Cohomology and Hodge Theory on Symplectic Manifolds: II

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Abstract

We show that the exterior derivative operator on a symplectic manifold has a natural decomposition into two linear differential operators, analogous to the Dolbeault operators in complex geometry. These operators map primitive forms into primitive forms and therefore lead directly to the construction of primitive cohomologies on symplectic manifolds. Using these operators, we introduce new primitive cohomologies that are analogous to the Dolbeault cohomology in the complex theory. Interestingly, the finiteness of these primitive cohomologies follows directly from an elliptic complex. We calculate the known primitive cohomologies on a nilmanifold and show that their dimensions can vary depending on the class of the symplectic form.

Contents

1 Introduction

This paper continues the study of differential cohomologies on smooth compact symplectic manifolds that we began in Paper I [11]. There, we introduced a number of new finite-dimensional cohomologies defined on the space of differential forms. These new cohomologies, dependent on the symplectic form, were shown in general to be distinct from the de Rham cohomolgy, and thus, they provide new symplectic invariants. Of particular interest for us here is the property noted in [11] that the new symplectic cohomologies can be equivalently described by cohomologies defined only on the subset of differential forms called primitive forms. We called this type of cohomologies "primitive cohomologies" and they are the main focus of this paper.

The fundamental nature of primitive cohomologies in symplectic geometry can be understood simply. Let us explain this via an analogy with complex geometry.

On a complex space, it is standard to decompose differential forms into its (p,q) components. Let $\mathcal{A}^{p,q}$ be the space of smooth differential (p,q)-forms. Then acting on it by the exterior derivative d, we have

$$d:\ \mathcal{A}^{p,q}\ \rightarrow\ \mathcal{A}^{p+1,q}\oplus\ \mathcal{A}^{p,q+1} \tag{1.1}$$

which of course encodes the complex decomposition, $d = \partial + \bar\partial$, where for instance the Dolbeault operator $\bar\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ projects $d(\mathcal{A}^{p,q})$ onto its $(p, q + 1)$ component. Thus, on a complex space, it is natural to decompose both the differential forms and exterior derivative in a complex structure dependent way. This raises the question in the symplectic context whether a symplectic structure dependent decomposition of differential forms and the exterior derivative are also possible.

For differential forms, the decomposition in the presence of a symplectic form ω is well-known

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$$\mathcal{L}^{0,4}$$

$$\mathcal{L}^{0,3} \qquad \mathcal{L}^{1,3}$$

$$\mathcal{L}^{0,2} \qquad \mathcal{L}^{1,2} \qquad \mathcal{L}^{2,2}$$

$$\mathcal{L}^{0,1} \qquad \mathcal{L}^{1,1} \qquad \mathcal{L}^{2,1} \qquad \mathcal{L}^{3,1}$$

$$\mathcal{L}^{0,0} \qquad \mathcal{L}^{1,0} \qquad \mathcal{L}^{2,0} \qquad \mathcal{L}^{3,0} \qquad \mathcal{L}^{4,0}$$
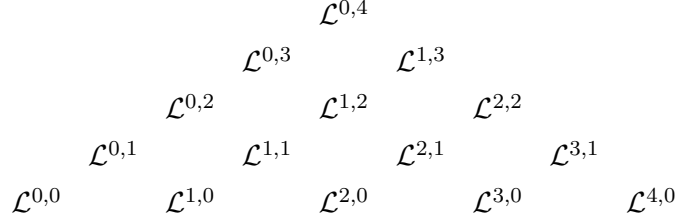
Figure 1: The (r,s) pyramid decomposition of differential forms in $d = 8$. The differential forms of $\mathcal{L}^{r,s}$ has degree $(2r + s)$. In the diagram, the degree increases in increment of one, from zero to $2n$ (left to right).

[13, 6]. This is commonly called the Lefschetz decomposition. The elemental components, which we shall label by two indices, (r,s), take the form $\frac{1}{r!}\omega^r \wedge B_s$, where $B_s \in \mathcal{P}^s$ is a primitive s-form. Recall that by definition, a primitive form satisfy $\Lambda\,B_s := \frac{1}{2}(\omega^{-1})^{ij}\,i_{\partial_{x^i}}i_{\partial_{x^j}}\,B_s = 0$. We shall denote the space of such (r,s)-forms by $\mathcal{L}^{r,s}$ with $0 \le r,s \le n$ for a symplectic space of dimension $d = 2n$. In Fig. 1, we have arranged the different $\mathcal{L}^{r,s}$'s into a pyramid diagram, representing the symplectic analog of the complex (p,q) diamond.

What may be a bit surprising is that a symplectic decomposition of the exterior derivative is also possible. (As far as we are aware, this has not been previously discussed in the literature.) Consider simply the action of d on $\mathcal{L}^{r,s}$. Since $d\omega = 0$, we have $d\left(\frac{1}{r!}\omega^r \wedge B_s\right) = \frac{1}{r!}\omega^r \wedge (dB_s)$. By this simple relation, we see clearly that the action of the exterior derivative on $\mathcal{L}^{r,s}$ is entirely determined by its action on the primitive part, i.e. B_s. And regarding the derivative of a primitive form, there is the useful formula (see e.g. [5, 11] or Section 2.2 below (2.18))

$$dB_s = B^0_{s+1} + \omega \wedge B^1_{s-1}\ , \tag{1.2}$$

where $B^0, B^1 \in \mathcal{P}^*$, and if $s = n$, then $B^0_{n+1} = 0$. Taking the exterior product of (1.2) with $\frac{1}{r!}\omega^r$, we find

$$d : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s+1} \oplus \mathcal{L}^{r+1,s-1}\ , \tag{1.3}$$

which gives us the symplectic analog of (1.1).

