# On the Efetov-Wegner terms by diagonalizing a Hermitian supermatrix

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The diagonalization of Hermitian supermatrices is studied. Such a change of coordinates is inevitable to find certain structures in random matrix theory. However it still poses serious problems since up to now the calculation of all Rothstein contributions known as Efetov-Wegner terms in physics was quite cumbersome. We derive the supermatrix Bessel function with all Efetov-Wegner terms. As applications we consider representations of generating functions for Hermitian random matrices with and without an external field as integrals over eigenvalues of Hermitian supermatrices. All results are obtained with all Efetov-Wegner terms which were unknown before.

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## I. INTRODUCTION

Many eigenvalue correlations for random matrix ensembles as matrix Green functions, the $k$-point correlation functions[1,2] as well as the free energy[3] can be derived by generating functions. These functions are averages over ratios of characteristic polynomials for random matrices.

A common approach to calculate generating functions is the supersymmetry method[1,4–6]. Another approach is the orthogonal polynomial method[7]. In the supersymmetry method one maps integrals over ordinary matrices to integrals over supermatrices. Its advantage is the drastic reduction of the number of integration variables. Nevertheless there is a disadvantage. Up to now it is not completely clear how to get the full structures found with the orthogonal polynomial method for factorizing probability densities. For such probability densities the $k$-point correlation functions can be written as determinants and Pfaffians of certain kernels. This property was extended to the generating functions[8–11]. Unfortunately, the determinants and Pfaffians were not found for the full generating function after mapping the integrals over ordinary matrices to integrals over supermatrices. Only the determinantal expression of the $k$-point correlation function for rotation invariant Hermitian random matrix ensembles could be regained with help of the supersymmetry method[1,12]. Grönqvist, Guhr and Kohler studied this problem from another point of view in Ref. 10. They started from the determinantal and Pfaffian expressions of the $k$-point correlation functions for Gaussian orthogonal, unitary and symplectic random matrix ensembles and showed that the kernels of the generating functions with two characteristic polynomials as integrals in superspace yield the known result.

Changing coordinates in superspaces causes serious problems since the Berezinian[13] playing the role of the Jacobian in superspace incorporates differential operators. These differential operators have no analog in ordinary space and are known in the mathematical literature as Rothstein's vector fields[14]. For supermanifolds with boundaries such differential operators yield boundary terms. In physics these boundary terms are called Efetov-Wegner terms[15–18]. They can be understood as corrections to the Berezinian without Rothstein's vector fields. There were several attempts to calculate these vector fields or the corresponding Efetov-Wegner terms for diagonalizations of Hermitian supermatrices but this was only successful for low dimensional supermatrices, e.g. $(1+1)\times(1+1)$ Hermitian supermatrices[19–21]. For higher dimensions the calculation of all Efetov-Wegner terms becomes cumbersome. Also for other sets of supermatrices such changes in the coordinates were studied[22,23]. However they came never beyond low dimensional examples. One approach to derive all Efetov-Wegner terms is the one to consider these terms as boundary terms resulting from partial integrations of differential operators which are equivalent to the integration over the Grassmann variables.[15] Though a quite compact form for a differential operator for the Hermitian supermatrices was found we were unable to calculate all Efetov-Wegner terms. The order of such a differential operator is the number of pairs of Grassmann variables.

We derive the supermatrix Bessel function for Hermitian supermatrices with all Efetov-Wegner terms. Thereby we apply a method called "supersymmetry without supersymmetry"[8] on Hermitian matrix ensembles whose characteristic function factorizes in the eigenvalues of the random matrix. This approach uses determinantal structures of Berezinians without mapping into superspace. We combine this method with the supersymmetry method described in Refs. [1,12]. The supermatrix Bessel function is then obtained by simply identifying the left hand side with the right hand side of the resulting matrix integrals. As a simple application we calculate the generating function of arbitrary, rotation invariant Hermitian matrix ensembles with and without an external field. Hence we generalize known results[1,8,12,24] because they now contain all Efetov-Wegner terms which guarantee the correct normalizations.

We organize the article as follows. In Sec. II, we will give an outline of our approach and introduce some basic quantities. Using the method "supersymmetry without supersymmetry" we derive a determinantal structure for generating functions of rotation invariant Hermitian random matrices without an external field and with a factorizing characteristic function, in Sec. III. In Sec. IV, we briefly present the results of the supersymmetry method. We also calculate the supermatrix representation of the generating function with one determinant in the numerator as in the denominator. For this generating function the corresponding supermatrix Bessel function with all Efetov-Wegner terms is known[25] since it only depends on $(1+1)\times(1+1)$ Hermitian supermatrices. In Sec. V, we plug the result of Sec. IV into the result of Sec. III and obtain the supermatrix Bessel function with all Efetov-Wegner terms for arbitrary dimensional Hermitian supermatrices. Moreover we discuss this result with respect to double Fourier transformations and, thus, Dirac-distributions in superspace. In Sec. VI, we, first, apply our result on arbitrary, rotation invariant Hermitian random matrices without an external field and, then, in an external field. Details of the proofs and the calculations are given in the appendices.

## II. OUTLINE

The main idea of our approach is the comparison of results for generating functions,

$$Z_{k_1/k_2}^{(N)}(\kappa)=\int\limits_{\mathrm{Herm}\,(N)}P^{(N)}(H)\frac{\prod\limits_{j=1}^{k_2}\det(H-\kappa_{j2}\mathbb{1}_N)}{\prod\limits_{j=1}^{k_1}\det(H-\kappa_{j1}\mathbb{1}_N)}d[H]\,, \tag{2.1}$$

obtained with and without supersymmetry. The matrix $\mathbb{1}_N$ is the $N$ dimensional unit matrix. The integration domain in Eq. (2.1) is the set $\mathrm{Herm}\,(N)$ of $N\times N$ Hermitian matrices and the parameter $\kappa=\mathrm{diag}\,(\kappa_{11},\ldots,\kappa_{k_11},\kappa_{12},\ldots,\kappa_{k_22})=\mathrm{diag}\,(\kappa_1,\kappa_2)$ are chosen in such a way that the integral is convergent. One common choice of them is as real energies $x_j$ with a small imaginary part $\imath\varepsilon$ and source variables $J_j$, i.e. $\kappa_{j1/2}=x_j-J_j\mp\imath\varepsilon$.[1,12] The differentiation with respect to $J_j$ of $Z_{k_1/k_2}^{(N)}$ generates the matrix Green function which are intimately related to the $k$-point correlation functions. The measure $d[H]$ is defined as

$$d[H]=\prod_{n=1}^{N}dH_{nn}\prod_{1\leq m<n\leq N}^{N}d\mathrm{Re}\;H_{mn}d\mathrm{Im}\;H_{mn}\,. \tag{2.2}$$

The matrix $\mathbb{1}_N$ is the $N\times N$ unit matrix and $P^{(N)}$ is a probability density over $N\times N$ Hermitian matrices.

In Sec. III, we show that $Z_{k_1/k_2}^{(N)}$ is a determinant,

$$Z_{k_1/k_2}^{(N)}(\kappa)\sim\det\left[\left.\frac{Z_{1/1}^{(N)}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right|Z_{1/0}^{(b)}(\kappa_{a1})\right], \tag{2.3}$$

if the Fourier-transform

$$\mathcal{F}P^{(N)}(\widetilde{H})=\int\limits_{\mathrm{Herm}\,(N)}P^{(N)}(H)\exp\left[\imath\mathrm{tr}\,H\widetilde{H}\right]d[H] \tag{2.4}$$

factorizes in the eigenvalues of the Hermitian matrix $\widetilde{H}$. These determinantal structures are similar to those found for generating functions with factorizing $P^{(N)}$ in the eigenvalues of the Hermitian matrix $H$.[8]

Moreover, the integral of $Z_{k_1/k_2}^{(N)}$ can be easily mapped into superspace if the characteristic function $\mathcal{F}P^{(N)}$ is rotation invariant, see Refs. [1,12]. In this representation one does not integrate over ordinary matrices but over Wick-rotated Hermitian supermatrices. The integrand is almost rotation invariant apart from an exponential term which reflects a Fourier-transformation in superspace. We aim at the full integrand with all Efetov-Wegner terms appearing by a diagonalization of a supermatrix. In particular we want to find the distribution $\widehat{\varphi}_{k_1/k_2}$ which satisfies

$$\int\exp\left[-\imath\mathrm{Str}\,\kappa\rho\right]F(\rho)d[\rho]=\int\mathrm{Ber}\,_{k_1/k_2}^{(2)}(r)\widehat{\varphi}_{k_1/k_2}(-\imath r,\kappa)F(r)d[r] \tag{2.5}$$

for an arbitrary rotation invariant superfunction $F$. In Eq. (2.5) we diagonalize the $(k_1+k_2)\times(k_1+k_2)$ Hermitian supermatrix $\rho$ to its eigenvalues $r$. This diagonalization does not only yield the Berezinian $\mathrm{Ber}\,_{k_1/k_2}^{(2)}$ but also the distribution $\widehat{\varphi}_{k_1/k_2}$. The distribution $\widehat{\varphi}_{k_1/k_2}$ is the integral over the supergroup $\mathrm{U}(k_1/k_2)$ and, additionally, all Efetov-Wegner terms.

We derive $\widehat{\varphi}_{k_1/k_2}$ in two steps. In the first step we combine the mapping into superspace with the determinant (2.3) for factorizing $\mathcal{F}P^{(N)}$. This is a particular case of the identity Eq. (2.5) but it is sufficient to generalize its result to an arbitrary rotation invariant superfunction $F$ in the second step.

The procedure described above incorporates determinants derived in Ref. [8,26]. Let $p \geq q$. Then, these determinants are

$$\sqrt{\mathrm{Ber}^{(2)}_{p/q}(\kappa)} = \frac{\prod\limits_{1\leq a<b\leq p}(\kappa_{a1}-\kappa_{b1})\prod\limits_{1\leq a<b\leq q}(\kappa_{a2}-\kappa_{b2})}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(\kappa_{a1}-\kappa_{b2})} \tag{2.6}$$

$$= (-1)^{p(p-1)/2}\det\begin{bmatrix}\left\{\dfrac{\kappa_{b1}^{p-q}\kappa_{a2}^{q-p}}{\kappa_{b1}-\kappa_{a2}}\right\}_{\substack{1\leq a\leq q\\1\leq b\leq p}}\\ \left\{\kappa_{b1}^{a-1}\right\}_{\substack{1\leq a\leq p-q\\1\leq b\leq p}}\end{bmatrix} \tag{2.7}$$

$$= (-1)^{p(p-1)/2}\det\begin{bmatrix}\left\{\dfrac{1}{\kappa_{b1}-\kappa_{a2}}\right\}_{\substack{1\leq a\leq q\\1\leq b\leq p}}\\ \left\{\kappa_{b1}^{a-1}\right\}_{\substack{1\leq a\leq p-q\\1\leq b\leq p}}\end{bmatrix}. \tag{2.8}$$

All three expressions find their applications in our discussion. For $p = q$ we obtain the Cauchy-determinant[25]

$$\sqrt{\mathrm{Ber}^{(2)}_{p/p}(\kappa)} = (-1)^{p(p-1)/2}\det\left[\frac{1}{\kappa_{a1}-\kappa_{b2}}\right]_{1\leq a,b\leq p}, \tag{2.9}$$

whereas for $p = 0$ we have the Vandermonde-determinant

$$\sqrt{\mathrm{Ber}^{(2)}_{0/q}(\kappa)} = \Delta_q(\kappa) = (-1)^{q(q-1)/2}\det\left[\kappa_{b2}^{a-1}\right]_{1\leq a,b\leq q}. \tag{2.10}$$

These determinants appear as square roots of Berezinians by diagonalizing Hermitian supermatrices. This also explains our notation for them. The upper index 2 results from the Dyson index $\beta$ which is two throughout this work. Thus we are consistent with our notation used in other articles [8,9,12,27].

## III. THE DETERMINANTAL STRUCTURE AND THE CHARACTERISTIC FUNCTION

We consider the generating function (2.1). Let the probability density $P^{(N)}$ be rotation invariant. Then, the characteristic function (2.4) inherits this symmetry. We consider such characteristic functions which factorize in the eigenvalues $\widetilde{E} = \mathrm{diag}\,(\widetilde{E}_1,\ldots,\widetilde{E}_N)$ of the matrix $\widetilde{H}$,

$$\mathcal{F}P^{(N)}(\widetilde{E}) = \prod_{j=1}^{N} f(\widetilde{E}_j)\,. \tag{3.1}$$

Due to the normalization of $P^{(N)}$, the function $f : \mathbb{R} \to \mathbb{C}$ is unity at zero. We restrict us to the case

$$k_2 \leq k_1 \leq N\,. \tag{3.2}$$

This case is sufficient for our purpose.

**Theorem III.1** *Let $f$ be conveniently integrable and $d = N + k_2 - k_1$. The generating function in Eq.* (2.1) *has for factorizing characteristic function* (3.1) *under the condition* (3.2) *the determinantal expression*

$$Z^{(N)}_{k_1/k_2}(\kappa) = \frac{(-1)^{k_2(k_2+1)/2+k_2k_1}}{\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}}\det\left[\left\{\frac{Z^{(\widetilde{N})}_{1/1}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq k_1\\1\leq b\leq k_2}}\quad\left\{Z^{(b)}_{1/0}(\kappa_{a1})\right\}_{\substack{1\leq a\leq k_1\\d+1\leq b\leq N}}\right] \tag{3.3}$$

*for all $\widetilde{N} \in \{d, d+1,\ldots,N\}$.*

Notice that indeed Eq. (3.3) is for all values $\widetilde{N} \in \{d, d+1, \ldots, N\}$ true because the determinant is skew symmetric. Hence, we may add any linear combination of the last $k_1 - k_2$ columns to the first $k_2$ columns.

With this theorem we made the first step to obtain all Efetov-Wegner terms for the diagonalization of a supermatrix. We prove it in App. A. For non-normalized probability densities, we find Eq. (3.3) multiplied by $f^\lambda(0)$ where $\lambda = d(d+1)/2 - (N-1)N/2 - \widetilde{N}k_2$. We need the non-normalized version to analyze the integral in Sec. V.

