Large-N limits of 2d CFTs, Quivers and AdS_3 duals

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Abstract: We explore the large-N limits of 2d CFTs, focusing mostly on the WZW model and cosets. The $SU(N)_k$ theory is parametrized in this limit by a 't Hooft-like coupling. We show a duality between strong coupling where the theory is described by almost free fermions and weak coupling where the theory is described by bosonic fields by an analysis of the spectra and correlators. The AdS_3 dual is described, and several quantitative checks are performed. We find ground states with large degeneracy that can dominate the standard Cardy entropy at weak coupling and should correspond to bulk black holes.

Keywords: ..

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1. Introduction and Outlook

Two dimensional conformal field theories (CFT's) have been studied extensively over the years. There are many solvable examples that admit a large-N limit. The exact solvability of these theories in generic regimes of parameter space makes them very attractive as toy models for questions that are typically very hard for analogous field theories in higher dimensions. One can compute the exact spectrum and in some cases also exact correlation functions. In this paper we revisit a special set of two dimensional CFT's: WZW models (and cosets thereof) based on the $SU(N)$ group. Our interest in these models stems from the following observations:

(i) On general grounds it is expected that conformal field theories with a large-N expansion have a dual description in terms of a string theory on AdS. For the AdS_3/CFT_2 correspondence the canonical example is provided by the symmetric orbifold of N copies of the 2d sigma model with target space $\mathcal{M}_4$, which has been argued to be dual to type IIB string theory on $AdS_3 \times S^3 \times \mathcal{M}_4$ (the large-N diagrammatic expansion of the symmetric orbifold theory has been discussed recently in [1]). In order to gain a deeper insight into the general principles underlying the AdS/CFT correspondence it is important to extend the list of known examples. Situations involving solvable CFT's are of obvious interest and hence a natural question is whether it is possible to identify the string theory duals of well-known exact 2d CFT's. In the process one hopes to uncover qualitatively new features of the AdS/CFT correspondence.

(ii) We will discuss 2d theories that have interesting analogies/connections to 3d(4d) field theories. For example, WZW models are well-known to be connected to 3d Chern-Simons (topological) theories via a bulk-boundary correspondence, albeit not a holographic one [2, 3]. Roughly, quantization of a Chern-Simons (CS) theory with (simple) group G and coefficient $k \in Z$ on $R \times \Sigma$ where Σ is a closed 2d Riemann surface, provides a Hilbert space with states that in one-to-one correspondence with the conformal blocks of the G_k current algebra on Σ.

CS theories coupled to matter also give rise to a generic class of 3d CFT's. Examples with $\mathcal{N} = 2, 4, 6, 8$ supersymmetries have been argued to describe the (low-energy) world-volume theories on multiple M2 branes (see [4] for a prototype example). M2 branes have two dimensional boundaries when they end on M5 branes. The theory on these intersections (self-dual strings) is expected to be a 2d CFT. Recent work [5] indicates that this theory involves a WZW model part.

The AdS_4/CFT_3 duality for the CSM theories on M2-branes predicts that there is a drastic reduction of the degrees of freedom as one moves from weak to strong 't Hooft coupling. We will observe a qualitatively similar reduction of the degrees of freedom at strong 't Hooft coupling in two dimensional large-N CFT's.

A good 2d analogue of the (conformal) CS gauge interactions in three dimensions are the quadratic non-derivative gauge interactions exemplified by two dimensional gauged WZW models. We will discuss a gauged WZW model that exhibits full level-rank duality [6]. This particular duality bears many similarities with the Seiberg-like duality of the one-adjoint A_k CSM theories in three dimensions [7] and the Seiberg-Kutasov duality of corresponding one-adjoint SQCD theories in four dimensions [8]. A relation between level-rank duality in $SU(N)_k$ WZW models and a Seiberg-like duality in topological $\mathcal{N} = 2$ CS theories was also pointed out in [9]. The distinguishing feature of the level-rank duality that we will discuss here is that it extends to the full theory and not just to the level of conformal blocks. Accordingly, it bears similarities with 3d (and 4d) Seiberg duality in non-topological theories.

The theories we will be focusing on are the WZW model $SU(N)_k$, [10, 11] and its avatars, namely the coset theories, [12, 13] , $SU(M+N)_k/SU(N)_k$ and $SU(N)_{k_1} \times SU(N)_{k_2}/(SU(N)_{k_1+k_2}$. There are more general cosets, as well as large generalizations of the coset construction, [14, 15] but we will not consider them here.

We will show that the generating theory, the $SU(N)_k$ WZW model, has an interesting 't Hooft (or Veneziano-like) large N limit where $N \to \infty$, $k \to \infty$ with $\frac{N}{k} = \lambda$ fixed.

Moreover we will show that this theory has two dual descriptions. At weak coupling, $\lambda \to 0$, the weakly coupled description is the conventional WZW model, written in terms of a bosonic field g that transforms as a bifundamental under the current algebra symmetry $SU(N)_L \times SU(N)_R$.

At strong coupling $\lambda \to \infty$, the weakly coupled description is in terms of the IR limit of N copies of massless Dirac fermions transforming in the fundamental representation of a $U(k)$ gauge group. This looks like a conventional gauge theory[1] and in this language k

[1]Although the YM action is irrelevant in the IR in two dimensions.

is color, N is flavor and $\lambda = \frac{N}{k}$ is the Veneziano ratio. The bosonic field g corresponds the fermion mass operator in the strongly coupled regime. In this sense this theory can be thought of as a gauge theory with quarks where although there is confinement, the theory is conformal in the IR and there is no chiral symmetry breaking.

This picture is corroborated by studies of the spectrum and four-points functions. This study also gives a concrete example of the non-commutativity of the two limits $N \to \infty$ and $\lambda \to \infty$. The central charge scales as $\mathcal{O}(N^2)$, but also depends on λ. At strong coupling there is a drastic reduction of the number of degrees of freedom as attested by the value of c, not unlike a similar effect in AdS_4/CFT_3.

The spectrum is comprised by affine primary ground-states and excitations over them generated by the current modes. Scaling dimensions in the large N-limit take values from $\mathcal{O}(1)$ to $\mathcal{O}(N^2)$ for ground-states corresponding to representations with about $kN/2$ boxes in the Young tableau. Such states have multiplicities that can compete with the Cardy formula.

Such a theory is expected to have an AdS_3 dual with $SU(N)_L \times SU(N)_R$ symmetry. The closed string sector is expected to be trivial and the dependence on the metric is induced on the boundary by the proper boundary conditions on the open string sector as recognized already in [16]. The open string sector is expected to be realized in a way similar to the one advocated for flavor in higher dimensions. D_2 and $\bar{D}_2$ branes generate the gauge symmetry which at low energy is realized by two CS actions with couplings of opposite sign. A direct computation of the effective action for currents is in agreement with the CFT calculation using the WZW model.

The study of scaling dimensions indicates that in the weak coupling limit, the spectrum of ground states can be made to have a large gap from the stringy states. This suggests that in that limit, a "gravity" description of the physics is possible. In the strong coupling limit all dimensions are of the same order and therefore a stringy bulk description is necessary.

The ground states are generated by a bifundamental field T that should correspond to an open string stretching between D_2 and $\bar{D}_2$. This is a picture analogous to the one in [17]. Other ground states in the CFT correspond to multi-particle states. Sources for T correspond to a mass matrix in the fermionic language. They generate a flow that drives the theory to an IR fix point equivalent to $SU(N-r)_k$ where r the rank of the source of T.

The states with scaling dimensions of $\mathcal{O}(N^2)$, have masses that can be comparable to M_p. We find that they are of the order of M_{planck} at strong coupling and much larger at weak coupling. Therefore, at weak coupling these states should correspond to macroscopic smooth solutions of the bulk theory, with associated flavor hair. This hair is responsible for their entropy, and in this case it dominates the Cardy entropy.

The cosets are also very interesting CFTs with several possible large-N limits we analyze. They also have dual versions , between gauged-WZW models and quiver gauge theories coupled to massless quarks.

The $SU(M+N)_k/SU(N)_k$ theory can be thought of also as a quiver. The gauge group is $U(N)\times U(k)$ and the $k(N+M)$ Dirac fermions transform as $(\square,\overline{\square})$ under $(SU(N),U(k))$ and as M copies of $(\square,1)$, having global chiral symmetry $SU(M)_L\times SU(M)_R$.

Finally the $U(k_1+k_2)_N/U(k_1)_N\times U(k_2)_N$ CFT can be thought of as a quiver gauge theory with gauge group $U(k_1)\times U(k_2)\times SU(N)$ and massless quarks transforming as $(\square,1,\overline{\square})$ and $(1,\square,\overline{\square})$ under the gauge group. This description automatically explains the rank-level duality symmetry that states that

$$\frac{U(k_1+k_2)_N}{U(k_1)_N\times U(k_2)_N}\sim\frac{SU(N)_{k_1}\times SU(N)_{k_2}}{SU(N)_{k_1+k_2}} \tag{1.1}$$

The subclass of coset models $\frac{SU(N)_{k_1}\times SU(N)_1}{SU(N)_{k_1+1}}$, giving rise to the W_N minimal models has been analyzed recently in [18]. Its closed string sector has been argued to correspond to the quantum hamiltonian reduction $SL(N,C)\to W_N$. Such theories provide more complex examples of large N limits but we will only touch upon them in this paper.

There are several issues that remain open in this direction. The first concerns a more organized control of the spectrum via the partition function. The AdS/CFT correspondence is fundamentally a relation between partition functions that reads

$$Z_{\rm AdS}=Z_{\rm CFT}\ . \tag{1.2}$$

In the canonical formulation for the partition function on the torus

$$Z_{\rm CFT}={\rm Tr}\left[e^{2\pi i\tau(L_0-\frac{c}{24})}e^{-2\pi i\bar\tau(\bar L_0-\frac{\bar c}{24})}\right]\ . \tag{1.3}$$

If fermions are present in the theory we also need to specify their periodicities around the two cycles of the torus. The imaginary part of τ plays the role of inverse temperature in the bulk, and the real part is a chemical potential for angular momentum. There is a reasonably good understanding of this partition function for the WZW model and the interesting question is whether it can be recast in a way that makes it interpretable as the partition function on AdS, $Z_{\rm AdS}$. At low energies the Hilbert space of the gravitational theory comprises of a gas of particles moving on AdS. At high energies we encounter black holes. There have been several attempts to make sense of $Z_{\rm AdS}$ as a sum over all saddle points of the full bulk effective action I (including in principle all string and loop corrections), *i.e.* recast Z_{AdS} as

$$Z_{\text{AdS}}(\tau,\bar{\tau}) = \sum e^{-I} \ . \tag{1.4}$$

The most concrete realization of this programme, is the Farey tail expansion of [19]. In this case, instead of considering the full partition function one focuses on a BPS subsector and computes the elliptic genus. Then, one observes that the elliptic genus admits an expansion that is suggestive of a supergravity interpretation in terms of a sum over geometries. It would be interesting to explore if there is a similar expansion of the *full* partition function of the $SU(N)_k$ WZW model. A more promising context for this idea is to analyze the elliptic genus of $\mathcal{N}=2$ gauged WZW models, *e.g.* $\mathcal{N}=2$ Kazama-Suzuki models (the supersymmetric Grassmannian coset may be an interesting example).

As we have seen there are many configurations in such CFTs that have macroscopic entropy and will be therefore expected to correspond to smooth bulk solutions currying flavor hair. It would be interesting to investigate the existence of such solutions. Moreover, our analysis indicates that at least for the $SU(N)_k$ theory, at strong coupling such solutions should be thought of as describing a fermionic ground state (fermi surface) of almost free fermions.

CFTs like the WZW model have a class of features that are interesting to explore in an AdS setup. They contain current algebra null vectors that are responsible for the truncation of the spectrum (affine cutoff). It is interesting that although such non-trivial relations are counter-intuitive in the weak coupling limit, they are simply explained in the strong coupling limit where the theory can be described in terms of kN Dirac fermions. From the basic formula $g_{a,b} = \sum_{i=1}^{k} \psi^i_a \bar{\psi}^i_b$. It is then obvious that fermi statistics forbids symmetrized powers $(g_{ab})^p$ with $p > k$. The current algebra null vectors are responsible for the existence of "instanton" corrections to the partition function (namely terms that behave as e^{-N}). The study of these effects is interesting as they should match with D-instanton effects in the dual string theory.

A related set of null vectors are the Knizhnik-Zamolodchikov ones, whose content is based on the fact that the stress tensor is quadratic in the currents. They are the key tools in computing the correlation functions. We have seen that the affine-Sugawara construction is an avatar of the proper boundary conditions for the CS theory in 3d. It should be possible to derive the analogue of KZ equations from the bulk.

The bulk description once developed will provide a concrete tool for the study of RG flows between different fixed points. A lot is known in 2d CFTs about such flows, but it is expected the bulk description will provide more efficient tools in this analysis. An intermediate step in this program is the understanding of the bulk effective actions for the scalar fields, something that needs further study. As a byproduct, this approach would allow a more thorough study of the thermodynamics of 2d QFTs.

