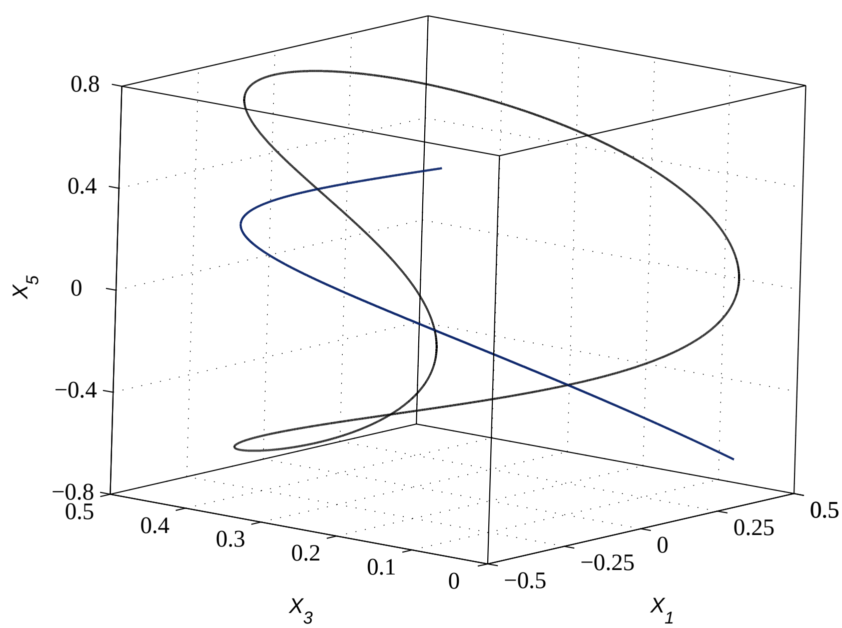

Figure 1: Shape of the string for $n = 1,\ m = 1,\ \gamma = \frac{\pi}{4}$: a bended circle at $\tau = 0$ and folded (in projection on $X_1X_3X_5$) at $\tau = \frac{\pi}{2}$

The energy and spins are (see Appendix C):

$$
\begin{aligned}
&\mathcal{E} = 2m\sin\gamma \\
&\mathcal{J}_1 = \mathcal{J}_{12} = \frac{1}{2} m \sin(2\gamma)\ \mathrm{BesselJ}_1\left[2\frac{m}{n}\tan\gamma\right] \\
&\mathcal{J}_3 = \mathcal{J}_{56} = m\sin^2\gamma
\end{aligned}
\tag{5.21}
$$

For fixed n, m we find

$$\mathcal{E} = 2\sqrt{m\mathcal{J}_3}, \qquad \mathcal{J}_1 = \sqrt{\mathcal{J}_3(m-\mathcal{J}_3)}\text{BesselJ}_1\left[2\frac{m}{n}\sqrt{\frac{\mathcal{J}_3}{m-\mathcal{J}_3}}\right], \tag{5.22}$$

or, restoring λ,

$$E = 2\sqrt{m\sqrt{\lambda}\ J_3}, \qquad J_1 = \sqrt{m\sqrt{\lambda}\ J_3\left(1-\frac{J_3}{m\sqrt{\lambda}}\right)}\text{BesselJ}_1\left[2\frac{m}{n}\sqrt{\frac{\frac{J_3}{m\sqrt{\lambda}}}{1-\frac{J_3}{m\sqrt{\lambda}}}}\right]. \tag{5.23}$$

The dependance $\mathcal{J}_1(\mathcal{J}_3)$ is presented on figure 2.