The symplectic decomposition of the exterior derivative becomes now just a projection onto the two different spaces on the right hand side of (1.3). But as already mentioned, the derivative action on $\mathcal{L}^{r,s}$ is completely encoded in its action on the primitive forms; thus, we really only need to consider the primitive components, $\mathcal{L}^{0,s} = \mathcal{P}^s$. (A complete discussion taking into

account of all $\mathcal{L}^{r,s}$ will be given in Section 2.) With (1.2), we are led to write the decomposition of d as follows,

$$d = \partial_+ + \omega \wedge \partial_- \tag{1.4}$$

where $\partial_\pm : \mathcal{P}^s \to \mathcal{P}^{s\pm1}$. Hence, we have seen the importance of the primitive subspace of differential forms and have defined a pair of new first-order symplectic differential operators (∂_+, ∂_-) which preserve the primitive property of forms.

Just like $(\partial, \bar{\partial})$ on a complex space, (∂_+, ∂_-) has a number of desirable properties that follow directly from their definition in (1.4). In fact, it follows from $d^2 = 0$ and the two decompositions - Lefschetz and the exterior derivative (1.4) - that both ∂_+ and ∂_- square to zero and anticommute with each other. These facts suggest defining on a compact symplectic manifold (M^{2n}, ω) the following two cohomologies

$$PH_{\partial_+}^k(M) = \frac{\ker \partial_+ \cap \mathcal{P}^k(M)}{\mathrm{im}\ \partial_+ \cap \mathcal{P}^k(M)} \tag{1.5}$$

$$PH_{\partial_-}^k(M) = \frac{\ker \partial_- \cap \mathcal{P}^k(M)}{\mathrm{im}\ \partial_- \cap \mathcal{P}^k(M)} \tag{1.6}$$

for $k < n$. The two cohomologies are not well-defined for $k = n$ since, by the definition of primitivity, there are no degree $n+1$ primitive forms. Note that these new cohomologies $PH_{\partial_+}^*(M)$ and $PH_{\partial_-}^*(M)$ are very different from the de Rham cohomology. For instance, a ∂_+-closed form may not be d-closed, and moreover, $d\,\partial_+ = \omega \wedge (\partial_- \partial_+)$, which is not identically zero. Nevertheless, we will show that the two cohomologies above are indeed finite-dimensional on a compact manifold and in general they give new symplectic invariants. Interestingly, the finiteness follows directly by associating the two cohomologies with the single differential complex,

$$0 \longrightarrow \mathcal{P}^0 \xrightarrow{\partial_+} \mathcal{P}^1 \xrightarrow{\partial_+} \cdots \xrightarrow{\partial_+} \mathcal{P}^{n-1} \xrightarrow{\partial_+} \mathcal{P}^n \xrightarrow{\partial_+\partial_-} \mathcal{P}^n \xrightarrow{\partial_-} \tag{1.7}$$

$$\overset{\partial_-}{\hookrightarrow} \mathcal{P}^{n-1} \xrightarrow{\partial_-} \cdots \xrightarrow{\partial_-} \mathcal{P}^1 \xrightarrow{\partial_-} \mathcal{P}^0 \xrightarrow{\partial_-} 0$$

which we will prove is elliptic (in Section 2, Proposition 2.8). This elliptic complex can be thought of as the symplectic analog of the Dolbeault complex. Now, if we introduce a Riemannian metric, the elliptic complex implies that the second-order Laplacians, $\Delta_{\partial_\pm}$, associated with $PH_{\partial_\pm}^*(M)$ are also elliptic, and hence the primitive cohomologies have Hodge theoretic properties. Moreover, we will also show that these two cohomologies are actually isomorphic, i.e. $PH_{\partial_+}^k(M) \cong PH_{\partial_-}^k(M)$.

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Primitive Cohomologies

$k < n$	$k \leq n$
(1) $PH^k_{\partial_+}(M) = \dfrac{\ker \partial_+ \cap \mathcal{P}^k(M)}{\operatorname{im} \partial_+ \cap \mathcal{P}^k(M)}$	(3) $PH^k_{d+d^\Lambda}(M) = \dfrac{\ker(\partial_+ + \partial_-) \cap \mathcal{P}^k(M)}{\partial_+ \partial_- \mathcal{P}^k(M)}$
(2) $PH^k_{\partial_-}(M) = \dfrac{\ker \partial_- \cap \mathcal{P}^k(M)}{\operatorname{im} \partial_- \cap \mathcal{P}^k(M)}$	(4) $PH^k_{dd^\Lambda}(M) = \dfrac{\ker \partial_+ \partial_- \cap \mathcal{P}^k(M)}{\partial_+ \mathcal{P}^{k-1} + \partial_- \mathcal{P}^{k+1}}$
	(5) $PH^k_{d \cap d^\Lambda}(M) = \dfrac{\ker(\partial_+ + \partial_-) \cap \mathcal{P}^k(M)}{\partial_+ \tilde{\mathcal{P}}^{k-1} + \partial_- \tilde{\mathcal{P}}^{k+1}}$

Table 1: The primitive cohomologies defined on a symplectic manifold (M^{2n}, ω) introduced here (1-2) and in Paper I (3-5) [11], expressed in terms of ∂_+ and ∂_-. $\tilde{\mathcal{P}}$ in (5) denotes the space of primitive forms which are $\partial_+ \partial_-$-closed.