Theorem III.1 is also a very handy intermediate result. It yields an easier result than the one for factorizing probability densities[8]. Thus, all eigenvalue correlations for probability densities stemming from a characteristic function with the property (3.1) are determined by one and two point averages.

## IV. INTEGRAL REPRESENTATION IN THE SUPERSPACE OF THE GENERATING FUNCTION AND THE EFETOV–WEGNER TERM OF $Z_{1/1}^{(N)}$

In Refs. [1,4,12,27], it was shown that the generating function (2.1) can be mapped to an integral over supermatrices. Let $\Sigma_{k_1/k_2}^{(0)}$ be the set of supermatrices with the form

$$\rho = \begin{bmatrix} \rho_1 & \eta^\dagger \\ \eta & \rho_2 \end{bmatrix} . \tag{4.1}$$

The Boson–Boson block $\rho_1$ is an ordinary $k_1 \times k_1$ positive definite Hermitian matrix and the Fermion–Fermion block $\rho_2$ is an ordinary $k_2 \times k_2$ Hermitian matrix. The off-diagonal block $\eta$ comprises $k_1 \times k_2$ independent Grassmann variables. We recall that $(\eta^\dagger)^\dagger = -\eta$ and the integration over one Grassmann variable is defined by

$$\int \eta_{nm}^j d\eta_{nm} = \int \eta_{nm}^{*\,j} d\eta_{nm}^* = \delta_{j1} \,, \quad j \in \{0,1\} . \tag{4.2}$$

We need the Wick–rotated set $\Sigma_{k_1/k_2}^{(\psi)} = \widetilde{\Pi}_\psi \Sigma_{k_1/k_2}^{(0)} \widetilde{\Pi}_\psi$ to regularize the integrals below. The matrix $\widetilde{\Pi}_\psi = (\mathbb{1}_{k_1}, e^{\imath\psi/2}\mathbb{1}_{k_2})$ with $\psi \in ]0,\pi[$ is the generalized Wick–rotation[12,15]. We assume that a Wick–rotation exists such that the characteristic function is a Schwartz function on the Wick–rotated real axis.

We define the supersymmetric extension $\Phi$ of the characteristic function $\mathcal{F}P$ with help of a representation

$$\mathcal{F}P_1(\operatorname{tr} H^m, m \in \mathbb{N}) = \mathcal{F}P^{(N)}(H) \tag{4.3}$$

as a function in matrix invariants,

$$\Phi^{(k_1/k_2)}(\rho) = \mathcal{F}P_1(\operatorname{Str} \rho^m, m \in \mathbb{N}) \,, \tag{4.4}$$

see Ref. [12]. Let the signs of the imaginary parts for all $\kappa_{j1}$ be negative. Assuming that $\Phi$ is analytic in the Fermion–Fermion block $\rho_2$, the generalized Hubbard-Stratonovich transformation[1,12,27] tells us that the integral Eq. (2.1) with the condition (3.2) is

$$Z_{k_1/k_2}^{(N)}(\kappa) = C_{k_1/k_2}^{(N)} \int\limits_{\Sigma_{k_1/k_2}^{(\psi)}} \Phi^{(k_1/k_2)}(\hat{\rho}) \exp[-\imath \operatorname{Str} \kappa\hat{\rho}] \det{}^d \rho_1 \prod_{j=1}^{k_2} \left( e^{-\imath\psi} \frac{\partial}{\partial r_{j2}} \right)^{d-1} \delta\left(e^{\imath\psi} r_{j2}\right) d[\rho] \,, \tag{4.5}$$

where $e^{-\imath\psi} r_{j2}$ are the eigenvalues of $\rho_2$ and the matrix $\hat{\rho}$ is given by

$$\hat{\rho} = \left[ \begin{array}{c|c} \rho_1 & e^{\imath\psi/2}\eta^\dagger \\ \hline e^{\imath\psi/2}\eta & e^{\imath\psi}(\rho_2 + \eta\rho_1^{-1}\eta^\dagger) \end{array} \right] . \tag{4.6}$$

The constant is

$$C_{k_1/k_2}^{(N)} = \frac{(-1)^{k_2(k_2+2N-1)/2} \imath^{N(k_2-k_1)} \pi^{N(k_2-k_1)+k_2} 2^{k_2 k_1 + k_2 - k_1}}{[(d-1)!]^{k_2}} \frac{\operatorname{Vol}(\mathrm{U}\,(N))}{\operatorname{Vol}(\mathrm{U}\,(d))} f^d(0) \,. \tag{4.7}$$

Here, the volume of the unitary group is

$$\operatorname{Vol}(\mathrm{U}\,(N)) = \prod_{j=1}^{N} \frac{2\pi^j}{(j-1)!} \,. \tag{4.8}$$

The term $f(0)$ vanishes if we consider normalized probability densities. The definition of the measure $d[\rho] = d[\rho_1]d[\rho_2]d[\eta]$ is equal to the one in Ref. [27],

$$d[\rho_1] = \prod_{n=1}^{k_1} d\rho_{nn1} \prod_{1\leq m<n\leq k_1} d\mathrm{Re}\,\rho_{mn1} d\mathrm{Im}\,\rho_{mn1}\,, \tag{4.9}$$

$$d[\rho_2] = e^{\imath k_2^2\psi} \prod_{n=1}^{k_2} d\rho_{nn2} \prod_{1\leq m<n\leq k_2} d\mathrm{Re}\,\rho_{mn2} d\mathrm{Im}\,\rho_{mn2}\,, \tag{4.10}$$

$$d[\eta] = e^{-\imath k_1 k_2\psi} \prod_{n=1}^{k_1}\prod_{m=1}^{k_2} d\eta_{mn} d\eta_{mn}^*\,. \tag{4.11}$$

We use the conventional notation for the supertrace "Str" and superdeterminant "Sdet".

Let $\rho \in \Sigma_{k_1/k_2}^{(\psi)}$ with the form (4.1). As long as the eigenvalues of the Boson-Boson block $\rho_1$ are pairwise different with those of the Fermion-Fermion block $\rho_2$, we may diagonalize the whole supermatrix $\rho$ by an element $U \in \mathrm{U}\,(k_1/k_2)$. The corresponding diagonal eigenvalue matrix is $r = \mathrm{diag}\,(r_{11},\ldots,r_{k_1 1}, e^{\imath\psi}r_{12},\ldots,e^{\imath\psi}r_{22}) = \mathrm{diag}\,(r_1, e^{\imath\psi}r_2)$, i.e. $\rho = UrU^\dagger$. Due to Rothstein's[14] vector field resulting from such a change of coordinates in the Berezin measure, we have

$$d[\rho] \neq \mathrm{Ber}_{k_1/k_2}^{(2)}(r) d[r] d\mu(U)\,, \tag{4.12}$$

where $d[r]$ is the product of all eigenvalue differentials and $d\mu(U)$ is the supersymmetric Haar–measure of the unitary supergroup $\mathrm{U}\,(k_1/k_2)$. We have to consider some boundary terms since the Berezin integral is fundamentally connected with differential operators[14,15,28,29].

An arbitrary supersymmetric extension $\Phi$ of a factorizing characteristic function (3.1) has not always to factorize. However, we want to consider only such extensions which have this property. In particular, we only use the extension

$$\Phi^{(k_1/k_2)}(\rho) = \prod_{a=1}^{k_1} f(r_{a1}) \prod_{b=1}^{k_2} \frac{1}{f(e^{\imath\psi} r_{b2})}\,. \tag{4.13}$$

We consider the generating function $Z_{1/1}^{(N)}$. Equation (4.5) is an integral over Dirac distributions. Hence, we cannot simply apply a Cauchy-like theorem[15–18,30,31]. The following lemma states that also for this integration domain we obtain an Efetov–Wegner term for $Z_{1/1}^{(N)}$.

**Lemma IV.1** *Let the function* $1/f$ *be analytic at the zero point. Then, we have*

$$\begin{aligned}\frac{Z_{1/1}^{(N)}(\kappa)}{f^N(0)} &= \left[\frac{f(r_1)}{f\,(e^{\imath\psi}r_2)}\exp[-\imath\mathrm{Str}\,\kappa r]\right]\Bigg|_{r=0} + \frac{\imath(-1)^N}{(N-1)!} \\ &\times \int\limits_{\mathbb{R}_+\times\mathbb{R}} \frac{\kappa_1-\kappa_2}{r_1 - e^{\imath\psi}r_2}\frac{f(r_1)}{f\,(e^{\imath\psi}r_2)}\exp[-\imath\mathrm{Str}\,\kappa r] r_1^N \left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1}\delta\,(r_2)\,dr_2 dr_1 \end{aligned} \tag{4.14}$$

$$\begin{aligned} &= \frac{(-1)^N}{(N-1)!}\int\limits_{\mathbb{R}_+\times\mathbb{R}} \left[\frac{1}{r_1 - e^{\imath\psi}r_2}\left(\frac{\partial}{\partial r_1} + e^{-\imath\psi}\frac{\partial}{\partial r_2}\right) + \imath\frac{\kappa_1-\kappa_2}{r_1 - e^{\imath\psi}r_2}\right] \\ &\times \frac{f(r_1)}{f\,(e^{\imath\psi}r_2)}\exp[-\imath\mathrm{Str}\,\kappa r] r_1^N \left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1}\delta\,(r_2)\,dr_2 dr_1\,. \end{aligned} \tag{4.15}$$

*In Eq.* (4.15), *we integrate first over* $r_2$ *and then over* $r_1$.

We prove this lemma in App. B. The first summand of the equality (4.14) is 1 which is the Efetov-Wegner term. The second equality (4.15) is more convenient than equality (4.14) for the discussions in the ensuing section.

## V. SUPERMATRIX BESSEL FUNCTION WITH ALL EFETOV–WEGNER TERMS

Let supermatrices in $\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}$ be similar to those in $\Sigma_{k_1/k_2}^{(\psi)}$ without the positive definiteness of the Boson-Boson block. We want to find the distribution $\widehat{\varphi}_{k_1/k_2}$ which satisfies

$$\int\limits_{\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}} \exp\left[-\imath \mathrm{Str}\,\kappa\rho\right] F(\rho) d[\rho] = \int\limits_{\mathbb{R}^{k_1+k_2}} \mathrm{Ber}\,_{k_1/k_2}^{(2)}(r)\widehat{\varphi}_{k_1/k_2}(-\imath r,\kappa)F(r)d[r]\,, \tag{5.1}$$

for an arbitrary sufficiently integrable, rotation invariant superfunction $F$ analytic at the zero point. Recognizing that the integral expression (4.5) includes the supersymmetric Ingham–Siegel integral[1,12,27], the generating function is apart from a shift in the Fermion-Fermion block and analyticity a particular example of this type of integral.

It is convenient to derive $\widehat{\varphi}_{k_1/k_2}$ in Eq. (5.1) for factorizing superfunctions (4.13) since the function $F$ can be expressed in terms of such functions. Due to the analyticity of $F$, this superfunction can be chosen as a function of supertraces,

$$F(\rho) = F_0(\mathrm{Str}\,\rho^m, m \in \mathbb{N})\,. \tag{5.2}$$

As for the ordinary determinant and trace, the relation

$$\mathrm{Sdet}\,(\alpha\rho + \mathbb{1}_{k_1+k_2}) = \exp\left[\mathrm{Str}\,\ln(\alpha\rho + \mathbb{1}_{k_1+k_2})\right] \tag{5.3}$$

holds, $\alpha \in \mathbb{C}$. By expansion of the logarithm at zero, we regain all supertraces from the superdeterminant with the formula

$$\mathrm{Str}\,\rho^m = (-1)^m \alpha^{1-m} \frac{\partial}{\partial\alpha} \exp\left[\sum_{n=1}^{m-1} \frac{(-\alpha)^n}{n} \mathrm{Str}\,\rho^n\right] \mathrm{Sdet}\,(\alpha\rho + \mathbb{1}_{k_1+k_2})\Bigg|_{\alpha=0}\,,\ m \in \mathbb{N}\,. \tag{5.4}$$

The superfunction under the differentiation factorizes and has a generalized Wick–rotation to regularize the integral. The superfunction $F$ may consist of products and sums of those functions and can be approximated by polynomials in the traces with help of Weierstraß approximation theorem. Thus, we can first restrict us to factorizing superfunctions (4.13) and, then, extend to arbitrary $F$.

**Lemma V.1** *The distribution* $\widehat{\varphi}_{k_1/k_2}$ *defined by*

$$\begin{aligned}&\int\limits_{\Sigma_{k_1/k_2}^{(\psi)}} F(\hat{\rho}) \exp[-\imath \mathrm{Str}\,\kappa\hat{\rho}] \mathrm{det}^d \rho_1 \prod_{j=1}^{k_2}\left(e^{-\imath\psi}\frac{\partial}{\partial r_{j2}}\right)^{d-1} \delta\left(e^{\imath\psi} r_{j2}\right) d[\rho] \\ &= \int\limits_{\mathbb{R}_+^{k_1}\times\mathbb{R}^{k_2}} \mathrm{Ber}\,_{k_1/k_2}^{(2)}(r)\widehat{\varphi}_{k_1/k_2}(-\imath r,\kappa)F(r)\mathrm{det}^d r_1 \prod_{j=1}^{k_2}\left(e^{-\imath\psi}\frac{\partial}{\partial r_{j2}}\right)^{d-1} \delta\left(e^{\imath\psi} r_{j2}\right) d[r]\end{aligned} \tag{5.5}$$

*is*

$$\begin{aligned}\widehat{\varphi}_{k_1/k_2}(-\imath r,\kappa) &= \frac{(-1)^{(k_1+k_2)(k_1+k_2-1)/2}(\imath\pi)^{(k_2-k_1)^2/2-(k_1+k_2)/2}}{2^{k_1k_2}k_1!k_2!\sqrt{\mathrm{Ber}\,_{k_1/k_2}^{(2)}(\kappa)\mathrm{Ber}\,_{k_1/k_2}^{(2)}(r)}} \sum_{\substack{\omega_1\in\mathfrak{S}(k_1)\\ \omega_2\in\mathfrak{S}(k_2)}} \exp\left[-\imath\sum_{j=1}^{k_1}\kappa_{j1}r_{\omega_1(j)1} + \imath e^{\imath\psi}\sum_{j=1}^{k_2}\kappa_{j2}r_{\omega_2(j)2}\right] \\ &\times \det\left[\begin{array}{c}\left\{\dfrac{-\imath}{(\kappa_{b1}-\kappa_{a2})(r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2})}\left(\dfrac{\partial}{\partial r_{\omega_1(b)1}} + e^{-\imath\psi}\dfrac{\partial}{\partial r_{\omega_2(a)2}}\right)\right\}_{\substack{1\le a\le k_2\\ 1\le b\le k_1}} \\ \left\{r_{\omega_1(b)1}^{a-1}\right\}_{\substack{1\le a\le k_1-k_2\\ 1\le b\le k_1}}\end{array}\right]\end{aligned} \tag{5.6}$$

*for an arbitrary sufficiently integrable, rotation invariant superfunction* $F(\hat{\rho})$ *analytic at zero. The set of permutations over* $k$ *elements is* $\mathfrak{S}(k)$*.*

The Wick-rotated Dirac-distribution is defined by $\delta(e^{\imath\psi}r_{b1})=e^{-\imath\psi}\delta(r_{b1})$. We prove this lemma in App. C. The proof incorporates theorem III.1 and lemma IV.1.