A related issue is the holographic dual of a c-function, a fact that is firmly established for 2d CFTs. It would be interesting to find classes of examples where one could follow the flows between many fixed points, exploring in such a way the landscape of 2d CFTs. Such large landscapes of flows exist in 2d CFTs, [20], with the flows and c-function known exactly, although the intermediate non-conformal theories in that case are non-relativistic Hamiltonian theories that flow to standard relativistic CFTs at the fixed points. Recent ideas on "order" and distance in such a landscape [21] may prove useful.

2. Solvable 2d CFTs

In two dimensions we have both the largest class of known CFTs as well as the largest class of solvable CFTs.

Solvable CFTs are composed essentially of WZW models based on compact affine Lie groups, [10] and cosets, [12, 13]. All of these can be solved exactly. Solvable extensions include some non-compact theories, like Liouville theory, [22, 23] and some semisimple groups [24, 25]. Larger classes of irrational CFTs were found by generalizations of the GKO construction, [14, 15] but no non-trivial CFT in this class has been solved so far.

We will consider below the simplest classes of WZW and coset models which posses large-N limits. But before doing this it is appropriate to make some general comments on "color" vs "flavor" degrees of freedom.

We define color as degrees of freedom which are gauged. Gauge fields in two dimension have two types of IR dynamics. The standard, namely YM kinetic terms, are irrelevant in the IR, and never play role in CFTs. The quadratic gauge terms that play a role are non-propagating, and the gauged WZW models provide an example of these. As a result pure gauge dynamics in 2d is trivial and the main role of the gauge group is kinematic confinement and removal or colored degrees of freedom from the spectrum. In this sense, gauge interactions in two dimensions reduce the most the number of degrees of freedom because of confinement, compared with 3d or 4d gauge dynamics.

On the other hand we will define flavor degrees of freedom as those that are not affected by gauge interactions. For example a set of N_f free Majorana fermions for example has a maximal flavor symmetry $O(N_f)_L \times O(N_f)_R$ that is promoted to an affine current algebra in two dimensions.

As is usual for color degrees of freedom being gauged, the relevant operators are gauge singlets and give rise in a dual string theory to closed strings. Flavor degrees of freedom being un-gauged give rise in a dual string theory to open strings/D-branes. The fact that 2d pure gauge theories are trivial, imply the absence of Regge trajectories. Their dynamical role is to remove matter degrees of freedom. Therefore the holographic closed sector dynamics is typically field theoretic and involves a finite set of states (one of which is the graviton) living on an $AdS_3 \times M_p$ space. On the other hand flavor degrees of freedom have also an associated closed sector: Flavor singlet operators should be though of as dual to closed string states. These are the closed string states that consistently interact with the open string degrees of freedom.

We will now present a few of the examples that we will analyze in the paper, albeit at different depth.

2.1 The $SU(N)_k$ WZW model

This is an interesting case which seems to contain only flavor. The flavor symmetry is $SU(N)_L \times SU(N)_R$ and as it usually happens with non-trivial compact global symmetries in CFTs, it is extended to a full affine algebra. Note that this not the flavor symmetry we would obtain from N massless Dirac fermions in 2d dimensions. That symmetry is larger and its is $O(2N)_L \times O(2N)_R$.

This theory has two parameters, N which is the number of flavors and k that plays the role of the σ-model coupling constant. We will see in the subsequent section that there are several possible large-N limits that can be defined here.

This is a CFT whose spectrum is conveniently represented using current algebra representations. There are "ground states" that coincide with the primary affine representations with spin zero, transforming as $(R, \bar{R}) \in SU(N)_L \times SU(N)_R$ with R an integrable representations of the $SU(N)_k$ affine algebra. All other states are build on the primary states from the action of current operators. They should be thought of as the oscillators of an appropriate open string in AdS_3, with the zero mode sector generated by an appropriate CS theory. The closed string states are traces of the flavor degrees of freedom. The stress tensor in particular is composite in the currents and it is therefore not an independent operator. Closed string states therefore can be probably thought as multiparticle (non-Fock) states of open string states.

As it will explained in section 4, the $SU(N)_k$ theory can be thought of as the IR limit of a theory of N copies of massless Dirac fermions transforming in the fundamental of the gauge group $U(k)$.

2.2 The $SU(N)_k/SU(N)_k$ gauged WZW model

This is the simplest theory that contains SU(N) color but no flavor. The gauge degrees of freedom remove essentially all states in this theory. In particular the theory is topological and has central charge $c = 0$. It has a finite number of ground states that are in one to one correspondence with all integrable representations of the $SU(N)_k$ affine algebra. In this sense it should be thought of as the space of states of a point particle moving on a fuzzy group manifold.

Its AdS dual theory is the topological $SU(N)$ Chern-Simons (CS) theory at level k. This theory is topological and has a finite number of states that is also in one to one correspondence with all integrable representations of the $SU(N)_k$ affine algebra, [2]. This is the simplest topological open string theory in 3d. Its related closed sector is a trivial topological string: it has no propagating states. From the interpretation of the previous section we gather than the present theory can be thought of as a simple quiver: two gauge groups U(k) and U(N) and Dirac fermions in the bifundamental. There are no Regge trajectories in this case.

2.3 The $SU(N+M)_k/SU(N)_k$ gauged WZW model

This theory has an SU(N) gauge group and therefore N stands for color degrees of freedom. The (un-gauged) commuting subgroup of $SU(M+N)$, namely $SU(M)$ should be thought of as a flavor group, while k is a coupling constant. The theory has the following central charge

$$c = k\left(\frac{(N+M)^2-1}{k+N+M} - \frac{N^2-1}{k+N}\right) . \tag{2.1}$$

In the 't Hooft limit, $N, k \gg 1$ with the ratio $\lambda = \frac{N}{k}$ fixed, and for M fixed the central charge becomes to leading order in M/N

$$c \sim \frac{2+\lambda}{(1+\lambda)^2} NM . \tag{2.2}$$

This is analogous to the quenched limit of 4d gauge theories where the number of flavors is kept finite as the number of colors becomes large. At weak 't Hooft coupling, $c \sim 2NM$, and the theory looks like a (perturbative) QCD theory with M quarks. Its string theory dual could be identified as an open+closed string theory that arises by adding M branes in the topological closed string sector of the $M = 0$ case.

Another interesting limit of this theory is a Veneziano-type limit where $M/N = m$ is kept fixed and finite. In the limit of large m we are approaching our next and most general example: the Grassmannian coset models.

This theory can be thought of also as a quiver. The gauge group is $U(N) \times U(k)$ and the $k(N+M)$ Dirac fermions transform as $(\square, \overline{\square})$ under $(U(N), U(k))$ and as M copies of $(\square, 1)$, having global chiral symmetry $SU(M)_L \times SU(M)_R$.

2.4 The $\frac{U(k_1+k_2)_N}{U(k_1)_N \times U(k_2)_N}$ gauged WZW model

This theory can be thought of as a theory with $U(k_1 + k_2)$ flavor symmetry whose $U(k_1) \times U(k_2)$ part is gauged. The coupling constant is N. This CFT is dual to the

$$\frac{SU(N)_{k_1} \times SU(N)_{k_2}}{SU(N)_{k_1+k_2}}$$

coset by rank-level duality, [6]. In this dual version of the CFT one starts from an $SU(N) \times SU(N)$ flavor symmetry and then gauges the diagonal subgroup.

From the parameters we can build two independent 't Hooft couplings, $\lambda_i = N/k_i$. The central charge becomes to leading order in the 't Hooft couplings $c = N^2 + subleading$ indicating that this model is similar to a gauge theory coupled to an adjoint scalar. It is interesting to compare it to the $\mathcal{N} = 2$ one-adjoint A_{n+1} CSM theories of [7]. Some parallels between these theories are:

(a) Both are controlled by three discrete parameters. In the CSM case, these three parameters are: k the level of the CS interaction, N the rank of the $U(N)$ gauge group, and $n+1$ the power of the single-trace operator $\mathrm{Tr}X^{n+1}$ that appears in the action as a superpotential deformation.

(b) A crucial effect of the superpotential deformation in the CSM theory is that it truncates the chiral ring. The levels k_i play an analogous role in the WZW model truncating the spectrum.

(c) Both theories exhibit a non-trivial duality. The $U(N)_k$ A_{n+1} CSM theory is Seiberg-dual to the $U(nk-N)_k$ A_{n+1} CSM theory. The $\frac{SU(N)_{k_1} \times SU(N)_{k_2}}{SU(N)_{k_1+k_2}}$ gauged WZW model is dual, by level-rank duality, to the $\frac{SU(k_1+k_2)_N}{SU(k_1)_N \times SU(k_2)_N \times U(1)}$ gauged WZW model.

This theory can be thought of as a quiver gauge theory with gauge group $U(k_1) \times U(k_2) \times U(N)$ and massless quarks transforming as $(\square, 1, \overline{\square})$ and $(1, \square, \overline{\square})$ under the gauge group. This description automatically explains the rank-level duality symmetry.

3. On large N limits

There are several large N limits that are possible in the CFTs we have mentioned above. They have been discussed in different contexts in the literature and we will go through them for comparison. As the CFTs we are analyzing are solvable, we will be able to characterize explicitly the nature of each of these limits.

3.1 The 't Hooft large-N limit

The characteristic feature of the 't Hooft limit is that the coupling constant is rescaled so that it compensates for the increase of degrees of freedom. Another characteristic is that for adjoint theories the normalized n-point functions behave as $N^{1-\frac{n}{2}}$. This implies in particular that the central charge $c \sim \mathcal{O}(N^2)$. For the $SU(N)_k$ theory the 't Hooft limit implies $N \to \infty$, $k \to \infty$ with

$$\lambda = \frac{N}{k} \tag{3.1}$$

kept fixed, [26, 18]. When $\lambda \ll 1$ we are in a perturbative regime. This implies in particular that α'-perturbation theory is applicable. Since in the perturbative limit, $k >> N$, the affine cutoff [27] is not visible when we consider representations with $\sim \mathcal{O}(N)$ columns in the Young tableau or less. Therefore the fusion algebra of such low-lying representations is "perturbative": it coincides with the classical Glebsch-Gordan decomposition.

In the opposite limit $\lambda \gg 1$ we are in the strong coupling regime. The σ-model semiclassical expansion breaks down and since $k \ll N$ the affine cutoff is felt at relatively low representations. This implies algebraic relations between primary fields (the vanishing of fields with spin higher than k) well before the distinction between single trace and double trace operators sets in.

The central charge can be written in this limit as

$$c = \frac{N^2}{1+\lambda} + \mathcal{O}(1) \tag{3.2}$$

and it is indeed $\mathcal{O}(N^2)$ as advertised. It does remain so at weak 't Hooft coupling but at strong coupling

$$c \sim \frac{N^2}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \sim kN + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \tag{3.3}$$

is parametrically smaller than N^2 and behaves as $\mathcal{O}(N)$ for finite k. As the central charge is a quantum measure of the number of degrees of freedom, this indicates that there is a drastic reduction of degrees of freedom at strong 't Hooft coupling mimicking a similar situation predicted by AdS_4/CFT_3 for three dimensional conformal field theories. The only difference is that here this is explicitly calculable.[2] Another difference is that the reduction here is by a factor of N while in three dimensions it is by a factor of $\sqrt{N}$.[3]

Focusing at the conformal dimensions with an $\mathcal{O}(1)$ number of boxes of the Young tableau, we obtain (see appendix 7)

$$\Delta_R = \frac{\lambda}{1+\lambda}\Delta_R(\infty) + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.4}$$

To leading order at $1/N$ and at weak coupling they all asymptote to zero

$$\Delta_R = \lambda\Delta_R(\infty) + \mathcal{O}\left(\lambda^2\right) \tag{3.5}$$

in agreement with the fact that in the classical theory all primary operators have vanishing scaling dimension.

At strong 't Hooft coupling on the other hand they asymptote to half-integers

$$\Delta_R = \Delta_R(\infty) + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{3.6}$$

Note that this is the same spectrum as in the naive large-N limit discussed in section 3.2

In general, in the 't Hooft limit the dimensions of primaries are

$$\Delta_R = \frac{\lambda}{2(\lambda+1)}\sum_{i=1}^{k} m_i \tag{3.7}$$

where m_i are the Dynkin indices provided the sums are $\mathcal{O}(1)$. Otherwise the full formula (4.7) should be used. All m_i take values $0 \le m_i \le N/2$. In all cases, $\sum_{i=1}^{k} m_i$ is the total number of boxes in the Young tableau of the associated representation.

The maximal dimension is obtained when $m_i = N/2 \ \forall\ i$. In that case

$$C_2 = \frac{kN(k+N)}{8} \quad , \quad \Delta_{max} = \frac{\lambda}{2(\lambda+1)}\frac{kN}{2} = \frac{N^2}{8\lambda} \tag{3.8}$$

As we will later see, this state is one of those related to a macroscopic black hole.