The primitive cohomologies $PH^k_{\partial_\pm}(M)$ are calculable. Let us denote the space of primitive ∂_--closed k-form by $\mathcal{P}'^k(M)$ (with an additional prime). By demonstrating the validity of the local ∂_--Poincaré lemma, we shall show the isomorphism of $PH^k_{\partial_-}(M)$ with the Cech cohomology $\breve{H}^{n-k}(M, \mathcal{P}'^n)$, for $0 \leq k < n$. (This and other properties of $PH^k_{\partial_\pm}(M)$ will be worked out in Section 3.) It is interesting to note here that the middle degree primitive ∂_--closed forms plays a special role. As pointed out in [11], the Poincaré dual currents of closed lagrangians are precisely d-closed (or equivalently ∂_--closed) middle degree primitive currents.

Concerning the middle degree, observe that in the elliptic complex (1.7) above, the middle degree primitive forms are curiously connected by the second-order differential operator, $\partial_+ \partial_-$. With its presence, two middle-dimensional primitive cohomologies can be read off from the elliptic complex:

$$PH^n_{dd^\Lambda} = \frac{\ker \partial_+ \partial_- \cap \mathcal{P}^n(M)}{\operatorname{im} \partial_+ \cap \mathcal{P}^n(M)} \ , \qquad PH^n_{d+d^\Lambda} = \frac{\ker \partial_- \cap \mathcal{P}^n(M)}{\operatorname{im} \partial_+ \partial_- \cap \mathcal{P}^n(M)} \ .$$

These two middle-dimensional cohomologies are actually special cases of the three primitive cohomologies - $PH^k_{d+d^\Lambda}(M)$, $PH^k_{dd^\Lambda}(M)$, $PH^k_{d \cap d^\Lambda}(M)$ - introduced in Paper I [11]. These three were obtained by Lefschetz decomposing their corresponding symplectic cohomologies -

$H^*_{d+d^\Lambda}(M)$, $H^*_{dd^\Lambda}(M)$, $H^*_{d\cap d^\Lambda}(M)$ - which are defined on the space of all differential forms. The three primitive cohomologies from Paper I are well-defined for all $k \leq n$, which includes the middle degree, and can be expressed in terms of ∂_+ and ∂_- , as presented in Table 1, where we have collected all the primitive cohomologies. It is interesting to note that though we have argued that both $PH^n_{\partial_+}(M)$ and $PH^n_{\partial_-}(M)$ are ill-defined (and in fact infinite-dimensional) cohomologies, their intersection gives the known cohomology

$$PH^n_{d\cap d^\Lambda}(M) = \frac{\ker \partial_- \cap \mathcal{P}^n(M)}{\partial_+ \tilde{\mathcal{P}}^{n-1}(M)} \tag{1.8}$$

where $\tilde{\mathcal{P}}^n$ denotes the space of primitive n-forms which are $\partial_-\partial_+$-closed. This cohomology of course is well-defined and finite-dimensional.

We should emphasize that the dimensions of all the primitive cohomologies are invariant under symplectomorphisms. However, they can vary along with the de Rham class of the symplectic form. In Section 4, we have calculated the various primitive symplectic cohomologies for a six-dimensional symplectic nilmanifold. As can be seen clearly in Table 2 in Section 4, primitive cohomologies on a symplectic manifold do contain more information than the de Rham cohomology. In particular, we will show explicitly that the dimension of $PH^2_{\partial_\pm}(M)$ do vary in this specific nilmanifold as the class of the symplectic form varies.

This paper for the most part will be focused on introducing $PH^k_{\partial_\pm}(M)$ and developing their properties. A fuller discussion of applications and relations between the different primitive cohomologies will be given elsewhere [12].

2 Primitive Cohomologies

We begin by discussing the primitive structures on symplectic spaces that arise due to the presence of a symplectic form. We then proceed to develop the primitive cohomologies and show their finiteness on compact symplectic manifolds.

2.1 Primitive structures on symplectic manifolds

Let (M^{2n}, ω) be a smooth symplectic manifold. There is a natural sl_2 representation (L, Λ, H) that acts on the space of differential forms, $\Omega(M)$. On a differential form $A \in \Omega^*(M)$, the operators act as follows:

$$L: \quad L(A) = \omega \wedge A$$

$$\Lambda: \quad \Lambda(A_k) = \frac{1}{2}(\omega^{-1})^{ij} \, i_{\partial_{x^i}} i_{\partial_{x^j}} A$$

$$H: \quad H(A) = \sum_k (n - k) \, \Pi^k A$$

where $\wedge$ and i respectively denote the wedge and interior product, $(\omega^{-1})^{ij}$ is the inverse matrix of ω_{ij}, and $\Pi^k : \Omega^*(M) \to \Omega^k(M)$ projects onto forms of degree k. These actions result in the sl_2 algebra

$$[\Lambda, L] = H, \qquad [H, \Lambda] = 2\Lambda, \qquad [H, L] = -2L, \tag{2.1}$$

With this sl_2 action, the space of all differential forms $\Omega(M)$ can be arranged in terms of irreducible modules of the sl_2 representation [13]. From this perspective, the primitive forms are precisely the highest weight representatives of the sl_2 modules. More concretely, a differential s-form is called a primitive form, i.e. $B_s \in \mathcal{P}^s(M)$ with $s \leq n$, if it satisfies the condition $\Lambda B_s = 0$, or equivalently $L^{n-s+1} B_s = 0$.