In the next theorem we extend this lemma to arbitrary rotation invariant superfunctions without the additional Dirac-distributions in the integral, cf. Eq. (4.5).

**Theorem V.2** *The distribution $\widehat{\varphi}_{k_1/k_2}$ defined by Eq. (5.1) is the one defined in Eq. (5.6) for an arbitrary sufficiently integrable, rotation invariant superfunction $F(\rho)$ analytic at zero. For such superfunctions and for generalized Wick-rotations $\psi\in]0,\pi[$ this distribution has alternatively the form*

$$\widehat{\varphi}_{k_1/k_2}(-\imath r,\kappa)=\frac{(-1)^{(k_1+k_2)(k_1+k_2-1)/2}(\imath\pi)^{(k_2-k_1)^2/2-(k_1+k_2)/2}}{2^{k_1k_2}k_1!k_2!\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)\mathrm{Ber}_{k_1/k_2}^{(2)}(r)}} \tag{5.7}$$
$$\times\sum_{\substack{\omega_1\in\mathfrak{S}(k_1)\\ \omega_2\in\mathfrak{S}(k_2)}}\det\left[\begin{array}{c}\left\{\dfrac{-2\pi\delta(r_{\omega_1(b)1})\delta(e^{\imath\psi}r_{\omega_2(a)2})}{\kappa_{b1}-\kappa_{a2}}+\dfrac{\exp\left(-\imath\kappa_{b1}r_{\omega_1(b)1}+\imath e^{\imath\psi}\kappa_{a2}r_{\omega_2(a)2}\right)}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}\chi(\kappa_{b1}-\kappa_{a2})\right\}_{\substack{1\leq a\leq k_2\\ 1\leq b\leq k_1}}\\ \left\{r_{\omega_1(b)1}^{a-1}\exp\left(-\imath\kappa_{b1}r_{\omega_1(b)1}\right)\right\}_{\substack{1\leq a\leq k_1-k_2\\ 1\leq b\leq k_1}}\end{array}\right]$$

*with the distribution*

$$\chi(x)=\left\{\begin{array}{ll}0 & ,\ x=0,\\ 1 & ,\ \textit{else}.\end{array}\right. \tag{5.8}$$

Equation (5.7) is true because of the Cauchy-like theorem for $(1+1)\times(1+1)$ Hermitian supermatrices, see Refs. [15, 16,31]. One has to pay caution on which half of the complex plane the general Wick-rotation is lying. If $\psi\in]\pi,2\pi[$ then the minus changes to a plus infront of the Dirac-distributions.

We notice that the distribution $\widehat{\varphi}_{k_1/k_2}(r,\kappa)$ is not symmetric in exchanging its arguments $r$ and $\kappa$. Apart from the characteristic function $\chi$ such a symmetry exists for the supermatrix Bessel functions[15,25,32] which is $\widehat{\varphi}_{k_1/k_2}(r,\kappa)$ without the Dirac-distributions, i.e.

$$\begin{aligned}\varphi_{k_1/k_2}(-\imath r,\kappa) &= \frac{(-1)^{(k_1+k_2)(k_1+k_2-1)/2}(\imath\pi)^{(k_2-k_1)^2/2-(k_1+k_2)/2}}{2^{k_1k_2}k_1!k_2!\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)\mathrm{Ber}_{k_1/k_2}^{(2)}(r)}}\\ &\times\sum_{\substack{\omega_1\in\mathfrak{S}(k_1)\\ \omega_2\in\mathfrak{S}(k_2)}}\det\left[\begin{array}{c}\left\{\dfrac{\exp\left(-\imath\kappa_{b1}r_{\omega_1(b)1}+\imath e^{\imath\psi}\kappa_{a2}r_{\omega_2(a)2}\right)}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}\chi(\kappa_{b1}-\kappa_{a2})\right\}_{\substack{1\leq a\leq k_2\\ 1\leq b\leq k_1}}\\ \left\{r_{\omega_1(b)1}^{a-1}\exp\left(-\imath\kappa_{b1}r_{\omega_1(b)1}\right)\right\}_{\substack{1\leq a\leq k_1-k_2\\ 1\leq b\leq k_1}}\end{array}\right]\\ &= \frac{(-1)^{k_2(k_2-1)/2+k_1k_2}(\imath\pi)^{(k_2-k_1)^2/2-(k_1+k_2)/2}}{2^{k_1k_2}k_1!k_2!}\prod_{\substack{1\leq a\leq k_1\\ 1\leq a\leq k_2}}\chi(\kappa_{a1}-\kappa_{b2})\\ &\times\frac{\det\left[\exp(-\imath\kappa_{a1}r_{b1})\right]_{1\leq a,b\leq k_1}\det\left[\exp(\imath e^{\imath\psi}\kappa_{a2}r_{b2})\right]_{1\leq a,b\leq k_2}}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(r)}}\,.\end{aligned} \tag{5.9}$$

The asymmetry, $\widehat{\varphi}_{k_1/k_2}(r,\kappa)\neq\widehat{\varphi}_{k_1/k_2}(\kappa,r)$, is mainly due to the diagonalization of $\rho$ to $r$ whereas the supermatrix $\kappa$ is already diagonal. The characteristic function $\chi$ in Eqs. (5.7) and (5.9) is crucial because of the commutator

$$\begin{aligned}&\left[\frac{\partial}{\partial r_{\omega_1(b)1}}+e^{-\imath\psi}\frac{\partial}{\partial r_{\omega_2(a)2}},\exp\left(-\imath\kappa_{b1}r_{\omega_1(b)1}+\imath e^{\imath\psi}\kappa_{a2}r_{\omega_2(a)2}\right)\right]_-\\ =\ &-\imath(\kappa_{b1}-\kappa_{a2})\exp\left(-\imath\kappa_{b1}r_{\omega_1(b)1}+\imath e^{\imath\psi}\kappa_{a2}r_{\omega_2(a)2}\right)\chi(\kappa_{b1}-\kappa_{a2})\,.\end{aligned} \tag{5.10}$$

Indeed the set which is cutted out by $\chi$ is a set of measure zero and does not play any role when one integrates $\varphi_{k_1/k_2}(-\imath r,\kappa)$ or $\widehat{\varphi}_{k_1/k_2}(-\imath r,\kappa)$ over $\kappa$ with a conveniently smooth function. However it becomes important for the integral

$$\int\limits_{\mathbb{R}^{k_1+k_2}} \varphi_{k_1/k_2}(\imath s,r)\varphi_{k_1/k_2}(-\imath r,\widetilde{\kappa})\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)d[r] \tag{5.11}$$
$$= \frac{\pi^{(k_2-k_1)^2}}{2^{2k_1k_2-k_1-k_2}k_1!k_2!}\prod_{\substack{1\leq a\leq k_1\\ 1\leq a\leq k_2}}\chi(\kappa_{a1}-e^{-\imath\psi}\kappa_{b2})\frac{\det\left[\delta(\kappa_{a1}-s_{b1})\right]_{1\leq a,b\leq k_1}\det\left[e^{\imath\psi}\delta(\kappa_{a2}-s_{b2})\right]_{1\leq a,b\leq k_2}}{\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\widetilde{\kappa})}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(s)}}\,.$$

where $s=\mathrm{diag}\,(s_{11},\ldots,s_{k_11},e^{-\imath\psi}s_{12},\ldots,e^{-\imath\psi}s_{k_22})$ and $\widetilde{\kappa}=\mathrm{diag}\,(\kappa_{11},\ldots,\kappa_{k_11},e^{-\imath\psi}\kappa_{12},\ldots,e^{-\imath\psi}\kappa_{k_22})$ with $s_{ab},\kappa_{ab}\in\mathbb{R}$. This result is the correct one for the supermatrix Bessel function. The difference to other results[15,25,32] is the distribution $\chi$ which guarantees that the Dirac-distribution (5.11) in the eigenvalues $s$ of a Hermitian supermatrix $\sigma$ vanishes if a bosonic eigenvalue of $\widetilde{\kappa}$ equals with a fermionic one. The reasoning becomes clear when we interpret Eq. (5.11) as an integral over the supergroup $\mathrm{U}\,(k_1/k_2)$, i.e.

$$\int\limits_{\mathbb{R}^{k_1+k_2}} \varphi_{k_1/k_2}(\imath s,r)\varphi_{k_1/k_2}(-\imath r,\widetilde{\kappa})\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)d[r]\sim\int\limits_{\mathrm{U}\,(k_1/k_2)}\delta(UsU^\dagger-\widetilde{\kappa})d\mu(U)\,. \tag{5.12}$$

The measure $d\mu(U)$ is the Haar measure on $\mathrm{U}\,(k_1/k_2)$ and the Dirac-distribution is defined by two Fourier transformations

$$\delta(UsU^\dagger-\widetilde{\kappa})\sim\int\exp\left[\imath\mathrm{Str}\,\rho(UsU^\dagger-\widetilde{\kappa})\right]d[\rho]\,. \tag{5.13}$$

The Haar measure $d\mu$ of the supergroup $\mathrm{U}\,(k_1/k_2)$ can not be normalized as it can be done for the ordinary unitary groups since the volume of $\mathrm{U}\,(k_1/k_2)$ is zero for $k_1k_2\neq 0$. This is also the reason why Eq. (5.11) has to vanish if one bosonic eigenvalue of $\widetilde{\kappa}$ equals to a fermionic one. Then the integral (5.12) is rotation invariant under the subgroup $\mathrm{U}\,(1/1)$ which has zero volume, too. This cannot be achieved without the distribution $\chi$ as it was done in the common literature [15,25,32]. Interestingly the replacement of $\varphi_{k_1/k_2}$ by $\widehat{\varphi}_{k_1/k_2}$ in Eq. (5.11) yields the Dirac-distribution

$$\int\limits_{\mathbb{R}^{k_1+k_2}} \widehat{\varphi}_{k_1/k_2}(\imath s,r)\widehat{\varphi}_{k_1/k_2}(-\imath r,\widetilde{\kappa})\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)d[r] \tag{5.14}$$
$$= \frac{\pi^{(k_2-k_1)^2}}{2^{2k_1k_2-k_1-k_2}k_1!k_2!}\frac{\det\left[\delta(\kappa_{a1}-s_{b1})\right]_{1\leq a,b\leq k_1}\det\left[e^{\imath\psi}\delta(\kappa_{a2}-s_{b2})\right]_{1\leq a,b\leq k_2}}{\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\widetilde{\kappa})}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(s)}}$$

which is similar to the one in Eq. (5.11). Equation (5.14) is derived in Appendix E.

We want to finish this section by a remark about the relation of the result for the supermatrix Bessel function, see Eq. (5.7), and the differential operator derived by the author in an earlier work [15]. This differential operator is defined by

$$D_r^{(k_1k_2)}F(r)=\int F(\rho)d[\eta] \tag{5.15}$$

for an arbitrary sufficiently integrable superfunction $F$ on the $(k_1+k_2)\times(k_1+k_2)$ supermatrices invariant under $\mathrm{U}\,(k_1/k_2)$. It has the form

$$D_r^{(k_1k_2)} = \frac{1}{(k_1k_2)!(4\pi)^{k_1k_2}}\frac{1}{\Delta_{k_1}(r_1)\Delta_{k_1}(e^{\imath\psi}r_2)}$$
$$\times\sum_{n=0}^{k_1k_2}\binom{k_1k_2}{n}\left(\mathrm{Str}\,\frac{\partial^2}{\partial r^2}\right)^{k_1k_2-n}\prod_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}}(r_{a1}-e^{\imath\psi}r_{b2})\left(-\mathrm{Str}\,\frac{\partial^2}{\partial r^2}\right)^n\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)}\,, \tag{5.16}$$

where we define

$$\mathrm{Str}\,\frac{\partial^2}{\partial r^2}=\sum_{a=1}^{k_1}\frac{\partial^2}{\partial r_{a1}^2}-e^{-2\imath\psi}\sum_{b=1}^{k_2}\frac{\partial^2}{\partial r_{b2}^2}\,. \tag{5.17}$$

Due to Eq. (5.15) the differential operator $D_r^{(k_1k_2)}$ is equivalent to the integration over all Grassmann variables of the supermatrix $\rho$.

The comparison of Eqs. (5.15) and (5.16) with Eqs. (5.1) and (5.7) for an arbitrary sufficiently integrable, rotation invariant superfunction $F$ and arbitrary diagonal supermatrix $\kappa$ yields

$$D_r^{(k_1k_2)} = \frac{1}{(2\pi)^{k_1k_2}}\frac{1}{\Delta_{k_1}(r_1)\Delta_{k_1}(e^{\imath\psi}r_2)}\prod_{\substack{1\leq a\leq k_1\\1\leq b\leq k_2}}\left(\frac{\partial}{\partial r_{a1}}+e^{-\imath\psi}\frac{\partial}{\partial r_{b2}}\right)\sqrt{\mathrm{Ber}\,_{k_1/k_2}^{(2)}(r)}\,. \tag{5.18}$$

Thus, we have found a quite compact form for $D_r^{(k_1k_2)}$ which is easier to deal with as the one in Eq. (5.16).