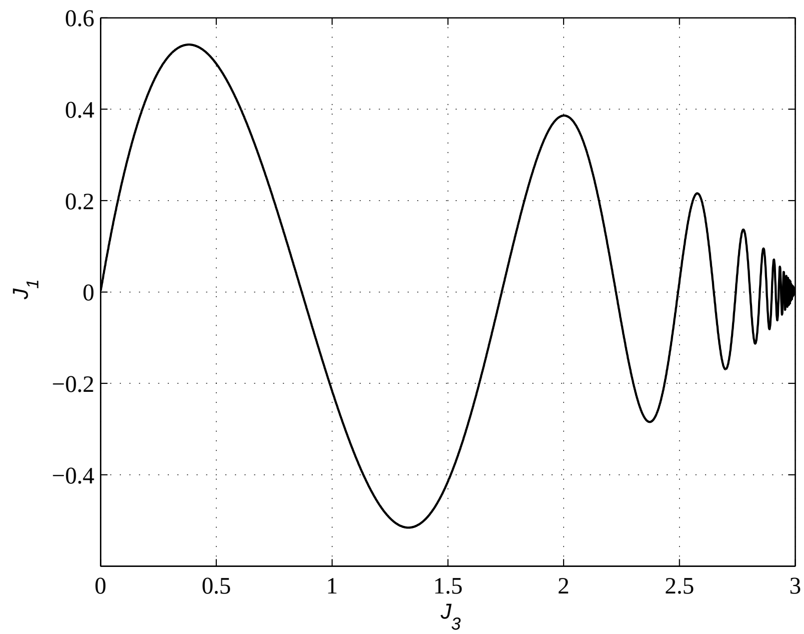

Figure 2: $\mathcal{J}_1(\mathcal{J}_3)$ for $n=1,\ m=3$.

Solution (5.20) does not admit large-spin limit, as the values of spins are bounded above:

$$J_3 \le J_3^{max} = m\sqrt{\lambda}, \qquad J_1 \le J_1^{max} \approx 0,6\frac{n\sqrt{\lambda}}{1+\frac{n^2}{m^2}}. \tag{5.24}$$

In the small-spin limit, expanding the Bessel function in the expression for $\mathcal{J}_1$, we obtain

$$E = 2\sqrt{m\sqrt{\lambda}\ J_3}, \qquad J_1 = J_3\frac{m}{n}\left(1-\frac{1}{2}\frac{m^2}{n^2}\frac{J_3}{m\sqrt{\lambda}} + O(\lambda^{-1})\right). \tag{5.25}$$

Let us show that in the limit $n \to 0$, certain solutions of type (5.20) reduce to (5.14).

In that limit from (5.20) one finds

$$\begin{gathered}\kappa = 2m\sin\gamma \\ X_1 = \frac{1}{\sqrt{2}}\cos\gamma, \qquad X_3 = \frac{1}{\sqrt{2}}\cos\gamma \\ X_2 = \frac{1}{\sqrt{2}}\cos\gamma\ \sin\left[\sqrt{2}\ m\tan\gamma\sigma_+\right], \qquad X_4 = \frac{1}{\sqrt{2}}\cos\gamma\ \cos\left[\sqrt{2}\ m\tan\gamma\,\sigma_+\right] \\ X_5 = \sin\gamma\cos(m\sigma_-), \qquad X_6 = \sin\gamma\sin(m\sigma_-)\end{gathered} \tag{5.26}$$

Here we omitted the infinite phases in $X_1,\ X_3$, coming from $\frac{\cos(n\sigma_+)}{n}$, as they do not contribute into the consideration.

Relations (5.26) look like ones describing rigid chiral solution with two spins. Indeed one could rewrite them as

$$\begin{gathered}\kappa = 2a_3m \\ X_1 = a, \qquad X_3 = a, \\ X_2 = a\sin(k\sigma_+), \qquad X_4 = a\cos(k\sigma_+) \\ X_5 = c\cos(m\sigma_-), \qquad X_6 = c\sin(m\sigma_-)\end{gathered} \tag{5.27}$$

where $a = \frac{1}{\sqrt{2}}\cos\gamma$, $c = \sin\gamma$, $k = \sqrt{2}m\tan\gamma$. However k is in general arbitrary, so (5.27) corresponds to the rigid chiral solutions only when

$$k = \sqrt{2}m\tan\gamma \quad \text{is integer.}$$

The expression for the AdS_5 energy does not changes in $n \to 0$ limit. The spins transform to (use integral expressions from appendix C and restore $m, n \neq 1$)

