

# On Solutions to the "Faddeev-Niemi" Equations

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Recently it has been pointed out that the "Faddeev-Niemi" equations that describe the Yang-Mills equations of motion in terms of a decomposed gauge field, can have solutions that obey the standard Yang-Mills equations with a source term. Here we present a general class of such gauge field configurations. They might have physical relevance in a strongly coupled phase, where the Yang-Mills theory can not be described in terms of a Landau liquid of asymptotically free gluons.

## I. INTRODUCTION

In the weak coupling limit a Yang-Mills theory describes a Landau liquid of weakly interacting, asymptotically free gluons. But it has been proposed by several authors that in the strong coupling regime where the physical excitations are different from asymptotically free gluons, it might become more effective to describe the gauge field in a decomposed representation. A proper field decomposition might also help to identify those field degrees of freedom that become relevant when the theory enters its strongly coupled phase. In particular, the Cho-Duan-Ge [1], [2] decomposition describes the gauge field in a manner that directly relates to magnetic monopoles, widely presumed to be responsible for the confining properties of the theory. An on-shell refinement of this decomposition was presented in [3], where it was also shown how the decomposition modifies the Yang-Mills equations. In [3], [4] it was argued that for *generic* field configurations the ensuing on-shell decomposed Yang-Mills equations coincide with the conventional Yang-Mills equations. See however [5] for an off-shell completion of the decomposition, and [6], [7] for a different kind of manifestly complete decomposition based on the concept of spin-charge separation; for a properly spin-charge separated gauge field the decomposed equations coincide with the original Yang-Mills equations.

In [4] it was pointed out that besides the generic solutions to the "Faddeev-Niemi" (FN) equations in [3], there can also be *non-generic* field configurations. In general these give rise to a source term in the original Yang-Mills equations. In a recent article [8] it has been pointed out that these non-generic field configurations include a constant strength color-electric field that has an obvious attractive physical appeal. Here we report on a more general class of such non-generic solutions of the FN equations.

## II. DECOMPOSING YANG-MILLS

The decomposed four dimensional  $SU(2)$  gauge field introduced in [3] is

$$\mathbf{A}_\mu = C_\mu \mathbf{n} + \partial_\mu \mathbf{n} \times \mathbf{n} + \rho \partial_\mu \mathbf{n} + \sigma \partial_\mu \mathbf{n} \times \mathbf{n} \quad (1)$$

Here  $\mathbf{n}$  is a three component  $SU(2)$  Lie-algebra valued unit length vector field,  $C_\mu$  is a vector field in  $\mathbb{R}^4$  (we use Euclidean signature) and  $\rho$ ,  $\sigma$  are two real scalar fields. The reason for the separation between the second and the fourth term in (1) is that it allows us to identify the first two terms with the Cho-Duan-Ge connection [1], [2].

When (1) is substituted into the Yang-Mills action the ensuing equations of motion obtained by varying the variables  $(C_\mu, \mathbf{n}, \rho, \sigma)$  yield the following FN equations [3],

$$\mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \quad (2)$$

$$\partial_\nu \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \quad (3)$$

$$\partial_\nu \mathbf{n} \times \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \quad (4)$$

$$(\nabla_\nu \rho + \nabla_\nu \sigma \cdot \mathbf{n} \times) \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \quad (5)$$

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In order to get a better understanding of relation between solutions of this set of equations and the original Yang-Mills field equations we follow [4] to introduce a more geometrical framework. Namely, we use a right-handed orthonormal triplet  $(\mathbf{e}_\theta, \mathbf{e}_\varphi, \mathbf{n})$  and define

$$\kappa_\mu^a = \mathbf{e}_a \cdot \partial_\mu \mathbf{n} \quad (6)$$

so that

$$\mathbf{A}_\mu = C_\mu \mathbf{n} + (\kappa_\mu^2 \mathbf{e}_1 - \kappa_\mu^1 \mathbf{e}_2) + (\rho \kappa_\mu^1 + \sigma \kappa_\mu^2) \mathbf{e}_1 + (\rho \kappa_\mu^2 - \sigma \kappa_\mu^1) \mathbf{e}_2$$