## VI. SOME APPLICATIONS FOR HERMITIAN MATRIX ENSEMBLES

In random matrix theory generating functions as

$$Z_k^{(N)}(\kappa,\alpha H_0) = \int\limits_{\mathrm{Herm}\,(N)} P^{(N)}(H)\prod_{j=1}^{k}\frac{\det(H+\alpha H_0-\kappa_{j2}\mathbf{1}_N)}{\det(H+\alpha H_0-\kappa_{j1}\mathbf{1}_N)}d[H]\,. \tag{6.1}$$

are paramount important since they model Hermitian random matrices in an external potential[24,33] or intermediate random matrix ensembles[34–39]. The matrix $H_0$ is a $N\times N$ Hermitian matrix and can be an arbitrary matrix or can also be drawn from another random matrix ensemble. The external parameter $\alpha\in\mathbb{R}$ is the coupling constant between the two matrices $H$ and $H_0$ and yields the generating function (2.1) for $\alpha=0$, i.e. $Z_k^{(N)}(\kappa,0)=Z_{k/k}^{(N)}(\kappa)$. The variables $\kappa_{j1}$ have to have an imaginary increment to guarantee the convergence of the integral, i.e. $\kappa_{j1}=x_{j1}-J_j-\imath\varepsilon$.

In subsection VI A we consider the generating function (6.1) with $\lambda=0$. We will use the mapping of this integral to a representation in superspace which is shown in a previous work by the authors[8] and diagonalize the supermatrix. In a formalism similar to the case $\alpha=0$ we will treat the more general case $\alpha\neq0$ in subsection VI B.

### A. Hermitian matrix ensembles without an external source ($\alpha=0$)

Omitting the index $N$ in $P^{(N)}$ we consider a normalized rotation invariant probability density $P$ with respect to $N\times N$ Hermitian random matrices.

Following the derivations made in Refs. [1,12] we have to calculate the characteristic function $\mathcal{F}P$, see Eq. (2.4). Assuming that this can be done we recall the rotation invariance of $\mathcal{F}P$. This allows us to choose a representation of $\mathcal{F}P$ as a function in a finite number of matrix invariants, i.e. $\mathcal{F}P(\widetilde{H})=\mathcal{F}P_1(\mathrm{tr}\,\widetilde{H},\ldots,\mathrm{tr}\,\widetilde{H}^N)$. A straightforward supersymmetric extension $\Phi$ of the characteristic function is the one in Eq. (4.3) and its Fourier transform is

$$\mathcal{F}\Phi(\sigma) = 2^{2k(k-1)}\int\limits_{\widetilde{\Sigma}_{k/k}^{(\psi)}}\Phi(\rho)\exp(-\imath\mathrm{Str}\,\rho\sigma)d[\rho]\,. \tag{6.2}$$

The Fourier-transform is denoted by $Q$ in Ref. [1]. The supermatrices $\rho$ and $\sigma$ are Wick-rotated with the phases $e^{\imath\psi}$ and $e^{-\imath\psi}$, respectively.

The supersymmetric integral for $Z_k^{(N)}(\kappa,\alpha H_0)$ is[1,12]

$$Z_k^{(N)}(\kappa,\alpha H_0) = \int\limits_{\widetilde{\Sigma}_{k/k}^{(-\psi)}}\mathcal{F}\Phi(\sigma)\mathrm{Sdet}^{\,-1}(\sigma\otimes\mathbf{1}_N+\alpha\mathbf{1}_{k+k}\otimes H_0-\kappa)d[\sigma]\,. \tag{6.3}$$

Setting the coupling constant $\alpha$ to zero we also find

$$Z_k^{(N)}(\kappa) = 2^{2k(k-1)}\int\limits_{\widetilde{\Sigma}_{k/k}^{(\psi)}}\int\limits_{\widetilde{\Sigma}_{k/k}^{(-\psi)}}\mathcal{F}\Phi(\sigma)\exp[\imath\mathrm{Str}\,\rho(\sigma-\kappa)]I_N(\rho)d[\sigma]d[\rho]\,, \tag{6.4}$$

where the supersymmetric Ingham-Siegel integral is[1]

$$
\begin{aligned}
I_k^{(N)}(\rho) &= \int\limits_{\widetilde{\Sigma}_{k/k}^{(-\psi)}} \mathrm{Sdet}^{-N}(\sigma + \imath\varepsilon \mathbf{1}_{k+k}) \exp(-\imath \mathrm{Str}\, \rho\sigma + \varepsilon \mathrm{Str}\, \rho) d[\sigma] \\
&= (-1)^{k(k+1)/2} 2^{-k(k-1)} \prod_{j=1}^{k} \left[ \frac{2\pi}{(N-1)!} r_{j1}^N \Theta(r_{j1}) \left( -e^{-\imath\psi} \frac{\partial}{\partial r_{j2}} \right)^{N-1} \delta(e^{\imath\psi} r_{j2}) \right], \qquad (6.5)
\end{aligned}
$$

The distribution $\Theta$ is the Heavyside distribution and $r_{j1}$ and $e^{\imath\psi} r_{j2}$ are the bosonic and fermionic eigenvalues of $\rho$, respectively. In App. F we perform the integration (6.5) with help of the results in Sec. V and find

$$
\begin{aligned}
Z_k^{(N)}(\kappa) &= \frac{\imath^k}{2^{k(k-1)}} \int\limits_{\mathbb{R}^{2k}} \frac{d[s]}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \mathcal{F}\Phi(s) \qquad (6.6) \\
&\times \det\left[ \frac{\delta(s_{b1})\delta(e^{-\imath\psi} s_{a2})}{\kappa_{b1} - \kappa_{a2}} \left( \frac{\kappa_{a2}}{\kappa_{a1}} \right)^N + \frac{N \left( e^{-\imath\psi} s_{a2} - \kappa_{a2} \right)^{N-1}}{2\pi\imath \left( e^{-\imath\psi} s_{b1} - \kappa_{b1} \right)^{N+1} (s_{b1} - e^{-\imath\psi} s_{a2})} \right]_{1 \leq a,b \leq k}.
\end{aligned}
$$

The first term in the determinant is the Efetov-Wegner term whereas the second term can be understood as integrals over supergoups.

### B. Hermitian matrix ensembles in the presence of an external source ($\alpha \neq 0$)

For Hermitian matrix ensembles in an external field it is convenient to consider the integral representation

$$
\begin{aligned}
Z_k^{(N)}(\kappa, \alpha H_0) &= 2^{2k(k-1)} \int\limits_{\widetilde{\Sigma}_{k/k}^{(\psi)}} \Phi(\rho) \exp[-\imath \mathrm{Str}\, \kappa\rho] \\
&\times \left[ \int\limits_{\widetilde{\Sigma}_{k/k}^{(-\psi)}} \exp(-\imath \mathrm{Str}\, \rho\sigma + \varepsilon \mathrm{Str}\, \rho) \mathrm{Sdet}^{-1}(\sigma \otimes \mathbf{1}_N + \alpha \mathbf{1}_{k+k} \otimes H_0 + \imath\varepsilon \mathbf{1}_{N(k+k)}) d[\sigma] \right] d[\rho] \qquad (6.7)
\end{aligned}
$$

for the generating function, see Eqs. (6.1) and (6.3). In App. G we integrate this representation in two steps and get

$$
Z_k^{(N)}(\kappa, \alpha H_0) = \frac{(-1)^{k(k-1)/2}}{\Delta_N(\alpha E_0) \sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \int\limits_{\mathbb{R}^{2k}} \det \begin{bmatrix} \{B_1(r_{b1}, r_{a2}, \kappa_{b1}, \kappa_{a2})\}_{1 \leq a,b \leq k} & \{B_{b2}(r_{a2}, \kappa_{a2})\}_{\substack{1 \leq a \leq k \\ 1 \leq b \leq N}} \\ \left\{ B_3(r_{b1}, \kappa_{b1}, \alpha E_a^{(0)}) \right\}_{\substack{1 \leq a \leq N \\ 1 \leq b \leq k}} & \left\{ (-\alpha E_a^{(0)})^{b-1} \right\}_{1 \leq a,b \leq N} \end{bmatrix} \Phi(r) d[r], \qquad (6.8)
$$

where

$$
B_1(r_{b1}, r_{a2}, \kappa_{b1}, \kappa_{a2}) = \frac{\delta(r_{b1})\delta(e^{\imath\psi} r_{a2})}{\kappa_{b1} - \kappa_{a2}} + \imath \frac{I_1^{(N)}(r_{b1}, e^{\imath\psi} r_{a2})}{2\pi(r_{b1} - e^{\imath\psi} r_{a2})} \exp\left( -\imath\kappa_{b1} r_{b1} + \imath e^{\imath\psi} \kappa_{a2} r_{a2} \right) \chi(\kappa_{b1} - \kappa_{a2}), \qquad (6.9)
$$

$$
B_{b2}(r_{a2}, \kappa_{a2}) = \exp\left( \imath e^{\imath\psi} \kappa_{a2} r_{a2} \right) \left( -\imath e^{-\imath\psi} \frac{\partial}{\partial r_{a2}} \right)^{b-1} \delta(e^{\imath\psi} r_{a2}) \prod_{j=1}^{k} \chi(\kappa_{j1} - \kappa_{a2}), \qquad (6.10)
$$

$$
B_3(r_{b1}, \kappa_{b1}, \alpha E_a^{(0)}) = -\imath \exp\left( -\imath\kappa_{b1} r_{b1} \right) \Theta(r_{b1}) \sum_{n=N}^{\infty} \frac{(\imath\alpha E_a^{(0)} r_{b1})^n}{n!} \prod_{j=1}^{k} \chi(\kappa_{b1} - e^{\imath\psi} \kappa_{j2}). \qquad (6.11)
$$

The case $\alpha = 0$ can be easily deduced from the result (6.8),

$$
Z_k^{(N)}(\kappa) = \frac{(-1)^{k(k-1)/2}}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \int\limits_{\mathbb{R}^{2k}} \det\left[ \frac{\delta(r_{b1})\delta(e^{\imath\psi} r_{a2})}{\kappa_{b1} - \kappa_{a2}} + \imath \frac{I_1^{(N)}(r_{b1}, e^{\imath\psi} r_{a2})}{2\pi(r_{b1} - e^{\imath\psi} r_{a2})} \chi(\kappa_{b1} - \kappa_{a2}) \right]_{1 \leq a,b \leq k} \exp\left( -\imath \mathrm{Str}\, \kappa r \right) \Phi(r) d[r]. \qquad (6.12)
$$

Again we are able to distinguish the Efetov-Wegner terms from those terms corresponding to supergroup integrals. In the determinant of Eq. (6.12) and in the left upper block of Eq. (6.8), see Eq. (6.9), the Dirac-distributions are the contribution from the Efetov-Wegner terms. When we expand the determinants in the Dirac-distribution we obtain the leading terms of $Z_k^{(N)}$, $Z_{k-1}^{(N)}$, …, $Z_0^{(N)}$ which are exactly those found in Refs. [1,12]. Thus, we have found an expression which can be understood as a generator for all generating functions $Z_k^{(N)}$.

The external matrix $H_0$ can also be drawn from another random matrix ensemble as it was done in Refs. [34–39]. However, we do not perform the calculation, here, since it is straightforward to those in an application of Ref. [8].

## VII. REMARKS AND CONCLUSIONS

We derived the supermatrix Bessel function with all Efetov-Wegner terms for Hermitian supermatrices of arbitrary dimensions. We arrived at an expression from which one can easily deduce what the Efetov-Wegner terms are and which terms result from supergroup integrals. With this result we showed that the completeness and orthogonality relation for the supermatrix Bessel function without Efetov-Wegner terms slightly differs from the formerly assumed one[35]. It is has to be zero on a set of measure zero and, thus, does not matter for smooth integrands but plays an important role if the integrand has singularities on this set.

We applied the supermatrix Bessel function with all Efetov-Wegner terms to arbitrary, rotation invariant Hermitian random matrix ensembles with and without an external field. The already known leading terms[1,8,12,24,33–39] were obtained plus all Efetov-Wegner terms. The correction terms were unknown before and yield new insights in the supersymmetric representation of the generating functions. For example the Efetov-Wegner terms become important for the matrix Green functions.

We also found an integral identity for the generating functions whose integrand can be easily expanded in the Efetov-Wegner terms. In such an expansion one obtains correlation functions related to $k$-point correlation functions which are of lower order than those corresponding to the originally considered generating function. Thus, it reflects the relation of Mehta's definition[7] for the $k$-point correlation function and the one commonly used in the supersymmetry method[1] which was explained in Ref. [40].

We expect that similar results may also be derived for other supermatrices, e.g. diagonalization of complex supermatrices[41]. Nevertheless we guess that the knowledge about the supergroup integrals as well as about the ordinary group integrals is crucial. We could only obtain these compact results due to this knowledge.