$$\begin{gathered}\mathcal{J}_1 = \mathcal{J}_{12} \to 0, \qquad \mathcal{J}_2 = \mathcal{J}_{34} \to 0, \qquad \mathcal{J}_3 = \mathcal{J}_{56} = mc^2, \\ \mathcal{J}_{24} = \frac{\sqrt{2}}{2}m\tan\gamma\cos^2\gamma\frac{\sin(n\sigma_+)}{2\pi n}\Bigg|_0^{2\pi} \to ka^2\end{gathered} \tag{5.28}$$

in exact agreement with (5.17).

Being expressed, as harmonic functions with the argument (5.8), solutions (5.7) are not easy to analyze in terms of stability. Straight forward analysis may be performed only for the rigid chiral solutions (5.14), which were proved stable in [12]. Thus one may also expect (5.20) being stable, due to their relation to (5.14).

One may hope that generalization of $F_1(\sigma_+) = \alpha\cos n\sigma_+$ to

$$F_1(\sigma_+) = m_1\sigma_+ + \beta_1\sin(n_1\sigma_+) \tag{5.29}$$

would also give a simple solution. However, in this case, from (5.19) we find that F_2 is expressed via elliptic functions E, F and Π all together. Finding anther simple solutions from ansatz (5.20) is an open question.

Summary

In this paper we have discussed several classical solutions for a closed bosonic string in the $AdS_5 \times S^5$.

First, we considered small rigid strings with two spins in AdS_5 part of $AdS_5 \times S^5$. Starting from the flat-space solutions (3.15) and using perturbation theory in the curvature of AdS_5 space, we constructed leading terms in the small two-spin solution and found corrections to the leading Regge term in the classical string energy (3.38) and (3.47). We uncovered a discontinuity in the spectrum of classical strings with equal and unequal winding numbers in the Y_1Y_2 and Y_3Y_4 planes (n_1 and n_2). In the limit $n_1 = n_2$ the expression for $E_{n_1 \neq n_2}(S_1, S_2; \lambda)$ does not coincide with $E_{n_1 = n_2}(S_1, S_2; \lambda)$. We then investigated the connection between small-spin (flat-space) and large-spin limits of two-spin string solutions in AdS_5. For the $\omega_1 = \omega_2$ (i.e. $S_1 = S_2$) case we found that, apart from the trivial cases of folded and circular strings, the general rigid solution with $S_1 = S_2$ in AdS_5 admitting the large-spin limit does not have a small-spin limit.

In the second part of the paper we constructed a new class of chiral solutions in $R_t \times S^5$ for which the embedding coordinates of S^5 satisfy the linear Laplace equations (5.4). We used the ansatz (5.7) and obtained the general solution for it in the form (5.12). These solutions generalize the previously studied rigid string chiral solutions (5.14) [12]. We studied in detail a simple non-trivial example of these solutions (5.20).

There are a number of open questions that we leave for future investigation. It is of interest to find solutions in full AdS_5which correspond to more general flat-space solutions than rigid and folded ones. The relation between small-spin and large-spin limits should be clarified. So far, it looks plausible that there is no connection between them apart in the trivial limits. The origin of the discontinuity in the spectrum of small-string solutions with $n_1 = n_1$ and $n_1 \neq n_1$ is also not quite clear. Another direction is to study possible applications of chiral solutions (5.12).

