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String Theory and The Velo–Zwanziger Problem

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Abstract

We examine the behavior of the leading Regge trajectory of the open bosonic string in a uniform electromagnetic background and present a consistent set of Fierz–Pauli conditions for these symmetric tensors that generalizes the Argyres–Nappi spin-2 result. These equations indicate that String Theory does bypass the Velo–Zwanziger problem, *i.e.* the loss of causality experienced by a massive high–spin field minimally coupled to electromagnetism. Moreover, we provide some evidence that *only* the first Regge trajectory can be described in isolation and show that the open–string spectrum is free of ghosts in weak constant backgrounds. Finally, we comment on the roles of the critical dimension and of the gyromagnetic ratio.

Contents

1 Introduction

In Quantum Field Theory, the available types of fundamental particles reflect the irreducible unitary representations of the Poincaré group [1, 2], which exist for arbitrary (integer or half–integer) values of the spin, not only for the handful of choices that underlie the Standard Model of Electroweak and Strong Interactions or General Relativity. One is thus confronted with a challenging problem, since higher–spin systems [3] are apparently fraught with grave difficulties, so much so that in Minkowski space minimal interactions of *massless* particles of high spin with electromagnetism (EM) or gravity are not allowed [4, 5, 6] [1]. *Massive* high–spin particles certainly exist, in the form of hadronic resonances. Truly enough, these particles are composite, so that the actual form factors describing their interactions are complicated functions of the exchanged momenta. Still, in the quasi–collinear regime, when the exchanged momenta are small compared to the particle masses, one expects that their dynamics is governed by consistent *local* actions. Moreover, massive higher–spin modes play a role in (open) string spectra, where they describe excitations that are generically unstable, and where the finite string size puts them again somewhat on the par with extended composite systems. Some of the most spectacular novelties of String Theory [9, 10], however, including (planar) duality, modular invariance and open–closed duality, rest heavily on their presence, and this is by itself a compelling motivation to take a closer look at their properties.

Even if one restricts the attention to *massive* higher–spin fields, a number of known actions readily exhibit pathological behavior in the simplest possible settings, and in particular in constant external backgrounds [11, 12, 13]. A notorious example is provided by a charged massive spin-2 field in a constant EM background in flat space, and it was indeed an early analysis of this problem that led Fierz and Pauli [14] to stress the importance of a Lagrangian formulation for higher–spin systems. Their suggestion actually opened a wide avenue of research, with first complete results in the 1970's [15, 16] and new additions up to recent times [17, 18, 19, 20], but even the resulting Lagrangians, as we have anticipated, do not come to terms with the original problem. Rather, in general they do not propagate the correct number of degrees of freedom (DoF) in the presence of *minimal* EM couplings, nor do they propagate their own DoFs only within the light cone. For spin $s = 2$, for instance, although the first difficulty can be overcome by an apparently unique choice of the gyromagnetic ratio [21], $g = \frac{1}{2}$, some of the modes suffer from lack of hyperbolicity or faster–than–light

[1] Minimal–like interactions do become available for massive fields or in the presence of a cosmological constant [7], but in both circumstances they ought to be regarded as byproducts of higher–derivative "seeds", as recently stressed in [8].

propagation. This is the vexing "Velo–Zwanziger problem" [11], which generally shows up when massive charged fields with spin $s > 1$ are minimally coupled to an EM background. The problem actually persists for a wide class of non-minimal extensions, so that constructing consistent interactions for charged massive higher–spin fields with EM from a field–theory vantage point appears to be a challenging task.

On the other hand, String Theory was originally meant to describe hadronic resonances, a plethora of massive particles that, as we have already mentioned, typically carry high spins, and actually electric charges as well, so that it should provide a valuable laboratory to investigate these exotic types of EM interactions. And indeed, starting from the open bosonic string, Argyres and Nappi built long ago a consistent Lagrangian [22] for a massive charged spin-2 field coupled to a constant EM background. In this case the interactions reflect rather basic properties of String Theory, since they are induced by the EM deformation of the free string developed in [23]. The resulting Lagrangian is nonetheless highly non-minimal, but both its equations of motion (EoMs) and the constraints they give rise to are strikingly simple: they mimic those of the free theory after some field redefinitions, which makes their consistency almost manifest.

The Lagrangian formulation attained via the BRST technique as in [22, 24, 25] requires that the Fock space be extended to include world–sheet (anti)ghosts. Hence, it involves in general a host of auxiliary fields, and the procedure becomes rather cumbersome already for $s = 3$ [25]. A result of this complication is that it is not even clear, as of yet, whether the open bosonic string cures the Velo–Zwanziger problem of its massive modes. And even if this were the case, a number of related questions still await a proper answer, including the following two. Does consistency call for physical fields belonging to all Regge trajectories present at a given string mass level, or could a (sub)leading Regge trajectory be consistent in isolation? Could one attain a consistent description in non-critical dimensions as well? One would definitely like to arrive at a better understanding of these issues, and to some extent we shall succeed. At the same time, while the BRST method gives gauge–invariant Lagrangians for the modes of the open bosonic string in $d = 26$, it is important to stress that gauge invariance alone *does not* guarantee that the resulting description be consistent, since after all any action can be made gauge invariant via the Stückelberg formalism. In fact, the classical consistency of a dynamical system, and of the Argyres–Nappi system in particular, rests on the behavior of the EoMs in a unitary gauge. Truly enough, a Lagrangian formulation does guarantee that the resulting EoMs be algebraically consistent, but this key property can be also verified directly, taking the EoMs themselves at face value.

In view of these considerations, we begin by formulating physical state conditions in the presence of a constant EM background, without introducing any (anti)ghosts. These give rise to (partially gauge–fixed) EoMs that the string fields must obey, and after removing some leftover modes that are pure gauge one can investigate directly their consistency. One can work at any given mass level, because one is actually dealing with deformed free strings, and in this fashion it is possible to identify particular sets of fields that are required for algebraic consistency. Given these EoMs, one can also analyze explicitly both the actual propagating DoFs and their causal properties. The main results of this paper are thus a relatively concise description of the consistent (non-minimal) EM interactions of massive totally symmetric tensors of arbitrary spin that are present in String Theory and an explicit proof that they provide a remedy for the Velo–Zwanziger problem, at least in $d = 26$. More in detail, we show that any symmetric tensor belonging to the first Regge trajectory of the open bosonic string can propagate independently, in a constant EM background, the correct number of DoFs, and that these develop properly within the light cone. In addition, we provide some evidence that fields belonging to subleading trajectories do not propagate consistently by themselves. Let us emphasize, however, that our claims apply insofar as the EM field invariants, among which $F_{\mu\nu}\, F^{\mu\nu}$ is but one, are small in units of m^2/e, where m is mass of the higher–spin field and e is its electric charge. This is an important qualification: if some invariant were $\mathcal{O}(1)$ in those units, a number of new phenomena would present themselves, including Schwinger pair production [26] and Nielsen–Olesen instabilities [27]. Their very existence implies precisely that any effective Lagrangian for a charged particle interacting with EM fields can be reliable, even well below its own cutoff scale, only with this further proviso. The Velo–Zwanziger problem is particularly important precisely because it appears well within the expected range of validity of the effective theory.

The paper is organized as follows. In Section 2 we reconsider the world–sheet description of a charged bosonic open string in a constant EM background and perform a careful analysis of the mode expansion and the Virasoro generators, with emphasis on the behavior in the limit of vanishing total charge. Armed with this knowledge, in Sections 3 and 4 we translate the physical state conditions for string states into the language of string fields. Section 3 is actually devoted to free strings, but it is meant to make the reader better equipped for understanding the more complicated case of charged strings, which we consider in Section 4, where we show explicitly that String Theory indeed cures the Velo–Zwanziger problem for the symmetric tensors of the first Regge trajectory. In Section 5 we present a proof of the corresponding no–ghost theorem, showing that the Hilbert space of string states has a non-negative inner product even in a (weak) constant EM background, which is crucial for consistency. In Section 6 we take a closer look at spin-2 Lagrangians: in particular, Section 6.1 investigates the role of the critical dimension in the consistency of the Argyres–Nappi construction, while Section 6.2 elaborates on a possible route for its generalization to arbitrary dimensions, and then solves a conundrum and clears up a misconception about the gyromagnetic ratio of spin-2 particles. Finally, Section 7 contains some concluding remarks and the two Appendices collect useful material on the massive $s = 2$ system and on the bosonic string.

2 Open strings in a constant EM background

In this section we review in detail the mode expansion and the Virasoro algebra for a charged bosonic open string in a constant EM background. The program originally started in [23] and has received a wide attention in the literature, giving rise also to a number of applications (see, for instance, [28, 29, 30], for recent discussions). The novelty of our treatment in this section is a careful definition of the expansion that makes it possible to reach smoothly the limits of neutral or free strings.

It will suffice to consider an open bosonic string whose endpoints lie on a space–filling D–brane. A Maxwell field A_μ living in the world–volume of the D–brane couples to charges e_0 and e_π at the string endpoints, and this turns the string action into

$$S \ = \ \frac{1}{4\pi\alpha'} \int d\tau d\sigma (\dot{X}_\mu \dot{X}^\mu - X'_\mu X'^\mu) + \int d\tau d\sigma \left[e_0 \delta(\sigma) + e_\pi \delta(\sigma - \pi) \right] A_\mu(X) \dot{X}^\mu \ , \qquad (2.1)$$

where the world sheet is chosen to be a strip of width π with a conformally flat metric of signature $(-, +)$ and α' is the string Regge slope. The X^μ are coordinates in the $d = 26$ target space, which we take to be Minkowski, while "dot" and "prime" denote derivatives with respect to the world–sheet coordinates τ and σ.

In this paper we only consider electromagnetic backgrounds whose field strength $F_{\mu\nu}$ is constant, so that one can choose the potential

$$A_\mu \ = \ - \tfrac{1}{2} \, F_{\mu\nu} \, X^\nu \ . \qquad (2.2)$$

In units with $\alpha' = \frac{1}{2}$, the string sigma model then reads

$$S = \frac{1}{2\pi}\int d\tau d\sigma(\dot{X}_\mu \dot{X}^\mu - X'_\mu X'^\mu) + \frac{1}{2}\int d\tau d\sigma \left[e_0 \delta(\sigma) + e_\pi \delta(\sigma - \pi)\right] F_{\mu\nu} X^\mu \dot{X}^\nu \ . \tag{2.3}$$

As a result, the EoMs are those of the usual free string,

$$\ddot{X}_\mu - X''_\mu = 0 \ , \tag{2.4}$$

while the boundary conditions are affected by the new terms and become

$$X'_\mu \ = \ -\pi e_0 F_{\mu\nu} \dot{X}^\nu \qquad (\sigma = 0) \ , \tag{2.5}$$

$$X'_\mu \ = \ +\pi e_\pi F_{\mu\nu} \dot{X}^\nu \qquad (\sigma = \pi) \ . \tag{2.6}$$

Let us also define, for later use,

$$e \ \equiv \ e_0 \ + \ e_\pi \ . \tag{2.7}$$

2.1 Mode expansion

In solving this boundary value problem, one should take into account that the $F_{\mu\nu} \to 0$ limit ought to recover the *free* string mode expansion, which is recalled in Appendix B along with some useful facts about the free mode functions. We denote with $\mathbb{N}_0$ ($\mathbb{N}_1$) the set of all natural numbers including (excluding) 0, and we adopt for matrix multiplications a concise notation, so that $A^{\mu\nu}u_\nu = A^\mu{}_\nu u^\nu \equiv (Au)^\mu$, $u_\mu A^{\mu\nu} = u^\mu A_\mu{}^\nu \equiv (uA)^\nu$, $A^{\mu\rho}B_\rho{}^\nu = A^\mu{}_\rho B^{\rho\nu} \equiv (AB)^{\mu\nu}$, and we write $\eta^{\mu\nu}$ as $\mathbf{1}^{\mu\nu}$, and δ^μ_ν as $\mathbf{1}^\mu_\nu$.

We can thus present the solution [23] of Eqs. (2.4)–(2.6) in the form

$$\begin{aligned} X^\mu(\tau,\sigma) \ &= \ x^\mu + \left[\left(\frac{\mathrm{e}^{-G_0}}{2}\cdot\frac{\mathrm{e}^{G(\tau+\sigma)} - M_+}{G} \ + \ \frac{\mathrm{e}^{+G_0}}{2}\cdot\frac{\mathrm{e}^{G(\tau-\sigma)} - M_-}{G}\right)\alpha_0\right]^\mu \\ &\quad + \frac{i}{2}\sum_{m\neq 0}\left[\left(\tfrac{1}{m\mathbf{1}+iG}\right)\left\{\mathrm{e}^{-i(m\mathbf{1}+iG)(\tau+\sigma)-G_0} + \mathrm{e}^{-i(m\mathbf{1}+iG)(\tau-\sigma)+G_0}\right\}\alpha_m\right]^\mu , \end{aligned} \tag{2.8}$$

where the matrices

$$G_0 \ = \ \tanh^{-1}(\pi e_0 F) \ , \qquad G_\pi \ = \ \tanh^{-1}(\pi e_\pi F) \ , \qquad G \ = \ \frac{1}{\pi}\left[\,G_0 + G_\pi\,\right] \ , \tag{2.9}$$

are uniquely determined by the boundary conditions (2.5) and (2.6), while the matrices $M_\pm$

are additional functions of F whose forms will be specified shortly. No ambiguities are met in these expressions, since $G, G_0, M_\pm$ and their inverses are all functions of F only, and are thus mutually commuting. Note, finally, that the matrices $(m\mathbf{1} \pm iG)$, with $m \neq 0$, are always invertible whenever the EM field invariants are sufficiently small.

In writing the mode expansion (2.8) we required that, in the $F \to 0$ limit, the α_m^μ reduce for any given m to the modes of the free string, so that the same must hold true for the mode function matrices. This is readily seen to be the case for the "oscillator" modes in the second line of Eq. (2.8), since G_0 and G tend to zero in this limit. On the other hand, the requirement that the coefficient matrix of α_0^μ reduces, in the limit, to $\bar{\Psi}_0 = \tau$, poses on $M_\pm$ the non-trivial condition that

$$M_\pm \;=\; \mathbf{1} \pm \gamma\, G \;+\; \mathcal{O}(G^2)\ . \tag{2.10}$$

As we shall see shortly, the constant γ, along with the whole $\mathcal{O}(G^2)$ term, can be completely determined requiring that the x^μ's in (2.8) be *standard commuting center–of–mass coordinates.* This choice will also lead to a smooth limit of the resulting expressions in the dipole $(e_0 + e_\pi = 0)$ or free $(e_0 = e_\pi = 0)$ cases, contrary to some claims that have appeared in [30].