Now we can of course also decompose any differential form $A_k \in \Omega^k(M)$ into components of different sl_2 modules. This is commonly called the Lefschetz decomposition (from the Kähler geometry literature). Specifically, we can write

$$A_k = \sum_{r \geq \max(k-n, 0)} \frac{1}{r!} L^r B_{k-2r} \tag{2.2}$$

We emphasize that the Lefschetz decomposition is unique as the B_{k-2r}'s in above are solely determined by A_k. By a straightforward calculation, we find

$$B_{k-2r} = \left(\sum_{l=0} a_{r,l} \frac{1}{l!} L^l \Lambda^{r+l} \right) A_k. \tag{2.3}$$

where $a_{r,l}$ are rational coefficients given by the expression

$$a_{r,l} = (-1)^l (n - k + 2r + 1)^2 \prod_{i=0}^{r} \frac{1}{n - k + 2r + 1 - i} \prod_{j=0}^{l} \frac{1}{n - k + 2r + 1 + j} \tag{2.4}$$

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$$A_0 = B_{0,0}$$

$$A_1 = \quad\quad\quad B_{0,1}$$

$$A_2 = LB_{1,0} \quad\quad\quad + B_{0,2}$$

$$A_3 = \quad\quad\quad LB_{1,1} \quad\quad\quad + B_{0,3}$$

$$A_4 = \frac{1}{2!}L^2 B_{2,0} \quad\quad + LB_{1,2} \quad\quad + B_{0,4}$$

$$A_5 = \quad\quad\quad \frac{1}{2!}L^2 B_{2,1} \quad\quad + LB_{1,3}$$

$$A_6 = \frac{1}{3!}L^3 B_{3,0} \quad\quad + \frac{1}{2!}L^2 B_{2,2}$$

$$A_7 = \quad\quad\quad \frac{1}{3!}L^3 B_{3,1}$$

$$A_8 = \frac{1}{4!}L^4 B_{4,0}$$

Figure 2: Lefschetz decomposition of differential forms in dimension $d = 8$. Here, $B_{r,k-2r}$ denotes a primitive $(k-2r)$-form associated with the $\frac{1}{r!}L^r$ term.

Thus for example, it follows from (2.3) and (2.4) that the first primitive form term B_k in the decomposition $A_k = B_k + L\,B_{k-2} + \ldots$ for $k \le n$ has the form

$$B_k = \left\{ 1 - \frac{1}{n-k+2}L\Lambda + \frac{1}{2!}\frac{1}{(n-k+2)(n-k+3)}L^2\Lambda^2 - \ldots \right\} A_k \tag{2.5}$$

To fully appreciate the decomposition, it is useful to see the Lefschetz decomposition applied to all differential forms of a given dimension $d = 2n$. We write out the decomposition for $d = 8$ in Fig. 2 having arranged the terms in a suggestive manner.

Clearly, each term of the decomposition can be labeled by a pair (r, s) corresponding to the space

$$\mathcal{L}^{r,s}(M) = \left\{ A \in \Omega^{2r+s}(M) \,\big|\, A = \frac{1}{r!}L^r B_s \ \text{with} \ \Lambda B_s = 0 \right\} \tag{2.6}$$

Notice that the indices r and s each takes value between 0 and n. And we can naturally arrange all $\mathcal{L}^{r,s}$'s into a *pyramid* as in Fig. 1 (in Section 1), having rotated the terms in Fig. 2 counterclockwise by $90°$. The symplectic pyramid is the analog of the (p, q) diamond of complex geometry.

To distinguish the different $\mathcal{L}^{r,s}(M)$ spaces, we shall make use of the operator H and also introduce the operator R which picks out the r index.

Definition 2.1 On a symplectic manifold, (M, ω), the R operator acts on an element $L^{r,s} \in \mathcal{L}^{r,s}(M)$ as

$$R\,L^{r,s} = r\,L^{r,s} \,. \tag{2.7}$$

The s index is discerned by the $(H + 2R)$ operator

$$(H + 2R)\,L^{r,s} = (n - s)L^{r,s} \tag{2.8}$$

where again $L^{r,s} \in \mathcal{L}^{r,s}(M)$. Note that acting on $\mathcal{L}^{r,s}(M)$, L and Λ raises and lower R by one, respectively. More precisely, we have the following useful relations relating (L, Λ, H, R).

Lemma 2.2 *On a symplectic manifold (M, ω), the following relations hold:*

(i) $[\Lambda\,,\,L^r] = (H + r - 1)\,r\,L^{r-1}$ *for $r \geq 1$;*

(ii) $L\Lambda = (H + R + 1)R\,;$

(iii) $\Lambda L = (H + R)(R + 1)\,.$

Proof. (i) follows straightforwardly from repeated applications of the sl_2 algebra commutation relations in (2.1). (ii) and (iii) can be checked by acting on $\mathcal{L}^{r,s}(M)$ and using (i).