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A Appendix: Circular and folded strings in AdS_5

A.1 The circular string solution

A particular simple solution of equations (2.5), (2.3), (2.4) is the rigid circular rotating string [4, 12, 16]:

$$Y_{05} = \frac{\sqrt{m^2 + w^2}}{\sqrt{2}m} e^{i\kappa\tau}, \qquad Y_{12} = \frac{\kappa}{2m} e^{i\omega\tau + im\sigma}, \qquad Y_{34} = \frac{\kappa}{2m} e^{i\omega\tau - im\sigma}, \tag{A.1}$$

where $w = \sqrt{m^2 + \kappa^2}$. It can also be rewritten in the form

$$Y_{05} = \sqrt{1 + r^2} e^{i\kappa\tau}, \qquad \tilde{Y}_{12} = r\cos(m\sigma) e^{i\omega\tau}, \qquad \tilde{Y}_{34} = r\sin(m\sigma) e^{i\omega\tau}, \tag{A.2}$$

where $\omega = m\sqrt{1 + 2r^2}$ and $r = \sinh\rho_0 = \frac{\kappa}{\sqrt{2}m}$ is a radius of the string. This is a consistent closed-string solution periodic in $O \leq \sigma < 2\pi$.

The two spins of the string are equal $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$ and are related to the energy by

$$\mathcal{E} = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{\kappa^2 + m^2}} , \qquad \mathcal{S} = \frac{\kappa^2}{4m^2}\sqrt{m^2+\kappa^2} . \tag{A.3}$$

In the small-string limit ($\mathcal{S} \to 0$) the profile of the string reads

$$Y_{05} \approx (1+\frac{1}{2}\,\epsilon^2\, a^2)\, e^{i\sqrt{2}\ \epsilon\ am\tau}, \qquad Y_{12} \approx a\cos(\sigma)\, e^{im(1+\epsilon^2 a^2)\ \tau}, \qquad Y_{34} \approx a\sin(\sigma)\, e^{im(1+\epsilon^2 a^2)\ \tau}. \tag{A.4}$$

The expression for the classical energy in this limit is

$$\mathcal{E} = 2\sqrt{m\mathcal{S}}\left(1+\frac{\mathcal{S}}{m} + O(\mathcal{S}^2)\right) \qquad \text{or} \qquad E = 2\sqrt{m\sqrt{\lambda}S}\left(1+\frac{S}{m\sqrt{\lambda}} + O(\lambda^{-1})\right). \tag{A.5}$$

Here the classical energy contains a non-trivial curvature corrections which modify the leading-order flat-space Regge behavior.

A.2 Folded string solution

Another simple solution of equations (2.5), (2.3), (2.4) is the classical solution for the folded string spinning in the AdS_3 part of AdS_5

$$ds^2 = -\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\phi^2$$

described by [2, 17]

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma), \tag{A.6}$$

where

$$\rho'^2 = \kappa^2\cosh^2\rho - w^2\sinh^2\rho . \tag{A.7}$$

ρ varies from 0 to its maximal value ρ_*

$$\coth^2\rho_* = \frac{w^2}{\kappa^2} \equiv 1 + \frac{1}{l^2} . \tag{A.8}$$

Thus l measures the length of the string. The solution of the differential equation (A.7), i.e.

$$\rho' = \pm\kappa\sqrt{1 - l^{-2}\sinh^2\rho} , \qquad \rho(0) = 0 \tag{A.9}$$

can be written in terms of the Jacobi function sn

$$\sinh\rho = l\ \mathrm{sn}(\kappa l^{-1}\sigma,\ -l^2) . \tag{A.10}$$

The periodicity in σ implies the following condition on the parameters [2]

$$\kappa = l\ {}_2F_1(\frac{1}{2},\frac{1}{2};1;-l^2) . \tag{A.11}$$

The classical energy $E = \sqrt{\lambda}\mathcal{E}$ and the spin $S = \sqrt{\lambda}\mathcal{S}$ are found to be

$$\mathcal{E} = l\ {}_2F_1(-\frac{1}{2},\frac{1}{2};1;-l^2), \qquad \mathcal{S} = \frac{l^2}{2}\sqrt{1+l^2}\ {}_2F_1(\frac{1}{2},\frac{3}{2};2;-l^2) . \tag{A.12}$$

Here we will be interested in the short string limit $0 < \epsilon \ll 1$, $l = a\epsilon$ in which

$$\rho_* = a\epsilon - \frac{1}{6}\epsilon^3 a^3 + O(\epsilon^5) \ . \tag{A.13}$$

In the strict limit $a = 0$ or $\kappa = 0$ we get $\rho = \rho_* = 0$, so that the string shrinks to a point with $E = 0$.