In these variables the FN equations read [4]

$$\begin{aligned} \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} &= 0 \\ \kappa_\nu^+ \mathbf{e}_+ \cdot \nabla_\mu \mathbf{F}_{\mu\nu} &\equiv \kappa_\nu^+ (\nabla_\mu \mathbf{F}_{\mu\nu})^+ = 0 \\ \nabla_\nu \phi \mathbf{e}_- \cdot \nabla_\mu \mathbf{F}_{\mu\nu} &\equiv \nabla_\nu \phi (\nabla_\mu \mathbf{F}_{\mu\nu})^- = 0 \end{aligned} \quad (7)$$

where the corresponding field strength tensor is

$$\mathbf{F}_{\mu\nu} = \mathbf{n} \{ (\partial_\mu C_\nu - \partial_\nu C_\mu) - [1 - (\rho^2 + \sigma^2)] \mathbf{n} \cdot \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n} \} + \frac{1}{4} \nabla_\mu \phi [\kappa_\nu^+ (\mathbf{e}^+ + \mathbf{e}^-) + \kappa_\nu^- (\mathbf{e}^+ - \mathbf{e}^-)] - (\mu \leftrightarrow \nu) + c.c. \quad (8)$$

with

$$\kappa_\mu^+ = \kappa_\mu^1 + i \kappa_\mu^2$$

In [3] it was argued that these equations reproduce the original four dimensional Yang-Mills equations, for *generic*  $\kappa_\nu^+$  and  $\nabla_\nu \phi$ . Subsequently this was shown to be the case in two dimensions (coordinates  $x_1 = x$ ,  $x_2 = y$ ) [4], where using antisymmetry of  $\mathbf{F}_{\mu\nu}$  it was shown that the last two equations in (7) can be written as the following homogeneous linear system,

$$\mathcal{M}^\alpha{}_\beta (\nabla \mathbf{F}_{yx})^\beta \equiv \begin{pmatrix} \kappa_x^1 & -\kappa_y^1 & -\kappa_x^2 & \kappa_y^2 \\ \kappa_x^2 & -\kappa_y^2 & \kappa_x^1 & -\kappa_y^1 \\ \nabla_x \rho & -\nabla_y \rho & \nabla_x \sigma & -\nabla_y \sigma \\ \nabla_x \sigma & -\nabla_y \sigma & -\nabla_x \rho & \nabla_y \rho \end{pmatrix} \begin{pmatrix} (\nabla_y \mathbf{F}_{yx})^1 \\ (\nabla_x \mathbf{F}_{yx})^1 \\ (\nabla_y \mathbf{F}_{yx})^2 \\ (\nabla_x \mathbf{F}_{yx})^2 \end{pmatrix} = 0 \quad (9)$$

Consequently, for *generic* two dimensional field configurations that is field configurations for which the determinant of the  $4 \times 4$  matrix  $\mathcal{M}$  in (9) does not vanish, the FN equations (7) reproduce the original Yang-Mills equations [4]

$$\nabla_\mu \mathbf{F}_{\mu\nu} = 0$$

Therefore, one may expect that for *non-generic* field configurations leading to the vanishing determinant in (9), solutions of the FN equations may possibly not obey the original Yang-Mills equations. Indeed, recently [8] have investigated solutions of (2)-(5) that do not obey the original four dimensional Yang-Mills equations. As an Ansatz the authors considered essentially two dimensional (*e.g.*  $x_3$  and  $x_4$  independent) gauge fields in  $\mathbb{R}^4$ . The ensuing solutions of (7) that fail to satisfy the original Yang-Mills equations in  $\mathbb{R}^4$  are then obtained by looking for such two dimensional decomposed gauge fields (1) for which the determinant of the  $4 \times 4$  matrix  $\mathcal{M}$  in (9) vanishes: In the explicit Ansatz in [8] the elements on the second and fourth columns of this matrix are all zero. This leads to a constant strength color-electric Yang-Mills field, a solution of the original Yang-Mills equations in the presence of an external source [8]. Since a constant strength color-electric field has an obvious physical appeal, it is worth while to study further the properties of the non-generic solutions of the FN equations.