$\square$

Let us now introduce the symplectic star operator $*_s : \Omega^k(M) \to \Omega^{2n-k}(M)$ introduced in [3, 1]. It is defined by the local inner product

$$A \wedge *_s A' = (\omega^{-1})^k (A, A')\, d\,\mathrm{vol}$$
$$= \frac{1}{k!}(\omega^{-1})^{i_1 j_1}(\omega^{-1})^{i_2 j_2} \ldots (\omega^{-1})^{i_k j_k}\, A_{i_1 i_2 \ldots i_k}\, A'_{j_1 j_2 \ldots j_k}\, \frac{\omega^n}{n!}\,. \tag{2.9}$$

We note that $*_s *_s = 1$ which follows from Weil's relation [13, 4],

$$*_s \frac{L^r}{r!} B_s = (-1)^{s(s+1)/2} \frac{L^{n-r-s}}{(n - r - s)!} B_s \,. \tag{2.10}$$

Therefore, acting on $\mathcal{L}^{r,s}(M)$, we have that

$$*_s : \mathcal{L}^{r,s}(M) \to \mathcal{L}^{n-r-s,s}(M).$$

In particular, for forms of middle degree $k = 2r + s = n$ or equivalently $r = \frac{1}{2}(n - s)$, the action of the $*_s$ operator leaves them invariant up to a -1 factor. And consider all $\mathcal{L}^{r,s}(M)$ elements together as in the pyramid diagram in Fig. 1, the action of $*_s$ is a reflection with respect to the central vertical axis.

Finally, let us briefly discuss the linear structure - the primitive exterior vector space. Let V be a real symplectic vector space of dimension $d = 2n$. We write $\bigwedge^k V$ for the k-exterior product of V. Let $e_1, e_2, \ldots, e_{2n}$ be a basis for V and take the symplectic form to be $\omega = e_1 \wedge e_2 + \ldots + e_{2n-1} \wedge e_{2n}$. Let $P\bigwedge^k V$ denote the primitive elements of $\bigwedge^k V$. The symplectic pyramid as in Fig. 1 allows us to relate the dimension of $P\bigwedge^k V$ with the dimension of $\bigwedge^k V$. Specifically, for $k \le n$, it easy to see from the pyramid diagram that

$$\dim P\bigwedge^k V = \dim \bigwedge^k V - \dim \bigwedge^{k-2} V = \binom{2n}{k} - \binom{2n}{k-2}. \tag{2.11}$$

Moreover, the sum of the dimensions of all primitive exterior vector space is given by

$$\sum_{k=0}^n \dim P\bigwedge^k V = \dim \bigwedge^{n-1} V + \dim \bigwedge^n V = \binom{2n}{n-1} + \binom{2n}{n}. \tag{2.12}$$

Let us also give a canonical recursive method to write down the set of basis elements of $P\bigwedge^k V$. The idea is to construct the basis elements of dimension $d = 2n$ from those of dimension $d = 2(n-1)$. For instance, selecting out the e_1 and e_2 elements, we have the following decomposition.

Lemma 2.3 *Let V be a symplectic vector space with the non-degenerate form $\omega = e_{12} + e_{34} + \ldots + e_{2n-1,2n}$ (where the notation $e_{12} = e_1 \wedge e_2$). Then any element of the primitive exterior vector space $\mu_k \in P\bigwedge^k V$ can be expressed as*

$$\mu_k = e_1 \wedge \beta_1 + e_2 \wedge \beta_2 + (e_{12} - \frac{1}{H+1} \sum_{j=2}^n e_{2j-1,2j}) \wedge \beta_3 + \beta_4 \tag{2.13}$$

where $\beta_1, \beta_2 \in P\bigwedge^{k-1}V$, $\beta_3 \in P\bigwedge^{k-2}V$, and $\beta_4 \in P\bigwedge^k V$ and $\beta_1, \beta_2, \beta_3, \beta_4$ do not contain e_1 and e_2.

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Proof. Generally, we can write

$$\mu = e_1 \wedge \alpha_1 + e_2 \wedge \alpha_2 + e_{12} \wedge \alpha_3 + \alpha_4 \tag{2.14}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are exterior products of $e_3, e_4, \ldots, e_{2n-1}, e_{2n}$. Primitivity of μ implies

$$\Lambda\mu = e_1 \wedge \Lambda\alpha_1 + e_2 \wedge \Lambda\alpha_2 + e_{12} \wedge \Lambda\alpha_3 + \alpha_3 + \Lambda\alpha_4 = 0$$

giving the conditions

$$\Lambda\alpha_1 = \Lambda\alpha_2 = \Lambda\alpha_3 = 0 \tag{2.15}$$

$$\alpha_3 + \Lambda\alpha_4 = 0 \tag{2.16}$$

Hence, α_i for $i = 1, 2, 3$ must be primitive and we will denote these α_i by $\beta_i \in P\bigwedge^* V$ to highlight their primitive property. Now, α_4 is not primitive. But with (2.16) and $\alpha_3 = \beta_3$ being primitive, we have $\Lambda^2\alpha_4 = 0$. Thus we can write $\alpha_4 = \beta_4 + L\beta'_4$ with $\beta_4, \beta'_4 \in P\bigwedge^* V$, and moreover, using (2.16) again, we have $\beta'_4 = -H^{-1}\beta_3$. Altogether, (2.14) becomes

$$\mu_k = e_1 \wedge \beta_1 + e_2 \wedge \beta_2 + e_{12} \wedge \beta_3 + \beta_4 - LH^{-1}\beta_3$$

$$= e_1 \wedge \beta_1 + e_2 \wedge \beta_2 + \frac{H+1}{H+2}\left[e_{12} - \frac{1}{H+1}(e_{34} + \ldots + e_{2n-1,2n})\right] \wedge \beta_3 + \beta_4 \ .$$

Lastly, scaling $\beta_3 \to \dfrac{H+2}{H+1}\beta_3$, we obtain (2.13). $\qquad\square$

With Lemma 2.34, we have at hand a recursive algorithm to write down a basis for $P\bigwedge^k V$ for V of any arbitrary even dimension.