On the other hand for the maximal symmetric tensor $m_i = 1 \ \forall\ i \le k$

$$C_2 = \frac{k(k+N)}{2} \quad , \quad \Delta_{sym} = \frac{N}{2\lambda} + \cdots \tag{3.9}$$

For the maximal antisymmetric representation $m_1 = N/2$, all others zero

$$C_2 = \frac{N^2}{8} \quad , \quad \Delta_a = \frac{\lambda}{8(\lambda+1)}N + \cdots \tag{3.10}$$

[2]Recently the analogous result in three dimensions was computed from a reduced matrix model, [29].

[3]An analogous analysis in the AdS_7/CFT_6 correspondence for M5 branes indicates that the CFT_6 at strong 't Hooft coupling has more rather than less degrees of freedom from an equivalent weakly coupled theory.

3.2 The simple large-N limit

This amounts to taking $N \to \infty$ while keeping k fixed. It has been studied in specific examples, in [30, 31], in order to produce representations of the W_∞ algebra. In this limit

$$c \simeq kN + \mathcal{O}(1) \tag{3.11}$$

and in this sense this look like a vectorial large-N theory. The large-N limit of the dimensions of primary fields gives

$$\Delta_R \simeq \Delta_R(\infty) + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.12}$$

Therefore the primary dimensions become half integers and this hints at a free-fermionic formulation in terms of 2kN free fermions (as also suggested by the central charge). This is indeed true as analyzed in appendix 8.

The formula (3.12) applies to representations with an $\mathcal{O}(1)$ number of boxes in the Young-tableau. However, if one considers representations with $\mathcal{O}(N)$ boxes then things are different. For example the antisymmetric representation with $\frac{N}{2}+m$ boxes has dimension

$$\Delta_{A_{\frac{N}{2}+m}} = \frac{N+1}{8} - \frac{m^2}{2N} + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{3.13}$$

On the other hand the maximal symmetric representation can only have k boxes because of the affine cutoff. We therefore have

$$\Delta_{S_k} = \frac{k}{2} + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.14}$$

Therefor we have dimensions scaling as $\mathcal{O}\left(N\right)$ and dimensions scaling as $\mathcal{O}\left(1\right)$

In coset theories primary dimensions can also be of $\mathcal{O}\left(\frac{1}{N}\right)$ as shown in [30, 18]. Consider the coset

$$CFT \equiv \frac{SU(N)_{k_1} \times SU(N)_{k_2}}{SU(N)_{k_1+k_2}} \quad , \quad c = \frac{k_1 k_2 (k_1+k_2+2N)(N^2-1)}{(k_1+N)(k_2+N)(k_1+k_2+N)} \tag{3.15}$$

In the naive large-N limit we may rewrite the central charge as

$$c = 2k_1k_2 + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.16}$$

which is finite in the large-N limit.

There is an interesting symmetry in this theory, stemming from rank-level duality, that indicates that this CFT is equivalent to a dual one[4]

$$CFT \sim C\tilde{F}T \equiv \frac{SU(k_1+k_2)_N}{SU(k_1)_N \times SU(k_2)_N \times U(1)} \tag{3.17}$$

[4]This has been explicitly checked in the associated supersymmetric models, [6] although it is also plausible here.

Most of the dimensions of the coset are associated with three representations: $R_1 \in G_1$, $R_2 \in G_2$ and $R_3 \in R_1 \otimes R_2$. Since

$$\Delta_{R_1,R_2;R_3} = \Delta_{R_1} + \Delta_{R_2} - \Delta_{R_3} + \text{integer} \tag{3.18}$$

we obtain in the large-N limit

$$\Delta_{R_1,R_2;R_3} = \Delta_{R_1}(\infty) + \Delta_{R_2}(\infty) - \Delta_{R_3}(\infty) + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.19}$$

For representations with an $\mathcal{O}(1)$ number of boxes in the Young tableau this dimension is of $\mathcal{O}\left(\frac{1}{N}\right)$ For example consider the case $R_1 = A_{m_1}$, $R_2 = A_{m_2}$, $R_3 = A_{m_1+m_2}$. We obtain

$$\Delta_{A_{m_1},A_{m_2};A_{m_1+m_2}} = \frac{m_1 m_2}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{3.20}$$

We will still have also dimensions that scale like $\mathcal{O}\left(N\right)$. For example

$$\Delta_{A_{\frac{N}{2}-m_1},A_{\frac{N}{2}-m_2};A_{N-m_1-m_2}} = \frac{N+1}{4} - \frac{m_1+m_2}{2} + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.21}$$

Therefore in such CFTs, there are primary operators in class A with dimensions $\mathcal{O}\left(N\right)$, operators in class B with dimensions $\mathcal{O}\left(1\right)$ and operators in class C with $\mathcal{O}\left(\frac{1}{N}\right)$. Moreover the maximum dimension is obtained with $m_1 = m_2 = \cdots m_{k-1} = \frac{N}{2}$ with

$$\Delta_m \simeq \frac{kN}{8} + \cdots \tag{3.22}$$

It was shown in [30] that one can make a class of operators of dimension $\mathcal{O}\left(1\right)$ out of operators of the class C. The way is to take operators as in (3.20) and take the limit $m_1 = q_1\sqrt{N}$, $m_2 = q_2\sqrt{N}$. Such operators were shown to have abelian OPE's and generate the analogue of the discrete series operators in pp-wave CFTs as shown in [25].

Note that this large-N limit has a dual version in the CFT (3.17) as a weak coupling limit where all current algebra levels go to infinity. Therefore the σ-model is a flat space to leading order with $2k_1k_2$ dimensions, and the states in this theory, correspond to the class B operators as well as the class C operators that can be made to have $\mathcal{O}(1)$ dimensions as explained earlier.

Note also that the $U(N)_1$ theory is equivalent to a collection of 2N free fermions $\sim O(2N)_1$, as described in appendix 8. The only integrable representations are the anti-symmetric ones and are constructed from products of fermions. The $SU(N)_1$ is obtained from the free fermion theory by coseting by the overall U(1).

3.3 The BMN large-N limit

This is a large-N limit in which a tuning is done so that some dimensions stay finite, [32, 25]. We will consider an example of this: $SU(N)_k \times U(1)^{N-1}$ where the U(1)'s are time-like and have level $2N$. The BMN limit is a large-N limit at k fixed, but which ties together specific combinations of U(1) and SU(N) reps. This will generate a contraction of the group to a non-semi-simple one where one linear combination of the U(1)'s and SU(N) Cartan generators becomes a set of N-1 central currents, the other linear combination becomes a set of N-1 rotation operators, and all raising and lowering operators become transverse pp-wave operators.

The dimension of a generic primary is

$$\Delta_{\vec{q},R} = -\frac{\vec{q}^2}{2N} + \frac{\mathcal{C}_2(R)}{k+N} \tag{3.23}$$

Consider a representation of type A with $x_i N + \xi_i$ boxes in the $i-th$ column, with $0 < x_i < 1$, $i = 1, 2, \cdots, k$ and ξ_i of order $\mathcal{O}(1)$. This representation has a dimension that is $\mathcal{O}(N)$. We also pick the U(1) charges q_i so that they cancel the $\mathcal{O}(N)$ piece of the previous dimension

$$q_i = N\sqrt{x_i(1-x_i)} \tag{3.24}$$

We compute in the large-N limit

$$\Delta = \sum_{i=1}^{k} \xi_i(1-2x_i) + (1-k)x_i(1-x_i) \tag{3.25}$$

Such states provide highest weight or lowest weight representations. If the CFT has operators of type C, then their continuous limit with U(1) charges of order $\mathcal{O}(1)$ provides continuous series representations [25].

4. The $SU(N)_k$ WZW model

This is a prototypical unitary CFT, realizing current algebra and depending on two natural numbers, N, k. The global symmetry is $SU(N)_L \times SU(N)_R$ that is enhanced to the full affine left-moving and right-moving algebra $SU(N)_k$.

It can appear as an IR fixed point in many CFTs, including the SU(N) chiral model modified by the addition of a WZ term, [10]. It can also appear as the IR fixed point of 2-dimensional massless QCD with gauge group U(k) and N Dirac flavors of quarks, [37] as explained in appendix 8. In such a description the YM action is becoming irrelevant in the

IR and the theory flows to the $U(kN)_1/U(k)_N$ coset that is equivalent to the $SU(N)_k$ CFT. Therefore k can be identified as the number of colors and N as the number of massless quark flavors.

We will denote the ratio of flavors to colors as

$$\lambda = \frac{N}{k} \tag{4.1}$$

and we will call it the 't Hooft coupling although from the fermionic point of view, it is the Veneziano ratio $\frac{N_f}{N_c}$. In the σ-model picture this ratio does look more like the conventional 't Hooft coupling.

In the gauge theory picture the global flavor symmetry is $SU(N)_L \times SU(N)_R$ and is manifest in the theory. This is an example of a confining gauge theory without chiral symmetry breaking. The reason that such a theory can be still a CFT is that in two dimensions non-abelian gauge fields carry no propagating degrees of freedom and confinement is essentially kinematical. It can be implemented by the (generalized) Gauss law of the appropriate current algebra on the states of the un-gauged theory.

Denoting the currents of the $SU(N)_k$ WZW model as $J^a(z)$, $\bar{J}^a(\bar{z})$ we have the OPE's

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{z-w} \tag{4.2}$$

with a corresponding expression for the right-moving currents $\bar{J}^a$ (which we will systematically omit).

The stress-energy tensor satisfies the affine Sugawara construction

$$T(z) = \frac{1}{2(k+N)}\sum_a (J^aJ^a)(z) \ . \tag{4.3}$$

The central charge in this theory is given by

$$c = \frac{k(N^2-1)}{k+N} = \frac{(N^2-1)}{\lambda+1} \sim \mathcal{O}(N^2) \sim \mathcal{O}(k^2) \tag{4.4}$$

In the large 't Hooft limit $\lambda \gg 1$, $c \simeq \frac{N^2}{\lambda} = kN$. In this case one can observe the same phenomenon that has been observed in 3d CFTs dual to M2 brane geometries. At strong 't Hooft coupling there is a reduction of the number of degrees of freedom from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$. As shown in appendix 8, in this regime, $N \gg k$ the theory is approximately described by kN free massless Dirac fermions and YM interaction can be treated perturbatively. In the opposite regime, $\lambda \ll 1$ or $N \ll k$, the theory is well described by the (weakly coupled) WZW theory, over a large volume group manifold.

We now proceed to analyze the spectrum and conformal dimensions. The spectrum is composed of affine (spinless) primary states transforming in the $(R, \bar{R})$ integrable representation of the global $SU(N)_L \times SU(N)_R$ group, with one copy per representation. On top of these primary states the whole affine representation is build by acting with the lowering operators of the Current algebra, the negative J^a_{-n} current modes. Were it not for the existence of non-trivial current algebra null vectors, the multiplicity of the states inside representations would be the same as the related $U(1)^{(N^2-1)}$ free theory. The current algebra null vectors provide non-perturbative effects in k, and disappear as $k \to \infty$.

The (left) conformal dimensions for the affine primary fields of the SU(N)$_k$ theory, transforming in the R irreducible unitary representation of the SU(N) algebra are given by

$$\Delta_R = \frac{C_2(R)}{k+N} \tag{4.5}$$

where $C_2(R)$ is the quadratic Casimir, defined and analyzed in appendix 7. To find the large-N limits we must analyze the scaling of the quadratic Casimir for SU(N) representations, with the result

$$C_2(R) \simeq N\Delta_R(\infty) + \mathcal{O}\left(1\right) \tag{4.6}$$

in the $N \to \infty$ limit for representations with an $\mathcal{O}(1)$ number of boxes in their Young tableau.

We can give a general formula for the quadratic Casimir $C_2(R) = C_2(\bar{R})$, corresponding to a Young tableau with m_1 boxes in the first column, m_2 boxes in the second column etc, with the m_i ordered, $m_1 \geq m_2 \geq m_3 \cdots$. We also use the prescription that the proper Young tableau is the one with the minimum number of boxes. In this notation the Casimir is

$$C_2(m_1, m_2, \cdots, m_n) = \frac{(\sum_{i=1}^n m_i)N^2 - ((\sum_{i=1}^n m_i^2) - \sum_{i=1}^n (2i-1)m_i)N - (\sum_{i=1}^n m_i)^2}{2N} \tag{4.7}$$

For $m_i \sim O(1)$ and $n \sim \mathcal{O}(1)$ we obtain

$$\Delta_R(\infty) = \frac{1}{2}(\sum_i m_i) \tag{4.8}$$

which are integers or half integers. In particular, this number is the total number of boxes in the Young tableau, divided by two.