Now notice that the matrix–valued functions (of τ and σ),

$$\Psi'_m(\tau,\sigma) \;=\; \frac{i/2}{\sqrt{m\mathbf{1}+iG}}\ \mathrm{e}^{-i(m\mathbf{1}+iG)\tau}\big[\,\mathrm{e}^{-i(m\mathbf{1}+iG)\sigma - G_0} + \mathrm{e}^{i(m\mathbf{1}+iG)\sigma + G_0}\,\big] \qquad m \in \mathbb{N}_0\ , \tag{2.11}$$

form an orthonormal set, since

$$(\Psi'_m, \Psi'_n) \;\equiv\; \frac{1}{\pi}\int_0^\pi d\sigma \Psi'^{\dagger}_m(\tau,\sigma) \star \Psi'_n(\tau,\sigma) \;=\; \delta_{mn} \qquad m,n \in \mathbb{N}_0\ , \tag{2.12}$$

if $\star$ is defined as

$$\star \;\equiv\; i\overleftrightarrow{\partial}_\tau - i\tanh G_0\, \delta(\sigma) - i\tanh G_\pi\, \delta(\sigma - \pi)\ . \tag{2.13}$$

A constant has thus a non-vanishing norm, and moreover it is orthogonal to all other functions in Eq. (2.11),

$$(1,1) \;=\; -ieF\ , \qquad (1, \Psi'_m) \;=\; 0 \qquad m \in \mathbb{N}_0\ , \tag{2.14}$$

where e is the total string charge defined in Eq. (2.7). Therefore, in view of the mode expansion (2.8) one can write

$$\alpha_m^\mu \;=\; (\sqrt{m\mathbf{1}+iG})^\mu{}_\nu (\Psi'_m, X)^\nu \qquad m \in \mathbb{N}_0 \;, \tag{2.15}$$

taking into account the reality of X^μ and the relations

$$(\Psi'^*_m, \Psi'^*_n) \;=\; \delta_{mn}, \qquad (\Psi'_m, \Psi'^*_n) \;=\; -(\Psi'^*_m, \Psi'_n) \;=\; i\delta_{m0} \qquad m,n \in \mathbb{N}_0 \;, \tag{2.16}$$

and one can let

$$\alpha_{-m}^\mu \;=\; (\sqrt{m\mathbf{1}-iG})^\mu{}_\nu (\Psi'^*_m, X)^\nu \qquad m \in \mathbb{N}_0 \;. \tag{2.17}$$

Upon quantization, the string modes α_m^μ with $m \in \mathbb{Z}$ of Eqs. (2.15) and (2.17) become operators that obey non-trivial commutation relations. In order to quantize the system, one can first note that the canonical momentum for the sigma model (2.3) is

$$P^\mu(\tau,\sigma) \;=\; \frac{1}{\pi}\,\dot{X}^\mu(\tau,\sigma) \;-\; \frac{1}{2}\,[\,e_0\delta(\sigma)+e_\pi\delta(\sigma-\pi)\,]\,F^{\mu\nu}X_\nu(\tau,\sigma) \;, \tag{2.18}$$

and then require that X and P satisfy the equal time commutation relations

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] \;=\; i\eta^{\mu\nu}\delta(\sigma-\sigma') \;, \tag{2.19}$$

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] \;=\; [P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\,] \;=\; 0 \;. \tag{2.20}$$

Using Eqs. (2.15), (2.17) and (2.18)–(2.20) it is then simple to show that

$$[\,\alpha_m^\mu, \alpha_n^\nu\,] \;=\; (m\,\eta^{\mu\nu} \;+\; i\,G^{\mu\nu})\,\delta_{m,-n} \qquad m,n \in \mathbb{Z} \;. \tag{2.21}$$

Before computing the other commutators, let us elaborate on the meaning of these results. In physically interesting situations, away from instabilities, the matrices $\sqrt{m\mathbf{1}\pm iG}$ are always invertible when $m \in \mathbb{N}_1$ so that, on account of Eq. (2.21),

$$a_m^\mu \;\equiv\; \left[\frac{1}{\sqrt{m\mathbf{1}+iG}}\;\alpha_m\right]^\mu, \qquad a_m^{\dagger\mu} \;\equiv\; \left[\frac{1}{\sqrt{m\mathbf{1}-iG}}\;\alpha_{-m}\right]^\mu \qquad m \in \mathbb{N}_1 \tag{2.22}$$

are an infinite set of creation and annihilation operators:

$$[\,a_m^\mu, a_n^{\dagger\nu}\,] \;=\; \eta^{\mu\nu}\delta_{mn}, \qquad [\,a_m^\mu, a_n^\nu\,] \;=\; [\,a_m^{\dagger\mu}, a_n^{\dagger\nu}\,] \;=\; 0 \qquad m,n \in \mathbb{N}_1 \;. \tag{2.23}$$

When $m=0$, however, one cannot reach this point starting from (2.21), since $\sqrt{G}$ is not invertible when $F \neq 0$ in some Lorentz frame (and obviously so when $eF = 0$). The $\alpha_0^\mu = \alpha_0^{*\mu}$, on the other hand, are well–defined, and their commutation relations read

$$[\,\alpha_0^\mu, \alpha_0^\nu\,] \;=\; i\,G^{\mu\nu} \; . \tag{2.24}$$

Naively, one would expect that α_0^μ play the role of a covariant momentum, since after all it reduces to the string momentum $\bar{\alpha}_0^\mu \;=\; p^\mu$ when F vanishes. Furthermore, the string Hamiltonian is

$$H \;\equiv\; \int_0^\pi d\sigma[\,P^\mu \dot{X}_\mu - \mathcal{L}\,] \;=\; \frac{1}{2\pi}\int_0^\pi d\sigma(\dot{X}^2 + X'^2) \;=\; \tfrac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_m \; , \tag{2.25}$$

where the last equality is due to a consequence of the mode expansion (2.8), namely

$$(\dot{X} \pm X')^\mu(\tau,\sigma) \;=\; \sum_{m\in\mathbb{Z}} \left[\mathrm{e}^{-i(m\mathbf{1}+iG)(\tau\pm\sigma)\mp G_0}\right]^\mu{}_\nu\,\alpha_m^\nu \qquad \sigma \in [0,\pi] \; . \tag{2.26}$$

In view of Eqs. (2.23) and (2.25), therefore,

$$H \;=\; \tfrac{1}{2}\,\alpha_0^2 + (\text{Stringy Oscillator Contributions}) \; , \tag{2.27}$$

whereas for a charged point particle the Hamiltonian would read

$$H \;=\; \tfrac{1}{2}\,p_{\rm cov}^2 \; , \tag{2.28}$$

where $p_{\rm cov}^\mu$ is the covariant momentum. This vindicates the identification of α_0^μ with the covariant momentum, up to a matrix redefinition that is needed since

$$[\,p_{\rm cov}^\mu, p_{\rm cov}^\nu\,] \;=\; ieF^{\mu\nu} \; . \tag{2.29}$$

Comparing Eqs. (2.24) and (2.29), one is thus led to conclude that

$$\alpha_0^\mu \;=\; Q^\mu{}_\nu\, p_{\rm cov}^\nu \; , \qquad QQ^T \;=\; \frac{G}{eF} \; , \tag{2.30}$$

so that the covariant derivative,

$$D^\mu \;\equiv\; ip_{\rm cov}^\mu \; , \tag{2.31}$$

finally obeys the desired commutation relation:

$$[\,D^\mu, D^\nu\,] \;=\; -ieF^{\mu\nu}\ . \tag{2.32}$$

Notice that the matrix Q reduces to unity in the $F \to 0$ limit, and that one can choose it to be symmetric, letting

$$Q \;=\; \sqrt{\frac{G}{eF}}\ . \tag{2.33}$$

For future reference, it is actually convenient to define a different covariant derivative,

$$\mathcal{D}^\mu \;\equiv\; \left(\sqrt{G/eF}\right)^{\mu\nu} D_\nu \;=\; i\,\alpha_0^\mu\ , \tag{2.34}$$

such that

$$[\mathcal{D}^\mu, \mathcal{D}^\nu] \;=\; -\,i\,G^{\mu\nu}\ . \tag{2.35}$$

Now one can easily compute the commutators $[x^\mu, \alpha_m^\nu]$, starting from the relation

$$[X^\mu(\tau,\sigma), (\dot{X}+X')^\nu(\tau,\sigma')] \;=\; [X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] \;=\; i\pi\eta^{\mu\nu}\delta(\sigma-\sigma')\ , \tag{2.36}$$

and from Eqs. (2.8) and (2.26), with the end result

$$[\,x^\mu, \alpha_m^\nu\,] \;=\; \frac{i}{2}\left[\,\mathrm{e}^{-G_0}M_+ + \mathrm{e}^{G_0}M_-\,\right]^{\mu\nu}\delta_{m0}\ . \tag{2.37}$$

The identification of x^μ as center–of–mass coordinates now demands that

$$[x^\mu, p^\nu_{\mathrm{cov}}] \;=\; i\eta^{\mu\nu}\ , \tag{2.38}$$

and in view of Eqs. (2.30), (2.33) and (2.37) this leads to

$$\mathrm{e}^{-G_0}M_+ + \mathrm{e}^{G_0}M_- \;=\; 2\sqrt{\frac{G}{eF}}\ . \tag{2.39}$$

Finally, starting from the fundamental commutator

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] \;=\; 0\ , \tag{2.40}$$

and making use of the mode expansion (2.8) and of Eqs. (2.21) and (2.37), one can see that

$$[\, x^\mu, x^\nu \,] \;=\; 0 \;, \tag{2.41}$$

so that the center–of–mass coordinates are indeed mutually commuting, on account of Eq. (2.39).

In order to find $M_\pm$ explicitly, one can now demand that the mode expansion (2.8) be symmetric under the flip operation $\sigma \to (\pi - \sigma)$, to be combined with the interchange of the charges at the endpoints, $e_{0,\pi} \to e_{\pi,0}$, and with the flip properties of the oscillators, $\alpha_m \to (-1)^m \alpha_m$. In view of (2.10) it is clear that the $M_\pm$ do not transform under such a flip. Therefore, Eq. (2.39) gives

$$\mathrm{e}^{-G_\pi} M_+ + \mathrm{e}^{G_\pi} M_- \;=\; 2\sqrt{\frac{G}{eF}} \;, \tag{2.42}$$

and thus Eqs. (2.39) and (2.42) finally lead to

$$M_\pm \;=\; \sqrt{\frac{G}{eF}}\, \mathrm{sech}\left[\, \tfrac{1}{2}(G_\pi - G_0)\right] \, \mathrm{e}^{\pm \pi G/2} \;=\; \mathbf{1} \pm \tfrac{1}{2}\pi G + \mathcal{O}(G^2) \;. \tag{2.43}$$

Let us note that, if the string is a dipole, so that $e_0 = -e_\pi$, G vanishes but G_0 remains finite, while the covariant momentum reduces to the ordinary momentum. From Eqs. (2.30), (2.33) and (2.43), one can see that the mode expansion (2.8) of the charged string reduces smoothly to a corresponding expression for the neutral one,

$$\begin{aligned} X^\mu_{\mathrm{neut}}(\tau,\sigma) \;&=\; x^\mu + \left[\frac{\tau \,-\, (\sigma - \pi/2)\, \pi e_0\, F}{\sqrt{\mathbf{1} - (\pi e_0 F)^2}}\right]^{\mu\nu} \left[\frac{1}{\sqrt{\mathbf{1} - (\pi e_0 F)^2}}\right]_{\nu\sigma} p^\sigma \\ &+\; \frac{i}{2} \sum_{m \neq 0} \tfrac{1}{m} \left[\mathrm{e}^{-im(\tau+\sigma) - G_0} + \mathrm{e}^{-im(\tau-\sigma) + G_0}\,\right]^\mu{}_\nu \;\alpha^\nu_m \;. \end{aligned} \tag{2.44}$$

It is important to notice that this expression differs from Eq. (2.26) of [30], and moreover that the charged string mode functions are not quite given by (2.11). Rather, they are

$$\Psi_m(\tau,\sigma) \;=\; \tfrac{i/2}{\sqrt{m\mathbf{1}+iG}}\, \mathrm{e}^{-i(m\mathbf{1}+iG)\tau} \left[\,\mathrm{e}^{-i(m\mathbf{1}+iG)\sigma - G_0} + \mathrm{e}^{i(m\mathbf{1}+iG)\sigma + G_0}\,\right] \qquad m \in \mathbb{N}_1 \;, \tag{2.45}$$

$$\Psi_0(\tau,\sigma) \;=\; \frac{\mathrm{e}^{-G_0}}{2} \cdot \frac{\mathrm{e}^{G(\tau+\sigma)} - M_+}{G} \;+\; \frac{\mathrm{e}^{+G_0}}{2} \cdot \frac{\mathrm{e}^{G(\tau-\sigma)} - M_-}{G} \;, \tag{2.46}$$

together, of course, with a constant mode. While $\Psi_m = \Psi'_m$ for all $m \in \mathbb{N}_1$, Ψ_0 is in fact a linear combination of Ψ_0' and the constant mode. The latter fact is crucial, in that it guarantees a smooth $F \to 0$ limit. The inner product is still defined according to Eqs. (2.12) and (2.13), so that the following orthogonality relations for $m, n \in \mathbb{N}_0$ hold:

$$(\Psi_m, \Psi_n) \;=\; \delta_{mn}(1-\delta_{m0}) \;, \qquad (1,1) \;=\; -ieF \;, \qquad (1,\Psi_m) \;=\; i\sqrt{eF/G}\,\delta_{m0} \;. \tag{2.47}$$

These mode functions naturally split once more into two mutually orthogonal subsets, particle–like $\{1, \Psi_0\}$, and string–like $\{\Psi_{m\in\mathbb{N}_1}\}$, and the infinitely many string–like modes form an orthonormal set of functions with respect to the inner product of Eq. (2.12). The two particle–like modes have a non-vanishing inner product, and the norm of Ψ_0 vanishes while that of 1 is $\mathcal{O}(F)$. Everything thus parallels the relations (B.4) and (B.5) for free strings, because of the particular linear combination appearing in (2.46), and in the present case Eq. (B.6) generalizes to

$$x^\mu \;=\; i(\Psi_0, X)^\mu \;, \qquad p^\mu_{\rm cov} \;=\; -i\,(\mathbf{1} + eF\Psi_0, X)^\mu \;. \tag{2.48}$$

2.2 Virasoro generators

As we have seen already, the world–sheet action under consideration differs from the free string Polyakov action only by boundary terms. The latter, however, do not depend on the world–sheet metric, since they are obtained via the pullback of target–space one–forms. Therefore, the constraints that are to be imposed after gauge fixing take the same form as in the free theory:

$$(\dot X \pm X')^2 \;=\; 0 \;. \tag{2.49}$$

With the Virasoro generators L_n defined as

$$\sum_{n\in\mathbb{Z}} L_n\,\mathrm{e}^{-in(\tau\pm\sigma)} \;\equiv\; \tfrac{1}{2}(\dot X \pm X')^2(\tau,\sigma) \;, \tag{2.50}$$

one can formally extend the range of σ to $[-\pi, \pi]$ with the help of Eq. (2.26), to write

$$L_n \;=\; \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma\,\mathrm{e}^{in(\tau+\sigma)}(\dot X \pm X')^2(\tau,\sigma) \;=\; \tfrac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m \;. \tag{2.51}$$

The final expression is identical to that of the free theory, but as we have seen the α_m's now have the commutation relations (2.21). One can work out the commutation relations obeyed by the Virasoro generators, paying attention as usual to the central extension. The end result is the emergence, in the constant EM background, of an additive contribution to L_0, so that

$$L_0 \;\to\; L_0 + \tfrac{1}{4}\mathrm{Tr}G^2 \;, \tag{2.52}$$

but up to this shift the Virasoro algebra,

$$[L_m, L_n] \;=\; (m-n)L_{m+n} \;+\; \tfrac{1}{12}\, d(m^3-m)\delta_{m,-n} \;, \tag{2.53}$$

remains precisely as in the free theory. The shift, however, has an important effect, since it reflects itself in deformed masses for the open–string excitations.