From (A.12) in the $\epsilon \ll 1$ or the small $\mathcal{S}$ limit we obtain

$$\mathcal{E} = \sqrt{2\mathcal{S}} \left(1 + \frac{3}{8}\mathcal{S} + O(\mathcal{S}^2) \right) , \tag{A.14}$$

so the short string limit corresponds to $\mathcal{S} \ll 1$ and the expansion of the energy looks like

$$E = \sqrt{2\sqrt{\lambda}S} \left(1 + \frac{3}{8\sqrt{\lambda}}S + O(\lambda^{-1}) \right) . \tag{A.15}$$

Expanding the exact solution (A.10) in powers of ϵ we obtain

$$\sinh\rho = \epsilon \ a \ \sin\sigma - \epsilon^3 \ \frac{a^3}{16}(\sin(3\sigma) + \sin\sigma) + O(\epsilon^5) \tag{A.16}$$

or equivalently, changing phase $\sigma \to \frac{\pi}{2} - \sigma$

$$\sinh\rho = \epsilon \ a \ \cos\sigma - \epsilon^3 \ \frac{a^3}{16}(-\cos(3\sigma) + \cos\sigma) + O(\epsilon^5). \tag{A.17}$$

For the frequencies we have

$$\omega = 1 + \epsilon^2 \ \frac{a^2}{4} + O(\epsilon^4), \qquad \kappa = \epsilon \ a - \epsilon^3 \ \frac{1}{4} \ a^3 + O(\epsilon^4). \tag{A.18}$$

B Appendix: Folded string displaced from the AdS_5 center ($n_2 = 0$).

The possibility omitted in section 3.3 is when one of the frequencies of the original flat-space solutions (n_i) is zero, while the "amplitude" $y_i = \text{const} \neq 0$[5]. We will look for the solutions of (3.8), (3.9) in the form:

$$\begin{aligned} y_1(\sigma) &= \epsilon \ a\sin(\sigma n) + \epsilon^3 z_1(\sigma) \\ y_2(\sigma) &= \epsilon \ b + \epsilon^3 z_2(\sigma), \end{aligned} \tag{B.1}$$

where $n \in Z$ and

$$\begin{aligned} \omega_1 &= n(1 + \epsilon^2 \ \tilde{\omega}_1), \qquad & \omega_2 &= \epsilon \ \tilde{\omega}_2 \\ \kappa &= \epsilon \ \kappa_0 + \epsilon^3 \ \kappa_1, \qquad & \kappa_0^2 &= a^2 n^2. \end{aligned} \tag{B.2}$$

It follows from (3.13), that expansion of ω_i^2 must consist of the even powers of ϵ. So if $n_2 = 0$ the leading order of ω_2 is ϵ.

From (3.8), (3.9) one obtains the set of equations:

$$-b \ (n^2 z_1 + z_1'') + a\sin(n\sigma) \ z_2'' = 2ab\sin(n\sigma)(\tilde{\omega}_1 \ n^2 - \tilde{\omega}_2^2) \tag{B.3}$$

$$\begin{aligned} 2an \ [\sin(n\sigma)nz_1 + \cos(n\sigma)z_1'] &= 2\chi - b^2\tilde{\omega}_2^2 + a^2b^2n^2 \\ &\quad -2a^2 \ \tilde{\omega}_1 \ n^2\sin^2(n\sigma) + 2a^4n^2\sin^2(n\sigma) - a^4n^2\sin^4(n\sigma). \end{aligned} \tag{B.4}$$

[5] One may also consider perturbations under a flat-space solution with $n_1 = n_2 = 0$, i.e. a point-like string displaced from the center of AdS_5. There are no closed string solutions in this limit.