We finally obtain

$$\Delta_R \to \frac{\lambda}{\lambda+1}\Delta_R(\infty) + \mathcal{O}\left(\frac{1}{N}\right) \tag{4.9}$$

In the strong coupling limit $\lambda \to \infty$, we obtain that the dimensions are half integers, reflecting the fact that theory asymptotes to kN free massless fermions (see appendix 8 and the next section). In that case one can think of a rectagular box of dimensions $N \times k$ divided into kN compartments (boxes). A choice of "occupied" boxes defines a Young tableau and therefore an integrable representation R of $SU(N)_k$. Not surprisingly, the set of "ground-states" of the theory, namely the primary fields, is in one to one correspondence with filling some of the boxes using the kN fermions and abiding to the Pauli principle.

We will now consider reps with large scaling dimensions. The maximum scaling dimension is given for representations around the "half-box",

$$m_i = \frac{N}{2} + n_i \quad , \quad n_i \sim \mathcal{O}(1) \tag{4.10}$$

with

$$C_2 = \frac{Nk(N+k)}{8} - \frac{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (k+1-2i)n_i + \frac{1}{N}(\sum_{i=1}^k n_i)^2}{2} \tag{4.11}$$

with scaling dimension

$$\Delta = \frac{N^2}{8\lambda} - \frac{\lambda}{2(\lambda+1)}\left[\frac{1}{N}\sum_{i=1}^k n_i^2 + \frac{1}{N}\sum_{i=1}^k (k+1-2i)n_i + \frac{1}{N^2}(\sum_{i=1}^k n_i)^2\right] \tag{4.12}$$

Note that the scaling dimension is of $\mathcal{O}(N^2)$, and is 1/8 times the central charge in (4.4).

The spectrum of the theory is therefore consisting of "ground-states" associated with the affine primary fields and "stringy excitations" associated with the affine descendants, generating Regge trajectories on top of the ground-states.

The ground state transforming under the representation $(R, \bar{R})$ of the global group $SU(N)_L \times SU(N)_R$ has multiplicity $D(R)^2$ where $D(R)$ is the dimension of the associated representations. The dimensions start at $\mathcal{O}(1)$ but the multiplicity they generate can be substantially higher.

For example the maximal representation corresponding to the "half-box" in (4.10) with $n_i = 0$ has dimension

$$D = \frac{\prod_{i=1}^{N/2} \frac{(N+k-i)!}{(N-i)!}}{\prod_{i=1}^{k} \frac{\left(\frac{N}{2}+i-1\right)!}{(i-1)!}} \tag{4.13}$$

The logarithm of the multiplicity of this ground state is derived in appendix 7 as

$$\begin{aligned}\log D^2 = &\left[4(\lambda+1)\log 2 - \lambda^2 \log\lambda + 2(\lambda+1)^2\log(\lambda+1) - (\lambda+2)^2\log(\lambda+2)\right]\frac{N^2}{2\lambda^2} + \\ &-\frac{N}{2}\log N + \left[2\log(\lambda+2) - 4(\lambda+1)\log(\lambda+1) + 2(\lambda-1)\log\lambda + 2\lambda\log 2\right]\frac{N}{2\lambda} -\end{aligned} \tag{4.14}$$

$$-\frac{1}{6}\log\frac{\lambda+2}{\lambda}+\mathcal{O}(N^{-1})$$

At strong coupling, $\lambda \gg 1$

$$\log D^2 \simeq (2\lambda\log 2-\log\lambda+\cdots)\frac{N^2}{\lambda^2}+\mathcal{O}(N\log N)\sim\mathcal{O}\left(\frac{N^2}{\lambda}\right)\simeq 2c\log 2+\cdots \tag{4.15}$$

while at weak coupling, $\lambda \ll 1$

$$\log D^2 \simeq \left[\frac{\log\frac{1}{\lambda}}{2}+\mathcal{O}(\lambda^2)\right]N^2+\mathcal{O}(N\log N)\sim\mathcal{O}\left(N^2\log\frac{1}{\lambda}\right)\simeq c\log\frac{1}{\lambda}+\cdots \tag{4.16}$$

We should also consider states whose scaling dimensions scale as $\mathcal{O}(N)$ in the large-N limit. In section 3.1 we have mentioned two such representations the maximal symmetrized representation with all Dynkin indices $m_{0<i<k+1}=1$, and the "maximal" antisymmetrized representation, $m_1=\frac{N}{2}$ and all others zero. The scaling dimension of the maximal symmetric tensor is

$$C_2=\frac{k(k+N)}{2}\quad,\quad\Delta_{sym}=\frac{N}{2\lambda}+\cdots \tag{4.17}$$

while for the antisymmetric one

$$C_2=\frac{N^2}{8}\quad,\quad\Delta_a=\frac{\lambda}{8(\lambda+1)}N+\cdots \tag{4.18}$$

Note that Δ_{sym} vanishes in the strong coupling limit. This is a consequence of fermi-statistics, as this representation is built out of fermions as $\prod_{i=1}^{k}\psi_i^{a_i}\times cc$, and therefore one cannot obtain a symmetric object in the flavor indices, a_i. On the other hand $\Delta_a\to\frac{N}{8}$ in the strongly coupled limit as in this case a similar state, can be achieved in terms of free fermions. Such states resemble baryons, and indeed their "masses" are of order N. Of course there are many more representations of this type, beyond the ones discussed above.

Note that the N dependence of scaling dimensions that arises in (4.5,4.7) contains contributions from tree level, disk level and one loop. On the other hand the central charge has only a single one-loop contribution. The λ dependence although simple indicates the presence of a full perturbative series of corrections both at small and large λ.

We have seen that the Hilbert space decomposes into a finite number of unitary irreducible representations of the current algebra. For each representation there is a ground state, transforming in the $(R,\bar{R})$ of $SU(N)_L\times SU(N)_R$ corresponding to the primary field, and the rest of the states are generated from the primary field by the action of the current oscillators. The different primary operators can be thought of as products of the basic WZW field $g_{a,b}(z,\bar{z})$. Using appropriate normal ordering and symmetrizations or antisymmetrizations of the indices we may construct any integrable primary of the algebra. This fact will be important when we analyze the AdS$_3$ dual.

4.1 The four-point function

An important observable in CFT are correlation functions.

We will analyze the large-N limits of the four-point function of the basic primary fields of the $SU(N)_k$ WZW model, namely the fundamental $g_{a,b}(z,\bar{z})$ and its conjugate $g^{-1}_{b,a}(z,\bar{z})$ following [11] and then study its large N-limit.

The four-point was calculated solving the KZ equation and reads

$$G(x,\bar{x}) \equiv \langle g_{a_1,b_1}(\infty) g^{-1}_{b_2,a_2}(1) g_{a_3,b_3}(x,\bar{x}) g^{-1}_{b_4,a_4}(0)\rangle = \sum_{A,B=1}^{2} I^A \bar{I}^B \; G_{AB}(x,\bar{x}) \tag{4.19}$$

$$I^1 = \delta_{a_1,a_2}\delta_{a_3,a_4} \quad , \quad \bar{I}^1 = \delta_{b_1,b_2}\delta_{b_3,b_4} \quad , \quad I^2 = \delta_{a_1,a_4}\delta_{a_2,a_3} \quad , \quad \bar{I}^2 = \delta_{b_2,b_4}\delta_{b_1,b_3} \tag{4.20}$$

with

$$G_{AB}(x,\bar{x}) = \mathcal{F}^{(1)}_A(x)\mathcal{F}^{(1)}_B(\bar{x}) + h\; \mathcal{F}^{(2)}_A(x)\mathcal{F}^{(2)}_B(\bar{x}) \tag{4.21}$$

There are two group invariants corresponding to the two representations, the singlet and the adjoint appearing in the product of fundamental with an anti-fundamental. Taking into account the left-moving and right-moving group structure, the total number of invariants is four and they appear in (4.20). h in (4.21) is the only non-trivial quantum Glebsch-Gordan (OPE) coefficient coupling a fundamental, an anti-fundamental and the adjoint.

The detailed analysis is done in appendix 9. We will summarize the results here.

The associated conformal blocks are

$$\mathcal{F}^{(1)}_1(x) = x^{-2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}\; F\left(-\frac{1}{2\kappa}, \frac{1}{2\kappa}; 1 + \frac{N}{2\kappa}, x\right) \tag{4.22}$$

$$\mathcal{F}^{(1)}_2(x) = -\frac{x^{1-2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}}{2\kappa + N}\; F\left(1 - \frac{1}{2\kappa}, 1 + \frac{1}{2\kappa}; 2 + \frac{N}{2\kappa}, x\right) \tag{4.23}$$

$$\mathcal{F}^{(2)}_1(x) = x^{\Delta_A - 2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}\; F\left(-\frac{N-1}{2\kappa}, -\frac{N+1}{2\kappa}; 1 - \frac{N}{2\kappa}, x\right) \tag{4.24}$$

$$\mathcal{F}^{(2)}_2(x) = -N x^{\Delta_A - 2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}\; F\left(-\frac{N-1}{2\kappa}, -\frac{N+1}{2\kappa}; -\frac{N}{2\kappa}, x\right) \tag{4.25}$$

where F is the hypergeometric function and

$$\Delta_\Box = \frac{N^2-1}{2N(N+k)} \quad , \quad \Delta_A = \frac{N}{N+k} \quad , \quad 2\kappa = -N - k \tag{4.26}$$

and

$$h = \frac{1}{N^2}\frac{\Gamma\left[\frac{N-1}{N+k}\right]\Gamma\left[\frac{N+1}{N+k}\right]\Gamma^2\left[\frac{k}{N+k}\right]}{\Gamma\left[\frac{k+1}{N+k}\right]\Gamma\left[\frac{k-1}{N+k}\right]\Gamma^2\left[\frac{N}{N+k}\right]} \tag{4.27}$$

In the 't Hooft limit we obtain

$$G_{11} = |x|^{-\frac{2\lambda}{1+\lambda}} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad , \quad G_{22} = |1-x|^{-\frac{2\lambda}{1+\lambda}} + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{4.28}$$

$$G_{12} = \frac{\lambda\bar{x}}{N}|x|^{-\frac{2\lambda}{1+\lambda}}\ F\left(1,1;\frac{2+\lambda}{1+\lambda},\bar{x}\right) - \frac{(1-x)}{N}|1-x|^{-\frac{2\lambda}{1+\lambda}}\ F\left(1,1;1+\frac{\lambda}{1+\lambda},x\right) + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{4.29}$$

Note that in the symmetric channels G_{11} and G_{22} are powerlike at leading order in $1/N$ suggesting the existence of a single block. In the asymmetric channel G_{12} there is non-trivial structure at subleading order in $1/N$

Finally in the strong coupling limit $\lambda \to \infty$ the result simplifies to

$$G_{11} = \frac{1}{|x|^2} + \cdots \quad , \quad G_{22} = \frac{1}{|1-x|^2} + \cdots \quad , \quad G_{12} = \frac{\lambda}{N}\frac{1}{x(1-\bar{x})} + \cdots \tag{4.30}$$

This is indeed compatible with the claim that the theory is described by free fermions in that limit. In particular the properly normalized fundamental field of the WZW can be written in the strong coupling limit as

$$g_{ab}(z,\bar{z}) = \frac{1}{k}\sum_{i=1}^{k}\psi_i^a(z)\tilde{\psi}_i^b(\bar{z}) \tag{4.31}$$

where the tilde indicates right-movers. From (4.31) the 4-point function in (4.30) follows.

4.2 Non-commutativity of large-N and large-λ limits.

An interesting question in any large-N theory is the commutativity of the large-N and the large 't Hooft coupling limit. In the conventional definition we first take the large-N limit, and we then let λ become large. In our example we can study these limits explicitly and we will show that they do not commute. More details can be found in appendix 9.

The particular observable to study is the OPE coefficient in (4.27) This OPE coefficient has a double expansion

$$h = \sum_{n=1}^{\infty}\sum_{m=-2}^{\infty}\frac{W_{n,m}}{N^{2n}\lambda^m} \tag{4.32}$$

If we first take the large-N limit while keeping λ fixed we will reorganize the double expansion as

$$\lim_{N\to\infty} h = \sum_{n=1}^{\infty}\frac{Z_n(\lambda)}{N^{2n}} \tag{4.33}$$

We will now take the large λ limit. The functions Z_n have the following $\lambda \to \infty$ limit

$$\lim_{\lambda\to\infty} Z_2 = -\lambda^2 + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.34}$$

$$\lim_{\lambda\to\infty} Z_3 = 2\lambda\psi''(1) - 6\psi''(1) + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.35}$$

from which we read in the range $1 \le n \le 3$, $-2 \le m \le 0$

$$W_{1,0} = 1 \quad , \quad W_{2,-2} = -1 \quad , \quad W_{3,-1} = 2\psi''(1) \quad , \quad W_{3,0} = -6\psi''(1) \tag{4.36}$$

all other being zero in that range.