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### Appendix A: Proof of theorem III.1

We plug the characteristic function (2.4) in Eq. (2.1) and diagonalize $\widetilde{H}$. This yields

$$Z_{k_1/k_2}^{(N)}(\kappa) = \frac{V_N}{2^N \pi^{N^2}} \int\limits_{\mathrm{Herm}\,(N)} \int\limits_{\mathbb{R}^N} \exp\left[-\imath \mathrm{tr}\, H\widetilde{E}\right] \frac{\prod\limits_{j=1}^{k_2} \det(H - \kappa_{j2} \mathbb{1}_N)}{\prod\limits_{j=1}^{k_1} \det(H - \kappa_{j1} \mathbb{1}_N)} \prod_{j=1}^{N} f(\widetilde{E}_j) \Delta_N^2(\widetilde{E}) d[\widetilde{E}] d[H] \,, \tag{A1}$$

where the constant is

$$V_N = \frac{1}{N!} \prod_{j=1}^{N} \frac{\pi^{j-1}}{(j-1)!} \,. \tag{A2}$$

The diagonalization of $H$ gives the matrix Bessel function[42,43] according to the unitary group $\mathrm{U}(N)$,

$$\begin{aligned}\varphi_N(E,\widetilde{E}) &= \int\limits_{\mathrm{U}(N)} \exp\left[-\imath \mathrm{tr}\, EU\widetilde{E}U^\dagger\right] d\mu(U) \\ &= \prod_{j=1}^{N} \imath^{j-1}(j-1)! \frac{\det\left[\exp(-\imath E_a\widetilde{E}_b)\right]_{1\leq a,b\leq N}}{\Delta_N(\widetilde{E})\Delta_N(E)} \end{aligned} \tag{A3}$$

The measure $d\mu$ is the normalized Haar-measure. Thus, we find

$$Z_{k_1/k_2}^{(N)}(\kappa) = \frac{\imath^{N(N-1)/2}}{(2\pi)^N N! \prod\limits_{j=0}^{N} j!} \int\limits_{\mathbb{R}^{2N}} \det\left[\exp(-\imath E_a\widetilde{E}_b)\right]_{1\leq a,b\leq N} \tag{A4}$$

$$\times \prod_{a=1}^{N} f(\widetilde{E}_a) \frac{\prod\limits_{b=1}^{k_2}(E_a-\kappa_{b2})}{\prod\limits_{b=1}^{k_1}(E_a-\kappa_{b1})} \Delta_N(\widetilde{E})\Delta_N(E) d[\widetilde{E}]d[E]\,. \tag{A5}$$

Here, one can easily check that the normalization is $Z_{k_1/k_2}^{(N)}(0) = f^N(0)$. Since determinants are skew-symmetric, we first expand the Vandermonde determinant $\Delta_N(\widetilde{E})$ and then the determinant of the exponential functions. We have

$$Z_{k_1/k_2}^{(N)}(\kappa) = \frac{(-\imath)^{N(N-1)/2}}{(2\pi)^N \prod\limits_{j=1}^{N}(j-1)!} \int\limits_{\mathbb{R}^{2N}} \prod_{a=1}^{N} \left[ f(\widetilde{E}_a)\exp(-\imath E_a\widetilde{E}_a)\widetilde{E}_a^{a-1} \frac{\prod\limits_{b=1}^{k_2}(E_a-\kappa_{b2})}{\prod\limits_{b=1}^{k_1}(E_a-\kappa_{b1})}\right] \Delta_N(E) d[\widetilde{E}]d[E]\,. \tag{A6}$$

Following the ideas in Ref. [8], we extend the integrand by a square root Berezinian and find with help of (2.8) the determinant

$$Z_{k_1/k_2}^{(N)}(\kappa) = \frac{(-1)^{k_2(k_2-1)/2+(k_2+1)k_1}\imath^{N(N-1)/2}}{(2\pi)^N \prod\limits_{j=1}^{N}(j-1)!} \frac{1}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}} \tag{A7}$$

$$\times \det\begin{bmatrix} \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}} & \left\{\displaystyle\int\limits_{\mathbb{R}^2} \dfrac{f(E_2)E_2^{b-1}}{\kappa_{a1}-E_1}\exp[-\imath E_2E_1]d[E]\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq N}} \\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq k_2}} & \left\{\displaystyle\int\limits_{\mathbb{R}^2} f(E_2)E_2^{b-1}E_1^{a-1}\exp[-\imath E_2E_1]d[E]\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq N}} \end{bmatrix}.$$

We define the sign of the imaginary parts of $\kappa_{j1}$ by

$$-L_j = \frac{\mathrm{Im}\,\kappa_{j1}}{|\mathrm{Im}\,\kappa_{j1}|}\,. \tag{A8}$$

Integrating over $E_1$, Eq. (A7) reads

$$Z_{k_1/k_2}^{(N)}(\kappa) = \frac{(-1)^{k_2(k_2-1)/2+(k_2+1)k_1}\imath^{N(N-1)/2}}{\prod\limits_{j=1}^{N}(j-1)!} \frac{1}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}} \tag{A9}$$

$$\times \det\begin{bmatrix} \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}} & \left\{\imath L_a\displaystyle\int\limits_{\mathbb{R}} f(E)E^{b-1}\exp[-\imath\kappa_{a1}E]\Theta(L_aE)dE\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq N}} \\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq k_2}} & \left\{\displaystyle\int\limits_{\mathbb{R}} f(E)E^{b-1}\left(\imath\dfrac{\partial}{\partial E}\right)^{a-1}\delta(E)dE\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq N}} \end{bmatrix}.$$

In the lower right block we use the following property of the integral

$$\int\limits_{\mathbb{R}} f(E)E^{b-1}\left(\imath\frac{\partial}{\partial E}\right)^{a-1}\delta(E)dE=0\quad\text{for } b>a\,. \tag{A10}$$

Since $d=N+k_2-k_1\leq N$, cf. Eq. (3.2), the last $N-d$ columns in the lower right block in the determinant (A9) is zero. The matrix

$$\mathbf{M}=\left[\int\limits_{\mathbb{R}} f(E)E^{b-1}\left(\imath\frac{\partial}{\partial E}\right)^{a-1}\delta(E)dE\right]_{1\leq a,b\leq d} \tag{A11}$$

is a lower triangular matrix with diagonal elements

$$M_{jj}=\int\limits_{\mathbb{R}} f(E)E^{j-1}\left(\imath\frac{\partial}{\partial E}\right)^{j-1}\delta(E)dE=(-\imath)^{j-1}(j-1)!\,. \tag{A12}$$

Thus, the determinant of this matrix is

$$\det\mathbf{M}=(-\imath)^{N(N-1)/2}\prod_{j=1}^{N}(j-1)!\,. \tag{A13}$$

We pull the Matrix $\mathbf{M}$ out the determinant (A9) and find

$$Z_{k_1/k_2}^{(N)}(\kappa)=\frac{(-1)^{k_2(k_2-1)/2+(k_2+1)k_1}}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}} \tag{A14}$$

$$\times\ \det\left[\left\{K^{(d)}(\kappa_{a1},\kappa_{b2})\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}}\ \left\{\imath L_a\int\limits_{\mathbb{R}} f(E)E^{b-1}\exp[-\imath\kappa_{a1}E]\Theta(L_aE)dE\right\}_{\substack{1\leq a\leq k_1\\ d+1\leq b\leq N}}\right],$$

where

$$K^{(d)}(\kappa_{a1},\kappa_{b2})=\frac{1}{\kappa_{a1}-\kappa_{b2}}-\imath L_a\sum_{m,n=1}^{d}\int\limits_{\mathbb{R}} f(E)E^{m-1}\exp[-\imath\kappa_{a1}E]\Theta(L_aE)dEM_{mn}^{-1}\kappa_{b2}^{n-1}\,. \tag{A15}$$

Again we use the fact that the determinant is skew-symmetric which allows also to write

$$Z_{k_1/k_2}^{(N)}(\kappa)=\frac{(-1)^{k_2(k_2-1)/2+(k_2+1)k_1}}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}} \tag{A16}$$

$$\times\ \det\left[\left\{K^{(\widetilde{N})}(\kappa_{a1},\kappa_{b2})\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}}\ \left\{\imath L_a\int\limits_{\mathbb{R}} f(E)E^{b-1}\exp[-\imath\kappa_{a1}E]\Theta(L_aE)dE\right\}_{\substack{1\leq a\leq k_1\\ d+1\leq b\leq N}}\right],$$

for an arbitrary $\widetilde{N}\in\{d,d+1,\ldots,N\}$. For the cases $(k_1/k_2)=(1/1)$ and $(k_1/k_2)=(1/0)$, we identify

$$Z_{1/1}^{(N)}(\kappa_{a1},\kappa_{b2}) = (\kappa_{a1}-\kappa_{b2})K^{(N)}(\kappa_{a1},\kappa_{b2})\,, \tag{A17}$$

$$Z_{1/0}^{(N)}(\kappa_{a1}) = -\imath L_a\int\limits_{\mathbb{R}} f(E)E^{N-1}\exp[-\imath\kappa_{a1}E]\Theta(L_aE)dE\,. \tag{A18}$$

This yields the theorem.

### Appendix B: Proof of lemma IV.1

The proof of this Lemma is similar to the derivation of the supermatrix Bessel function with the Efetov–Wegner term in Sec. V.A of Ref. [15]. We consider the integral

$$\frac{Z_{1/1}^{(N)}(\kappa)}{f^N(0)} = \frac{(-1)^N 2\pi}{(N-1)!} \int\limits_{\Sigma_{1/1}^{(\psi)}} \Phi^{(1/1)}(\hat{\rho}) \exp[-\imath \mathrm{Str}\, \kappa\hat{\rho}] r_1^N \left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1} \delta\left(e^{\imath\psi} r_2\right) d[\rho]\,. \tag{B1}$$

As in Ref. [15], we exchange the integration over the Grassmann variables by a differential Operator which yields

$$\begin{aligned}\frac{Z_{1/1}^{(N)}(\kappa)}{f^N(0)} &= \frac{(-1)^N}{(N-1)!} \int\limits_{\mathbb{R}_+\times\mathbb{R}} r_1^N \left[\left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1} \delta(r_2)\right] \left[\imath \frac{\kappa_1-\kappa_2}{r_1 - e^{\imath\psi} r_2}\right.\\ &+ \left.\frac{1}{r_1 - e^{\imath\psi}r_2}\left(\frac{\partial}{\partial r_1} + e^{-\imath\psi}\frac{\partial}{\partial r_2}\right) - \frac{e^{-\imath\psi}}{r_1}\frac{\partial}{\partial r_2}\right] \left[\frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)} \exp[-\imath \mathrm{Str}\, \kappa r]\right] dr_1 dr_2\,.\end{aligned} \tag{B2}$$

The term

$$Z_1 = \frac{\imath(-1)^N}{(N-1)!} \int\limits_{\mathbb{R}_+\times\mathbb{R}} \frac{\kappa_1-\kappa_2}{r_1 - e^{\imath\psi}r_2} \frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)} \exp[-\imath \mathrm{Str}\, \kappa r] r_1^N \left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1} \delta(r_2)\, dr_1 dr_2 \tag{B3}$$

contains the supermatrix Bessel function with respect to $\mathrm{U}(1/1)$[15,25,32]. The second term

$$\begin{aligned}Z_2 &= \frac{(-1)^N}{(N-1)!} \int\limits_{\mathbb{R}_+\times\mathbb{R}} r_1^N \left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1} \delta(r_2)\\ &\times \left[\frac{1}{r_1 - e^{\imath\psi}r_2}\left(\frac{\partial}{\partial r_1} + e^{-\imath\psi}\frac{\partial}{\partial r_2}\right) - \frac{e^{-\imath\psi}}{r_1}\frac{\partial}{\partial r_2}\right] \left[\frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)} \exp[-\imath \mathrm{Str}\, \kappa r]\right] dr_1 dr_2\,.\end{aligned} \tag{B4}$$

has to yield the Efetov–Wegner term. By partial integration, we evaluate the Dirac distribution and omit the generalized Wick–rotation. Thus, Eq. (B2) becomes

$$\begin{aligned}Z_2 &= \frac{-1}{(N-1)!} \int\limits_{\mathbb{R}_+} \left[\sum_{j=0}^{N-1} \frac{(N-1)!}{j!} r_1^j \left(\frac{\partial^{j+1}}{\partial r_1 \partial r_2^j} + \frac{\partial^{j+1}}{\partial r_2^{j+1}}\right) - r_1^{N-1}\frac{\partial^N}{\partial r_2^N}\right]\\ &\times \left.\left[\frac{f(r_1)}{f(r_2)} \exp[-\imath \mathrm{Str}\, \kappa r]\right]\right|_{r_2=0} dr_1\,.\end{aligned} \tag{B5}$$

For all terms up to $j = 0$ we perform a partial integration in $r_1$ and find a telescope sum. Hence, we have

$$Z_2 = -\int\limits_{\mathbb{R}_+} \frac{\partial}{\partial r_1} \left.\left[\frac{f(r_1)}{f(r_2)} \exp[-\imath \mathrm{Str}\, \kappa r]\right]\right|_{r_2=0} dr_1 = 1\,. \tag{B6}$$

This is indeed the Efetov–Wegner term.

The second equality (4.15) follows from

$$\int\limits_{\mathbb{R}_+\times\mathbb{R}} \frac{1}{r_1-e^{\imath\psi}r_2}\left(\frac{\partial}{\partial r_1}+e^{\imath\psi}\frac{\partial}{\partial r_2}\right)\left(\frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)}\exp[-\imath\mathrm{Str}\,\kappa r]r_1^N\left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1}\delta(r_2)\right)dr_1dr_2$$
$$= \int\limits_{\mathbb{R}_+\times\mathbb{R}} r_1^N\left[\left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1}\delta(r_2)\right]\frac{1}{r_1-e^{\imath\psi}r_2}\left(\frac{\partial}{\partial r_1}+e^{\imath\psi}\frac{\partial}{\partial r_2}\right)\left(\frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)}\exp[-\imath\mathrm{Str}\,\kappa r]\right)dr_1dr_2$$
$$+ \int\limits_{\mathbb{R}_+\times\mathbb{R}} \frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)}\exp[-\imath\mathrm{Str}\,\kappa r]\left[\frac{Nr_1^{N-1}}{r_1-e^{\imath\psi}r_2}\left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1}+\frac{r_1^N}{r_1-e^{\imath\psi}r_2}\left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N}\right]\delta(r_2)dr_1dr_2$$
$$= \int\limits_{\mathbb{R}_+\times\mathbb{R}} r_1^N\left[\left(e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)^{N-1}\delta(r_2)\right]\frac{1}{r_1-e^{\imath\psi}r_2}\left(\frac{\partial}{\partial r_1}+e^{\imath\psi}\frac{\partial}{\partial r_2}\right)\left(\frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)}\exp[-\imath\mathrm{Str}\,\kappa r]\right)dr_1dr_2$$
$$+ (-1)^{N-1}\int\limits_{\mathbb{R}_+}\left[\sum_{j=0}^{N-1}\frac{N!}{j!}r_1^{j-1}\frac{\partial^j}{\partial r_2^j}-\sum_{j=0}^{N}\frac{N!}{j!}r_1^{j-1}\frac{\partial^j}{\partial r_2^j}\right]\left(\frac{f(r_1)}{f\left(e^{\imath\psi}r_2\right)}\exp[-\imath\mathrm{Str}\,\kappa r]\right)\Bigg|_{r_2=0}dr_1\,. \tag{B7}$$

Both sum cancel each other up to the term $j=N$ which is the term $r_1^{N-1}\partial^N/\partial r_2^N$ in Eq. (B5).