2.2 Differential operators and cohomologies

In this subsection, we consider the action of differential operators on $\mathcal{L}^{r,s}(M)$. We start with the exterior derivative, d. We obtain

$$d\left(\frac{L^r}{r!}B_s\right) = \frac{L^r}{r!}d(B_s)$$

$$= \frac{L^r}{r!}B^0_{s+1} + \frac{L^{r+1}}{r!}B^1_{s-1} \tag{2.17}$$

where we noted the symplectic condition $[d, L] = 0$ and the useful formula [1]

$$dB_s = B^0_{s+1} + L\, B^1_{s-1} \tag{2.18}$$

where $B^0_{s+1} \in \mathcal{P}^{s+1}(M)$ and $B^1_{s-1} \in \mathcal{P}^{s-1}(M)$ are primitive forms, and moreover, $B^0_{s+1} = 0$ if $s = n$. (2.18) follows simply (see e.g. [5, 11]) from first writing down the Lefschetz decomposition for dB_s,

$$dB_s = B^0_{s+1} + L\, B^1_{s-1} + \frac{1}{2!} L^2\, B^2_{s-3} + \frac{1}{3!} L^3\, B^3_{s-5} + \dots\ . \tag{2.19}$$

and then applying L^{n-s+1} to both sides of (2.19). Since the left hand side would then be zero by the primitive condition, $L^{n-s+1}B_s = 0$, each term on the right hand side (with an additional L^{n-s+1}) must also be zero. This results in the requirement that $B^2_{s-3}, B^3_{s-5}, \dots$ in (2.19) must be identically zero.

In all, we have the result that d acting on $\mathcal{L}^{r,s}$ leads to at most two terms,

$$d:\ \mathcal{L}^{r,s}\ \to\ \mathcal{L}^{r,s+1} \oplus\ \mathcal{L}^{r+1,s-1}\ . \tag{2.20}$$

As explained in the Introduction (Section 1) via an analogy to the complex geometry case, (2.20) naturally gives us a decomposition of the exterior derivative operator in symplectic geometry. And indeed we can define the decomposition of d into two linear differential operators (∂_+, ∂_-) by writing

$$d = \partial_+ + L\, \partial_-\ . \tag{2.21}$$

By comparing (2.17) and (2.21), we have the following definition.

Definition 2.4 On a symplectic manifold (M, ω), the first order differential operators $\partial_+ : \mathcal{L}^{r,s}(M) \to \mathcal{L}^{r,s+1}(M)$ and $\partial_- : \mathcal{L}^{r,s}(M) \to \mathcal{L}^{r,s-1}(M)$ are defined by the property

$$\partial_+ \left(\frac{L^r}{r!} B_s \right) = \frac{L^r}{r!} B^0_{s+1} \tag{2.22}$$

$$\partial_- \left(\frac{L^r}{r!} B_s \right) = \frac{L^r}{r!} B^1_{s-1} \tag{2.23}$$

where $B_s, B^0_{s+1}, B^1_{s-1} \in \mathcal{P}^*(M)$ and $dB_s = B^0_{s+1} + L\, B^1_{s-1}$.

Note that we can restrict to the primitive subspace of differential forms by setting $r = 0$ above. Then ∂_+ and ∂_- become the projections of (dB_s) to the primitive terms, B^0_{s+1} and

---

[1](2.18) is just (1.2) with $\omega\wedge$ replaced here by L.

B^1_{s-1}, respectively. Therefore, $\partial_\pm : \mathcal{P}^s(M) \to \mathcal{P}^{s\pm1}(M)$ preserve primitivity and are the natural operators on the space of primitive differential forms $\mathcal{P}^*(M)$.

With Definition 2.4, we have the following properties.

Lemma 2.5 *On a symplectic manifold, (M^{2n}, ω), the symplectic differential operators (∂_+, ∂_-) satisfy the following: (i) $[\partial_+, L] = [L\,\partial_-, L] = 0$; (ii) $(\partial_+)^2 = (\partial_-)^2 = 0$; (iii) $\partial_+\partial_- = -\,\partial_-\partial_+$.*

Proof. Using $d = \partial_+ + L\,\partial_-$ and the uniqueness of the Lefschetz decomposition, these relations follow directly from $[d, L] = 0$ for (i), and $d^2 = 0$ for (ii)-(iii). $\qquad\qquad\square$

We remark that the relation in Lemma 2.5(i), $[L\,\partial_-, L] = 0$, simplifies to $[\partial_-, L] = 0$ when acting on $\mathcal{L}^{r,s}(M)$ for $r + s < n$. When $r + s = n$, then the additional L operator is needed in $[L\,\partial_-, L] = 0$ since the primitive condition implies $L\,\mathcal{L}^{n-s,s} = 0$. So effectively, Lemma 2.5 does imply that both ∂_+ and ∂_- commute with L, besides squaring to zero and anticommuting with each other.