We will now take the opposite sequence of limits, taking the large-λ limit first. We rewrite the double expansion as

$$\lim_{\lambda\to\infty} h = \sum_{n=-2}^{\infty} H_n(N)\lambda^{-n} \tag{4.37}$$

and we subsequently take the large -N limit of the functions H_n. We obtain

$$\lim_{N\to\infty} H_{-1} = \frac{2\psi''(1)}{N^6} + \mathcal{O}(N^{-8}) \tag{4.38}$$

$$\lim_{N\to\infty} H_0 = \frac{1 - 4\gamma_E + \frac{\pi^2}{3}}{N^4} + \frac{15\pi^2 + \pi^4 - 180\psi''(1)}{45N^6} + \mathcal{O}(N^{-8}) \tag{4.39}$$

from which we deduce

$$W_{2,-2} = -1 \quad , \quad W_{3,-1} = 2\psi''(1) \quad , \quad W_{2,0} = 1 - 4\gamma_E + \frac{\pi^2}{3} \quad , \quad W_{3,0} = \frac{15\pi^2 + \pi^4 - 180\psi''(1)}{45} \tag{4.40}$$

while the rest are zero.

Comparing (9.30) and (9.36) we observe that the two limits do not commute.

4.3 The effective action for sources

We will derive here the effective action of the WZW theory once we couple the currents to sources. The WZW action is given by

$$I(g) = \frac{1}{16\pi}\int d^2\xi Tr[\partial_a g \partial^a g^{-1}] + \Gamma(g) \quad , \quad \Gamma(g) = \frac{i}{24\pi}\int d^3\xi Tr[g^{-1}\partial_a g g^{-1}\partial_b g g^{-1}\partial_c g]\epsilon^{abc} \tag{4.41}$$

where the second integral is over a 3d manifold with two dimensional space as its boundary. The action satisfies the Polyakov-Wiegmann relation, [28]

$$I(gh^{-1}) = I(g) + I(h) + \frac{1}{16\pi}\int d^2\xi Tr[g^{-1}\partial_{\bar{z}} g h^{-1}\partial_z h] \tag{4.42}$$

The associated path integral for $SU(N)_k$ is defined as

$$Z = \int \mathcal{D}g \; e^{-k \; I(g)} \tag{4.43}$$

The left and right currents for this theory are given by

$$J_z = \partial_z g g^{-1} \quad , \quad J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g \tag{4.44}$$

and we can couple external sources to them and define their effective action as

$$Z(A_z, A_{\bar{z}}) = e^{-W(A_z, A_{\bar{z}})} = \frac{1}{Z}\int \mathcal{D}g \; e^{-k \; I(g, A_z, A_{\bar{z}})} \tag{4.45}$$

with

$$I(g, A_z, A_{\bar{z}}) = I(g) + \frac{1}{16\pi} Tr \int d^2\xi \left[A_{\bar{z}} J_z - A_z J_{\bar{z}} + A_{\bar{z}} g A_z g^{-1}\right] \tag{4.46}$$

We may now parameterize without loss of generality the two sources $A_{z,\bar{z}}$ in terms of two scalar functions, $h, \bar{h}$.

$$A_z = \partial_z h \; h^{-1} \quad , \quad A_{\bar{z}} = \partial_{\bar{z}} \bar{h} \; \bar{h}^{-1} \tag{4.47}$$

Note that this does not imply that the sources have a flat field strength. Using this parametrization we may rewrite the source action as

$$I(g, A_z, A_{\bar{z}}) = I(\bar{h}^{-1} g h) - I(\bar{h}^{-1} h) + \frac{1}{16\pi} Tr \int d^2\xi A_z A_{\bar{z}} \tag{4.48}$$

We may now perform the path integral in (4.45), by changing variables from $g \to \bar{h}^{-1} g h$ and noting that the path integral measure is invariant under left and right group transformations to obtain

$$W(A_z, A_{\bar{z}}) = -kI(\bar{h}^{-1} h) + \frac{k}{8\pi} Tr \int d^2\xi A_z A_{\bar{z}} \tag{4.49}$$

$$= -k \; I(\bar{h}^{-1}) - k \; I(h) = W(A_{\bar{z}}) + W(A_z)$$

where in the second step we used (4.42). This is the final factorized action for the sources. Variation with respect to $A_z, A_{\bar{z}}$ will give the current correlators.

5. Holography on AdS_3

There has been a lot of evidence for the holographic correspondence between 2d CFTs and string theories on AdS_3 that is summarized in [16].

Gravity theories and solutions are characterized by a Planck scale $M_p = 1/(16\pi G_3)$ and the associated Planck length, $\ell_p = G_3$. Solutions to the gravity theory are characterized by their mass M and angular momentum J. The central charge of the CFT is related to the gravity data as

$$c = \frac{3\ell}{2\ell_p} \ , \tag{5.1}$$

The general formula for the central charge (5.1) was derived in [35] using low-energy gravity. Mass and angular momentum are related to the CFT data by

$$M = \frac{L_0 + \bar{L}_0}{\ell} \quad , \quad J = L_0 - \bar{L}_0 \tag{5.2}$$

where ℓ is the AdS$_3$ radius. The calculation of the central charge in the gravity theory is expected to be reliable in the semi-classical regime $\ell \gg \ell_p$. Therefore, the result (5.1) should be viewed as the leading term in an expansion in ℓ_p/ℓ. The formulae (5.2) however are expected to be a universal feature of the correspondence.

Assuming that the string theory dual is of the form

$$AdS_3 \times \mathcal{M}_p \ , \tag{5.3}$$

where $\mathcal{M}_p$ is a p-dimensional compact manifold with volume $V = \left(\frac{\ell_{\mathcal{M}}}{\ell_s}\right)^p$, the three-dimensional Planck length can be written in terms of the string coupling g_s and the string length ℓ_s as

$$\frac{1}{\ell_p} = \frac{V}{g_s^2 \ell_s} = \frac{1}{g_3^2 \ell_s} \ . \tag{5.4}$$

g_3 is the three-dimensional string coupling.

Before going further, it is worth discussing the more familiar situation of the $F1$-$NS5$ system (p $F1$'s and k $NS5$'s). In this case

$$c = 6kp = \frac{3\ell}{2\ell_p} \ , \quad \frac{\ell_s}{\ell_p} = \frac{V}{g_s^2} = \frac{1}{g_3^2} = 4p\sqrt{k} \ , \quad \frac{\ell}{\ell_s} = \sqrt{k} \ . \tag{5.5}$$

The volume of the internal manifold S^3 and g_s behave as

$$V \sim \left(\frac{\ell}{\ell_s}\right)^3 = k^{3/2} \quad , \quad g_s \sim \sqrt{\frac{k}{p}} \ . \tag{5.6}$$

Defining

$$N \equiv \sqrt{kp} \ \ (\text{so that } c \sim N^2) \ \ \text{and} \ \ \lambda \equiv g_s N \tag{5.7}$$

we obtain

$$\lambda = k \ \ \text{and} \ \ V(\lambda) \sim \lambda^{\frac{3}{2}} \ . \tag{5.8}$$

If we assume the validity of (5.1), for the present CFT we obtain

$$\frac{\ell}{\ell_p} = \frac{2}{3} c \simeq \frac{2N^2}{3(1+\lambda)} + \mathcal{O}(1) \ . \tag{5.9}$$

The next ingredient is the relation of string coupling to λ. To obtain some intuition we will try to establish first in which region we expect a field theory description. Note the scaling dimensions for the representations with finite number of boxes were given in (4.10) as

$$\Delta = \frac{\lambda}{\lambda+1}\Delta_R(\infty) + n + \bar{n} \tag{5.10}$$

where the integers $n, \bar{n}$ are the contributions of the current oscillators. We would like to study the gap in dimensions between the primaries and their descendants. In the limit $\lambda \to \infty$, the primary dimensions are half integers and therefore there is no adjustable gap separating them from the excited states. On the other hand as $\lambda \to 0$ the primary field dimensions vanish and this creates an adjustable gap. This suggests that the gravity limit will be reliable when $\lambda \to 0$. We will adjust ℓ_s so that the scaling dimensions in the field theory limit are constant while the excited states' dimensions vary with λ. This gives

$$\frac{\ell}{\ell_s} = \sqrt{\frac{1+\lambda}{\lambda}} \tag{5.11}$$

We may then estimate the string couplings as

$$g_s^2 \sim \frac{\ell_p}{\ell_s} \sim \frac{(1+\lambda)^{\frac{3}{2}}}{N^2\sqrt{\lambda}} \quad \to \quad g_s N \sim \left[\frac{(1+\lambda)^3}{\lambda}\right]^{\frac{1}{4}} \tag{5.12}$$

We have made several assumptions to derive the previous results, including the fact that the string theory dual is a three-dimensional non-critical string theory. This is motivated from the realization of the CFT and previous experience that suggests that extra adjoint matter will generate extra dimensions, whereas fundamental matter is induced by space-filling flavor branes.

We expect that the bulk string theory has a trivial closed string sector. The reason is two-fold. First the closed string sector should contain the pure gauge theory states and these are trivial in two dimensions. Second, as we will see, the open string sector associated with flavor will generate the necessary correlators of the stress tensor and other closed string fields. This does not imply that there is no non-trivial gravitational action for the metric but that there will be no non-trivial fluctuations here and no stringy states.

The non-trivial string sectors are associated with the flavor symmetry $SU(N)_L \times SU(N)_R$. This should be a symmetry that is realized as a bulk gauge symmetry. It will be realized by two sectors of open strings associated to N D_2 branes and N $\overline{D_2}$ as in higher-dimensional realizations of flavor. The gauge fields L_μ, R_μ associated with flavor symmetry will have an action that starts with the CS action as

$$S_{bulk} = \frac{ik}{8\pi} Tr \int (LdL + \frac{2}{3}L^3 - RdR - \frac{2}{3}R^3) + \cdots \tag{5.13}$$

where the ellipsis indicates higher derivative terms starting from with YM action and $L = L^a T^a$ with $Tr[T^a T^b] = \delta^{ab}$. The sign of the coupling constants is implied by the parity invariance of the CFT.

In global AdS$_3$

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\phi^2. \tag{5.14}$$

The asymptotic expansion near the boundary, $r \to \infty$ for solutions of the gauge fields are, [16]

$$A_i(r, \vec{x}) = A_i^{(0)}(\vec{x}) + \frac{A_i^{(1)}(\vec{x})}{r^2} + \cdots \tag{5.15}$$

where we chose the gauge $A_r = 0$. The equations of motion, including the higher order terms imply that $A^{(0)}$ is flat

$$F(A^{(0)})_{ij} = 0 \tag{5.16}$$

It also implies that it is only the CS terms that contribute to boundary terms. In particular the boundary current is

$$\Delta S = \frac{i}{2\pi} Tr \int d^2x \sqrt{g^{(0)}} J^i \delta A_i^{(0)} \tag{5.17}$$

where $g_{(0)}$ is the boundary metric.

The correct variational principle states that in complex boundary coordinates one of the two $A_{z,\bar{z}}$ should vanish at the boundary. The boundary action that imposes this condition is

$$S_{\text{boundary}} = -\frac{k}{16\pi} Tr \int d^2x \sqrt{g} g^{ab} (L_a L_b + R_a R_b) \tag{5.18}$$

We therefore have the boundary conditions

$$L_{\bar{z}}^{(0)} = R_z^{(0)} = 0 \quad , \quad J_z = \frac{1}{2} J^{\bar{z}} = \frac{ik}{2} L_z^{(0)} \quad , \quad J_{\bar{z}} = \frac{1}{2} J^z = -\frac{ik}{2} R_{\bar{z}}^{(0)} \tag{5.19}$$

The boundary value of the stress tensor can also be computed from (5.20) to be

$$T_{ab} = \frac{k}{8\pi} Tr[L_a^{(0)} L_b^{(0)} - \frac{1}{2} g_{ab}^{(0)} L_c^{(0)} L^{(0),c} + (L \leftrightarrow R)] \tag{5.20}$$

from where we obtain the affine Sugawara stress tensors

$$T_{zz} = \frac{k}{8\pi} Tr[L_z^{(0)} L_z^{(0)}] \quad , \quad T_{zz} = \frac{k}{8\pi} Tr[R_z^{(0)} R_z^{(0)}] \quad , \quad T_{z\bar{z}} = 0 \tag{5.21}$$

5.1 The effective action

We will now establish the effective action for the sources, and match eventually to the one obtained in section 4.3 equation (4.49). To do this we follow the analysis in [3]. In this work, it is shown how one can write the gauge fields on-shell, in terms of a two-dimensional flat connection. The end result will be that the on-shell value of the effective action agrees with (4.49).

The equations of motion of the CS theory imply that

$$F_L = F_R = 0 \tag{5.22}$$

in the bulk whose solution is

$$L_\mu = \partial_\mu h\ h^{-1} \quad , \quad R_\mu = \partial_\mu \bar{h}\ \bar{h}^{-1} \tag{5.23}$$

with $h, \bar{h}$ functions of r and the two dimensional coordinates $z, \bar{z}$.