## Appendix C: Proof of lemma V.1

Let the characteristic function and, hence, the superfunction $\Phi^{(k_1/k_2)}$ be factorizable, cf. Eq. (4.13). To prove lemma V.1 we plug Eqs. (4.5) and (4.15) into the result (3.3) for $\widetilde{N}=d$. We find

$$\int\limits_{\mathbb{R}_+^{k_1}\times\mathbb{R}^{k_2}} \mathrm{Ber}_{k_1/k_2}^{(2)}(r)\widehat{\varphi}_{k_1/k_2}(r,\kappa)\left[\frac{\prod\limits_{j=1}^{k_1}f(r_{j1})}{\prod\limits_{j=1}^{k_2}f(e^{\imath\psi}r_{j2})}\det{}^d r_1\prod_{j=1}^{k_2}\left(e^{-\imath\psi}\frac{\partial}{\partial r_{j2}}\right)^{d-1}\delta(r_{j2})d[r]\right]$$
$$= \frac{(-1)^{(k_1+k_2)(k_1+k_2-1)/2}(\imath\pi)^{(k_2-k_1)^2/2-(k_1+k_2)/2}}{2^{k_1k_2}\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}}$$
$$\times\ \det\left[\begin{array}{c}\left\{\int\limits_{\mathbb{R}_+\times\mathbb{R}}\frac{\exp(-\imath\kappa_{b1}r_1+\imath e^{\imath\psi}\kappa_{b2}r_2)}{(\kappa_{b1}-\kappa_{a2})(r_1-e^{\imath\psi}r_2)}\left(\frac{\partial}{\partial r_1}+e^{-\imath\psi}\frac{\partial}{\partial r_2}\right)\left[\frac{f(r_1)}{f(e^{\imath\psi}r_2)}r_1^d\left(e^{-\imath\psi}\frac{\partial}{\partial r_{j2}}\right)^{d-1}\delta(r_2)\right]d[r]\right\}_{\substack{1\leq a\leq k_2\\1\leq b\leq k_1}}\\ \left\{\int\limits_{\mathbb{R}_+}f(r_1)r_1^{a-1}e^{-\imath\kappa_{b1}r_1}dr_1\right\}_{\substack{d+1\leq a\leq N\\1\leq b\leq k_1}}\end{array}\right]. \tag{C1}$$

The next step is to pull all factors of $f$, the monomials $r_1^d$ and the distribution $\left(e^{-\imath\psi}\partial/\partial r_{j2}\right)^{d-1}\delta(r_2)$ out the determinant. This proves the lemma for factorizing superfunctions $F$.

Since all analytic and rotational invariant superfunctions $F(\rho)$ are analytic in the supertraces of $\rho$ these superfunctions are generated by factorizing ones. For example one can consider an arbitrary polynomial in the supertraces times a factorizing and integrable superfunction. With help of Weierstrass´ approximation theorem one can generate an arbitrary sufficiently integrable superfunction which may also be non-factorizable.

## Appendix D: Proof of theorem V.2

We use the result of lemma V.1 as an ansatz in Eq. (5.1). To prove that this ansatz is indeed the result we are looking for we construct a boundary value problem in a weak sense. We consider the left hand side of Eq. (5.1) with the supermatrix

$$\sigma = \begin{bmatrix} \{\sigma_{ab1}\}_{1\leq a,b\leq k_1} & \{\chi_{ba}^*\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}} \\ \{\chi_{ab}\}_{\substack{1\leq a\leq k_2\\ 1\leq b\leq k_1}} & \{\sigma_{ab2}\}_{1\leq a,b\leq k_2} \end{bmatrix} \tag{D1}$$

with non-zero entries everywhere instead of a diagonal supermatrix $\kappa$. Then the action of the differential operator

$$\mathrm{Str}\,\frac{\partial^2}{\partial\sigma^2} = \sum_{a=1}^{k_1}\frac{\partial^2}{\partial\sigma_{aa1}^2} + 2\sum_{1\leq a<b\leq k_1}\frac{\partial^2}{\partial\sigma_{ab1}\partial\sigma_{ab1}^*} - \sum_{a=1}^{k_2}\frac{\partial^2}{\partial\sigma_{aa2}^2} - 2\sum_{1\leq a<b\leq k_2}\frac{\partial^2}{\partial\sigma_{ab2}\partial\sigma_{ab2}^*} + 2\sum_{\substack{1\leq a\leq k_2\\ 1\leq b\leq k_1}}\frac{\partial^2}{\partial\chi_{ab}^*\partial\chi_{ab}} \tag{D2}$$

on the left hand side of Eq. (5.1) yields

$$\mathrm{Str}\,\frac{\partial^2}{\partial\sigma^2}\int\limits_{\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}} F(\rho)\exp[-\imath\mathrm{Str}\,\sigma\rho]d[\rho] = -\int\limits_{\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}} F(\rho)\mathrm{Str}\,\rho^2\exp[-\imath\mathrm{Str}\,\sigma\rho]d[\rho]\,. \tag{D3}$$

Since the integrand is rotation invariant the integral only depends on the eigenvalues of the supermatrix $\sigma$. This leads to a differential equation in the diagonal supermatrix $\kappa$. With the differential operator $\mathrm{Str}\,\partial^2/\partial\kappa^2$ defined similar to Eq. (5.17) we have

$$\frac{1}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}}\mathrm{Str}\,\frac{\partial^2}{\partial\kappa^2}\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}\int\limits_{\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}} F(\rho)\exp[-\imath\mathrm{Str}\,\kappa\rho]d[\rho] = -\int\limits_{\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}} F(\rho)\mathrm{Str}\,\rho^2\exp[-\imath\mathrm{Str}\,\kappa\rho]d[\rho]\,, \tag{D4}$$

cf. Ref. [35].

The boundaries of $\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}$ are given by $\widetilde{\Sigma}_{k_1-1/k_2-1}^{(\psi)}$ canonically embedded in $\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}$ if one bosonic eigenvalue of a supermatrix in $\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}$ equals to a fermionic one, i.e. there is $a\in\{1,\ldots,k_1\}$ and $b\in\{1,\ldots,k_2\}$ with $\kappa_{a1}=\kappa_{b2}$. For these cases we may use the Cauchy-like integral theorems for Hermitian supermatrices[15,16,31]. Without loss of generality we consider the case $\kappa_{k_11}=\kappa_{k_22}$ and have

$$\int\limits_{\widetilde{\Sigma}_{k_1/k_2}^{(\psi)}} F(\rho)\exp[-\imath\mathrm{Str}\,\kappa\rho]d[\rho] = (-1)^{k_1}2^{2-k_1-k_2}\imath\int\limits_{\widetilde{\Sigma}_{k_1-1/k_2-1}^{(\psi)}} F(\rho)\exp\left[-\imath\mathrm{Str}\,\left.\kappa\right|_{\kappa_{k_11}=\kappa_{k_22}=0}\rho\right]d[\rho]\,. \tag{D5}$$

Here we use the same symbol for the restriction of $F$ on $\widetilde{\Sigma}_{k_1-1/k_2-1}^{(\psi)}$.

The boundary condition (D5) for the distribution (5.7) can be readily checked. For the differential equation (D4) we expand the determinant (5.7) in $l\leq k_2$ rows and columns in the upper block. Apart from a constant prefactor each term is given by

$$\begin{aligned} g(\kappa,r) &= \frac{\sqrt{\mathrm{Ber}_{l/l}^{(2)}(\kappa_{\tilde\omega})}\sqrt{\mathrm{Ber}_{k_1-l/k_2-l}^{(2)}(r_\omega)}}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)\mathrm{Ber}_{k_1/k_2}^{(2)}(r)}}\prod_{j=1}^{l}\delta(r_{\omega(j)1})\delta(e^{\imath\psi}r_{\omega(j)1}) \\ &\times\prod_{a=l+1}^{k_1}\exp(-\imath\kappa_{\tilde\omega_1(a)1}r_{\omega_1(a)1})\prod_{a=l+1}^{k_2}\exp(\imath e^{\imath\psi}\kappa_{\tilde\omega_2(b)2}r_{\omega_2(b)2})\,, \end{aligned} \tag{D6}$$

where $\kappa_{\tilde\omega}=\mathrm{diag}\,(\kappa_{\tilde\omega_1(1)1},\ldots,\kappa_{\tilde\omega_1(l)1},\kappa_{\tilde\omega_2(1)2},\ldots,\kappa_{\tilde\omega_2(l)2})$ and $r_\omega=\mathrm{diag}\,(r_{\omega_1(l+1)1},\ldots,r_{\omega_1(k_1)1},r_{\omega_2(l+1)2},\ldots,r_{\omega_2(k_2)1})$ with the permutations $\omega_1,\tilde\omega_1\in\mathfrak{S}(k_1)$ and $\omega_2,\tilde\omega_2\in\mathfrak{S}(k_2)$. The action of the distribution $g(\kappa,r)$ on $\mathrm{Str}\,r^2$ is

$$\begin{aligned} g(\kappa,r)\mathrm{Str}\,r^2 &= \frac{\sqrt{\mathrm{Ber}_{l/l}^{(2)}(\kappa_{\tilde\omega})}\sqrt{\mathrm{Ber}_{k_1-l/k_2-l}^{(2)}(r_\omega)}}{\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)\mathrm{Ber}_{k_1/k_2}^{(2)}(r)}}\prod_{j=1}^{l}\delta(r_{\omega(j)1})\delta(e^{\imath\psi}r_{\omega(j)1}) \\ &\times\prod_{a=l+1}^{k_1}\exp(-\imath\kappa_{\tilde\omega_1(a)1}r_{\omega_1(a)1})\prod_{a=l+1}^{k_2}\exp(\imath e^{\imath\psi}\kappa_{\tilde\omega_2(b)2}r_{\omega_2(b)2})\mathrm{Str}\,r_\omega^2 \end{aligned} \tag{D7}$$

because all other terms are zero due to the Dirac-distributions. The differential operator in Eq. (D4) acts on $g(\kappa, r)$ as

$$\begin{aligned}
&\frac{1}{\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\kappa)}}\mathrm{Str}\,\frac{\partial^2}{\partial\kappa^2}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\kappa)}g(\kappa,r)\\
=\;&\frac{1}{\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\kappa)}}\mathrm{Str}\,\frac{\partial^2}{\partial\kappa^2}\frac{\sqrt{\mathrm{Ber}\,^{(2)}_{l/l}(\kappa_{\tilde{\omega}})}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1-l/k_2-l}(r_\omega)}}{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)}\prod_{j=1}^{l}\delta(r_{\omega(j)1})\delta(e^{\imath\psi}r_{\omega(j)1})\\
\times\;&\prod_{a=l+1}^{k_1}\exp(-\imath\kappa_{\tilde{\omega}_1(a)1}r_{\omega_1(a)1})\prod_{a=l+1}^{k_2}\exp(\imath e^{\imath\psi}\kappa_{\tilde{\omega}_2(b)2}r_{\omega_2(b)2})\,.
\end{aligned}\qquad\text{(D8)}$$

We split the differential operator $\mathrm{Str}\,\partial^2/\partial\kappa^2$ into a part acting on $\kappa_{\tilde{\omega}}$ and a part for the remaining variables in $\kappa$. The term for the latter variables acts on the exponential functions in Eq. (D8) and contributes the term $-\mathrm{Str}\,r_\omega^2$. For the term according to $\kappa_{\tilde{\omega}}$ we use the identity

$$\mathrm{Str}\,\frac{\partial^2}{\partial\kappa_{\tilde{\omega}}^2}\sqrt{\mathrm{Ber}\,^{(2)}_{l/l}(\kappa_{\tilde{\omega}})}=0\,.\qquad\text{(D9)}$$

Thus, the differential equation is also fulfilled by $\widehat{\varphi}_{k_1/k_2}$.

### Appendix E: Double Fourier-transform

We consider the integral

$$\begin{aligned}
I\;=\;&\int\limits_{\mathbb{R}^{k_1+k_2}}\det\left[\begin{array}{c}\left\{\dfrac{-2\pi\delta(r_{b1})\delta(e^{\imath\psi}r_{a2})}{\kappa_{b1}-e^{-\imath\psi}\kappa_{a2}}+\dfrac{\exp\left(-\imath\kappa_{b1}r_{b1}+\imath\kappa_{a2}r_{a2}\right)}{r_{b1}-e^{\imath\psi}r_{a2}}\chi(\kappa_{b1}-e^{-\imath\psi}\kappa_{a2})\right\}_{\substack{1\le a\le k_2\\1\le b\le k_1}}\\ \left\{r_{b1}^{a-1}\exp\left(-\imath\kappa_{b1}r_{b1}\right)\right\}_{\substack{1\le a\le k_1-k_2\\1\le b\le k_1}}\end{array}\right]\\
\times\;&\det\left[\begin{array}{c}\left\{\dfrac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}}-\dfrac{\exp\left(\imath r_{b1}s_{b1}-\imath r_{a2}s_{a2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right\}_{\substack{1\le a\le k_2\\1\le b\le k_1}}\\ \left\{s_{b1}^{a-1}\exp\left(\imath r_{b1}s_{b1}\right)\right\}_{\substack{1\le a\le k_1-k_2\\1\le b\le k_1}}\end{array}\right]\\
\times\;&\frac{d[r]}{\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\widetilde{\kappa})\mathrm{Ber}\,^{(2)}_{k_1/k_2}(s)}}\,.
\end{aligned}\qquad\text{(E1)}$$

We omit the two sums over the permutation groups, see Eq. (5.7).They do not contribute any additional new information of the calculation and the missing terms can be regained by permuting the indices of the eigenvalues in $s$ or $\widetilde{\kappa}$.