### III. SOLVING THE FN EQUATIONS

We shall now show that there are additional familiar and physically appealing field configurations that solve the FN equation but are described by the original Yang-Mills equations with a source term. In particular, it appears that many known classical solutions of Yang-Mills theory that give rise to a source term, are *sourceless* solutions of (7).

Since the structure of (7) is relatively simple, we expect that its non-generic solutions can be described in quite general terms. Here we look only for configurations that appear as solutions to

$$\nabla_\mu \phi = 0 \quad (10)$$

For these field configurations only the  $\mathbf{n}$ -component of (8) survives,

$$\mathbf{F}_{\mu\nu} \rightarrow \mathbf{n} G_{\mu\nu} \equiv \mathbf{n}\{(\partial_\mu C_\nu - \partial_\nu C_\mu) - [1 - |\phi|^2] \mathbf{n} \cdot \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}\} \equiv \mathbf{n}\{F_{\mu\nu} - (1 - |\phi|^2)H_{\mu\nu}\} \quad (11)$$

where

$$F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu, \quad (12)$$

$$H_{\mu\nu} = \mathbf{n} \cdot \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n} \quad (13)$$

$$G_{\mu\nu} = F_{\mu\nu} + H_{\mu\nu} \quad (14)$$

Let us now analyze how (10) impacts on the FN equations. Obviously, the third equation in (7) is identically fulfilled. The second equation leads to

$$\rho(\kappa_\nu^1 \kappa_\mu^2 - \kappa_\nu^2 \kappa_\mu^1) G_{\mu\nu} = 0, \quad (15)$$

$$(1 + \sigma)(\kappa_\nu^1 \kappa_\mu^2 - \kappa_\nu^2 \kappa_\mu^1) G_{\mu\nu} = 0 \quad (16)$$

which for nontrivial case i.e.,  $\rho \neq 0$  and  $\sigma \neq -1$  is equivalent to an orthogonality condition

$$H_{\mu\nu} G_{\mu\nu} = 0 \quad (17)$$

where we use

$$H_{\mu\nu} = \kappa_\mu^1 \kappa_\nu^2 - \kappa_\mu^2 \kappa_\nu^1$$

Finally, the first equation in (7) gives

$$\partial_\mu G_{\mu\nu} = 0 \quad (18)$$

To summarize, the FN equations in the sector defined by (10) are equivalent to the Maxwell equations for  $G_{\mu\nu}$  (18) and constrain (17).

Consider the full Yang-Mills equations for the choice (10)

$$\nabla_\mu \mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{F}_{\mu\nu} + \mathbf{A}_\mu \times \mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{n} G_{\mu\nu} + \mathbf{n} \partial_\mu G_{\mu\nu} + \mathbf{A}_\mu \times \mathbf{n} G_{\mu\nu} \quad (19)$$

However, assuming that fields obey the FN equations we get

$$\nabla_\mu \mathbf{F}_{\mu\nu} = \mathbf{J}_\nu \quad (20)$$

where we find the following generally non-vanishing external current

$$\mathbf{J}_\nu = (\rho \partial_\mu \mathbf{n} \times \mathbf{n} - \sigma \partial_\mu \mathbf{n}) G_{\mu\nu} \quad (21)$$