Evaluating the CS action on the solution of (5.22) we obtain

$$S_{bulk}^{\text{on-shell}} = -\frac{ik}{24\pi}\int (L^3 - R^3) = -k[\Gamma(h) - \Gamma(\bar{h})] = -k[\Gamma(h) + \Gamma(\bar{h}^{-1})] \tag{5.24}$$

Adding this to the boundary action (5.20) where $h, \bar{h}$ are evaluated at the boundary, we obtain perfect agreement with the effective action obtained from the CFT (4.49).

5.2 The bulk scalar

From the non-trivial ground states of the WZW theory, associated to primaries only one can be considered as the generating operator dual to a complex bulk scalar T_{ij}, that transforms in the bi-fundamental under the bulk gauge group $SU(N)_L \times SU(N)_R$. The reason is that all other primary ground states can be considered as composites (multi-particle states) of the fundamental scalar g under OPE, since they arise as appropriately regularized algebraic functions (products) of the fundamental operator.

Its mass is given by the standard formula that connects it to its scaling dimension

$$h = \Delta + \bar{\Delta} = \frac{\lambda}{\lambda+1} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad , \quad m^2\ell^2 = -\frac{\lambda(\lambda+2)}{(\lambda+1)^2} \tag{5.25}$$

The situation is similar to that of tachyon condensation in $D - \bar{D}$ systems realizing flavor in holography , [17]. The simplest two derivative action compatible with the symmetries is

$$S_T = \frac{1}{2}Tr\int \sqrt{g}\left(g^{\mu\nu}D_\mu T D_\nu T^\dagger - m^2 T T^\dagger\right) \tag{5.26}$$

with

$$D_\mu T = \partial_\mu T + iL_\mu T - iTR_\mu \quad , \quad D_\mu T^\dagger = \partial_\mu T^\dagger - iR_\mu T^\dagger + iT^\dagger L_\mu \tag{5.27}$$

The vev associated with the CFT vacuum is $T = 0$ which is the only choice that keeps the chiral symmetry unbroken. Turning on a source for T should correspond to perturbations of the WZW theory by g. In the strong coupling limit this corresponds to turning on a mass matrix for the fermions. Therefore, the theory is expected to flow to $SU(N-r)_k$ theory where r is the rank of the mass-matrix. This amounts to a reduction of the values of the 't Hooft coupling λ, therefore flowing towards weak coupling.

The flow is also visible in the the bulk theory, (5.13), (5.26). It is plausible that the non-linear theory is described by a DBI-like action along the lines of [33], [34].

5.3 Multiparticle states and entropy

We have seen in section 4 that ground states that are multiparticle states can have large multiplicities. Their scaling dimensions are of $\mathcal{O}(N^2)$ and therefore are expected to correspond to macroscopic solutions (black holes?) in the bulk theory.

The associated mass for the "half-box" states in (4.13) is given by

$$\frac{M}{M_p} = \frac{\Delta + \bar{\Delta}}{\frac{N^2}{24\pi(1+\lambda)}} \simeq 6\pi \frac{\lambda + 1}{\lambda} \tag{5.28}$$

where we have used (5.1,5.2). All other primary masses are suppressed by 1/N or more. Note that in the weak coupling limit, $\lambda \to 0$, the mass becomes much larger than the Planck scale, $\frac{M}{M_p} \sim \frac{1}{\lambda}$. therefore these are macroscopic configurations. Note that it is in this limit that the zero mode theory (gravity+CS) are expected to give a reliable description of the physics.

On the other hand for $\lambda \to \infty$, the mass of these states is bounded, $\frac{M}{M_p} \sim 6\pi$ by the Planck scale. Therefore, in this limit the ground states remain Planckian and do not generate macroscopic states.

The entropy of the "half-box" state was calculated in (4.14). We can compare it with the Cardy entropy for a state with the same scaling dimension as the conformal dimensions $\Delta = \bar{\Delta} = \frac{N^2}{8\lambda}$ in (4.12)

$$S_{Cardy} = 2 \cdot 2\pi \sqrt{\frac{c\Delta}{6}} \simeq \frac{\pi}{\sqrt{3}} \frac{N^2}{\lambda} + \cdots \tag{5.29}$$

We observe that at strong coupling the dimension multiplicity in (4.15) is subleading to (5.29) as $\frac{\pi}{\sqrt{3}} > 2\log 2$. On the other hand at weak coupling the Cardy entropy is subdominant. This suggests that in this case it is not the Regge trajectory of the vacuum module that dominates asymptotic entropy but this class of ground-states. As we saw above, such states have macroscopic energy so they should correspond to classical bulk solutions with large degeneracy. This degeneracy is coming from the large-N bulk gauge symmetry but it is also enhanced by the weak interactions. It is not clear what such solutions are.

In the opposite case of strong coupling such states have subdominant entropy, and never dominate the Cardy entropy.

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Appendices

7. Casimirs and dimensions for the SU(N) algebra

In this appendix we will tabulate some useful results on SU(N) representations namely dimensions, the Dynkin index and the quadratic Casimir that enters in conformal dimensions of primary fields.

We will denote the dimension of the representation R by $D(R)$. The irreducible representation R is completely specified with its Young tableau. Conjugation by an ϵ-tensor acts on any column of length $s \leq N$ of the Young tableau replacing it with a column of length $N-s$. We use the convention to have the minimal number of boxes for an $SU(n)$ representation. For example if we discuss the $\bar{\Box}$ that can be described also by $N-1$ antisymmetrized boxes we substitue instead the fundamental with one box. Therefore a representation represented by a generic Young tableau has m_1 boxes in the first column, m_2 boxes in the second column etc, with $m_1 \geq m_2 \geq m_3 \cdots \geq m_n$. We will denote this representation as $(m_1, m_2, \cdots, m_n)$.

The affine cutoff implies that only integrable representations can be present as primary states. For $SU(N)_k$, all representations with more than k columns are not integrable. Therefore $n \leq k$ above.

In a tensor product of irreducible representations $R_1 \otimes R_2 = \sum_k R_k$ we have the following relation for their dimensions

$$D(R_1)D(R_2) = \sum_k D(R_k) \tag{7.1}$$

from which the well known formulae for dimensions of an arbitrary Young tableau can be calculated.

The Dynkin index $S_2(R)$ of a representation R is defined as, [36]

$$Tr[T_R^a T_R^b] = S_2(R)\ \delta^{ab} \tag{7.2}$$

where T_R^a are the Lie algebra generators of the representation R. We will normalize them here so that $S_2(\Box) = \frac{1}{2}$. Again for a tensor product we have the following relations that allow the calculation of all Dynking indices

$$D(R_1)S_2(R_2) + D(R_2)S_2(R_1) = \sum_k S_2(R_k) \tag{7.3}$$

Finally the quadratic Casimir is defined as

$$\sum_a (T_R^a T_R^a)_{ij} = C_2(R)\delta_{ij} \tag{7.4}$$

and is related to the Dynking index by by

$$(N^2-1)S_2(R) = D(R)C_2(R) \tag{7.5}$$

that follows from (7.2) and (7.4).

The general formula for the Casimir for an arbitrary representation $(m_1, m_2, \cdots, m_n)$ is

$$C_2(m_1, m_2, \cdots) = \frac{(\sum_{i=1}^n m_i)N^2 - ((\sum_{i=1}^n m_i^2) - \sum_{i=1}^n (2i-1)m_i)N - (\sum_{i=1}^n m_i)^2}{2N} \tag{7.6}$$

The quadratic Casimir is related to the conformal dimensions of primary fields as

$$\Delta_R = \bar{\Delta}_R = \frac{C_2(R)}{k+N} \tag{7.7}$$

The large-N limit of dimension is obtained from the large limit of the Casimir

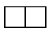
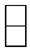
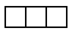
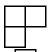
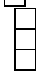
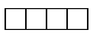
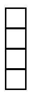
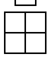
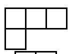
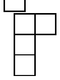
$$\lim_{N\to\infty} C_2(R) = N\Delta_R(\infty) + \mathcal{O}(1) \quad , \quad \lim \Delta_R = \frac{\lambda}{1+\lambda}\Delta_R(\infty) \tag{7.8}$$

with the 't Hooft coupling defined in (3.1). We obtain

$$\Delta_R(\infty) = \frac{1}{2}(\sum_i m_i) \tag{7.9}$$

for $m_i \sim \mathcal{O}(1)$.

The dimension, Dynkin index, Casimir and $\Delta_R(\infty)$ for some common representations are tabulated in table 7.

Representation	dimension	Dynkin Index S_2	Casimir C_2	$\Delta_R(\infty)$
	N	$\frac{1}{2}$	$\frac{N^2-1}{2N}$	$\frac{1}{2}$
	$\frac{N(N+1)}{2}$	$\frac{N+2}{2}$	$\frac{(N-1)(N+2)}{N}$	1
	$\frac{N(N-1)}{2}$	$\frac{N-2}{2}$	$\frac{(N+1)(N-2)}{N}$	1
Adjoint	N^2-1	N	N	1
	$\frac{N(N+1)(N+2)}{6}$	$\frac{(N+2)(N+3)}{4}$	$\frac{3(N-1)(N+3)}{2N}$	$\frac{3}{2}$
	$\frac{N(N^2-1)}{3}$	$\frac{N^2-3}{2}$	$\frac{3(N^2-3)}{2N}$	$\frac{3}{2}$
	$\frac{N(N-1)(N-2)}{6}$	$\frac{(N-2)(N-3)}{4}$	$\frac{3(N+1)(N-3)}{2N}$	$\frac{3}{2}$
	$\frac{N(N+1)(N+2)(N+3)}{24}$	$\frac{(N+2)(N+3)(N+4)}{12}$	$\frac{2(N-1)(N+4)}{N}$	2
	$\frac{N(N-1)(N-2)(N-3)}{24}$	$\frac{(N-2)(N-3)(N-4)}{12}$	$\frac{2(N+1)(N-4)}{N}$	2
	$\frac{N^2(N^2-1)}{12}$	$\frac{(N+2)(N+3)(N+4)}{12}$	$\frac{2(N^2-4)}{N}$	2
	$\frac{N(N-1)(N+1)(N+2)}{8}$	$\frac{(N+2)(N^2+N-4)}{4}$	$\frac{2(N^2+N-4)}{N}$	2
	$\frac{N(N+1)(N-1)(N-2)}{8}$	$\frac{(N-2)(N^2-N-4)}{4}$	$\frac{2(N^2-N-4)}{N}$	2
m-symmetric	$\binom{N+m-1}{m}$	$\frac{1}{2}\binom{N+m}{m-1}$	$\frac{m(N-1)(N+m)}{2N}$	$\frac{m}{2}$
m-antisymmetric	$\binom{N}{m}$	$\frac{1}{2}\binom{N-2}{m-1}$	$\frac{m(N-m)(N+1)}{2N}$	$\frac{Min[m,N-m]}{2}$

We will now consider reps with large dimensions and large multiplicities. The maximum is when

$$m_i = \frac{N}{2} + n_i \quad , \quad n_i \sim \mathcal{O}(1) \tag{7.10}$$

from where we compute

$$C_2 = \frac{Nk(N+k)}{8} - \frac{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (k+1-2i)n_i + \frac{1}{N}(\sum_{i=1}^k n_i)^2}{2} \tag{7.11}$$

which gives the scaling dimension

$$\Delta = \frac{C_2}{N+k} = \frac{Nk}{8} - \frac{\sum_{i=1}^k n_i^2 + \sum_{i=1}^k (k+1-2i)n_i + \frac{1}{N}(\sum_{i=1}^k n_i)^2}{2(N+k)} = \tag{7.12}$$

$$= \frac{N^2}{8\lambda} - \frac{\lambda}{2(\lambda+1)}\left[\frac{1}{N}\sum_{i=1}^k n_i^2 + \frac{1}{N}\sum_{i=1}^k (k+1-2i)n_i + \frac{1}{N^2}(\sum_{i=1}^k n_i)^2\right]$$

The dimension of the associated SU(N) representation when $n_i = 0$ for all i is

$$D = \frac{\prod_{i=1}^{N/2} \frac{(N+k-i)!}{(N-i)!}}{\prod_{i=1}^{k} \frac{\left(\frac{N}{2}+i-1\right)!}{(i-1)!}} \tag{7.13}$$

The logarithm is

$$\log D = \sum_{i=1}^{N/2} \left[\log(N+k-i)! - \log(N-i)!\right] - \sum_{i=1}^{k}\left[\log\left(\frac{N}{2}+i-1\right)! - \log(i-1)!\right] \tag{7.14}$$

Using Stirling's formula

$$\log(n!) = n\log n - n + \frac{1}{2}\log(2\pi n) + \mathcal{O}(n^{-1}) \tag{7.15}$$

we obtain

$$\log D = \sum_{i=1}^{N/2}\left[\left(N-i+\frac{1}{2}\right)\log\frac{(N+k-i)}{(N-i)} + k\log(N+k-i)\right] - \tag{7.16}$$
$$-\sum_{i=1}^{k}\left[\frac{N}{2}\log\left(\frac{N}{2}+i-1\right) + \left(i-\frac{1}{2}\right)\log\frac{(\frac{N}{2}+i-1)}{(i-1)}\right]$$

We may now use the summation formula

$$\sum_{i=1}^{N} f(i) = \int_1^{N+1} f(x)dx + \frac{1}{2}(f(N+1)-f(1)) + \frac{1}{12}(f'(N+1)-f'(1)) - \frac{1}{72}(f''(N+1)-f''(1)) + \mathcal{O}(f''') \tag{7.17}$$

to obtain

$$\log D = \left[4(\lambda+1)\log 2 - \lambda^2\log\lambda + 2(\lambda+1)^2\log(\lambda+1) - (\lambda+2)^2\log(\lambda+2)\right]\frac{N^2}{4\lambda^2} + \tag{7.18}$$
$$-\frac{N}{2}\log N + \left[2\log(\lambda+2) - 4(\lambda+1)\log(\lambda+1) + 2(\lambda-1)\log\lambda + 2\lambda\log 2\right]\frac{N}{4\lambda} -$$
$$-\frac{1}{12}\log\frac{\lambda+2}{\lambda} + \mathcal{O}(N^{-1})$$

8. Free fermions, $O(2N)_1$, $U(N)_1$, and $SU(N)_k$

Consider N complex free left-moving fermions $\psi^i, \bar{\psi}^i$, $i = 1, ..., N$, equivalent to 2N real (Majorana-Weyl) ones. They realize the $O(2N)_1$ current algebra, [10]. This is equivalent as a CFT to $U(N)_1 \sim U(1) \times SU(N)_1$.