3 Physical state conditions for free strings

Having spelled out the key properties of the (charged) open string modes, the corresponding Virasoro generators and their algebra, we can now turn to the physical state conditions for (charged) string states. In this section we actually begin by considering free strings, as this provides some valuable insights for the more complicated study of charged strings, which will be the subject of the next section.

Let us begin by recalling that a string state $|\Phi\rangle$ is called "physical" if it satisfies the conditions (see *e.g.* [9, 10])

$$(L_n - \delta_{n0})|\Phi\rangle \;=\; 0 \qquad n \in \mathbb{N}_0 \;. \tag{3.1}$$

Actually, in view of the Virasoro algebra, it suffices to demand that

$$\begin{aligned} (L_0 - 1)|\Phi\rangle &\;=\; 0 \;, && (3.2)\\ L_1|\Phi\rangle &\;=\; 0 \;, && (3.3)\\ L_2|\Phi\rangle &\;=\; 0 \;. && (3.4) \end{aligned}$$

Once Eqs. (B.13)–(B.15) are used, for the symmetric tensors of the leading Regge trajectory these physical state conditions translate into the well–known Fierz–Pauli conditions, namely the Klein–Gordon equation and the conditions that their divergences and traces vanish.

3.1 Massless level: $N = 1$

The generic state at this level is

$$|\Phi\rangle \;=\; A_\mu(x)\, a_1^{\dagger\mu}\, |0\rangle \; . \tag{3.5}$$

In this case Eq. (3.4) is empty, and one thus obtains the Maxwell equations in the Lorenz gauge,

$$\Box A_\mu \;=\; 0 \;, \qquad \partial^\mu A_\mu \;=\; 0 \;, \tag{3.6}$$

for the massless vector field A_μ. Notice that these equations are invariant under the *on–shell* gauge transformation

$$\delta A_\mu \;=\; \partial_\mu\, \alpha \;, \qquad \Box\, \alpha \;=\; 0 \; . \tag{3.7}$$

3.2 First massive level: $N = 2$

In this case a generic state takes the form

$$|\Phi\rangle \;=\; h_{\mu\nu}(x)\, a_1^{\dagger\mu} a_1^{\dagger\nu}\, |0\rangle \;+\; \sqrt{2}\, i\, B_\mu(x)\, a_2^{\dagger\mu}\, |0\rangle \;, \tag{3.8}$$

and Eqs. (3.2)–(3.4) give

$$(\Box - 2) h_{\mu\nu} \;=\; 0 \;, \qquad (\Box - 2) B_\mu \;=\; 0 \;, \tag{3.9}$$

$$\partial^\mu h_{\mu\nu} - B_\nu \;=\; 0 \;, \qquad h^\mu{}_\mu + 2 \partial^\mu B_\mu \;=\; 0 \; . \tag{3.10}$$

One can verify that these equations possess, in an arbitrary space–time dimension d, the *on–shell* gauge symmetry

$$\delta h_{\mu\nu} \;=\; (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \left(\tfrac{10}{d+4}\right) \eta_{\mu\nu}\, \partial \cdot \xi \;, \tag{3.11}$$

$$\delta B_\mu \;=\; 2\xi_\mu + \left(\tfrac{d-6}{d+4}\right) \partial_\mu\, \partial \cdot \xi \;, \tag{3.12}$$

where the gauge parameter ξ_μ satisfies the condition

$$(\Box - 2)\, \xi_\mu \;=\; 0 \; . \tag{3.13}$$

One could have arrived at the string field equations (3.9) and (3.10) via the BRST construction, following [31]. There would be more fields to begin with than those present in (3.8), but one could (partially) gauge fix the EoMs using the BRST symmetry, which holds only in the critical dimension, to finally recover Eqs. (3.9) and (3.10). The gauge symmetry (3.11)–(3.13) actually holds in an arbitrary number of dimensions, and is thus more general than what the BRST method would give. In fact, proceeding from a field theory perspective, one can also build rather naturally, at least for the first few mass levels, a generalization of the BRST symmetry that is available in an arbitrary number of space–time dimensions [32].

One can now gauge away the vector field B_μ, in any number of space–time dimensions, making use of the gauge parameter ξ_μ, and thus end up with the spin-2 Fierz–Pauli system

$$(\Box - 2)h_{\mu\nu} \;=\; 0 \;, \qquad \partial^\mu h_{\mu\nu} \;=\; 0 \;, \qquad h^\mu{}_\mu \;=\; 0 \;, \tag{3.14}$$

so that $h_{\mu\nu}$ is a massive spin-2 field, with $(\text{mass})^2 = 2$ (or $1/\alpha'$ taking into account our choice of units), that obeys the Fierz–Pauli conditions.

It is also possible to arrive at (3.14) by a gauge–fixing procedure that does not involve solving a differential equation for the gauge parameter. To this end, one can define a new vector field B'_μ whose gauge variation is algebraic in ξ_μ ,

$$B'_\mu \;\equiv\; B_\mu + \left(\tfrac{d-6}{10}\right)\left[\partial^\nu h_{\mu\nu} + \tfrac{1}{8}\left(\tfrac{d+4}{d-1}\right)\partial_\mu h^\nu{}_\nu\right] \;, \qquad \delta B'_\mu \;=\; \left(\tfrac{d+4}{5}\right)\xi_\mu \;. \tag{3.15}$$

It is now evidently possible to choose B'_μ in such a way that $B_\mu = 0$, whence Eq. (3.14) follows.

3.3 Second massive level: $N = 3$

A generic state at this mass level is

$$|\Phi\rangle \;=\; \phi_{\mu\nu\rho}(x)\, a_1^{\dagger\mu} a_1^{\dagger\nu} a_1^{\dagger\rho}\, |0\rangle \;+\; \tfrac{i}{\sqrt{2}}\,[\, h_{\mu\nu}(x) + A_{\mu\nu}(x)\,]\, a_2^{\dagger\mu} a_1^{\dagger\nu}\, |0\rangle - \tfrac{1}{\sqrt{3}}\, B_\mu(x)\, a_3^{\dagger\mu}\, |0\rangle \;, \tag{3.16}$$

where $A_{\mu\nu}$ is an antisymmetric tensor while both $\phi_{\mu\nu\rho}$ and $h_{\mu\nu}$ are symmetric tensors. Eqs. (3.2)–(3.4) give rise to

$$(\Box - 4)\phi_{\mu\nu\rho} \;=\; 0 \;, \qquad (\Box - 4)[h_{\mu\nu} + A_{\mu\nu}] \;=\; 0 \;, \qquad (\Box - 4)B_\mu \;=\; 0 \;, \tag{3.17}$$

$$3\partial^\mu \phi_{\mu\nu\rho} - h_{\nu\rho} \;=\; 0 \;, \qquad \partial^\mu A_{\mu\nu} - \partial^\mu h_{\mu\nu} + 2B_\nu \;=\; 0 \;, \tag{3.18}$$

$$3\phi^\mu{}_{\mu\nu} + \partial^\mu h_{\mu\nu} + \partial^\mu A_{\mu\nu} - B_\nu \;=\; 0 \;, \tag{3.19}$$

and in the critical dimension, $d = 26$, this system is invariant under the *on–shell* gauge transformations

$$\begin{aligned}
\delta\phi_{\mu\nu\rho} &= \partial_{(\mu}\lambda_{\nu\rho)} + \eta_{(\mu\nu}\xi_{\rho)} \ , && (3.20)\\
\delta h_{\mu\nu} &= 12\lambda_{\mu\nu} + 3\eta_{\mu\nu}\partial\cdot\xi + 3\partial_{(\mu}\xi_{\nu)} + 3\partial_{(\mu}(\partial\cdot\lambda)_{\nu)} \ , && (3.21)\\
\delta A_{\mu\nu} &= -15\partial_{[\mu}\xi_{\nu]} - 3\partial_{[\mu}(\partial\cdot\lambda)_{\nu]} \ , && (3.22)\\
\delta B_{\mu} &= 36\xi_\mu + 18(\partial\cdot\lambda)_\mu - \tfrac{9}{2}\partial_\mu\partial\cdot\xi \ , && (3.23)
\end{aligned}$$

where $\lambda_{\mu\nu} = \lambda_{\nu\mu}$ and the gauge parameters are subject to the conditions

$$(\Box - 4)\lambda_{\mu\nu} \ = \ 0 \ , \qquad (\Box - 4)\xi_\mu \ = \ 0 \ , \tag{3.24}$$

$$\lambda^{\mu}{}_{\mu} \ = \ -2\partial\cdot(\partial\cdot\lambda) - \tfrac{17}{2}\partial\cdot\xi \ . \tag{3.25}$$

Just as in the preceding $N = 2$ case, here it should be possible to render the gauge symmetry valid for arbitrary values of d by judicious modifications of the coefficients appearing in the gauge transformations and in the trace constraint (3.25). We hope to return to this point for the complete spectrum in a future publication [32].

It is possible to gauge away the vector field B_μ using the parameter ξ_μ. On the other hand, because of the trace constraint (3.25) on the gauge parameter $\lambda_{\mu\nu}$, one can only set to zero the traceless part of $h_{\mu\nu}$. This gauge fixing thus reduces the system to [2]

$$(\Box - 4)\phi_{\mu\nu\rho} \ = \ 0 \ , \qquad (\Box - 4)A_{\mu\nu} \ = \ 0 \ , \qquad (\Box - 4)h \ = \ 0 \ , \tag{3.26}$$

$$3\partial^\mu\phi_{\mu\nu\rho} \ = \ \eta_{\nu\rho}h \ , \qquad \partial^\mu A_{\mu\nu} \ = \ \partial_\nu h \ , \qquad 3\phi^{\mu}{}_{\mu\nu} + \partial^\mu A_{\mu\nu} \ = \ -\partial_\nu h \ , \tag{3.27}$$

where $h^{\mu}{}_{\mu} \equiv d\,h$. One can now show that h is an auxiliary scalar field: as is well known, it is essential for writing a local Lagrangian for a massive spin-3 field, but it is set to zero on–shell if Eqs. (3.26) and (3.27) are combined with their traces and divergences. Once h is eliminated, the system reduces to

$$(\Box - 4)\phi_{\mu\nu\rho} \ = \ 0 \ , \qquad \partial^\mu\phi_{\mu\nu\rho} \ = \ 0 \ , \qquad \phi^{\mu}{}_{\mu\nu} \ = \ 0 \ . \tag{3.28}$$

$$(\Box - 4)A_{\mu\nu} \ = \ 0 \ , \qquad \partial^\mu A_{\mu\nu} \ = \ 0 \ . \tag{3.29}$$

Eq. (3.28) are the Fierz–Pauli conditions for a massive spin-3 symmetric tensor field, $\phi_{\mu\nu\rho}$,

[2] In this step, one can forego the need to solve differential equations for the gauge parameters, as we have already seen for the $N = 2$ case.

that belongs to the first Regge trajectory for the open bosonic string and whose squared mass is $2/\alpha'$ (taking into account our choice of units), while Eq. (3.29) are the corresponding equations for an antisymmetric rank-2 field of the same mass [3]. These are the two physical fields that appear at the second massive level of the open bosonic string, and in the free theory their EoMs are completely decoupled.

4 Physical state conditions in an EM background

The construction of a generic string state in the presence of a constant EM background follows the same lines as in the free theory, because the two cases rest on identical sets of creation and annihilation operators. While the Virasoro generators themselves are different, their algebras are the same in both cases, so that the physical state conditions still read

$$
\begin{aligned}
(L_0 - 1)|\Phi\rangle &= 0 \,, & (4.1)\\
L_1|\Phi\rangle &= 0 \,, & (4.2)\\
L_2|\Phi\rangle &= 0 \,, & (4.3)
\end{aligned}
$$

where now

$$
\begin{aligned}
L_0 &= -\tfrac{1}{2}\mathcal{D}^2 + \sum_{m=1}^{\infty}(m+iG)_{\mu\nu}\, a_m^{\dagger\mu} a_m^{\nu} + \tfrac{1}{4}\,\mathrm{Tr}G^2 \\
&\equiv -\tfrac{1}{2}\mathcal{D}^2 + \left(\mathcal{N} + \tfrac{1}{4}\,\mathrm{Tr}G^2\right) + i\sum_{m=1}^{\infty} G_{\mu\nu}\, a_m^{\dagger\mu} a_m^{\nu} \,, \qquad (4.4)\\
L_1 &= -i\left[\sqrt{1+iG}\right]_{\mu\nu} \mathcal{D}^{\mu} a_1^{\nu} + \sum_{m=2}^{\infty}\left[\sqrt{(m+iG)(m-1+iG)}\right]_{\mu\nu} a_{m-1}^{\dagger\mu} a_m^{\nu} \,, \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
L_2 = -i\left[\sqrt{2+iG}\right]_{\mu\nu} \mathcal{D}^{\mu} a_2^{\nu} &+ \tfrac{1}{2}\left[\sqrt{1+G^2}\right]_{\mu\nu} a_1^{\mu} a_1^{\nu} \\
&+ \sum_{m=3}^{\infty}\left[\sqrt{(m+iG)(m-2+iG)}\right]_{\mu\nu} a_{m-2}^{\dagger\mu} a_m^{\nu} \,, \qquad (4.6)
\end{aligned}
$$

[3] In $d=4$, $A_{\mu\nu}$ is equivalent to a massive vector field, and Eq. (3.29) is clearly consistent with this fact.

and $\mathcal{D}_\mu$ was defined in Eq. (2.34). Notice that here we are going to define the number operator $\mathcal{N}$ in such a way that its eigenvalues N are integers, just as for free strings:

$$\mathcal{N} \equiv \sum_{n=1}^{\infty} n\, a_n^\dagger \cdot a_n \ . \tag{4.7}$$

This expression coincides indeed with its free–string counterpart when it is expressed in terms of the "a" operators, and as a result the two differ when expressed in terms of the "α" operators. In the presence of an EM background, our definition appears more convenient than the one used in [22, 24, 25].