Let us now consider the symplectic differential operator, d^Λ, and its action on $\mathcal{L}^{r,s}(M)$. Recall that acting on a differential k-form, it is defined as

$$d^\Lambda := d\,\Lambda - \Lambda\,d$$
$$= (-1)^{k+1} *_s\, d\, *_s \ .$$

where in the second line, d^Λ is expressed as the symplectic adjoint of d with respect to the symplectic star operator, $*_s$, defined by (2.9). Using Lemma 2.2(i) and (2.18), we can write

$$d\Lambda \frac{L^r}{r!} B_s = (H + R + 1)\frac{L^{r-1}}{(r-1)!} B^0_{s+1} + R(H + R)\frac{L^r}{r!} B^1_{s-1}$$

$$\Lambda d \frac{L^r}{r!} B_s = (H + R)\frac{L^{r-1}}{(r-1)!} B^0_{s+1} + (R + 1)(H + R)\frac{L^r}{r!} B^1_{s-1}$$

where for instance, $(R + 1)(H + R)\frac{L^r}{r!} B^1_{s-1} = (r+1)(n - r - s + 1)\frac{L^r}{r!} B^1_{s-1}$. Taking their difference, we obtain

$$d^\Lambda \frac{L^r}{r!} B_s = \frac{L^{r-1}}{(r-1)!} B^0_{s+1} - (H + R)\frac{L^r}{r!} B^1_{s-1} \tag{2.24}$$

which implies

$$d^\Lambda : \mathcal{L}^{r,s} \to \mathcal{L}^{r-1,s+1} \oplus \mathcal{L}^{r,s-1} \ . \tag{2.25}$$

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and the decomposition

$$d^\Lambda = \frac{1}{(H+R)} \Lambda\,\partial_+ - (H+R)\,\partial_- \tag{2.26}$$

where the notation $\frac{1}{H+R} = (H+R)^{-1}$ just inverts the constants, e.g. $(H+R)^{-1}\left(\frac{L^r}{r!}B_s\right) = (n-r-s)^{-1}\left(\frac{L^r}{r!}B_s\right)$.

We can now give an explicit expression for ∂_+ and ∂_- in terms of d and d^Λ. Comparing (2.21) with (2.26) and using Lemma (2.2), we obtain the following expressions.

Lemma 2.6 *On a symplectic manifold* (M,ω), ∂_+ *and* ∂_- *can be expressed as*

$$\partial_+ = \frac{1}{H+2R+1}[(H+R+1)d + L d^\Lambda]\ , \tag{2.27}$$

$$\partial_- = \frac{-1}{H+2R+1}[d^\Lambda - \frac{1}{H+R}\Lambda d\,]\ . \tag{2.28}$$

Let us point out first that the operator $(H+2R+1)$ always has a non-zero action on $\mathcal{L}^{r,s}$, since the corresponding eigenvalue $(n-s+1) > 0$ is always positive. Now, we could have equivalently defined ∂_+ and ∂_- using the expressions (2.27) and (2.28). It is straightforward to check that $\partial_\pm$ defined this way satisfy Definition 2.4. Moreover, since d^Λ is the symplectic adjoint of d, that is

$$d^\Lambda = (-1)^{k+1} *_s d *_s = (-1)^{k+1} *_s (\partial_+ + L\,\partial_-) *_s\ ,$$

it can also be verified using (2.27) and (2.28) that

$$\partial_+{}^{*_s} := (-1)^{k+1} *_s \partial_+ *_s = \frac{1}{H+R} \Lambda\,\partial_+$$

$$(L\,\partial_-)^{*_s} := (-1)^{k+1} *_s (L\,\partial_-) *_s = -(H+R)\,\partial_-\ .$$

which are consistent with (2.26).

With d and d^Λ, we can proceed to consider their composition, dd^Λ. This second-order differential operator appears naturally in symplectic cohomologies [11]. We can calculate dd^Λ : $\mathcal{L}^{r,s}(M) \to \mathcal{L}^{r,s}(M)$ using (2.21) and (2.26). We find

$$\begin{aligned}
dd^\Lambda &= (\partial_+ + L\,\partial_-)\left(\frac{1}{H+R}\Lambda\,\partial_+ - (H+R)\,\partial_-\right) \\
&= -\partial_+(H+R)\partial_- + \partial_- L\frac{1}{H+R}\Lambda\partial_+ \\
&= -\partial_+(H+R)\partial_- + \partial_- R\partial_+ \\
&= -(H+2R+1)\,\partial_+\partial_- \tag{2.29}
\end{aligned}$$

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Thus we have $dd^\Lambda \sim \partial_+\partial_-$.

As we have emphasized, the action of the differential operators $(\partial_+ , \partial_- , \partial_+\partial_-)$ on $\mathcal{L}^{r,s}$ reduces to their action on the primitive elements $\mathcal{L}^{0,s} = \mathcal{P}^s$. Acting on primitive forms, the expressions for the differential operators simplify.