The expansion in the first determinant yields

$$
\begin{aligned}
I \;=\; & \sum_{l=0}^{k_2} \sum_{\substack{\omega_1\in\mathfrak{S}(k_1)\\ \omega_2\in\mathfrak{S}(k_2)}} \frac{\operatorname{sign}\omega_1\omega_2}{(l!)^2(k_1-l)!(k_2-l)!} \int\limits_{\mathbb{R}^{k_1+k_2}} \det\left[\frac{-2\pi\delta(r_{\omega_1(b)1})\delta(e^{\imath\psi}r_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-e^{-\imath\psi}\kappa_{\omega_2(a)2}}\right]_{1\le a,b\le l} \qquad \text{(E2)}\\
\times\ & \det\left[\begin{array}{c} \left\{\dfrac{\exp\left(-\imath\kappa_{\omega_1(b)1}r_{\omega_1(b)1}+\imath\kappa_{\omega_2(a)2}r_{\omega_2(a)2}\right)}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}\chi(\kappa_{\omega_1(b)1}-e^{-\imath\psi}\kappa_{\omega_2(a)2})\right\}_{\substack{l+1\le a\le k_2\\ l+1\le b\le k_1}} \\ \left\{r_{\omega_1(b)1}^{a-1}\exp\left(-\imath\kappa_{\omega_1(b)1}r_{\omega_1(b)1}\right)\right\}_{\substack{1\le a\le k_1-k_2\\ l+1\le b\le k_1}} \end{array}\right]\\
\times\ & \det\left[\begin{array}{c} \left\{\dfrac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}}-\dfrac{\exp\left(\imath r_{b1}s_{b1}-\imath r_{a2}s_{a2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right\}_{\substack{1\le a\le k_2\\ 1\le b\le k_1}} \\ \left\{s_{b1}^{a-1}\exp\left(\imath r_{b1}s_{b1}\right)\right\}_{\substack{1\le a\le k_1-k_2\\ 1\le b\le k_1}} \end{array}\right]\\
\times\ & \frac{d[r]}{\sqrt{\operatorname{Ber}_{k_1/k_2}^{(2)}(r)}\sqrt{\operatorname{Ber}_{k_1/k_2}^{(2)}(\widetilde{\kappa})\operatorname{Ber}_{k_1/k_2}^{(2)}(s)}}\\
=\ & (-1)^{k_1(k_1-1)/2}\sum_{l=0}^{k_2} \sum_{\substack{\omega_1\in\mathfrak{S}(k_1)\\ \omega_2\in\mathfrak{S}(k_2)}} \frac{\operatorname{sign}\omega_1\omega_2}{(l!)^2(k_1-l)!(k_2-l)!} \det\left[\frac{-(2\pi)^2\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-e^{-\imath\psi}\kappa_{\omega_2(a)2}}\right]_{1\le a,b\le l}\\
\times\ & \prod_{a=l+1}^{k_1}\prod_{b=l+1}^{k_2}\chi(\kappa_{\omega_1(b)1}-e^{-\imath\psi}\kappa_{\omega_2(a)2}) \int\limits_{\mathbb{R}^{k_1+k_2-2l}} \prod_{a=l+1}^{k_1}\exp\left(-\imath\kappa_{\omega_1(a)1}r_{a1}\right)\prod_{b=l+1}^{k_2}\exp\left(\imath\kappa_{\omega_2(b)2}r_{b2}\right)\\
\times\ & \det\left[\begin{array}{c} \left\{\dfrac{2\pi\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{r_{b1}-e^{\imath\psi}r_{a2}}-\dfrac{\exp\left(\imath r_{b1}s_{\omega_1(b)1}-\imath r_{a2}s_{\omega_2(a)2}\right)}{s_{\omega_1(b)1}-e^{-\imath\psi}s_{\omega_2(a)2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right\}_{\substack{l+1\le a\le k_2\\ l+1\le b\le k_1}} \\ \left\{s_{\omega_1(b)1}^{a-1}\exp\left(\imath r_{b1}s_{\omega_1(b)1}\right)\right\}_{\substack{1\le a\le k_1-k_2\\ l+1\le b\le k_1}} \end{array}\right]\\
\times\ & \frac{d[r]}{\sqrt{\operatorname{Ber}_{k_1/k_2}^{(2)}(\widetilde{\kappa})\operatorname{Ber}_{k_1/k_2}^{(2)}(s)}}
\end{aligned}
$$

where the function “sign” yields 1 for an even permutation and −1 for an odd one. The permutations in the indices of the $r$ are absorbed in the integration. We remark that the remaining integral goes over $k_1+k_2-2l$ variables because we have already used the Dirac-distributions.

With help of the formula

$$
\begin{aligned}
& \int_{\mathbb{R}^2} \frac{2\pi\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{r_{b1}-e^{\imath\psi}r_{a2}}\exp\left(-\imath\kappa_{\omega_1(b)1}r_{b1}+\imath\kappa_{\omega_2(a)2}r_{a2}\right)d[r]\\
=\ & \int_{\mathbb{R}^2} \frac{2\pi\imath\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{(\kappa_{\omega_1(b)1}-e^{-\imath\psi}\kappa_{\omega_2(a)2})(r_{b1}-e^{\imath\psi}r_{a2})}\left(\frac{\partial}{\partial r_{b1}}+e^{-\imath\psi}\frac{\partial}{\partial r_{a2}}\right)\exp\left(-\imath\kappa_{\omega_1(b)1}r_{b1}+\imath\kappa_{\omega_2(a)2}r_{a2}\right)d[r]\\
=\ & \frac{(2\pi)^2\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-e^{-\imath\psi}\kappa_{\omega_2(a)2}} \qquad \text{(E3)}
\end{aligned}
$$

we integrate and sum the expression (E2) up. This yields

$$\int\limits_{\mathbb{R}^{k_1+k_2}} \widehat{\varphi}_{k_1/k_2}(\imath s,r)\widehat{\varphi}_{k_1/k_2}(-\imath r,\widetilde{\kappa})\mathrm{Ber}\,^{(2)}_{k_1/k_2}(r)d[r] \tag{E4}$$

$$= \frac{(-1)^{k_1(k_1-1)/2}\pi^{(k_2-k_1)^2}}{2^{2k_1k_2-k_1-k_2}k_1!k_2!\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\widetilde{\kappa})}\mathrm{Ber}\,^{(2)}_{k_1/k_2}(s)}\sum_{\substack{\omega_1\in\mathfrak{S}(k_1)\\ \omega_2\in\mathfrak{S}(k_2)}}$$

$$\times\ \det\left[\begin{array}{c}\dfrac{\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{\kappa_{b1}-e^{-\imath\psi}\kappa_{a2}}[1-\chi(\kappa_{b1}-e^{-\imath\psi}\kappa_{a2})]+e^{\imath\psi}\dfrac{\delta(\kappa_{b1}-s_{\omega_1(b)1})\delta(\kappa_{a2}-s_{\omega_2(a)2})}{s_{\omega_1(b)1}-e^{-\imath\psi}s_{\omega_2(a)2}}\chi(\kappa_{b1}-e^{-\imath\psi}\kappa_{a2})\\ \hline s^{a-1}_{\omega_1(b)1}\delta(\kappa_{b1}-s_{\omega_1(b)1})\end{array}\right]$$

which is the result (5.14). The index $a$ goes from 1 to $k_2$ in the upper block and from 1 to $k_1-k_2$ in the lower block whereas $b$ takes the values from 1 to $k_1$ in both blocks.

## Appendix F: Calculations for subsection VI A

We diagonalize the supermatrices $\sigma$ and $\rho$ in (6.4) and have for the generating function

$$Z_k^{(N)}(\kappa) = \frac{1}{(2\pi\imath)^{2k}}\int\limits_{\mathbb{R}^{4k}}\frac{d[r]d[s]}{\sqrt{\mathrm{Ber}\,^{(2)}_{k/k}(r)}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\kappa)}}\mathcal{F}\Phi(s)I_k^{(N)}(r) \tag{F1}$$

$$\times\ \det\left[\frac{-2\pi\delta(r_{b1})\delta(e^{\imath\psi}r_{a2})}{\kappa_{b1}-\kappa_{a2}}+\frac{\exp\left(-\imath\kappa_{b1}r_{b1}+\imath e^{\imath\psi}\kappa_{a2}r_{a2}\right)}{r_{b1}-e^{\imath\psi}r_{a2}}\chi(\kappa_{b1}-\kappa_{a2})\right]_{1\leq a,b\leq k}$$

$$\times\ \det\left[\frac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}}-\frac{\exp\left(\imath r_{b1}s_{b1}-\imath r_{a2}s_{a2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right]_{1\leq a,b\leq k}.$$

This expression is not well defined because the supersymmetric Ingham-Siegel is at zero not well defined. We recall that the supersymmetric Ingham-Siegel integral factorizes in each eigenvalue of the supermatrix $r$, cf. Eq. (6.5). To understand Eq. (F1) we have to know what $I_1^{(N)}(0)$ is. Since the supersymmetric Ingham-Siegel integral is a distribution we consider an arbitrary rotation invariant, sufficiently integrable superfunction $f$ on the set of $(1+1)\times(1+1)$ Hermitian supermatrices. Then we have

$$\begin{aligned}\int\limits_{\widetilde{\Sigma}^{(\psi)}_{1/1}} f(\rho)I_1^{(N)}(\rho)d[\rho] &= \int\limits_{\widetilde{\Sigma}^{(-\psi)}_{1/1}}\left(\int\limits_{\widetilde{\Sigma}^{(\psi)}_{1/1}} f(\rho)\exp(-\imath\mathrm{Str}\,\rho\sigma+\varepsilon\mathrm{Str}\,\rho)d[\rho]\right)\mathrm{Sdet}\,^{-N}(\sigma+\imath\varepsilon\mathbb{1}_{1+1})d[\sigma]\\ &= \imath\int\limits_{\widetilde{\Sigma}^{(\psi)}_{1/1}} f(\rho)\exp(\varepsilon\mathrm{Str}\,\rho)d[\rho]\\ &= f(0)\\ &\overset{!}{=} -\imath f(0)I_1^{(N)}(0)\end{aligned} \tag{F2}$$

with help of the Cauchy-like integral theorem for $(1+1)\times(1+1)$ Hermitian supermatrices, see Ref. [15,16]. Please notice that the constant resulting from the Cauchy-like integral theorem converts to the complex conjugate when the generalized Wick-rotation is complex conjugated. The last equality in Eq. (F2) is the Cauchy-like integral theorem formally applied to the left hand side of Eq. (F2). Hence we conclude that $I_1^{(N)}(0)=\imath$ in a distributional sense. Using this result we find

$$
\begin{aligned}
Z_k^{(N)}(\kappa) &= \frac{(-1)^{k(k+1)/2}}{2^{k(k+1)}\pi^{2k}}\sum_{l=0}^{k}\sum_{\omega_1,\omega_2\in\mathfrak{S}(k)}\frac{\operatorname{sign}\omega_1\omega_2}{[l!(k-l)!]^2}\int\limits_{\mathbb{R}^{4k}}\mathcal{F}\Phi(s)\det\left[\frac{-2\pi\imath\delta(r_{\omega_1(b)1})\delta(e^{\imath\psi}r_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2}}\right]_{1\leq a,b\leq l} \qquad \text{(F3)}\\
&\times\det\left[\frac{\exp\left(-\imath\kappa_{\omega_1(b)1}r_{\omega_1(b)1}+\imath e^{\imath\psi}\kappa_{\omega_2(a)2}r_{\omega_2(a)2}\right)}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}I_1^{(N)}(r_{\omega_1(b)1},e^{\imath\psi}r_{\omega_2(a)2})\chi(\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2})\right]_{l+1\leq a,b\leq k}\\
&\times\det\left[\frac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}}-\frac{\exp\left(\imath r_{b1}s_{b1}-\imath r_{a2}s_{a2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right]_{1\leq a,b\leq k}\frac{d[r]d[s]}{\sqrt{\operatorname{Ber}_{k_1/k_2}^{(2)}(r)}\sqrt{\operatorname{Ber}_{k/k}^{(2)}(\kappa)}}\\
&= \frac{(-1)^k}{2^{k(k+1)}\pi^{2k}}\sum_{l=0}^{k}\sum_{\omega_1,\omega_2\in\mathfrak{S}(k)}\frac{\operatorname{sign}\omega_1\omega_2}{[l!(k-l)!]^2}\prod_{a,b=l+1}^{k}\chi(\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2})\int\limits_{\mathbb{R}^{4k-2l}}\mathcal{F}\Phi(s)\\
&\times\det\left[\frac{-(2\pi)^2\imath\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2}}\right]_{1\leq a,b\leq l}\prod_{a,b=l+1}^{k}\exp\left(-\imath\kappa_{\omega_1(b)1}r_{b1}+\imath e^{\imath\psi}\kappa_{\omega_2(a)2}r_{a2}\right)I_1^{(N)}(r_{b1},e^{\imath\psi}r_{a2})\\
&\times\det\left[\frac{2\pi\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{r_{b1}-e^{\imath\psi}r_{a2}}-\frac{\exp\left(\imath r_{b1}s_{\omega_1(b)1}-\imath r_{a2}s_{\omega_2(a)2}\right)}{s_{\omega_1(b)1}-e^{-\imath\psi}s_{\omega_2(a)2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right]_{l+1\leq a,b\leq k}\frac{d[r]d[s]}{\sqrt{\operatorname{Ber}_{k/k}^{(2)}(\kappa)}}\\
&= \frac{1}{2^{k(k-1)}\imath^k}\sum_{l=0}^{k}\sum_{\omega_1,\omega_2\in\mathfrak{S}(k)}\frac{\operatorname{sign}\omega_1\omega_2}{[l!(k-l)!]^2}\prod_{a,b=l+1}^{k}\chi(\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2})\\
&\times\int\limits_{\mathbb{R}^{2k}}\frac{d[r]d[s]}{\sqrt{\operatorname{Ber}_{k/k}^{(2)}(\kappa)}}\mathcal{F}\Phi(s)\det\left[-\frac{\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2}}\right]_{1\leq a,b\leq l}\\
&\times\det\left[\frac{\delta(s_{\omega_1(b)1})\delta(e^{-\imath\psi}s_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2}}\left[1-\left(\frac{\kappa_{\omega_2(a)2}}{\kappa_{\omega_1(a)1}}\right)^N\right]\right.\\
&\left.\quad-\frac{N\left(e^{-\imath\psi}s_{\omega_2(a)2}-\kappa_{\omega_2(a)2}\right)^{N-1}}{2\pi\imath\left(e^{-\imath\psi}s_{\omega_1(b)1}-\kappa_{\omega_1(b)1}\right)^{N+1}\left(s_{\omega_1(b)1}-e^{-\imath\psi}s_{\omega_2(a)2}\right)}\right]_{l+1\leq a,b\leq k}
\end{aligned}
$$

We perform the sum and use the identity

$$
1-\left[1-\left(\frac{\kappa_{\omega_2(a)2}}{\kappa_{\omega_1(a)1}}\right)^N\right]\chi(\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2})=\left(\frac{\kappa_{\omega_2(a)2}}{\kappa_{\omega_1(a)1}}\right)^N. \qquad \text{(F4)}
$$

Then we have the result (6.6).