The $O(2N)_1$ theory contains the unit affine representation, the vector (V) representation (of dimension 2N) the spinor (S) of dimension 2^{N-1} and the conjugate spinor (C) with dimension also 2^{N-1}.

The left-right symmetric character-valued partition function is

$$Z_{O(2N)_1}(v,\bar{v}) = \frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{N}\frac{|\vartheta[^a_b](v_i)|^2}{|\eta|^2} = |\chi_0|^2 + |\chi_V|^2 + |\chi_S|^2 + |\chi_C|^2 \tag{8.1}$$

with $O(2N)_1$ characters

$$\chi_0(\vec{v}) = \frac{1}{2}\left[\frac{\prod_{i=1}^N \vartheta_3(v_i)}{\eta^N} + \frac{\prod_{i=1}^N \vartheta_4(v_i)}{\eta^N}\right] = q^{-N/24}[1+O(q)] \tag{8.2}$$

$$\chi_V(\vec{v}) = \frac{1}{2}\left[\frac{\prod_{i=1}^N \vartheta_3(v_i)}{\eta^N} - \frac{\prod_{i=1}^N \vartheta_4(v_i)}{\eta^N}\right] = 2N\, q^{-\frac{N}{24}+\frac{1}{2}}[1+O(q)] \tag{8.3}$$

$$\chi_S(\vec{v}) = \frac{1}{2}\left[\frac{\prod_{i=1}^N \vartheta_2(v_i)}{\eta^N} + \frac{\prod_{i=1}^N \vartheta_1(v_i)}{\eta^N}\right] = 2^{N-1}q^{-\frac{N}{24}+\frac{N}{8}}[1+O(q)] \tag{8.4}$$

$$\chi_C(\vec{v}) = \frac{1}{2}\left[\frac{\prod_{i=1}^N \vartheta_2(v_i)}{\eta^N} - \frac{\prod_{i=1}^N \vartheta_1(v_i)}{\eta^N}\right] = 2^{N-1}q^{-\frac{N}{24}+\frac{N}{8}}[1+O(q)] \tag{8.5}$$

We will write the theory in terms of the U(1) and $SU(N)_1$ degrees of freedom. The representations that descend from the unity and vector of $O(2N)$ are generated by the operators ψ^i, $\psi^i\psi^j$, ... $\prod_{k=1}^N \psi^{i_k}$, and their complex conjugates which correspond to the various antisymmetric reps of SU(N). These are the only integrable representations at level one.

This decomposition at the level of conformal dimensions becomes:

$$\Delta = \frac{Q^2}{2} + \Delta_R \tag{8.6}$$

where Q is the appropriately normalized U(1) charge (defined when the associated current has central term equal to one), and Δ_R is the conformal weight of the $SU(N)_1$ reps.

$$\Delta_R = \frac{C_R}{N+1}$$

For the j-index antisymmetric rep of SU(N), $C_R = (N+1)j(N-j)/2N$ so that

$$\Delta_j = \frac{j(N-j)}{2N} \tag{8.7}$$

Its (properly normalized) U(1) charge is $\pm j/\sqrt{N}$ Thus summing up we obtain conformal weight $j/2$ which is in agreement with the interpretation above.

Consider now the two spinor reps. The U(1) charges of those can be easily figured out by bosnizing pairwise the complex fermions

$$\psi^i = e^{i\phi_i} \quad , \quad \bar{\psi}^i = e^{-i\phi_i} \tag{8.8}$$

In this basis, the U(1) current is

$$J = \frac{i}{\sqrt{N}}\sum_{k=1}^N \partial\phi_k \tag{8.9}$$

The spinor and conjugate spinor are generated by the following vertex operators:

$$V_{C,S} = \prod_{k=1}^{N} exp\left[\frac{i}{2}\epsilon_k\phi_k\right] \tag{8.10}$$

with $\epsilon_k = \pm 1$. The spinor corresponds to $\prod_{K=1}^{N}\epsilon_k = 1$ and the conjugate spinor to $\prod_{K=1}^{N}\epsilon_k = -1$ The $U(1)$ charge is $Q = \sum_{k=1}^{N}\epsilon_k/2\sqrt{N}$. It is not difficult to see that the spectrum of U(1) charges coming from C and S is given by

$$Q_k = \frac{N-2k}{2\sqrt{N}} \quad , \quad k = 0,1,2,...,N \tag{8.11}$$

where k even corresponds to one spinor and k odd to the other. C,S decompose under $U(1)\times SU(N)$ to the antisymmetric reps. This can be confirmed by the conformal weights. When the U(1) charge is Q_k the accompanying $SU(N)$ rep is the k-index antisymmetric representation,

$$\frac{Q_k^2}{2} + \frac{k(N-k)}{2N} = \frac{N}{8} \tag{8.12}$$

which is the correct conformal weight of the spinors.

The quantum of U(1) charge is $\frac{1}{\sqrt{N}}$ in the unit-V sector and $\frac{1}{2\sqrt{N}}$ in the spinor sector. The number of states in the Dirac spinor is $2^N = 2\cdot 2^{N-1}$. This is the number of all antisymmetric representation states.

8.1 The fermionic current algebra

We define the U(N) currents J^{ij} in terms of the fermions that satisfy

$$\psi^i(z)\psi^j(w) \sim finite \quad , \quad \bar\psi^i(z)\bar\psi^j(w) \sim finite \quad , \quad \psi^i(z)\bar\psi^j(w) \sim \frac{\delta_{ij}}{z-w} + finite \tag{8.13}$$

$$J^{ij} = i :\psi^i\bar\psi^j: \quad , \quad J^{ij}(z)J^{kl}(w) = \frac{\delta^{il}\delta^{jk}}{(z-w)^2} + if^{ij,kl}{}_{mn}J^{mn}(w) + finite \tag{8.14}$$

$$f^{ij,kl}{}_{mn} = -\delta^{il}\delta^{mk}\delta^{jn} + \delta^{jk}\delta^{mi}\delta^{nl} \tag{8.15}$$

The properly normalized overall U(1) current is

$$J = \frac{1}{\sqrt{N}}\sum_i J^{ii} \quad , \quad J(z)J(w) = \frac{1}{(z-w)^2} + finite \tag{8.16}$$

The currents are uncharged under the zero mode J_0.

Consider now the antisymmetric operator $O^{i_1,\cdots,i_m} =: \psi^{i_1}\psi^{i_2}\cdots\psi^{i_m}$ with U(1) charge $Q = \frac{n}{\sqrt{N}}$.

$$J^{ij}(z)O^{i_1,\cdots,i_m}(w) = \sum_{n=1}^{m}\frac{(-1)^{n+1}\delta^{ji_n}}{z-w}O^{i_1,\cdots,i,\cdots,i_m} + finite \tag{8.17}$$

This indicates that the operators O are affine primaries, that transform as the m-index antisymmetric of $SU(N)$.

Of interest is the $m = N$ operator $O = \prod_{i=1}^{N} \psi^i$ that satisfies

$$J^{ij}(z)O(w) = finite, \quad i \neq j \tag{8.18}$$

Indeed it can be seen that this operator has charge $Q = \sqrt{N}$ and therefore dresses the trivial SU(N) representation.

8.2 $SU(N)_k$

We now consider k copies of N complex fermions, ψ^i_a, $\bar{\psi}^i_a$, $a = 1, 2, \cdots, k$, $i = 1, 2, \cdots, N$.

$$\psi^i_a(z)\psi^j_b(w) \sim finite \quad , \quad \bar{\psi}^i_a(z)\bar{\psi}^j_b(w) \sim finite \quad , \quad \psi^i_a(z)\bar{\psi}^j_b(w) \sim \frac{\delta_{ij}\delta^{ab}}{z-w} + finite \tag{8.19}$$

They realize the tensor product $U(N)_1^k$ CFT.

We may construct the $U(N)_k$ currents as

$$J^{ij} = \sum_{a=1}^{k} :\psi^i_a \bar{\psi}^j_a: \quad , \quad J^{ij}(z)J^{kl}(w) = k\frac{\delta^{il}\delta^{jk}}{(z-w)^2} + if^{ij,kl}{}_{mn}J^{mn}(w) + finite \tag{8.20}$$

as well as $U(k)_N$ currents

$$J^{ab} = \sum_{i=1}^{N} :\psi^i_a \bar{\psi}^i_b: \quad , \quad J^{ab}(z)J^{cd}(w) = N\frac{\delta^{ad}\delta^{bc}}{(z-w)^2} + if^{ab,cd}{}_{ef}J^{ef}(w) + finite \tag{8.21}$$

The two groups are not independent as they share the same overall U(1)

$$J = \frac{1}{\sqrt{kN}} \sum_{i=1}^{N}\sum_{a=1}^{k} :\psi^i_a \bar{\psi}^i_a \tag{8.22}$$

Moreover the two current algebras are not commuting. A general commutator gives other currents of the maximal $O(2kN)_1$ current algebra.

The global subalgebra of the $U(k)_N$ algebra acts non-trivially on the $(U(N)_1)^k/U(N)_k$ coset. To leading order it will be a symmetry of $U(N)_k$.

In this respect we have that the theory is given by $O(2kN)_1 \simeq U(kN)_1 \simeq U(N)_1^k$ and

$$O(2kN)_1 = SU(N)_k \otimes SU(k)_N \otimes U(1) \quad , \quad SU(N)_k = \frac{O(2kN)_1}{SU(k)_N \otimes U(1)} \tag{8.23}$$

In the limit $N \to \infty$, with k fixed, the coset $SU(k)_N$ has a central charge of $\mathcal{O}(1)$ therefore to leading order.

$$SU(N)_k \simeq O(2kN)_1 \tag{8.24}$$

The same is true in the 't Hooft limit with $\lambda \gg 1$. Therefore in these limits the WZW model reduces to a theory of kN free complex fermions.

Moreover (8.23) can be translated to the statement that the conformal U(k) gauge theory of N flavors of massless fermions is equivalent to the $SU(N)_k$ WZW model.

In this respect the 't Hooft limit $k \to \infty$, $N \to \infty$ with $N/k = \lambda$ fixed can be interpreted as a Veneziano limit with λ being the ratio of flavors to colors.

Several of the above issues were discussed in early papers on the realization of chiral symmetry in two dimensions, [37].

9. Analysis of a four-point function in $SU(N)_k$

We present in this appendix the details of the analysis of the $SU(N)_k$ four-point function of the fundamental, $g_{a,b}(z,\bar{z})$ and its conjugate $g^{-1}_{b,a}(z,\bar{z})$ following [11].

The result is

$$\langle g_{a_1,b_1}(z_1,\bar{z}_1) g^{-1}_{b_2,a_2}(z_2,\bar{z}_2) g_{a_3,b_3}(z_3,\bar{z}_3) g^{-1}_{b_4,a_4}(z_4,\bar{z}_4)\rangle = |z_{14}z_{23}|^{-4\Delta_\Box}\, G(x,\bar{x}) = \tag{9.1}$$

$$= \langle g_{a_1,b_1}(\infty) g^{-1}_{b_2,a_2}(1) g_{a_3,b_3}(x,\bar{x}) g^{-1}_{b_4,a_4}(0)\rangle$$

where $z_{ij} \equiv z_i - z_j$, x is the standard cross-ratio

$$x = \frac{z_{12}z_{34}}{z_{14}z_{32}} \tag{9.2}$$

and bars stand for complex conjugation.