4.1 Level $N = 1$

The generic state at this level is

$$|\Phi\rangle = A_\mu(x)\, a_1^{\dagger\mu}\, |0\rangle \ . \tag{4.8}$$

Eq. (4.3) is empty, while Eqs. (4.1) and (4.2) reduce to

$$\left[\mathcal{D}^2 - \tfrac{1}{2}\text{Tr}G^2\right] A_\mu \; - \; 2\,i\,G_\mu{}^\nu A_\nu \; = \; 0 \ , \tag{4.9}$$

$$\mathcal{D}^\mu \left(\sqrt{\mathbf{1} + iG} \cdot A\right)_\mu \; = \; 0 \ , \tag{4.10}$$

on account of the commutation relations (2.23) and of the usual definition of the oscillator vacuum. Letting

$$\mathcal{A}_\mu \equiv \left(\sqrt{\mathbf{1} + iG} \cdot A\right)_\mu \ , \tag{4.11}$$

these field equations can be cast in the form

$$\left[\mathcal{D}^2 - \tfrac{1}{2}\text{Tr}G^2\right] \mathcal{A}_\mu - 2iG_\mu{}^\nu \mathcal{A}_\nu \; = \; 0 \ , \tag{4.12}$$

$$\mathcal{D}^\mu \mathcal{A}_\mu \; = \; 0 \ , \tag{4.13}$$

and their algebraic consistency can be easily verified. Notice that in the presence of a constant EM background the spin-1 field has acquired a new contribution to its mass, $\frac{1}{4\alpha'}\text{Tr}G^2$. The divergence constraint (4.13) guarantees that the number of dynamical DoFs is not affected. That the propagation of the spin-1 field is causal can be shown along the lines of [22], but we postpone the proof until Section 4.4.

4.2 Level $N = 2$

A generic state at this mass level is

$$|\Phi\rangle \;=\; h_{\mu\nu}(x)\, a_1^{\dagger\mu} a_1^{\dagger\nu}\, |0\rangle \;+\; \sqrt{2}\, i\, B_\mu(x)\, a_2^{\dagger\mu}\, |0\rangle \;, \tag{4.14}$$

and after the field redefinitions

$$\mathcal{H}_{\mu\nu} \;\equiv\; \left(\sqrt{1+iG}\cdot h\cdot\sqrt{1-iG}\right)_{\mu\nu} \;, \tag{4.15}$$

$$\mathcal{B}_\mu \;\equiv\; \left(\sqrt{1+\tfrac{i}{2}G}\cdot B\right)_\mu \;, \tag{4.16}$$

one is led to

$$\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)\mathcal{H}_{\mu\nu} - 2\,i\,(G\mathcal{H} - \mathcal{H}G)_{\mu\nu} \;=\; 0 \;, \tag{4.17}$$

$$\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)\mathcal{B}_\mu - 2\,i\,G_\mu{}^\nu\mathcal{B}_\nu \;=\; 0 \;, \tag{4.18}$$

$$\mathcal{D}^\mu\mathcal{H}_{\mu\nu} - (1+iG)_\nu{}^\rho\mathcal{B}_\rho \;=\; 0 \;, \tag{4.19}$$

$$\mathcal{H}^\mu{}_\mu + 2\mathcal{D}^\mu\mathcal{B}_\mu \;=\; 0 \;. \tag{4.20}$$

Let us emphasize that, for consistency, in the presence of this non-trivial background the system should have the same number of dynamical fields as in the free case, and therefore the vector field $\mathcal{B}_\mu$ should continue to be non-dynamical. On the other hand, it is simple to see that the system (4.17)–(4.20) does not give rise to any algebraic relation between $\mathcal{B}_\mu$ and the other fields, while Eq. (4.18) contains second derivatives of $\mathcal{B}_\mu$. Therefore, $\mathcal{B}_\mu$ can be non-dynamical only if it is pure gauge.

In order to see that this is actually the case, let us begin by writing the most general *on–shell* gauge transformation for the system,

$$\delta\mathcal{H}_{\mu\nu} \;=\; [J\cdot\mathcal{D}]_\mu\,\xi_\nu + [J\cdot\mathcal{D}]_\nu\,\xi_\mu + \tfrac{1}{2}(K_{\mu\nu} + K_{\nu\mu})(\mathcal{D}\cdot\xi) \;, \tag{4.21}$$

$$\delta\mathcal{B}_\mu \;=\; (L\cdot\xi)_\mu + [M\cdot\mathcal{D}]_\mu(\mathcal{D}\cdot\xi) \;, \tag{4.22}$$

where the matrices J, K, L, and M are functions of G, and the gauge parameter ξ_μ satisfies the condition:

$$\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)\xi_\mu - 2\,i\,G_\mu{}^\nu\xi_\nu \;=\; 0 \;. \tag{4.23}$$

To see that Eqs. (4.21)–(4.22) define indeed the most general *on–shell* gauge transformation, let us take G to be small, so that one can restrict the attention to terms up to $\mathcal{O}(G)$, and let us examine the possible occurrence of additional higher–derivative terms in Eqs. (4.21)–(4.22). Any such term should be $\mathcal{O}(G)$, in view of the free limit of the transformations, Eqs. (3.11)–(3.12). It is then simple to realize that, in principle, Eqs. (4.21) and (4.22) can accommodate terms with odd and even numbers of derivatives, respectively. Moreover, in any such term G should be contracted with ξ, since otherwise one could eliminate it using the relation $G^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu \sim \mathrm{Tr}G^2$. One thus finds that any possible correction of (4.22) should contain $\mathcal{D}^2\xi$, which however would allow to lower the number derivatives by two units, due to the on–shell condition (4.23). In conclusion, the form of (4.22), with at most two derivatives, is the most general one. Once this is ascertained, looking at the field equations (4.17)–(4.20) one can conclude that no higher derivative terms can be present in Eq. (4.21) if the transformation is to be a symmetry.

One can verify that Eqs. (4.17)–(4.20) are invariant under the gauge transformation if

$$\begin{aligned} J &= \mathbf{1} + \tfrac{3}{2}\left(\tfrac{d-6}{d+4}\right) i\,G\,, && (4.24)\\ K &= -\left(\tfrac{10}{d+4}\right)\left[\mathbf{1} + \tfrac{1}{20}(d-6)G^2\right]\,, && (4.25)\\ L &= 2\cdot\mathbf{1} + \tfrac{3}{2}\left(\tfrac{d-6}{d+4}\right) i\,G\,, && (4.26)\\ M &= \left(\tfrac{d-6}{d+4}\right)\left[\mathbf{1} + \tfrac{1}{2}\,i\,G\right]\,. && (4.27)\end{aligned}$$

However, the gauge transformations (4.21)–(4.23) are a symmetry *only* in the critical dimension $d = 26$. In this case one can actually gauge away the vector field $\mathcal{B}_\mu$, ending up with

$$\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)\mathcal{H}_{\mu\nu} - 2\,i\,(G\mathcal{H} - \mathcal{H}G)_{\mu\nu} = 0\,, \qquad \mathcal{D}^\mu\mathcal{H}_{\mu\nu} = 0\,, \qquad \mathcal{H}^\mu{}_\mu = 0\,, \quad (4.28)$$

so that $\mathcal{H}_{\mu\nu}$ is a massive spin-2 field, with

$$(\mathrm{mass})^2 = \frac{1}{\alpha'}\left(1 + \tfrac{1}{4}\,\mathrm{Tr}G^2\right)\,, \qquad (4.29)$$

that possesses a suitable non-minimal coupling to the background EM field. It is manifest that the system (4.28) preserves the right number of DoFs, namely $\frac{1}{2}(d+1)(d-2)$. That this type of systems admit only causal propagation is shown in Section 4.4 for the more general case of spin-s tensors, retracing the arguments of Argyres and Nappi [22]. The Fierz–Pauli system (4.28) is indeed related to the Argyres–Nappi Lagrangian of [22], as we shall see in Section 6.1, but *only* in $d = 26$. In fact, we have already seen here that away from the critical dimension the vector field $\mathcal{B}_\mu$ cannot be gauged away. This is to be contrasted with the behavior in the absence of the EM background, since we have seen in the previous section that in the free theory the vector field is pure gauge in any number of space–time dimensions.

To conclude let us remark that, while setting to zero $\mathcal{B}_\mu$ via Eq. (4.22) results in a differential equation in ξ_μ, this can still be solved as a power series in $G\mathcal{D}^2$. One can indeed covariantize the redefinition (3.15), letting

$$\mathcal{B}'_\mu \;=\; \mathcal{B}_\mu + \left(\tfrac{d-6}{10}\right)\left[\mathcal{D}^\nu \mathcal{H}_{\mu\nu} + \tfrac{1}{8}\left(\tfrac{d+4}{d-1}\right)\mathcal{D}_\mu \mathcal{H}^\nu{}_\nu\right] . \tag{4.30}$$

The gauge variation of $\mathcal{B}'_\mu$ then becomes

$$\delta\mathcal{B}'_\mu \;=\; L'_{\mu\nu}\xi^\nu + \mathcal{O}(G\mathcal{D}^2\xi) \;, \tag{4.31}$$

where $L'_{\mu\nu}$ is algebraic and invertible for small enough G. In order to remove $\mathcal{B}_\mu$, one is thus led to an iterative definition of ξ_μ:

$$\xi_\mu \;=\; -L'^{-1}_{\mu\nu}\mathcal{B}'^\nu + \mathcal{O}(G\mathcal{D}^2\xi) \;. \tag{4.32}$$

4.3 Level $N = 3$

In Section 3 we have seen that at this mass level there are two distinct *physical* fields: a symmetric rank-3 tensor and an antisymmetric rank-2 tensor. The complete gauge fixing of the system, however, leaves an additional auxiliary scalar field. Our analysis of the $N = 2$ level leads one to expect that even in the presence of a non-trivial background one ought to be able to gauge away unphysical states, at least in the critical dimension. Given this premise, for the $N = 3$ level, one is entitled to begin by considering the state

$$|\Phi\rangle \;=\; \phi_{\mu\nu\rho}(x)\, a_1^{\dagger\mu} a_1^{\dagger\nu} a_1^{\dagger\rho}\,|0\rangle \;+\; \tfrac{i}{\sqrt{2}}\left[\,A_{\mu\nu}(x) + \eta_{\mu\nu}h(x)\,\right] a_2^{\dagger\mu} a_1^{\dagger\nu}\,|0\rangle \;, \tag{4.33}$$

where $\phi_{\mu\nu\rho}$ and $A_{\mu\nu}$ are, respectively, a symmetric 3-tensor and an antisymmetric 2-tensor, while h is a scalar. When applied to this state, after the field redefinitions

$$\varPhi_{\mu\nu\rho} \;\equiv\; (\sqrt{1+iG}\,)_\mu{}^\alpha(\sqrt{1+iG}\,)_\nu{}^\beta(\sqrt{1+iG}\,)_\rho{}^\gamma\,\phi_{\alpha\beta\gamma} \;, \tag{4.34}$$

$$\mathcal{A}_{\mu\nu} \;\equiv\; (\sqrt{1+iG}\,)_\mu{}^\alpha(\sqrt{1+iG}\,)_\nu{}^\beta\,A_{\alpha\beta} \;, \tag{4.35}$$

Eqs. (4.1)–(4.3) give

$$\left(\mathcal{D}^2 - 4 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)\varPhi_{\mu\nu\rho} + 2\,i\,G^{\alpha}{}_{(\mu}\varPhi_{\nu\rho)\alpha} \;=\; 0\;, \tag{4.36}$$

$$\left(\mathcal{D}^2 - 4 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)\mathcal{A}_{\mu\nu} - 2\,i\,(G_{\mu}{}^{\alpha}\mathcal{A}_{\alpha\nu} - \mathcal{A}_{\mu}{}^{\alpha}G_{\alpha\nu}) \;=\; 0\;, \tag{4.37}$$

$$\left(\mathcal{D}^2 - 4 - \tfrac{1}{2}\,\mathrm{Tr}G^2\right)h \;=\; 0\;, \tag{4.38}$$

$$3\mathcal{D}^{\mu}\varPhi_{\mu\nu\rho} \;=\; \tfrac{1}{2}\left[\left\{\sqrt{(1+\tfrac{i}{2}G)(1+iG)}\right\}_{\nu}{}^{\alpha}\{\mathcal{A}_{\alpha\rho} + (\sqrt{1+G^2}\,)_{\alpha\rho}\,h\} + (\nu\leftrightarrow\rho)\right]\;, \tag{4.39}$$

$$\mathcal{D}^{\mu}\mathcal{A}_{\mu\nu} \;=\; \mathcal{D}^{\mu}\left[(\sqrt{1+G^2}\,)_{\mu\nu}\,h\right]\;, \tag{4.40}$$

$$3\varPhi^{\mu}{}_{\mu\nu} + \mathcal{D}_{\mu}\left[\left(\sqrt{\frac{1+\frac{i}{2}G}{1+iG}}\right)^{\mu\rho}\{\mathcal{A}_{\rho\nu} + (\sqrt{1+G^2}\,)_{\rho\nu}\,h\}\right] \;=\; 0\;. \tag{4.41}$$

As expected, these equations reduce to those of the free theory, Eqs. (3.26)–(3.27), in the limit $G \to 0$, and one can verify their algebraic consistency making use of the commutation relations (2.35). One can also conclude that, just as in the free case, the scalar field h must be auxiliary. In order to see this, let us compute the trace of Eq. (4.39) and let us apply $\mathcal{D}^{\nu}$ from the left to Eq. (4.41). Subtracting the two resulting equations and making use of Eq. (4.40), one can then obtain

$$h \;=\; -\left[\mathrm{Tr}\sqrt{(1+\tfrac{i}{2}G)(1-iG)}\right]^{-1}\mathrm{Tr}\left[\sqrt{\frac{1+\frac{i}{2}G}{1+iG}}\cdot\mathcal{A}\right]\;. \tag{4.42}$$

Substituting this expression for h into the system (4.36)–(4.41), one is finally left with five independent equations, which are generalizations of Eqs. (3.28)–(3.29) in the presence of a constant EM background, and whose algebraic consistency can be directly verified.