Here $\chi^2 = \kappa_1^2 \kappa_0^2$. These system can be readily solved. The solution of (B.4) is straight forward:

$$\begin{aligned} z_1 = C_1 \cos(n\sigma) + an\ \sigma\ \cos(n\sigma) \left(\tilde{\omega}_1 - \frac{1}{4}a^2 \right) + \frac{\sin(n\sigma)}{2an^2}(2\chi - b^2 \tilde{\omega}_2^2) \\ -a \sin(n\sigma) \left(\tilde{\omega}_1 - \frac{1}{2}(a^2 + b^2) \right) - \frac{1}{4}a^3 \sin^2(n\sigma)\cos(n\sigma). \end{aligned} \tag{B.5}$$

Employing the closed string periodicity condition (2.4), one finds

$$\tilde{\omega}_1 = \frac{a^2}{4}. \tag{B.6}$$

Then the solution for z_2 is

$$z_2 = C_2 + \sigma C_3 - \frac{1}{4}a^2 b \cos(2n\sigma) + \sigma^2 \frac{b}{2} \left(a^2 n^2 - \tilde{\omega}_2^2 \right). \tag{B.7}$$

Making use of the closed string periodicity conditions, one finds

$$\tilde{\omega}_2 = \pm an, \qquad C_3 = 0\ . \tag{B.8}$$

There is no additional constraints on the parameters C_1, C_2, χ, so the solution of (B.3), (B.4) is

$$\begin{aligned} z_1 = C_1 \cos(n\sigma) + \frac{1}{16}a^3(3\sin(n\sigma) - \sin(3n\sigma)) + \frac{\kappa_1 \sin(n\sigma)}{n} \\ z_2 = C_2 - \frac{1}{4}a^2 b \cos(2n\sigma) \\ \omega_1 = n(1 + \epsilon^2\ \frac{a^2}{4}), \qquad \omega_2 = \pm\epsilon\ an, \qquad \kappa = \epsilon\ an + \epsilon^3\ \kappa_1. \end{aligned} \tag{B.9}$$

It is not hard to see, that due to $\kappa \approx \omega_2$, non-Cartan components of the spin $\mathcal{S}_{0i}$ do not vanish. This solution can be rotated by boost to a folded string one.

C Appendix: Spins for the Sine-Cosine solutions

In this section we will calculate the components of spin $\mathcal{J}_{ij} = \int\limits_0^{2\pi} \frac{d\sigma}{2\pi}[X_i \dot{X}_j - X_j \dot{X}_i]$ for the Sine-Cosine solutions (5.20). Set for the simplicity $n = m_3 = 1$.
Cartan components of the spin are

$$\begin{aligned} \mathcal{J}_1 = \mathcal{J}_{12} &= \int\limits_0^{2\pi} \frac{d\sigma}{2\pi}[X_1 \dot{X}_2 - X_2 \dot{X}_1] \\ &= \frac{1}{4}\sin(2\gamma) \int\limits_0^{2\pi} \frac{d\sigma}{2\pi} \left[\sin(\tau + \sigma + \pi/4) \sin(2\tan\gamma \sin(\tau + \sigma + \pi/4)) \right. \\ &\qquad \left. + \sin(\tau + \sigma - \pi/4) \sin(2\tan\gamma \sin(\tau + \sigma - \pi/4)) \right] \\ &= \frac{1}{2}\sin(2\gamma) \int\limits_0^{2\pi} \frac{d\zeta}{2\pi} \sin(\zeta) \sin(2\tan\gamma \sin(\zeta)) = \frac{1}{2}\sin(2\gamma) \mathrm{BesselJ}_1(2\tan\gamma) \end{aligned} \tag{C.1}$$