The function G can be decomposed into group channels as

$$G(x,\bar{x}) = \sum_{A,B=1}^{2} I^A \bar{I}^B\, G_{AB}(x,\bar{x}) \tag{9.3}$$

with

$$I^1 = \delta_{a_1,a_2}\delta_{a_3,a_4} \quad , \quad \bar{I}^1 = \delta_{b_1,b_2}\delta_{b_3,b_4} \quad , \quad I^2 = \delta_{a_1,a_4}\delta_{a_2,a_3} \quad , \quad \bar{I}^2 = \delta_{b_2,b_4}\delta_{b_1,b_3} \tag{9.4}$$

and conformal block channels as

$$G_{AB}(x,\bar{x}) = \mathcal{F}^{(1)}_A(x)\mathcal{F}^{(1)}_B(\bar{x}) + h\mathcal{F}^{(2)}_A(x)\mathcal{F}^{(2)}_B(\bar{x}) \tag{9.5}$$

The conformal blocks have been calculated by solving the Knizhnik-Zamolodchikov equations, [11] and are given by

$$\mathcal{F}^{(1)}_1(x) = x^{-2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}\; F\left(-\frac{1}{2\kappa}, \frac{1}{2\kappa}; 1+\frac{N}{2\kappa}, x\right) \tag{9.6}$$

$$\mathcal{F}_2^{(1)}(x) = -\frac{x^{1-2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}}{2\kappa+N}\ F\left(1-\frac{1}{2\kappa}, 1+\frac{1}{2\kappa}; 2+\frac{N}{2\kappa}, x\right) \tag{9.7}$$

$$\mathcal{F}_1^{(2)}(x) = x^{\Delta_A - 2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}\ F\left(-\frac{N-1}{2\kappa}, -\frac{N+1}{2\kappa}; 1-\frac{N}{2\kappa}, x\right) \tag{9.8}$$

$$\mathcal{F}_2^{(2)}(x) = -N x^{\Delta_A - 2\Delta_\Box}(1-x)^{\Delta_A - 2\Delta_\Box}\ F\left(-\frac{N-1}{2\kappa}, -\frac{N+1}{2\kappa}; -\frac{N}{2\kappa}, x\right) \tag{9.9}$$

where F is the hypergeometric function,

$$\Delta_\Box = \frac{N^2-1}{2N(N+k)} \quad , \quad \Delta_A = \frac{N}{N+k} \quad , \quad 2\kappa = -N-k \tag{9.10}$$

are the conformal dimensions of the fundamental and adjoint affine primaries and the $(\Box, \Box, \text{Adjoint})$ OPE coefficient h is given by

$$h = \frac{1}{N^2}\frac{\Gamma\left[\frac{N-1}{N+k}\right]\Gamma\left[\frac{N+1}{N+k}\right]\Gamma^2\left[\frac{k}{N+k}\right]}{\Gamma\left[\frac{k+1}{N+k}\right]\Gamma\left[\frac{k-1}{N+k}\right]\Gamma^2\left[\frac{N}{N+k}\right]} \tag{9.11}$$

We now take the t' Hooft limit of the correlation function, defined in section 3.1 to obtain

$$\Delta_\Box = \frac{\lambda}{2(1+\lambda)}\left[1-\frac{1}{N^2}\right] \quad , \quad \Delta_A = \frac{\lambda}{1+\lambda} \tag{9.12}$$

$$\Delta_A - 2\Delta_\Box = \frac{\lambda}{1+\lambda}\frac{1}{N^2} \tag{9.13}$$

$$h = \frac{1}{N^2}\frac{\Gamma\left[\frac{\lambda}{\lambda+1}\left(1-\frac{1}{N}\right)\right]\Gamma\left[\frac{\lambda}{\lambda+1}\left(1+\frac{1}{N}\right)\right]\Gamma^2\left[\frac{1}{1+\lambda}\right]}{\Gamma\left[\frac{1+\frac{\lambda}{N}}{\lambda+1}\right]\Gamma\left[\frac{1-\frac{\lambda}{N}}{\lambda+1}\right]\Gamma^2\left[\frac{\lambda}{1+\lambda}\right]} = \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \tag{9.14}$$

$$\begin{aligned}\mathcal{F}_1^{(1)}(x) &= x^{-\frac{\lambda}{1+\lambda}\left[1-\frac{1}{N^2}\right]}(1-x)^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}\ F\left(\frac{\lambda}{1+\lambda}\frac{1}{N}, -\frac{\lambda}{1+\lambda}\frac{1}{N}; \frac{1}{1+\lambda}, x\right) \\ &= x^{-\frac{\lambda}{1+\lambda}} + \mathcal{O}\left(\frac{1}{N^2}\right)\end{aligned} \tag{9.15}$$

$$\begin{aligned}\mathcal{F}_2^{(1)}(x) &= \frac{\lambda}{N}\ x^{\frac{1+\frac{\lambda}{N^2}}{1+\lambda}}(1-x)^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}\ F\left(1+\frac{\lambda}{1+\lambda}\frac{1}{N}, 1-\frac{\lambda}{1+\lambda}\frac{1}{N}; \frac{2+\lambda}{1+\lambda}, x\right) \\ &= \frac{\lambda}{N}\ x^{\frac{1}{1+\lambda}}\ F\left(1, 1; \frac{2+\lambda}{1+\lambda}, x\right) + \mathcal{O}\left(\frac{1}{N^2}\right)\end{aligned} \tag{9.16}$$

$$\begin{aligned}\mathcal{F}_1^{(2)}(x) &= x^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}(1-x)^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}\ F\left(\frac{\lambda}{1+\lambda}\left(1-\frac{1}{N}\right), \frac{\lambda}{1+\lambda}\left(1+\frac{1}{N}\right); 1+\frac{\lambda}{1+\lambda}, x\right) \\ &= F\left(\frac{\lambda}{1+\lambda}, \frac{\lambda}{1+\lambda}; 1+\frac{\lambda}{1+\lambda}, x\right) + \mathcal{O}\left(\frac{1}{N^2}\right)\end{aligned} \tag{9.17}$$

$$= (1-x)^{\frac{1}{1+\lambda}}\ F\left(1,1;1+\frac{\lambda}{1+\lambda},x\right)+\mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\mathcal{F}_2^{(2)}(x) = -N\ x^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}(1-x)^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}\ F\left(\frac{\lambda}{1+\lambda}\left(1-\frac{1}{N}\right),\frac{\lambda}{1+\lambda}\left(1+\frac{1}{N}\right);\frac{\lambda}{1+\lambda},x\right) \tag{9.18}$$

$$= -N\ x^{\frac{\lambda}{1+\lambda}\frac{1}{N^2}}(1-x)^{-\frac{\lambda}{1+\lambda}\left[1-\frac{1}{N^2}\right]}\ F\left(-\frac{\lambda}{1+\lambda}\frac{1}{N},\frac{\lambda}{1+\lambda}\frac{1}{N};\frac{\lambda}{1+\lambda},x\right)$$

$$= -N\ (1-x)^{-\frac{\lambda}{1+\lambda}}+\mathcal{O}\left(\frac{1}{N^2}\right)$$

Using the above we find

$$G_{11} = |x|^{-\frac{2\lambda}{1+\lambda}}+\mathcal{O}\left(\frac{1}{N^2}\right) \quad , \quad G_{22} = |1-x|^{-\frac{2\lambda}{1+\lambda}}+\mathcal{O}\left(\frac{1}{N^2}\right) \tag{9.19}$$

$$G_{12} = \frac{\lambda}{N}|x|^{-\frac{2\lambda}{1+\lambda}}\ \bar{x}\ F\left(1,1;\frac{2+\lambda}{1+\lambda},\bar{x}\right)-\frac{1}{N}(1-x)|1-x|^{-\frac{2\lambda}{1+\lambda}}\ F\left(1,1;1+\frac{\lambda}{1+\lambda},x\right)+\mathcal{O}\left(\frac{1}{N^2}\right) \tag{9.20}$$

In the limit $\lambda \to \infty$ of the leading order in 1/N result we obtain

$$G_{11} = \frac{1}{|x|^2}+\cdots \quad , \quad G_{22} = \frac{1}{|1-x|^2}+\cdots \tag{9.21}$$

$$G_{12} = \frac{\lambda}{N}\frac{1}{x(1-\bar{x})}+\cdots \tag{9.22}$$

where we have used $F(1,1,1,x) = \frac{1}{1-x}$.

This is the free-fermion four-point function.

In the limit $\lambda \to 0$ we obtain instead

$$G_{11} = 1-2\lambda\log|x|+\cdots \quad , \quad G_{22} = 1-2\lambda\log|1-x|+\cdots \quad , \quad G_{12} = -\frac{1}{N}+\cdots \tag{9.23}$$

9.1 On the large-N and large-λ limits.

We will now consider the commutativity of the two limits $N \to \infty$ and $\lambda \to \infty$, in one of the dynamical functions of the WZW model, namely the structure constant h, and show that the two limits do not commute.

We start from (9.11) and expand the OPE coefficient in a double expansion

$$h = \sum_{n=1}^{\infty}\sum_{m=-2}^{\infty}\frac{W_{n,m}}{N^{2n}\lambda^m} \tag{9.24}$$

Taking the large-N limit first while keeping λ fixed we obtain

$$\lim_{N\to\infty} h = \sum_{n=1}^{\infty} \frac{Z_n(\lambda)}{N^{2n}} \tag{9.25}$$

with

$$Z_1 = 1 \quad , \quad Z_2 = \frac{\lambda^2}{(\lambda+1)^2}\left[\psi'\left(\frac{\lambda}{1+\lambda}\right) - \psi'\left(\frac{1}{1+\lambda}\right)\right] \tag{9.26}$$

$$Z_3 = \frac{\lambda^4}{12(1+\lambda)^4}\left[6\left[\psi'\left(\frac{1}{1+\lambda}\right) - \psi'\left(\frac{\lambda}{1+\lambda}\right)\right]^2 - \psi'''\left(\frac{1}{1+\lambda}\right) + \psi'''\left(\frac{\lambda}{1+\lambda}\right)\right] \tag{9.27}$$

Taking then the $\lambda \to \infty$ limit we obtain

$$\lim_{\lambda\to\infty} Z_2 = -\lambda^2 + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{9.28}$$

$$\lim_{\lambda\to\infty} Z_3 = 2\lambda\psi''(1) - 6\psi''(1) + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{9.29}$$

from which we read in the range $1 \le n \le 3$, $-2 \le m \le 0$

$$W_{1,0} = 1 \quad , \quad W_{2,-2} = -1 \quad , \quad W_{3,-1} = 2\psi''(1) \quad , \quad W_{3,0} = -6\psi''(1) \tag{9.30}$$

all other being zero in that range.

On the other hand taking the large-λ limit first we obtain

$$\lim_{\lambda\to\infty} h = \sum_{n=-2}^{\infty} H_n(N)\lambda^{-n} \tag{9.31}$$

with

$$H_{-2} = -\frac{1}{N^4} \quad , \quad H_{-1} = \frac{2}{N^4}\left[2\gamma_E + \psi\left(\frac{1}{N}\right) + \psi\left(-\frac{1}{N}\right)\right] \tag{9.32}$$

$$\begin{aligned} H_0 = &-\frac{3\psi\left(1+\frac{1}{N}\right)^2 + 16\gamma_E\psi\left(-\frac{1}{N}\right) + \psi\left(-\frac{1}{N}\right)^2 + 16\gamma_E\psi\left(\frac{1}{N}\right) + \psi\left(\frac{1}{N}\right)^2}{2N^4} - \\ &-\frac{8\psi\left(1+\frac{1}{N}\right)\psi\left(1-\frac{1}{N}\right) + 3\psi\left(1-\frac{1}{N}\right)^2}{2N^4} + \frac{2-4\gamma_E(1+2\gamma_E)}{N^4} + \\ &+\frac{1}{N^2} - \frac{\pi}{N^5}\cot\left(\frac{\pi}{N}\right) + \frac{2\psi'\left(-\frac{1}{N}\right) - 2\psi'\left(\frac{1}{N}\right)}{N^5} \end{aligned} \tag{9.33}$$

We now take the large-N limit to obtain

$$\lim_{N\to\infty} H_{-1} = \frac{2\psi''(1)}{N^6} + \mathcal{O}(N^{-8}) \tag{9.34}$$

$$\lim_{N\to\infty} H_0 = \frac{1 - 4\gamma_E + \frac{\pi^2}{3}}{N^4} + \frac{15\pi^2 + \pi^4 - 180\psi''(1)}{45N^6} + \mathcal{O}(N^{-8}) \tag{9.35}$$

from which we deduce

$$W_{2,-2} = -1 \quad , \quad W_{3,-1} = 2\psi''(1) \quad , \quad W_{2,0} = 1 - 4\gamma_E + \frac{\pi^2}{3} \quad , \quad W_{3,0} = \frac{15\pi^2 + \pi^4 - 180\psi''(1)}{45} \tag{9.36}$$

while the rest are zero.

Comparing (9.30) and (9.36) we observe that the two limits do not commute.

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