Now, we are able to present several examples of sourceless configurations of the FN equations which are solutions to the Yang-Mills equations with the above source term. In the simplest case we assume

$$H_{\mu\nu} \equiv 0$$

which identically solves the constraint. Although this tensor identically vanishes, the unit vector field  $\mathbf{n}$  does not need to be trivial. It may for example simply depend *arbitrarily* on one single space-time coordinate, lets say  $\mathbf{n} = \mathbf{n}(x^\lambda)$ , where  $\lambda$  is a fixed index. Thus, we are left with  $U(1)$  gauge theory for  $C_\mu$

$$\partial_\mu F_{\mu\nu} = 0$$

Solution discussed in [8] belongs to this class., it can be easily generalized to a configuration for which the field tensor is independent of the coordinate  $F_{\mu\nu} = \text{const.}$  Namely,

$$C_\mu = a_\mu + b_{\mu\nu} x_\nu$$

where  $b_{\mu\nu}$  is an arbitrary four dimensional constant matrix. Then

$$F_{\mu\nu} = b_{\nu\mu} - b_{\mu\nu}$$

and the external current reads

$$\mathbf{J}_\nu = (\rho \partial_\lambda \mathbf{n} \times \mathbf{n} - \sigma \partial_\lambda \mathbf{n}) (b_{\nu\lambda} - b_{\lambda\nu}) \quad (22)$$

where no summation on  $\lambda$  is assumed.

The constancy of the field tensor is by no means essential to our construction. For example, one can consider the plane wave solution propagating along  $z$ -axis with frequency  $\omega$

$$\begin{aligned} C_0 &= C_z = 0 \\ C_1 &= a \sin \omega(z - t) \\ C_2 &= a \cos \omega(z - t) \end{aligned}$$

It solves the sourceless FN equations whereas for the full Yang-Mills equation there is a source

$$\mathbf{J}_\nu = a\omega (\rho \partial_x \mathbf{n} \times \mathbf{n} - \sigma \partial_x \mathbf{n}) \cos \omega(z - t) \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (23)$$

Observe that the external current vanishes if the so-called valence degrees of freedom are absent  $\rho = \sigma = 0$  [1]. This may indicate a particularly close relation in  $(2 + 1)$  dimensions between the FN and Yang-Mills equations, due to the fact that in  $(2+1)$  dimensional case the nonabelian gauge field has 3 on-shell degrees of freedom. Then, it is sufficient to use the Cho-Duan connection containing only the  $U(1)$  field  $C_\mu$  (one field degree of freedom) and  $\mathbf{n}$  (two field degrees of freedom).

#### IV. CONCLUSIONS

In conclusion, we have generalized the observation made in [8], that the "Faddeev-Niemi" equations have physically appealing solutions that solve the original Yang-Mills equation with a source term. The formalism that we have presented allows a more systematic analysis of such solutions. Since the decomposed representation of the Yang-Mills field is presumed to identify those excitations that become important in strongly coupled phases of the Yang-Mills theory that can not be described in terms of asymptotically free gluons and conventional weak coupling perturbation theory, it should be of interest to better understand the relevance of the solutions of the "Faddeev-Niemi" equations to the non-perturbative structure of Yang-Mills theories.

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- [1] Y.M. Cho, Phys. Rev. **D21** (1980) 1080; Y.M. Cho, Phys. Rev. Lett. **46** (1981) 302
  - [2] Y.S. Duan and M.L. Ge, Sinica Sci. **11** (1979) 1072
  - [3] L.D. Faddeev and A.J. Niemi, Phys. Rev. Lett. **82** (1999) 1624
  - [4] A.J. Niemi, Phys. Rev. **D70** (2004) 045017
  - [5] L.D. Faddeev and A.J. Niemi, Phys. Lett. **B464** (1999) 90
  - [6] L.D. Faddeev and A.J. Niemi, Phys. Lett. **B525** (2002) 195
  - [7] L.D. Faddeev and A.J. Niemi, Nucl. Phys. **B776** (2007) 38
  - [8] J. Evslin and S. Giacomelli, . e-Print: [arXiv:1010.1702](https://arxiv.org/abs/1010.1702) [hep-th]