Note that the right–hand side of (4.42) does not vanish for $G \neq 0$. Therefore, if one were to remove the scalar h from the generic state (4.33), and thus from (4.36)–(4.41), the system would be plagued by algebraic inconsistencies, in the form of an unwarranted constraint on the field $\mathcal{A}_{\mu\nu}$ that does not exist in the absence of the background. Therefore, any Lagrangian that can consistently describe the behavior of the two physical fields at level $N = 3$ in an EM background must contain an auxiliary scalar mode. String Theory of course takes care of the problem, since the construction of the spin-3 system includes, in the first place, a scalar field of this type.

The most important novelty introduced by the background is that the auxiliary scalar is connected, via Eq. (4.42), to the field $\mathcal{A}_{\mu\nu}$ of the second Regge trajectory. A consistent Lagrangian description of $\mathcal{A}_{\mu\nu}$ thus calls for the presence of h. On the other hand, h is also the auxiliary scalar for the spin-3 field of the first Regge trajectory, so that fields belonging to the second Regge trajectory *cannot* be described consistently without the leading trajectory. At the same time, Lagrangians are bound to mix the trajectories, and in order to see this in further detail let us set $\Phi_{\mu\nu\rho} = 0$ in Eqs. (4.36)–(4.41). The trace of Eq. (4.39) would then give

$$h \ = \ - \left[\mathrm{Tr}\sqrt{(1+\tfrac{i}{2}G)(1+iG)(1+G^2)} \right]^{-1} \mathrm{Tr}\left[\sqrt{(1+\tfrac{i}{2}G)(1+iG)} \cdot \mathcal{A} \right] \ , \tag{4.43}$$

and if Eqs. (4.42) and (4.43) were to hold one would be led to the unwarranted constraint

$$G^{\mu\nu}\mathcal{A}_{\mu\nu} + \mathcal{O}(G^2) \ = \ 0 \ . \tag{4.44}$$

The system (4.36)–(4.41), that involves fields from *both* Regge trajectories, is however algebraically consistent, and its Lagrangian [25] can also be obtained via the BRST method, integrating out auxiliary fields and performing a complete gauge fixing.

Next, one would like to know whether String Theory can consistently describe, in a constant EM background, a single spin-3 field of the leading Regge trajectory. With this in mind, let us set to zero the physical field $\mathcal{A}_{\mu\nu}$ of the subleading trajectory. Interestingly, this choice does not conflict in any way with the algebraic consistency of the system (4.36)–(4.41), which now reduces to the Fierz–Pauli system

$$\left(\mathcal{D}^2 - 4 - \tfrac{1}{2}\mathrm{Tr}G^2\right)\Phi_{\mu\nu\rho} \ + \ 2\,i\,G^{\alpha}{}_{(\mu}\Phi_{\nu\rho)\alpha} \ = \ 0 \ , \tag{4.45}$$

$$\mathcal{D}^{\mu}\Phi_{\mu\nu\rho} \ = \ 0 \ , \tag{4.46}$$

$$\Phi^{\mu}{}_{\mu\nu} \ = \ 0 \ . \tag{4.47}$$

This is indeed a consistent set of equations, which describes a massive spin-3 field with

$$(\mathrm{mass})^2 \ = \ \frac{1}{\alpha'}\left(2 \ + \ \tfrac{1}{4}\,\mathrm{Tr}G^2\right) \ , \tag{4.48}$$

and the proper number of propagating DoFs. String Theory also guarantees that this system comes from a Lagrangian, the Klishevich Lagrangian of [25], with $\mathcal{A}_{\mu\nu}$ set to zero and after a complete gauge fixing. We have not shown whether that Lagrangian yields the Fierz–Pauli system away from the critical dimension, but we do not expect it, since for spin 2 the answer is negative, as we shall see in Section 6.1.

4.4 Arbitrary mass level: $N = s$

Let us now focus on the first Regge trajectory, that at this level contains a symmetric rank-s tensor, and let us investigate the consistency of the string field equations that result from the physical state conditions (4.1)–(4.3). To begin with, we thus write the string state

$$|\Phi\rangle \;=\; \phi_{\mu_1\mu_2\ldots\mu_s}(x)\, a_1^{\dagger\mu_1} a_1^{\dagger\mu_2} \ldots\, a_1^{\dagger\mu_s}\,|0\rangle \;, \tag{4.49}$$

so that, on account of Eqs. (4.4)–(4.6), the physical state conditions (4.1)–(4.3) give

$$\left[\mathcal{D}^2 - 2(s-1) - \tfrac{1}{2}\,\mathrm{Tr}G^2\right] \phi_{\mu_1\ldots\mu_s} \;+\; 2\,i\,G^{\alpha}{}_{(\mu_1}\phi_{\mu_2\ldots\mu_s)\alpha} \;=\; 0 \;, \tag{4.50}$$

$$s\,\mathcal{D}^{\mu}\left[(\sqrt{1+iG}\,)_{\mu}{}^{\alpha}\phi_{\alpha\mu_2\ldots\mu_s}\right] \;=\; 0 \;, \tag{4.51}$$

$$\tfrac{1}{2}\,s(s-1)\left[(\sqrt{1+iG}\,)^{\alpha\mu_1}(\sqrt{1+iG}\,)_{\alpha}{}^{\mu_2}\phi_{\mu_1\mu_2\ldots\mu_s}\right] \;=\; 0 \;. \tag{4.52}$$

It is now convenient to define the symmetric field

$$\mathit{\Phi}_{\mu_1\mu_2\ldots\mu_s} \;\equiv\; (\sqrt{1+iG}\,)_{\mu_1}{}^{\alpha_1}(\sqrt{1+iG}\,)_{\mu_2}{}^{\alpha_2}\ldots(\sqrt{1+iG}\,)_{\mu_s}{}^{\alpha_s}\phi_{\alpha_1\alpha_2\ldots\alpha_s} \;, \tag{4.53}$$

since Eqs. (4.50)–(4.52) then give

$$\left[\mathcal{D}^2 - 2(s-1) - \tfrac{1}{2}\,\mathrm{Tr}G^2\right] \mathit{\Phi}_{\mu_1\ldots\mu_s} + 2\,i\,G^{\alpha}{}_{(\mu_1}\mathit{\Phi}_{\mu_2\ldots\mu_s)\alpha} \;=\; 0 \;, \tag{4.54}$$

$$\mathcal{D}^{\mu}\mathit{\Phi}_{\mu\mu_2\ldots\mu_s} \;=\; 0 \;, \tag{4.55}$$

$$\mathit{\Phi}^{\mu}{}_{\mu\mu_3\ldots\mu_s} \;=\; 0 \;. \tag{4.56}$$

One can easily show that these equations are algebraically consistent. They form a Fierz–Pauli system for a massive spin-s field, with a deformed mass, so that now

$$(\mathrm{mass})^2 \;=\; \frac{1}{\alpha'}\,\left(s - 1 + \tfrac{1}{4}\,\mathrm{Tr}G^2\right) \;, \tag{4.57}$$

and it is manifest that the system gives the correct count of DoFs. Eqs. (4.54)–(4.56) follow, at least in $d = 26$, from a Lagrangian determined by the BRST method. We have not derived it, since these relatively simple EoMs suffice for the analysis of the Velo–Zwanziger problem, to which we now turn.

The promised proof of causal propagation for generic spin s can be obtained adapting to our case the arguments of Argyres and Nappi [22], and thus resorting to the method of characteristic determinants reviewed briefly in Appendix A.2. In fact, we have seen that the highest–derivative terms appearing in the EoMs boil down to the scalar operator $\mathcal{D}^2$ acting on the fields. From the definition (2.34) of $\mathcal{D}_\mu$, it is then clear that the vanishing of the characteristic determinant is tantamount to the condition

$$(G/eF)^{\mu}{}_{\nu}\ n_\mu n^\nu \ = \ 0 \ , \tag{4.58}$$

where G is defined in Eq. (2.9).

One can perform a Lorentz transformation to reduce F to the block skew–diagonal form[4], $F^{\mu}{}_{\nu} \ = \ \mathrm{diag}\,(\,F_1\ ,F_2\ ,F_3\ ,\ldots\ \ldots)$, with the blocks given by

$$F_1 \ = \ a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad F_{i\neq 1} \ = \ b_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{4.59}$$

where a and b_i's are real–valued functions of the EM field invariants, such that in physically interesting cases their values are always small. Notice that because of the Lorentzian signature the first block F_1 is different from the $F_{i\neq 1}$'s. The same Lorentz transformation will clearly render G block skew–diagonal as well, $G^{\mu}{}_{\nu} \ = \ \mathrm{diag}\,(\,G_1\ ,G_2\ ,G_3\ ,\ldots\ \ldots)$, with

$$G_1 \ = \ f(a) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad G_{i\neq 1} \ = \ g(b_i) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{4.60}$$

where

$$f(a) \ \equiv \ \frac{1}{\pi}\left[\,\tanh^{-1}(\pi e_0 a) + \tanh^{-1}(\pi e_\pi a)\,\right] \ , \tag{4.61}$$

$$g(b_i) \ \equiv \ \frac{1}{\pi}\left[\,\tan^{-1}(\pi e_0 b_i) + \tan^{-1}(\pi e_\pi b_i)\,\right] \ . \tag{4.62}$$

Let us stress that if the EM field invariants are small, these functions are always well–defined and their absolute values are much smaller than unity. Given the forms (4.59) and (4.60), one can finally see that (G/eF), where $e = e_0 + e_\pi$, is the diagonal matrix

$$\left(\frac{G}{eF}\right)^{\mu}{}_{\nu} \ = \ \mathrm{diag}\left[\,\frac{f(a)}{ea}\ ,\frac{f(a)}{ea}\ ,\frac{g(b_2)}{eb_2}\ ,\frac{g(b_2)}{eb_2}\ ,\frac{g(b_3)}{eb_3}\ ,\frac{g(b_3)}{eb_3}\ ,\ldots\ \ldots\right] \ . \tag{4.63}$$

One can now notice that the functions (4.61) and (4.62) satisfy the inequalities

$$\frac{f(a)}{ea} \ \geq \ 1 \ , \qquad 0 \ < \ \frac{g(b_i)}{eb_i} \ \leq \ 1 \ , \tag{4.64}$$

[4]Leaving aside an exceptional set of nilpotent fields obeying $F^m = 0$ for some m.

so that, in view of (4.63), any solution n_μ of (4.58) must be space–like:

$$n^2 \ \geq \ 0 \ . \tag{4.65}$$

This is a direct transposition of the $s = 2$ argument of [22], and is of course a Lorentz invariant statement. We can thus conclude that the propagation of the first Regge trajectory is indeed causal, thanks to the special form of Eqs. (4.54)–(4.56), and in particular thanks to the structure of the non-minimal kinetic terms.

5 No–ghost theorem

The Argyres–Nappi [22] and Klishevich [25] Lagrangians, or for that matter the generalized Fierz–Pauli conditions of Section 4.4, contain non-standard kinetic contributions, so that it becomes interesting to investigate whether the flat–space no–ghost theorem extends to this case. We can now show that the no–ghost theorem (see *e.g.* [10]) continues to hold in the regime of physical interest that we identified in the Introduction.

No modifications of the standard arguments are needed in purely magnetic backgrounds, since in this case the two light–cone coordinates are still subject to standard Neumann boundary conditions. As a result, the Hilbert space spanned by the $\alpha^\pm_m$ operators maps exactly into that of the free string.

In generic backgrounds where electric fields are also present, matters are more subtle, since the light–cone directions are affected. However, a no–ghost theorem can still be proved via arguments that follow rather closely those used for the free string. To this end, it suffices to retrace the proof presented by Polchinski in [10], pp. 139–141. To begin with, one can block–diagonalize the external field strength, a step that can be carried out for generic constant $F_{\mu\nu}$ backgrounds. When an electric field is present, one can then modify Eq. (4.4.7) of [10], turning it into

$$[\alpha^\pm_m, \alpha^\mp_n] \ = \ - \, [m \pm i f(a)] \, \delta_{m,-n} \ , \qquad [\alpha^+_m, \alpha^+_n] \ = \ [\alpha^-_m, \alpha^-_n] \ = \ 0 \ , \tag{5.1}$$

where $f(a)$ is a skew eigenvalue of G, defined in Eq. (4.61). When $f(a) = 0$, the argument of [10] clearly holds directly, while when $f(a)$ does not vanish one can replace Eq. (4.4.8) of [10] with

$$N^{\rm lc} \;=\; \sum_{m\in\mathbb{Z},\, m\neq 0} \frac{1}{m - i f(a)}\; \alpha^+_{-m}\, \alpha^-_m \;. \tag{5.2}$$

The next step is to decompose Q_B as in Eq. (4.4.9) of [10], letting α_0^+ play the role of k^+. On the other hand, Q_1 and R can be defined exactly as in Eqs. (4.4.13) and (4.4.14) of [10], since α_0^- never appears in their definitions. This also means that a convenient (overcomplete) basis for the string states is provided by the Fock basis for $\alpha^{\pm}_{m\neq 0}$, together with the coherent states that are eigenstates of α_0^+. One thus finds that the first line of Eq. (4.4.15) of [10] should be replaced by

$$S \;\equiv\; \{Q_1, R\} \;=\; \sum_{m=1}^{\infty} \Big\{\, [m + i f(a)]\, b_{-m}\, c_m + [m - i f(a)]\, c_{-m}\, b_m - \alpha^+_{-m}\, \alpha^-_m - \alpha^-_{-m}\, \alpha^+_m \,\Big\} \;, \tag{5.3}$$

but the rest of the proof carries over verbatim.