(see Appendix D for a proof of the equality on the last line);

$$
\begin{aligned}
\mathcal{J}_2 = \mathcal{J}_{34} &= \int_0^{2\pi} \frac{d\sigma}{2\pi}[X_3\dot{X}_4 - X_4\dot{X}_3] \\
&= -\frac{1}{4}\sin(2\gamma)\int_0^{2\pi}\frac{d\sigma}{2\pi}\,[\sin(\tau+\sigma+\pi/4)\sin(2\tan\gamma\sin(\tau+\sigma+\pi/4)) \\
&\qquad\qquad - \sin(\tau+\sigma-\pi/4)\sin(2\tan\gamma\sin(\tau+\sigma-\pi/4))] \\
&= \frac{1}{4}\sin(2\gamma)\int_0^{2\pi}\frac{d\zeta}{2\pi}\,[\sin(\zeta)\sin(2\tan\gamma\sin(\zeta)) - \sin(\zeta)\sin(2\tan\gamma\sin(\zeta))] = 0\ ;
\end{aligned}
\tag{C.2}
$$

$$
\mathcal{J}_3 = \mathcal{J}_{56} = \int_0^{2\pi}\frac{d\sigma}{2\pi}[X_5\dot{X}_6 - X_6\dot{X}_5] = \sin^2\gamma\ . \tag{C.3}
$$

Non-Cartan components of the spin are

$$
\mathcal{J}_{13} = \int_0^{2\pi}\frac{d\sigma}{2\pi}[X_1\dot{X}_3 - X_3\dot{X}_1] = \frac{1}{2\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\sigma}{2\pi}\sin(\tau+\sigma) = 0\ ; \tag{C.4}
$$

$$
\mathcal{J}_{24} = \int_0^{2\pi}\frac{d\sigma}{2\pi}[X_2\dot{X}_4 - X_4\dot{X}_2] = -\frac{1}{2\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos(\tau+\sigma) = 0\ ; \tag{C.5}
$$

$$
\begin{aligned}
\mathcal{J}_{14} &= \int_0^{2\pi}\frac{d\sigma}{2\pi}[X_1\dot{X}_4 - X_4\dot{X}_1] \\
&= \frac{1}{2\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\sigma}{2\pi}\,[\sin(\tau+\sigma-\pi/4)\cos(2\tan\gamma\sin(\tau+\sigma-\pi/4)) \\
&\qquad\qquad + \sin(\tau+\sigma+\pi/4)\cos(2\tan\gamma\sin(\tau+\sigma+\pi/4))] \\
&= \frac{1}{\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\zeta}{2\pi}\sin(\zeta)\cos(2\tan\gamma\sin(\zeta)) \\
&= \frac{1}{\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\zeta}{2\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}(2\tan\gamma)^{2l}\sin^{2l+1}(\zeta) = 0\ ;
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
\mathcal{J}_{23} &= \int_0^{2\pi}\frac{d\sigma}{2\pi}[X_2\dot{X}_3 - X_3\dot{X}_2] = \\
&= \frac{1}{2\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\sigma}{2\pi}\,[\sin(\tau+\sigma-\pi/4)\cos(2\tan\gamma\sin(\tau+\sigma-\pi/4)) \\
&\qquad\qquad - \sin(\tau+\sigma+\pi/4)\cos(2\tan\gamma\sin(\tau+\sigma+\pi/4))] \\
&= \frac{1}{2\sqrt{2}}\sin(2\gamma)\int_0^{2\pi}\frac{d\zeta}{2\pi}\,[\sin(\zeta)\cos(2\tan\gamma\sin(\zeta)) - \sin(\zeta)\cos(2\tan\gamma\sin(\zeta))] = 0\ .
\end{aligned}
\tag{C.7}
$$

Here we used, that integral over period from odd powers of sine or cosine is zero [18]

$$\int_0^{2\pi} d\zeta \sin^{2l+1}\zeta = 0, \qquad \int_0^{2\pi} d\zeta \cos^{2l+1}\zeta = 0. \tag{C.8}$$

To prove that $\mathcal{J}_{5j} = \mathcal{J}_{6j} = 0$, $j = 1,2,3,4$, consider the following expansion of X_{ij} :

$$X_{12} = \sum_{l=0}^{\infty} g_l^{(1)} e^{il(\sigma+\tau)}, \qquad X_{34} = \sum_{l=0}^{\infty} g_l^{(2)} e^{il(\sigma+\tau)}, \qquad X_{56} = \sum_{l=0}^{\infty} h_l e^{il(\sigma-\tau)}. \tag{C.9}$$