## Appendix G: Calculations for subsection VI B

In the first step we derive the Fourier-transform of the superdeterminant in Eq. (6.7). Let the entries of the diagonal $(k+k)\times(k+k)$ supermatrix $r$ and the entries of diagonal $N\times N$ matrix $E_0$ be the eigenvalues of the supermatrix $\rho$ and the Hermitian matrix $H_0$, i.e. $\rho=UrU^\dagger$ with $U\in \mathrm{U}\,(k/k)$ and $H_0=VE_0V^\dagger$ with $V\in\mathrm{U}\,(N)$. Then the Fourier-transform is

$$
\begin{aligned}
J &= \int\limits_{\widetilde{\Sigma}_{k/k}^{(-\psi)}}\exp(-\imath\operatorname{Str}r\sigma+\varepsilon\operatorname{Str}\rho)\operatorname{Sdet}^{-1}(\sigma\otimes\mathbb{1}_N+\alpha\mathbb{1}_{k+k}\otimes H_0+\imath\varepsilon\mathbb{1}_{N(k+k)})d[\sigma]\\
&= \frac{\imath^k}{2^{k^2}\pi^k}\frac{1}{\sqrt{\operatorname{Ber}_{k/k}^{(2)}(r)}}\int\limits_{\mathbb{R}^{2k}}\det\left[\frac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}}+\frac{\exp\left(-\imath s_{b1}r_{b1}+\imath s_{a2}r_{a2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\chi(r_{b1}-e^{\imath\psi}r_{a2})\right]_{1\leq a,b\leq k}\\
&\times\operatorname{Sdet}^{-1}(s\otimes\mathbb{1}_N+\alpha\mathbb{1}_{k+k}\otimes H_0+\imath\varepsilon\mathbb{1}_{N(k+k)})d[s]\,. \qquad \text{(G1)}
\end{aligned}
$$

With help of identity (2.7) we find

$$
\begin{aligned}
J &= \frac{\imath^k}{2^{k^2}\pi^k}\frac{\exp(\varepsilon\mathrm{Str}\,r)}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(r)}}\sum_{l=0}^{k}\sum_{\omega_1,\omega_2\in\mathfrak{S}(k)}\frac{\mathrm{sign}\,\omega_1\omega_2}{[l!(k-l)!]^2}\prod_{a,b=l+1}^{k}\chi(r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2})\\
&\times\int\limits_{\mathbb{R}^{2k}}\mathrm{Sdet}^{-1}(s\otimes\mathbf{1}_N+\alpha\mathbf{1}_{k+k}\otimes E_0+\imath\varepsilon\mathbf{1}_{N(k+k)})\det\left[\frac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}\right]_{1\leq a,b\leq l}\\
&\times\det\left[\frac{\exp\left(-\imath s_{b1}r_{\omega_1(b)1}+\imath s_{a2}r_{\omega_1(a)2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\right]_{l+1\leq a,b\leq k} d[s]\\
&= \frac{\imath^k}{2^{k^2}\pi^k}\frac{\exp(\varepsilon\mathrm{Str}\,r)}{\Delta_N(\alpha E_0)\sqrt{\mathrm{Ber}_{k/k}^{(2)}(r)}}\sum_{l=0}^{k}\sum_{\omega_1,\omega_2\in\mathfrak{S}(k)}\frac{\mathrm{sign}\,\omega_1\omega_2}{[l!(k-l)!]^2}\prod_{a,b=l+1}^{k}\chi(r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2})\\
&\times\int\limits_{\mathbb{R}^{2k}}\det\left[\frac{2\pi\delta(s_{b1})\delta(e^{-\imath\psi}s_{a2})}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}\right]_{1\leq a,b\leq l}\\
&\times\det\left[\begin{array}{c|c}\frac{\exp\left(-\imath s_{b1}r_{\omega_1(b)1}+\imath s_{a2}r_{\omega_1(a)2}\right)}{s_{b1}-e^{-\imath\psi}s_{a2}}\left(\frac{e^{-\imath\psi}s_{a2}+\imath\varepsilon}{s_{b1}+\imath\varepsilon}\right)^N & \exp\left(\imath s_{a2}r_{\omega_1(a)2}\right)\left(e^{-\imath\psi}s_{a2}+\imath\varepsilon\right)^{b-1}\\ \hline \frac{\exp\left(-\imath s_{b1}r_{\omega_1(b)1}\right)}{s_{b1}+\imath\varepsilon+\alpha E_a^{(0)}}\left(\frac{-\alpha E_a^{(0)}}{s_{b1}+\imath\varepsilon}\right)^N & (-\alpha E_a^{(0)})^{b-1}\end{array}\right]d[s]\,. \quad (G2)
\end{aligned}
$$

In the left upper block both indices $a$ and $b$ run from $l+1$ to $k$ whereas in the right lower block the range is from 1 to $N$. In the right upper block $a$ goes from $l+1$ to $k$ and $b$ goes from 1 to $N$ whereas it is vice versa in the left lower block. We sum all terms in Eq. (G2) up and pull the integrations into the determinant. Then we have

$$
J = \frac{\imath^k}{2^{k^2}\pi^k}\frac{1}{\Delta_N(\alpha E_0)\sqrt{\mathrm{Ber}_{k/k}^{(2)}(r)}}\det\left[\begin{array}{cc}\left\{A_1(r_{b1},e^{\imath\psi}r_{a2})\right\}_{1\leq a,b\leq k} & \left\{A_{b2}(e^{\imath\psi}r_{a2})\right\}_{\substack{1\leq a\leq k\\ 1\leq b\leq N}}\\ \left\{A_3(r_{b1},\alpha E_a^{(0)})\right\}_{\substack{1\leq a\leq N\\ 1\leq b\leq k}} & \left\{(-\alpha E_a^{(0)})^{b-1}\right\}_{1\leq a,b\leq N}\end{array}\right], \quad (G3)
$$

where

$$
\begin{aligned}
A_1(r_{b1},e^{\imath\psi}r_{a2}) &= \exp\left[\varepsilon(r_{b1}-e^{\imath\psi}r_{a2})\right]\\
&\times\int\limits_{\mathbb{R}^2}\left(\frac{2\pi\delta(s_1)\delta(e^{-\imath\psi}s_2)}{r_{b1}-e^{\imath\psi}r_{a2}}+\frac{\exp\left(-\imath s_1r_{b1}+\imath s_2r_{a2}\right)}{s_1-e^{-\imath\psi}s_2}\left(\frac{e^{-\imath\psi}s_2+\imath\varepsilon}{s_1+\imath\varepsilon}\right)^N\prod\chi(r_{b1}-e^{\imath\psi}r_{a2})\right)d[s]\\
&= -2\pi\imath\frac{I_1^{(N)}(r_{b1},e^{\imath\psi}r_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}}\\
&= \frac{(2\pi)^2\imath}{(N-1)!}\frac{r_{b1}^N\Theta(r_{b1})}{r_{b1}-e^{\imath\psi}r_{a2}}\left(-e^{-\imath\psi}\frac{\partial}{\partial r_{a2}}\right)^{N-1}\delta(e^{\imath\psi}r_{a2})\,, \quad (G4)\\
A_{b2}(e^{\imath\psi}r_{a2}) &= \exp\left(-\varepsilon e^{\imath\psi}r_{a2}\right)\int\limits_{\mathbb{R}}\exp\left(\imath s_2r_{a2}\right)\left(e^{-\imath\psi}s_2+\imath\varepsilon\right)^{b-1}e^{-\imath\psi}ds_2\prod_{j=1}^{k}\chi(r_{j1}-e^{\imath\psi}r_{a2})\\
&= 2\pi\left(-\imath e^{-\imath\psi}\frac{\partial}{\partial r_{a2}}\right)^{b-1}\delta(e^{\imath\psi}r_{a2})\prod_{j=1}^{k}\chi(r_{j1}-e^{\imath\psi}r_{a2})\,, \quad (G5)\\
A_3(r_{b1},\alpha E_a^{(0)}) &= \exp\left(\varepsilon r_{b1}\right)\int\limits_{\mathbb{R}}\frac{\exp\left(-\imath s_1r_{b1}\right)}{s_1+\imath\varepsilon+\alpha E_a^{(0)}}\left(\frac{-\alpha E_a^{(0)}}{s_1+\imath\varepsilon}\right)^N ds_1\prod_{j=1}^{k}\chi(r_{b1}-e^{\imath\psi}r_{j2})\\
&= 2\pi\imath\Theta(r_{b1})\sum_{n=N}^{\infty}\frac{(\imath\alpha E_a^{(0)}r_{b1})^n}{n!}\prod_{j=1}^{k}\chi(r_{b1}-e^{\imath\psi}r_{j2})\,. \quad (G6)
\end{aligned}
$$

Surprisingly this part of our result agrees with the one in Ref. [8] (apart from a forgotten $2\pi\imath$ in the upper left block of Eq. (6.9) in Ref. [8] and the characteristic functions $\chi(r_{b1}-e^{\imath\psi}r_{a2})$) although we omitted all Efetov-Wegner terms in this work.

The second step is to diagonalize the supermatrix $\rho$ in Eq. (6.7),

$$\begin{aligned} & Z_k^{(N)}(\kappa,\alpha H_0) \\ = & \frac{(2\pi\imath)^{-k}}{\Delta_N(\alpha E_0)\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}}\int\limits_{\mathbb{R}^{2k}}\det\left[\frac{-2\pi\delta(r_{b1})\delta(e^{\imath\psi}r_{a2})}{\kappa_{b1}-\kappa_{a2}}+\frac{\exp\left(-\imath\kappa_{b1}r_{b1}+\imath e^{\imath\psi}\kappa_{a2}r_{a2}\right)}{r_{b1}-e^{\imath\psi}r_{a2}}\chi(\kappa_{b1}-\kappa_{a2})\right]_{1\leq a,b\leq k} \\ & \times\det\left[\begin{array}{c|c} -\dfrac{I_1^{(N)}(r_{b1},e^{\imath\psi}r_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}} & 2\pi\left(-\imath e^{-\imath\psi}\dfrac{\partial}{\partial r_{a2}}\right)^{b-1}\delta(e^{\imath\psi}r_{a2})\prod\limits_{j=1}^{k}\chi(r_{j1}-e^{\imath\psi}r_{a2}) \\ \hline \Theta(r_{b1})\sum\limits_{n=N}^{\infty}\dfrac{(\imath\alpha E_a^{(0)}r_{b1})^n}{n!}\prod\limits_{j=1}^{k}\chi(r_{b1}-e^{\imath\psi}r_{j2}) & (-\alpha E_a^{(0)})^{b-1} \end{array}\right]\frac{\Phi(r)d[r]}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(r)}}. \end{aligned} \tag{G7}$$

The range of the indices in the second determinant is the same as in Eq. (G3). Expanding the first determinant we have

$$\begin{aligned} & Z_k^{(N)}(\kappa,\alpha H_0) \\ = & \frac{(2\pi\imath)^{-k}}{\Delta_N(\alpha E_0)\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}}\sum_{l=0}^{k}\sum_{\omega_1,\omega_2\in\mathfrak{S}(k)}\frac{\mathrm{sign}\,\omega_1\omega_2}{[l!(k-l)!]^2}\prod_{a,b=l+1}^{k}\chi(\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2}) \\ & \times\int\limits_{\mathbb{R}^{2k}}\det\left[\frac{-2\pi\delta(r_{\omega_1(b)1})\delta(e^{\imath\psi}r_{\omega_2(a)2})}{\kappa_{\omega_1(b)1}-\kappa_{\omega_2(a)2}}\right]_{1\leq a,b\leq l}\det\left[\frac{\exp\left(-\imath\kappa_{\omega_1(b)1}r_{\omega_1(b)1}+\imath e^{\imath\psi}\kappa_{\omega_2(a)2}r_{\omega_2(a)2}\right)}{r_{\omega_1(b)1}-e^{\imath\psi}r_{\omega_2(a)2}}\right]_{l+1\leq a,b\leq k} \\ & \times\det\left[\begin{array}{c|c} -\dfrac{I_1^{(N)}(r_{b1},e^{\imath\psi}r_{a2})}{r_{b1}-e^{\imath\psi}r_{a2}} & 2\pi\left(-\imath e^{-\imath\psi}\dfrac{\partial}{\partial r_{a2}}\right)^{b-1}\delta(e^{\imath\psi}r_{a2})\prod\limits_{j=1}^{k}\chi(r_{j1}-e^{\imath\psi}r_{a2}) \\ \hline \Theta(r_{b1})\sum\limits_{n=N}^{\infty}\dfrac{(\imath\alpha E_a^{(0)}r_{b1})^n}{n!}\prod\limits_{j=1}^{k}\chi(r_{b1}-e^{\imath\psi}r_{j2}) & (-\alpha E_a^{(0)})^{b-1} \end{array}\right]\frac{\Phi(r)d[r]}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(r)}}. \end{aligned} \tag{G8}$$

When summing up we use the normalization (F2) of the supersymmetric Ingham-Siegel integral and arrive at the result (6.8).

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