6 Spin-2 Lagrangians

In the previous sections we have investigated the consistency of our systems at the level of EoMs. While Lagrangians can be built along the lines of String Theory, they are certainly more complicated than the Fierz–Pauli–like conditions that we have displayed in Section 4.4. For one matter, as we have seen, they are bound to mix the leading Regge trajectory with others. In the next subsection we follow the opposite path, and provide a corollary to [22], showing how their Lagrangian gives rise to a consistent spin-2 Fierz–Pauli system in the critical dimension $d = 26$. In Section 6.2 we then linearize the Lagrangian in the EM field strength and suggest a possible field theory program for building consistent Lagrangians in arbitrary dimensions. We also discuss the gyromagnetic ratio and conclude with a comparative study of the linearized Argyres–Nappi [22] and Federbush [21] Lagrangians.

6.1 Argyres–Nappi Lagrangian and Fierz–Pauli Conditions

The Argyres–Nappi Lagrangian [22] is

$$\begin{aligned} L_{\rm AN} \;=\;& \mathcal{H}^*_{\mu\nu}\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}{\rm Tr}G^2\right)\mathfrak{h}^{\mu\nu} - 2i\mathcal{H}^*_{\mu\nu}(G\mathfrak{h} - \mathfrak{h}G)^{\mu\nu} - \mathcal{H}^*\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}{\rm Tr}G^2\right)\mathcal{H} \\ & - \mathcal{H}^*_{\mu\nu}\left\{\mathcal{D}^\mu\mathcal{D}^\rho[(1+iG)\mathfrak{h}]_\rho{}^\nu - \tfrac{1}{2}\mathcal{D}^\mu\mathcal{D}^\nu\mathcal{H} + (\mu\leftrightarrow\nu)\right\} + \mathcal{H}^*\mathcal{D}^\mu\mathcal{D}^\nu\mathcal{H}_{\mu\nu} \;, \end{aligned} \tag{6.1}$$

where $\mathcal{D}^\mu$ was defined in Eq. (2.34),

$$\mathcal{H}_{\mu\nu} \equiv (1+iG)_\mu{}^\alpha (1+iG)_\nu{}^\beta \, \mathfrak{h}_{\alpha\beta} \ , \tag{6.2}$$

and for brevity we write $\mathcal{H}$ rather than $\mathcal{H}^\mu{}_\mu$. One can simply verify that this Lagrangian is Hermitian and that its variation gives rise to the equations of motion

$$\begin{aligned} \mathcal{R}_{\mu\nu} \ &\equiv \ \left(\mathcal{D}^2 - 2 - \tfrac{1}{2}\mathrm{Tr}G^2\right)\mathcal{H}_{\mu\nu} - 2i(G\mathcal{H} - \mathcal{H}G)_{\mu\nu} - (1+G^2)_{\mu\nu}\left(\mathcal{D}^2 - 2 - \tfrac{1}{2}\mathrm{Tr}G^2\right)\mathcal{H} \\ &\quad +\tfrac{1}{2}\left\{[(1+iG)\cdot\mathcal{D}]_\mu\,[(1+iG)\cdot\mathcal{D}]_\nu + [(1+iG)\cdot\mathcal{D}]_\nu\,[(1+iG)\cdot\mathcal{D}]_\mu\right\}\mathcal{H} \\ &\quad - \left\{[(1+iG)\cdot\mathcal{D}]_\mu\,\mathcal{D}^\rho\mathcal{H}_{\rho\nu} + [(1+iG)\cdot\mathcal{D}]_\nu\,\mathcal{D}^\rho\mathcal{H}_{\rho\mu}\right\} + (1+G^2)_{\mu\nu}\mathcal{D}^\alpha\mathcal{D}^\beta\mathcal{H}_{\alpha\beta} \\ &= \ 0 \ . \end{aligned} \tag{6.3}$$

One would like to know whether these equations can be turned into a Fierz–Pauli system in an *arbitrary* number of space–time dimensions. To this end, let us first take the trace of Eq. (6.3), which gives

$$\begin{aligned} \mathcal{R}^\mu{}_\mu \ &\equiv \ \left\{(d-2+\mathrm{Tr}G^2) - 2iG\right\}^{\alpha\rho}\mathcal{D}_\rho\mathcal{D}^\beta\mathcal{H}_{\alpha\beta} - \left\{(d-2+\mathrm{Tr}G^2) - G^2\right\}^{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta\mathcal{H} \\ &\quad +(d-1+\mathrm{Tr}G^2)\left(2+\tfrac{1}{2}\mathrm{Tr}G^2\right)\mathcal{H} \ = \ 0 \ . \end{aligned} \tag{6.4}$$

On the other hand, the divergence of Eq. (6.3) gives

$$\begin{aligned} \mathcal{D}^\mu\mathcal{R}_{\mu\nu} \ &\equiv \ -\left\{(1+iG)(2+iG)\right\}_\nu{}^\alpha\,\mathcal{D}^\beta\mathcal{H}_{\alpha\beta} - \left\{iG(1+iG)\right\}_\nu{}^\alpha\,\mathcal{D}_\alpha(\mathcal{D}^\rho\mathcal{D}^\sigma\mathcal{H}_{\rho\sigma}) \\ &\quad + \left\{(1+iG)\left[\left(2-\tfrac{1}{2}iG+\tfrac{3}{2}G^2\right) + iG\left(\mathcal{D}^2 - \tfrac{1}{2}\mathrm{Tr}G^2\right)\right]\right\}_\nu{}^\alpha\,\mathcal{D}_\alpha\mathcal{H} \\ &= \ 0 \ . \end{aligned} \tag{6.5}$$

One can then apply to Eq. (6.5) the operator $[2\mathcal{D}\cdot(1+iG)^{-1}]^\nu$ from the left to obtain

$$\begin{aligned} \left[2\mathcal{D}\cdot(1+iG)^{-1}\right]^\nu\mathcal{D}^\mu\mathcal{R}_{\mu\nu} \ &\equiv \ \left\{(4-\mathrm{Tr}G^2) - G^2\right\}^{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta\mathcal{H} + \tfrac{1}{2}\mathrm{Tr}G^2\left(1+\mathrm{Tr}G^2\right)\mathcal{H} \\ &\quad - \ \left\{(4-\mathrm{Tr}G^2) - 2iG\right\}^{\alpha\rho}\mathcal{D}_\rho\mathcal{D}^\beta\mathcal{H}_{\alpha\beta} \ = \ 0 \ . \end{aligned} \tag{6.6}$$

Matters simplify considerably if one adds Eqs. (6.4) and (6.6), obtaining

$$(d-6+2\mathrm{Tr}G^2)(D^\alpha\mathcal{D}^\beta\mathcal{H}_{\alpha\beta} - \mathcal{D}^2\mathcal{H}) + \left[2(d-1) + \tfrac{1}{2}\mathrm{Tr}G^2(d+4+2\mathrm{Tr}G^2)\right]\mathcal{H} = 0 \ . \tag{6.7}$$

Finally, applying to Eq. (6.5) the operator $[\mathcal{D}\cdot\{(1+iG)(2+iG)\}^{-1}]^{\nu}$ yields

$$\begin{aligned}\left[\mathcal{D}\cdot\{(1+iG)(2+iG)\}^{-1}\right]^{\nu}\mathcal{D}^{\mu}\mathcal{R}_{\mu\nu} \;&\equiv\; -\left[1+\left(\frac{iG}{2+iG}\right)^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}\right](D^{\alpha}\mathcal{D}^{\beta}\mathcal{H}_{\alpha\beta}-\mathcal{D}^2\mathcal{H})\\ &\quad -\left[\tfrac{1}{4}\mathrm{Tr}G^2+\tfrac{1}{2}(5+\mathrm{Tr}G^2)\left(\frac{iG}{2+iG}\right)^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}\right]\mathcal{H}\\ &=\; 0\;. \end{aligned} \tag{6.8}$$

One can now apply the operator $\left[1+\{iG(2+iG)^{-1}\}^{\mu\nu}\,\mathcal{D}_{\mu}\mathcal{D}_{\nu}\right]$ to Eq. (6.7), multiply Eq. (6.8) by $(d-6+2\mathrm{Tr}G^2)$ and add together the results, obtaining

$$\left[2(d-1)+\tfrac{1}{4}\,\mathrm{Tr}G^2(d+14+2\,\mathrm{Tr}G^2)-\tfrac{1}{2}(d-26)\left(\frac{iG}{2+iG}\right)^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}\right]\mathcal{H}\;=\;0\;. \tag{6.9}$$

Notice that in an EM background this reduces to an algebraic expression *only* in the critical space–time dimension $d=26$, whence one obtains the trace constraint

$$\mathcal{H}\;=\;0\quad, \tag{6.10}$$

so that the situation is quite different from the free case considered in Appendix A.1. When (6.10) holds, (6.7) sets to zero the double divergence of $\mathcal{H}$, which in its turn yields the divergence constraint from Eq. (6.5). Given the trace and divergence constraints, one can now obtain

$$\left(\mathcal{D}^2-2-\tfrac{1}{2}\mathrm{Tr}G^2\right)\mathcal{H}_{\mu\nu}-2i(G\cdot\mathcal{H}-\mathcal{H}\cdot G)_{\mu\nu}\;=\;0\;. \tag{6.11}$$

The end result is indeed the deformed Fierz–Pauli system (4.28). As we have seen, this follows from the Argyres–Nappi Lagrangian (6.1) *only* in the critical dimension $d=26$.

6.2 Space–time dimensionality and gyromagnetic ratio

It is important to notice that the constraint (6.9), which would follow from the Argyres–Nappi Lagrangian in an arbitrary number of dimensions, can be recast in the form

$$\begin{aligned}&\left[2(d-1)+\tfrac{1}{4}\,\mathrm{Tr}G^2(d+14+2\,\mathrm{Tr}G^2)+\tfrac{1}{2}(d-26)\,\mathrm{Tr}\left(\frac{G^2}{4+G^2}\right)\right]\mathcal{H}\\ &\qquad -\tfrac{1}{2}(d-26)\left(\frac{G^2}{4+G^2}\right)^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}\mathcal{H}\;=\;0\;,\end{aligned} \tag{6.12}$$

so that for $d \neq 26$ it actually fails to be purely algebraic only at $\mathcal{O}(G^2)$. As a result, if we restrict ourselves to terms that are at most linear in the EM field strength $F_{\mu\nu}$, the Argyres–Nappi Lagrangian still gives rise to correct constraints and causal propagation away from the critical dimension. More importantly, the number of space–time dimensions does not play any role in this case, up to $\mathcal{O}(F)$. One could thus argue that appropriate $\mathcal{O}(F^2)$ terms can always be added, pushing the desired features to $\mathcal{O}(F^2)$, and so on. While this is definitely possible when $d = 26$, there is no apparent reason why such corrections cannot be added for other dimensions as well. On top of this, one would need of course correction terms that contain derivatives of $F_{\mu\nu}$ if the latter were not constant [5].

Therefore, it becomes interesting to write explicitly the Argyres–Nappi Lagrangian up to terms linear in $F_{\mu\nu}$, restoring the dependence on α' and regarding $1/\alpha'$ as a generic value m^2. The result is

$$\begin{aligned} L_{\rm AN} &= -|D_\mu \mathfrak{h}_{\nu\rho}|^2 + 2|D_\mu \mathfrak{h}^{\mu\nu}|^2 + |D_\mu \mathfrak{h}|^2 + (\mathfrak{h}^*_{\mu\nu} D^\mu D^\nu \mathfrak{h} + \text{c.c.}) - m^2(\mathfrak{h}^*_{\mu\nu}\mathfrak{h}^{\mu\nu} - \mathfrak{h}^*\mathfrak{h}) \\ &\quad + 8ie\,\text{Tr}(\mathfrak{h} \cdot F \cdot \mathfrak{h}^*) + \delta L_{\rm kin} + \mathcal{O}(F^2) \ , \end{aligned} \tag{6.13}$$

where $\delta L_{\rm kin}$ is a kinetic deformation of $\mathcal{O}(F)$, given by

$$\begin{aligned} \delta L_{\rm kin} &= -\,i(e/m^2)(F\mathfrak{h}^* - \mathfrak{h}^* F)_{\mu\nu}\left[D^2\mathfrak{h}^{\mu\nu} - (D^\mu D^\rho \mathfrak{h}_\rho{}^\nu + D^\nu D^\rho \mathfrak{h}_\rho{}^\mu) + \tfrac{1}{2}D^{(\mu}D^{\nu)}\mathfrak{h}\right] \\ &\quad -\,i(e/m^2)D^\mu(F\mathfrak{h}^* - \mathfrak{h}^* F)_{\mu\nu}(D_\rho \mathfrak{h}^{\rho\nu} - D^\nu \mathfrak{h}) + \text{h.c.} \ . \end{aligned} \tag{6.14}$$

As was already mentioned, the Lagrangian (6.13) describes consistently a massive spin-2 system coupled to a constant EM background, up to $\mathcal{O}(F)$.

On the other hand, we note that the spin-2 Federbush Lagrangian of [21],

$$\begin{aligned} L_{\rm F} &= -|D_\mu \varphi_{\nu\rho}|^2 + 2|D_\mu \varphi^{\mu\nu}|^2 + |D_\mu \varphi|^2 + (\varphi^*_{\mu\nu} D^\mu D^\nu \varphi + \text{c.c.}) - m^2(\varphi^*_{\mu\nu}\varphi^{\mu\nu} - \varphi^*\varphi) \\ &\quad + ie\,\text{Tr}(\varphi \cdot F \cdot \varphi^*) \ , \end{aligned} \tag{6.15}$$

is, up to dimension-4 operators, the only Lagrangian that propagates the correct number of DoFs of a massive spin-2 field in a non-vanishing external EM field $F_{\mu\nu}$. However, it does not have the same "dipole" coefficient as the linearized Argyres–Nappi Lagrangian, nor does it contain, to begin with, dimension-6 kinetic deformations. And indeed, as shown in Appendix A.2, while Eq. (6.15) gives the correct DoF count, it does not take care of

[5] This type of construction would have a number of potential applications, which include the improvement of the holographic model for d-wave superconductors of [33].

hyperbolicity and/or causality. Therefore, it is interesting to investigate the connection between these two Lagrangians.