One can show that "cross-spins" (non-Cartan components of spins) between right- and left-chiral waves always vanish, i.e. for each pair of right- and left-chiral summands in (C.9):

$$Z_1 + iZ_2 = Ge^{in(\sigma+\tau)}, \qquad Z_3 + iZ_4 = He^{im(\sigma-\tau)}, \qquad n, m = \text{integer} \tag{C.10}$$

the correspondent contribution ($\mathcal{J}_{ij}^Z$) into $\mathcal{J}_{5j},\ \mathcal{J}_{6j},\ j = 1,2,3,4$ is zero.

Let us calculate the following values

$$\begin{aligned}
\mathcal{J}_+^Z &= \int_0^{2\pi} \frac{d\sigma}{2\pi}[z_1\dot{z}_2 - z_2\dot{z}_1] = [\mathcal{J}_{13}^Z - \mathcal{J}_{24}^Z] + i[\mathcal{J}_{23}^Z + \mathcal{J}_{14}^Z] \\
&= i\int_0^{2\pi} \frac{d\sigma}{2\pi} GH\ (m-n)e^{i\sigma(n-m)+i\tau(n+m)} = \begin{cases} 0, & \text{for} \quad \mathrm{m = n} \\ 0, & \text{for} \quad \mathrm{m \neq n} \end{cases} \\
\mathcal{J}_-^Z &= \int_0^{2\pi} \frac{d\sigma}{2\pi}[z_1\dot{z}_2^+ - z_2^+\dot{z}_1] = [\mathcal{J}_{13}^Z + \mathcal{J}_{24}^Z] + i[\mathcal{J}_{23}^Z - \mathcal{J}_{14}^Z] \\
&= i\int_0^{2\pi} \frac{d\sigma}{2\pi} GH\ (m-n)e^{i\sigma(n-m)+i\tau(n+m)} = \begin{cases} 0, & \text{for} \quad \mathrm{m = n} \\ 0, & \text{for} \quad \mathrm{m \neq n.} \end{cases}
\end{aligned} \tag{C.11}$$

"Cross-spins" for each left-right chiral pair in the expansion (C.9) vanish. We have

$$\mathcal{J}_{5j} = \mathcal{J}_{6j} = 0, \qquad j = 1,2,3,4. \tag{C.12}$$

D Appendix: Bessel functions

In this section we will prove the relation

$$\mathrm{BesselJ}_1(x) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \sin\alpha \sin(x\sin\alpha). \tag{D.1}$$

Two formulas from the theory of the Bessel functions are of use [19]:

- Integral representation of the Bessel functions

$$\mathrm{BesselJ}_n(x) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-ix\ \sin\alpha + in\alpha}. \tag{D.2}$$

- Recurrent formula

$$\frac{d}{dx}\left(\frac{\text{BesselJ}_\nu(x)}{x^\nu}\right) = \frac{\text{BesselJ}_{\nu+1}(x)}{x^\nu}. \tag{D.3}$$

Let us take a derivative from $\text{BesselJ}_0(x)$ in the integral representation:

$$\frac{d}{dx}\text{BesselJ}_0(x) = -i\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\sin\alpha\; e^{-ix\,\sin\alpha} = \int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\left[-i\sin\alpha\cos\left(x\sin\alpha\right) - \sin\alpha\sin\left(x\sin\alpha\right)\right]. \tag{D.4}$$

The Taylor expansion of $\sin\left(x\sin\alpha\right)$ and $\cos\left(x\sin\alpha\right)$ consist of odd and even powers of $\sin\alpha$, respectively. Making use of (C.8), one finds

$$\frac{d}{dx}\text{BesselJ}_0(x) = -\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}\sin\alpha\sin\left(x\sin\alpha\right). \tag{D.5}$$

Then employing (D.3):

$$\frac{d}{dx}\text{BesselJ}_0(x) = -\text{BesselJ}_1(x), \tag{D.6}$$

we end up with (D.1).

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