At first sight, the connection is simple: up to field redefinitions, one can conclude that the kinetic deformation present in the Argyres–Nappi Lagrangian is

$$\delta L_{\text{kin}} \;=\; -\,4\,i\,e\,\text{Tr}(\mathfrak{h}\cdot F\cdot \mathfrak{h}^*) \,+\, \mathcal{O}(F^2)\ , \tag{6.16}$$

so that to linear order in F the Argyres–Nappi and Federbush Lagrangians differ only in the coefficient of the dipole term. This coefficient is $4ie$ in the first case, while it is ie in the second. The first gives a gyromagnetic ratio $g = 2$, while the second gives $g = \frac{1}{2}$. Intriguingly enough, $g = 2$ is a special value that guarantees the absence of high–energy strong coupling in an important forward "Compton" scattering amplitude [34, 35], and amusingly all open–string charged states have $g = 2$ [34]. Upon reflection, this result seems paradoxical, because the Federbush dipole term is the only one that guarantees the correct number of propagating DoFs, and the number of DoFs cannot be changed by a local field redefinition! Actually there is no paradox here, simply because the extra propagating DoF shows up only at $\mathcal{O}(F^2)$ [11]. Thus, while the linearized Lagrangian is enough to determine the gyromagnetic ratio implied by the Argyres–Nappi model, it is insufficient to manifest the subtler problems associated with propagation in external fields. And indeed, expanding the Argyres–Nappi Lagrangian to $\mathcal{O}(F^2)$ one would find that the kinetic term of the extra DoF is pushed to higher orders, so that a complete cancelation of the offending mode is guaranteed only by the full, non-polynomial action.

The existence of a kinetic deformation in the Argyres–Nappi Lagrangian was overlooked in [13], where it was claimed that no spin-2 Lagrangian propagating the correct number of degrees of freedom could solve the Velo–Zwanziger problem. This conclusion follows if one assumes from the beginning a canonical spin-2 kinetic term, and is of course in contradiction with the explicit solution found in [22]. Since the problem with the number of DoFs first manifests itself at $O(F^2)$, it can be solved precisely via terms like those in Eq. (6.14), which are tantamount to a non-derivative Pauli coupling to linear order in F, but which alter the constraint equations to quadratic order.

7 Concluding Remarks

The main issue addressed in this paper is whether String Theory can cure the Velo–Zwanziger problem for a single massive charged spin-s particle in an external EM background. The answer is in the affirmative, at least for the first Regge trajectory of the open bosonic string, whose symmetric tensors can be exposed in isolation to constant EM backgrounds. In fact, we showed that all fields of this type can be described without including other dynamical fields, in that their generalized Fierz–Pauli conditions

$$\begin{aligned} \left[\mathcal{D}^2 \,-\, \tfrac{1}{\alpha'}\,\left(s-1+\tfrac{1}{4}\,\mathrm{Tr}G^2\right)\right] \varPhi_{\mu_1\ldots\mu_s} \,+\, \tfrac{1}{\alpha'}\, i\, G^{\alpha}{}_{(\mu_1}\varPhi_{\mu_2\ldots\mu_s)\alpha} \;&=\; 0 \;, \\ \mathcal{D}^{\mu}\,\varPhi_{\mu\mu_2\ldots\mu_s} \;&=\; 0 \;, \\ \varPhi^{\mu}{}_{\mu\mu_3\ldots\mu_s} \;&=\; 0 \;, \end{aligned} \tag{7.1}$$

are consistent (in an arbitrary number of space–time dimensions) even in the presence of a constant EM field strength. Moreover, thanks to the special form of their non-minimal kinetic contributions, these equations result in a causal propagation, thus providing a solution to the Velo–Zwanziger problem for this class of fields. On the other hand, we have seen in explicit examples that, in general, fields belonging to subleading trajectories cannot have consistent interactions with an external EM background without additional fields belonging to other trajectories. Our findings thus resonate with the fact that the Vasiliev systems [36], non-linear equations for symmetric tensors of arbitrary rank, can be formulated in an arbitrary number of dimensions. Conversely, it is natural to regard these systems as an effective description of the first Regge trajectory of the open bosonic string in a special regime where the remaining excitations decouple.

One may wonder whether the system (7.1) acquires a gauge symmetry when the mass, $\left(s-1+\frac{1}{4}\mathrm{Tr}G^2\right)/\alpha'$, is set to zero. With a finite α', within the regime of physical interest, this happens only for $s=1$ when $\mathrm{Tr}G^2=0$. On the other hand, for $s>1$ this would entail the $\alpha'\to\infty$ limit, but then a physically meaningful description would require that $eF\to 0$, so that $\alpha' eF$ approaches a finite limit. As a result, the higher–spin fields become free in the limit, consistently with the no–go theorems of [5, 6], which state that massless fields with $s>1$ cannot carry an electric charge.

How unique is the resolution of the original Fierz–Pauli problem that String Theory provides? After all, as was first noted in [22], the causality proof (that we retraced in Section 4.4 in order to extend it to spin-s fields) and other consistency issues are *not* affected if one makes the replacement $G\to 2\alpha' eF$. The complicated function G of the field strength reflects key properties of the string, which can be torn apart by "strong" electric fields ($2\pi\alpha' e|\vec{E}| \sim 1$), and has possibly important lessons in store on the interactions with non-constant backgrounds [22, 25]. Even slowly varying field strengths, however, are very difficult to study quantitatively since the string sigma model becomes non-linear in the first place.

String Theory provides a remedy for the Velo–Zwanziger problem but calls for kinetic deformations of the minimal Lagrangian. It does it in a judicious way, of course: kinetic deformations generically introduce extra DoFs or ghosts, but the ones present in String Theory do not. While proving this statement is relatively straightforward – it is essentially the free–string no–ghost theorem – we are not aware of any proof to this effect directly in the Lagrangian theory of massive higher–spin fields. At any rate, non-minimal terms are expected to lower the cutoff of the effective field theory from that implied by minimal ones, and non-constant external backgrounds would lower it even further. No simple improvement of the theory thus appears to bypass the upper bound for the cutoff proposed in [37].

How about the critical dimension, $d = 26$? For *free* massive higher spins, it is apparently possible to evade it rather naturally for low–lying excitations, proceeding from a field theory vantage point, at the price of making some terms in the Lagrangian or in the (Stückelberg) gauge transformations more complicated [32]. For charged fields in a constant EM background, if one looks *only* at the EoMs, the dimensionality of space–time does not play any role. What String Theory guarantees is rather that the EoMs come from a Lagrangian in $d = 26$, where the critical dimension is required in order that the BRST charge be nilpotent. Since nilpotency of the BRST charge is essential in proving the no–ghost theorem, we do not know if Eqs. (7.1) define a physical, ghost–free system in dimension other than 26.

In some sense, the very appearance of the tensor G may be regarded as evidence that the underlying theory is inherently non-local, since for instance *two* distinct charges, e_0 and e_π, enter Eq. (2.1), rather than the single charge that a point particle may possess. It is natural to expect that the fully interacting theory will not be local, and some indications to this effect can be extracted from the limiting behavior of string amplitudes, as in [38].

Finding similar models for massive charged higher–spin fermions starting directly from charged open superstrings (or from type-0 strings [28, 39], which contain a plethora of symmetric spinor–tensors) in an EM background, although possible in principle, seems rather complicated, and apparently no attempts have been made in this direction. It might be very instructive to look more closely at the indications provided by String Theory for the first few cases, and in particular for the massive spin-3/2 excitation of the superstring, to see whether they agree with the proposal of [40].

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A Higher–spin systems and EM backgrounds

In this Appendix we review some basic facts about massive higher–spin fields and their couplings with an EM background, with special emphasis on some difficulties that are encountered. We refer mostly to the case of a massive spin-2 field, and mention briefly about symmetric tensors of arbitrary rank. The reader can find more details and recent results on free higher–spin fields of mixed symmetry in [3, 20].

A.1 The Fierz–Pauli $s = 2$ system

Let us begin by reviewing the key properties of the free massive $s = 2$ case, which is described in any number of space–time dimensions by the Fierz–Pauli Lagrangian [14]

$$L_{\rm FP} = -\tfrac{1}{2}\,(\partial_\mu \varphi_{\nu\rho})^2 + (\partial_\mu \varphi^{\mu\nu})^2 + \tfrac{1}{2}\,(\partial_\mu \varphi)^2 - \partial_\mu \varphi^{\mu\nu} \partial_\nu \varphi - \tfrac{1}{2}\, m^2 [\varphi_{\mu\nu}^2 - \varphi^2] \,, \tag{A.1}$$

where for brevity we use the symbol φ rather than $\varphi^\mu{}_\mu$. The corresponding EoMs read

$$R_{\mu\nu} \;\equiv\; (\Box - m^2)\varphi_{\mu\nu} - \eta_{\mu\nu}(\Box - m^2)\varphi + \partial_\mu\partial_\nu\varphi - (\partial_\mu\partial^\rho\varphi_{\rho\nu} + \partial_\nu\partial^\rho\varphi_{\rho\mu}) + \eta_{\mu\nu}\partial^\alpha\partial^\beta\varphi_{\alpha\beta} \;=\; 0\,. \tag{A.2}$$

Taking divergences and the trace of Eq. (A.2) leads to

$$\partial^\mu R_{\mu\nu} \;\equiv\; -m^2(\partial^\mu\varphi_{\mu\nu} - \partial_\nu\varphi) \;=\; 0\,, \tag{A.3}$$

$$\partial^\mu\partial^\nu R_{\mu\nu} \;\equiv\; -m^2(\partial^\mu\partial^\nu\varphi_{\mu\nu} - \Box\varphi) \;=\; 0\,, \tag{A.4}$$

$$R^\mu{}_\mu \;\equiv\; (d-2)(\partial^\mu\partial^\nu\varphi_{\mu\nu} - \Box\varphi) + [(d-1)m^2]\,\varphi \;=\; 0\,, \tag{A.5}$$

and combining Eqs. (A.4) and (A.5) one arrives at an interesting consequence,

$$[(d-1)m^2]\,\varphi \;=\; 0\;, \tag{A.6}$$

so that for $m^2 \neq 0$ and $d > 1$ one is led to the dynamical trace constraint

$$\varphi \;=\; 0\;. \tag{A.7}$$

The transversality condition follows from Eq. (A.3), so that finally Eq. (A.2) reduces to the Klein–Gordon equation

$$(\Box - m^2)\varphi_{\mu\nu} \;=\; 0\;, \tag{A.8}$$

which is of course manifestly hyperbolic and causal. Along with the transversality condition, Eqs. (A.7)–(A.8) are the simplest instance of a Fierz–Pauli system, that draws its origin from the Lagrangian (A.1) in an arbitrary number of space–time dimensions, as we have seen.

Trace and divergence conditions are crucial to arrive at the correct number of propagating DoFs. In d dimensions, a symmetric rank-2 tensor has $\frac{1}{2}d(d+1)$ independent components; the divergence condition eliminates d of them and finally the trace condition removes one more. All in all, one is thus left with $\frac{1}{2}(d+1)(d-2)$ components, which is the correct number of propagating DoFs for a massive spin-2 field.

In general, for a symmetric tensor of arbitrary rank s, the Fierz–Pauli system takes the form

$$\eta^{\mu_1\mu_2}\,\phi_{\mu_1\mu_2\ldots\mu_s} \;=\; 0\;, \tag{A.9}$$
$$(\Box - m^2)\,\phi_{\mu_1\ldots\mu_s} \;=\; 0\;, \tag{A.10}$$
$$\partial^{\mu_1}\,\phi_{\mu_1\ldots\mu_s} \;=\; 0\;. \tag{A.11}$$

Its counterpart a for Fermi field, with spin $s = n+\frac{1}{2}$, contains a γ-trace condition, the Dirac equation and a divergence condition:

$$\gamma^{\mu_1}\,\psi_{\mu_1\mu_2\ldots\mu_n} \;=\; 0\;, \tag{A.12}$$
$$(\not{\partial} - m)\,\psi_{\mu_1\ldots\mu_n} \;=\; 0\;, \tag{A.13}$$
$$\partial^{\mu_1}\,\psi_{\mu_1\ldots\mu_n} \;=\; 0\;. \tag{A.14}$$

A.2 Massive $s = 2$ field and the Velo–Zwanziger problem

One can complexify the spin-2 field in the Lagrangian (A.1) and try to minimally couple it to a constant EM background, following [21]. Because covariant derivatives do not commute, the minimal coupling is ambiguous, so that one is actually led to a family of Lagrangians containing one parameter, which one can call the gyromagnetic ratio g (see *e.g.* [13]):

$$\begin{aligned} L \;=\; & -|D_\mu \varphi_{\nu\rho}|^2 + 2|D_\mu \varphi^{\mu\nu}|^2 + |D_\mu \varphi|^2 + (\varphi^*_{\mu\nu} D^\mu D^\nu \varphi + \text{c.c.}) - m^2(\varphi^*_{\mu\nu}\varphi^{\mu\nu} - \varphi^*\varphi) \\ & + 2ieg\,\text{Tr}(\varphi \cdot F \cdot \varphi^*)\ . \end{aligned} \tag{A.15}$$

The resulting EoMs are

$$\begin{aligned} 0 \;=\; \mathcal{R}_{\mu\nu} \;\equiv\; & (D^2 - m^2)\varphi_{\mu\nu} - \eta_{\mu\nu}(D^2 - m^2)\varphi + \tfrac{1}{2} D_{(\mu} D_{\nu)}\varphi - [D_\mu D^\rho \varphi_{\rho\nu} + D_\nu D^\rho \varphi_{\rho\mu}] \\ & + \eta_{\mu\nu} D^\alpha D^\beta \varphi_{\alpha\beta} + iegF_{\rho\mu}\varphi_\nu{}^\rho + iegF_{\rho\nu}\varphi_\mu{}^\rho\ . \end{aligned} \tag{A.16}$$

Combining the trace and the double divergence of Eq. (A.16) now gives

$$\left(\tfrac{d-1}{d-2}\right) m^4 \varphi \;=\; ie(2g-1)F^{\mu\nu} D_\mu D^\rho \varphi_{\rho\nu} + (g-2)e^2 F^{\mu\rho} F_\rho{}^\nu \varphi_{\mu\nu} - \tfrac{3}{4}\, e^2 F^{\mu\nu} F_{\mu\nu}\,\varphi\ . \tag{A.17}$$

The first term on the right–hand side signals a potential DoF breakdown, since a constraint of the free theory is turned into a propagating field equation unless $g = \frac{1}{2}$. The unique minimally coupled model that does not give rise to a wrong DoF count has therefore $g = \frac{1}{2}$, and the result is precisely the Federbush Lagrangian of [21]. With this choice, the divergences and the trace of Eq. (A.16) reduce to

$$D^\mu \varphi_{\mu\nu} - D_\nu \varphi \;=\; \tfrac{3}{2}(ie/m^2)\,[F^{\rho\sigma} D_\rho \varphi_{\sigma\nu} - F_{\nu\rho} D_\sigma \varphi^{\sigma\rho} + F_{\nu\rho} D^\rho \varphi]\ , \tag{A.18}$$

$$D^\mu D^\nu \varphi_{\mu\nu} - D^2 \varphi \;=\; \tfrac{3}{2}(1/m^2)\left[\text{Tr}(F \cdot \varphi \cdot F) - \tfrac{1}{2}\text{Tr}F^2 \varphi\right]\ , \tag{A.19}$$

$$\varphi \;=\; -\tfrac{3}{2}\left(\tfrac{d-2}{d-1}\right)(e/m^2)^2\left[\text{Tr}(F \cdot \varphi \cdot F) - \tfrac{1}{2}\text{Tr}F^2 \varphi\right]\ . \tag{A.20}$$

The trace constraint can also be recast in the form

$$\varphi \;=\; -\,\frac{\frac{3}{2}\left(\frac{d-2}{d-1}\right)(e/m^2)^2\,\text{Tr}(F \cdot \varphi \cdot F)}{1 - \frac{3}{4}\left(\frac{d-2}{d-1}\right)(e/m^2)^2\,\text{Tr}F^2}\ , \tag{A.21}$$

an expression that is never singular away from the instabilities of [26, 27], *i.e.* in the physically interesting situations where $|\text{Tr}F^2| \ll (m^2/e)^2$. Still, unlike in the free theory, the trace does not vanish in the presence of the EM background.

However, one still needs to see whether the dynamical DoFs propagate in the correct number and causally. To this end, let us isolate the terms in Eqs. (A.16) that are of second order in derivatives,

$$\mathcal{R}^{(2)}_{\mu\nu} \;=\; D^2\varphi_{\mu\nu} - [D_\mu(D^\rho\varphi_{\rho\nu} - D_\nu\varphi) + (\mu \leftrightarrow \nu)] - \tfrac{1}{2}D_{(\mu}D_{\nu)}\varphi + \eta_{\mu\nu}(D^\alpha D^\beta\varphi_{\alpha\beta} - D^2\varphi) \;, \tag{A.22}$$

where the last can be actually dropped in view of (A.19) while the constraint equations (A.18) and (A.20) can be substituted in the second and third terms. The end result,

$$\begin{aligned}\mathcal{R}^{(2)}_{\mu\nu} \;&=\; \Box\varphi_{\mu\nu} - \tfrac{3}{2}(ie/m^2)\left[F^{\rho\sigma}\partial_\rho\partial_{(\mu}\varphi_{\nu)\sigma} + F_{\rho(\mu}\partial_{\nu)}(\partial_\sigma\varphi^{\sigma\rho} - \partial^\rho\varphi)\right] \\ &\quad + \tfrac{3}{2}\left(\tfrac{d-2}{d-1}\right)(e/m^2)^2\left[F^{\rho\sigma}F_\sigma{}^\lambda\partial_\mu\partial_\nu\varphi_{\rho\lambda} - \tfrac{1}{2}\mathrm{Tr}F^2\partial_\mu\partial_\nu\varphi\right] \;,\end{aligned} \tag{A.23}$$

is the counterpart, for the model, of the Klein–Gordon equation.

Following [11], one can now resort to the characteristic determinant method to investigate the causal properties of the system, replacing $i\partial_\mu$ with n_μ, the normal to the characteristic hypersurfaces, in the highest derivative terms of the EoMs. The determinant $\Delta(n)$ of the resulting coefficient matrix determines in fact the causal properties of the system, and in particular if the algebraic equation $\Delta(n) = 0$ has real solutions for n_0 for any $\vec{n}$, the system is hyperbolic, with maximum wave speed $n_0/|\vec{n}|$. On the other hand, if there are time–like solutions n_μ for $\Delta(n) = 0$, the system admits acausal propagation. Note that the procedure is akin to solving the EoMs in the eikonal approximation, letting

$$\varphi_{\mu\nu} \;=\; \hat{\varphi}_{\mu\nu}\exp(itn\cdot x) \;, \qquad t \to \infty \;. \tag{A.24}$$

The coefficient matrix determined by (A.23) takes the form

$$\begin{aligned}M_{(\mu\nu)}{}^{(\alpha\beta)}(n) \;&=\; -\tfrac{1}{2}n^2\delta_\mu^{(\alpha}\delta_\nu^{\beta)} + \tfrac{3}{4}(ie/m^2)\left[n_\rho F^{\rho(\alpha}n_{(\mu}\delta_{\nu)}^{\beta)} - n_{(\mu}F_{\nu)}{}^{(\alpha}n^{\beta)} + 2n_{(\mu}F_{\nu)}{}^\rho n_\rho\eta^{\alpha\beta}\right] \\ &\quad - \tfrac{3}{2}\left(\tfrac{d-2}{d-1}\right)(e/m^2)^2\, n_\mu n_\nu\left[F^{\alpha\rho}F_\rho{}^\beta - \tfrac{1}{2}\mathrm{Tr}F^2\eta^{\alpha\beta}\right] \;.\end{aligned} \tag{A.25}$$

This expression should indeed be regarded as a matrix whose $\frac{1}{2}d(d+1)$ rows and columns are labeled by pairs of Lorentz indices $(\mu\nu)$ and $(\alpha\beta)$. In particular in four dimensions its determinant reads

$$\Delta(n) \;=\; (n^2)^8\left[n^2 - \left(\tfrac{e}{m^2}\right)^2\left(\tilde{F}\cdot n\right)^2\right]\left[n^2 + \left(\tfrac{3e}{2m^2}\right)^2\left(\tilde{F}\cdot n\right)^2\right] \;, \tag{A.26}$$

where $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, so that

$$(\tilde{F} \cdot n)^2 \;\equiv\; (n_0 \vec{B} + \vec{n} \times \vec{E})^2 - (\vec{n} \cdot \vec{B})^2 \; . \tag{A.27}$$

Let us now consider four–dimensional EM invariants such that $\vec{B} \cdot \vec{E} = 0$, $\vec{B}^2 - \vec{E}^2 > 0$, which entails that $\vec{B}^2$ is non-vanishing in all Lorentz frames. One can always find a vector $\vec{n}$, perpendicular to $\vec{B}$, for which the characteristic determinant vanishes if

$$\frac{n_0}{|\vec{n}|} \;=\; \frac{1}{\sqrt{1 - \left(\frac{3e}{2m^2}\right)^2 \vec{B}^2}} \; , \tag{A.28}$$

thanks to the last factor appearing in (A.26). As a result, superluminal propagation can occur even for infinitesimally small values of $\vec{B}^2$, and the propagation itself ceases to occur whenever $\vec{B}^2 \geq \left(\frac{2}{3}m^2/e\right)^2$. Actually, one can always find a Lorentz frame where the pathology shows up, and in particular the magnetic field $\vec{B}^2$ can reach the critical value in a frame where $\vec{E}^2 = \left(\frac{2}{3}m^2/e\right)^2 - \epsilon$, with ϵ arbitrarily small. This is the most serious aspect of the problem: it persists even for very small values of the EM field invariants, far away from the instabilities of [26, 27], where one would expect to be dealing with well–behaved and long–lived propagating particles. This is the so–called Velo–Zwanziger problem [11], which as we have just recalled arises already at the classical level [6]. It is important to note that the pathology is not a special property of the spin-2 case, but it is expected to persist for all charged massive particles with $s \geq 3/2$, since it originates from the very existence of longitudinal modes of massive high–spin particles [43].

B The bosonic string: notation and conventions

In this Appendix we spell out a few basic facts about the open bosonic string that have some bearing on our derivations. As in the main body of the paper we work with signature $(-,+)$ on the world sheet and in units such that $\alpha' = \frac{1}{2}$. The standard open strings with Neumann boundary conditions are then described by the coordinate functions

$$X^\mu_{\text{free}}(\tau,\sigma) \;=\; x^\mu \;+\; \tau\, \bar{\alpha}^\mu_0 \;+\; \frac{i}{2}\sum_{m\neq 0} \frac{1}{m}\left[\mathrm{e}^{-im(\tau+\sigma)} \;+\; \mathrm{e}^{-im(\tau-\sigma)}\right]\, \bar{\alpha}^\mu_m \; , \tag{B.1}$$

[6]The pathology in the corresponding quantum mechanical theory, for $s = 3/2$, was found much earlier by Johnson and Sudarshan [41]. From a canonical viewpoint, the equal time commutation relations become ill–defined in an EM background. That the Johnson–Sudarshan and Velo–Zwanziger problems have a common origin was later shown in [42].

where x^μ is the center–of–mass coordinate while the term proportional to τ is the corresponding momentum. The mode functions are thus, aside from a constant,

$$\bar{\Psi}_m(\tau,\sigma) \;=\; \frac{i/2}{\sqrt{m}}\,\mathrm{e}^{-im\tau}[\,\mathrm{e}^{-im\sigma}+\mathrm{e}^{im\sigma}\,] \qquad m\in\mathbb{N}_1\ , \tag{B.2}$$

$$\bar{\Psi}_0(\tau,\sigma) \;=\; \tau\ , \tag{B.3}$$

which obey orthogonality relations that can be presented in the convenient form

$$(\bar{\Psi}_m,\bar{\Psi}_n) \;\equiv\; \frac{1}{\pi}\int_0^\pi d\sigma\bar{\Psi}^*_m(\tau,\sigma)\star\bar{\Psi}_n(\tau,\sigma) \;=\; \delta_{mn}(1-\delta_{m0}) \qquad m,n\in\mathbb{N}_0\ , \tag{B.4}$$

where $\star\equiv i\overleftrightarrow{\partial_\tau}=i\overrightarrow{\partial_\tau}-i\overleftarrow{\partial_\tau}$. The constant mode, just like $\bar{\Psi}_0$, is orthogonal to all $\bar{\Psi}_{m\in\mathbb{N}_1}$ and has a vanishing norm for the inner product (B.4), but is not orthogonal to $\bar{\Psi}_0$

$$(1,1) \;=\; 0\ , \qquad (1,\bar{\Psi}_m) \;=\; i\delta_{m0}\ . \tag{B.5}$$

The mode functions split naturally into two mutually orthogonal subsets, particle–like $\{1,\bar{\Psi}_0\}$, and string–like $\{\bar{\Psi}_{m\in\mathbb{N}_1}\}$, and the infinitely many string–like modes form an orthonormal set of functions. For the free string their orthonormality relation is usually presented in a more familiar form that does not involve τ, but this form extends naturally to the case of a constant EM background, as reviewed in Section 2.1. The two particle–like modes have zero norm and a non-vanishing mutual inner product, so that

$$x^\mu \;=\; i(\bar{\Psi}_0,X^\mu)\ , \qquad p^\mu \;=\; -i(1,X^\mu)\ . \tag{B.6}$$

The free string modes $\bar{\alpha}_m$ in (B.1) obey the commutation relations

$$[\,\bar{\alpha}^\mu_m,\bar{\alpha}^\nu_n\,] \;=\; m\eta^{\mu\nu}\delta_{m,-n} \qquad m,n\in\mathbb{Z}\ , \tag{B.7}$$

that correspond to the $G\to 0$ limit of Eq. (2.21). The oscillators $\bar{\alpha}_{m\neq 0}$ define an infinite set of creation and annihilation operators,

$$a^\mu_m \;=\; \frac{1}{\sqrt{m}}\,\bar{\alpha}^\mu_m\ , \qquad a^{\dagger\mu}_m \;=\; \frac{1}{\sqrt{m}}\,\bar{\alpha}^\mu_{-m} \qquad m\in\mathbb{N}_1\ , \tag{B.8}$$

so that one can consider general linear combinations of terms obtained applying creation operators to the ground state:

$$|\Psi\rangle \ = \ \sum_{s=1}^{\infty} \sum_{m_i=1}^{\infty} \psi^{(m_1 m_2 ... m_s)}_{\mu_1 \mu_2 ... \mu_s} \, a^{\dagger \mu_1}_{m_1} a^{\dagger \mu_2}_{m_2} \, ... \, a^{\dagger \mu_s}_{m_s} \, |0\rangle \ . \tag{B.9}$$

Given a set of integers $(m_1, m_2, ..., m_s)$, the coefficient function $\psi^{(m_1 m_2 ... m_s)}_{\mu_1 \mu_2 ... \mu_s}$ is a rank-s Lorentz tensor, generically of mixed symmetry, that is interpreted as a field associated to the corresponding string state, and as such is a function of the string center–of–mass coordinates.

Of particular interest are the string states that are eigenstates of the number operator,

$$\mathcal{N} \ \equiv \ \sum_{n=1}^{\infty} n \, a^{\dagger}_n \cdot a_n \ , \tag{B.10}$$

whose integer eigenvalues N determine the masses of the open–string states according to

$$(\text{mass})^2 \ = \ 2(N-1) \ = \ \frac{1}{\alpha'} \, (N-1) \ , \tag{B.11}$$

where in the last step we have reinstated α'.

A "physical" string state is to satisfy some conditions that involve the Virasoro generators

$$L_n \ = \ \tfrac{1}{2} \sum_{m \in \mathbb{Z}} \bar{\alpha}_{n-m} \cdot \bar{\alpha}_m \qquad n \in \mathbb{N}_0 \ . \tag{B.12}$$

L_0, L_1, and L_2 are particularly important, since all other generators with $n \in \mathbb{N}_0$ can be recovered from their commutators. In terms of the "a" operators they read

$$L_0 \ = \ -\tfrac{1}{2} \, \Box + \sum_{m=1}^{\infty} m \, a^{\dagger}_m \cdot a_m \ \equiv \ -\tfrac{1}{2} \, \Box + \mathcal{N} \ , \tag{B.13}$$

$$L_1 \ = \ - \, i \, \partial \cdot a_1 \ + \ \sum_{m=2}^{\infty} \sqrt{m(m-1)} \, a^{\dagger}_{m-1} \cdot a_m \ , \tag{B.14}$$

$$L_2 \ = \ -\sqrt{2} \, i \, \partial \cdot a_2 \ + \ \tfrac{1}{2} \, a_1 \cdot a_1 + \sum_{m=3}^{\infty} \sqrt{m(m-2)} \, a^{\dagger}_{m-2} \cdot a_m \ . \tag{B.15}$$

As we have pointed out, demanding that L_0, L_1 and L_2 annihilate a physical state translates precisely, in terms of the coefficient fields, into the Fierz–Pauli conditions (A.9)–(A.11).

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