Lemma 2.7 *Acting on primitive differential forms, the operators, $(\partial_+ , \partial_- , \partial_+\partial_-)$ have the following expressions*
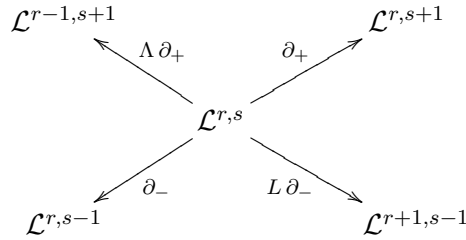
$$\partial_+ = d - L\,H^{-1}\Lambda\,d \; , \tag{2.30}$$

$$\partial_- = \frac{1}{H}\,\Lambda\,d \; , \tag{2.31}$$

$$\partial_+\partial_- = -(H+1)\,d\,d^\Lambda = (H+1)\,d\Lambda d \; , \tag{2.32}$$

and moreover, $d^\Lambda = -H\partial_-$.

And finally, to conclude this subsection, let us note that the elements on the symplectic pyramid can be connected by first-order differential operators as follows.

In the above diagram, the right-pointing arrows with operators $(\partial_+ , \, L\,\partial_-)$ are associated with d , while the left-pointing ones $(\Lambda\,\partial_+ , \, \partial_-)$ are associated with d^Λ . From the diagram, we have two natural sets of differential complexes.

$$\mathcal{L}^{r,0} \xrightarrow{\partial_+} \mathcal{L}^{r,1} \xrightarrow{\partial_+} \quad \cdots \quad \xrightarrow{\partial_+} \mathcal{L}^{r,n-r-1} \xrightarrow{\partial_+} \mathcal{L}^{r,n-r}$$

$$\mathcal{L}^{r,0} \xleftarrow{\partial_-} \mathcal{L}^{r,1} \xleftarrow{\partial_-} \quad \cdots \quad \xleftarrow{\partial_-} \mathcal{L}^{r,n-r-1} \xleftarrow{\partial_-} \mathcal{L}^{r,n-r}$$

We can construct cohomologies with them. Define

$$H^{r,s}_{\partial_+}(M) = \frac{\ker \partial_+ \cap \mathcal{L}^{r,s}}{\partial_+ \mathcal{L}^{r,s-1}} \; ,$$

and

$$H^{r,s}_{\partial_-}(M) = \frac{\ker \partial_- \cap \mathcal{L}^{r,s}}{\partial_- \mathcal{L}^{r,s+1}} \; ,$$

for $r < n - s$. But by the commutativity of $\partial_{\pm}$ with L, we have $H^{r,s}_{\partial_+}(M) \cong H^{0,s}_{\partial_+}(M)$ and $H^{r,s}_{\partial_-}(M) \cong H^{0,s}_{\partial_-}(M)$ for any $r < n - s$. Hence, we will focus on the two primitive cohomologies

$$PH^s_{\partial_+}(M) = \frac{\ker \partial_+ \cap \mathcal{P}^s}{\partial_+ \mathcal{P}^{s-1}} \ ,$$

and

$$PH^s_{\partial_-}(M) = \frac{\ker \partial_- \cap \mathcal{P}^s}{\partial_- \mathcal{P}^{s+1}} \ ,$$

for $s < n$.

Besides these two cohomologies, let us just note that three other primitive cohomologies were introduced in Paper I [11]. They can be found in Table 1 in Section 1, expressed in terms of ∂_+ and ∂_-.

2.3 A symplectic elliptic complex

We now show that $PH^*_{\partial_{\pm}}(M)$ is finite dimensional. Since we have naturally the two differential complexes,

$$0 \longrightarrow \mathcal{P}^0 \xrightarrow{\partial_+} \mathcal{P}^1 \xrightarrow{\partial_+} \ \ldots \ \xrightarrow{\partial_+} \mathcal{P}^{n-1} \xrightarrow{\partial_+} \mathcal{P}^n$$

$$0 \xleftarrow{\partial_-} \mathcal{P}^0 \xleftarrow{\partial_-} \mathcal{P}^1 \xleftarrow{\partial_-} \ \ldots \ \xleftarrow{\partial_-} \mathcal{P}^{n-1} \xleftarrow{\partial_-} \mathcal{P}^n$$

it is interesting to ask whether they are elliptic. Unfortunately, these two complexes are not elliptic: the ellipticity property breaks down at $\mathcal{P}^n$ since ∂_+ maps all primitive n-forms to zero, and for ∂_-, there is no primitive $\mathcal{P}^{n+1}$ space. We may try to consider connecting the two complexes by joining them as follows

$$\ldots \xrightarrow{\partial_+} \mathcal{P}^{n-1} \xrightarrow{\partial_+} \mathcal{P}^n \xrightarrow{\partial_-} \mathcal{P}^{n-1} \xrightarrow{\partial_-} \ \ldots$$

But such a combined complex is unfortunately no longer a differential complex as $\partial_- \partial_+ \neq 0$. Fortunately, there is a way to obtain a differential elliptic complex if we utilize the second-order differential operator $\partial_+ \partial_-$.

Proposition 2.8 *The following complex is elliptic.*[2]

$$
0 \xrightarrow{\partial_+} \mathcal{P}^0 \xrightarrow{\partial_+} \mathcal{P}^1 \xrightarrow{\partial_+} \cdots \xrightarrow{\partial_+} \mathcal{P}^{n-1} \xrightarrow{\partial_+} \mathcal{P}^n \qquad (2.33)
$$

$$
\downarrow \partial_+ \partial_-
$$

$$
0 \xleftarrow{\partial_-} \mathcal{P}^0 \xleftarrow{\partial_-} \mathcal{P}^1 \xleftarrow{\partial_-} \cdots \xleftarrow{\partial_-} \mathcal{P}^{n-1} \xleftarrow{\partial_-} \mathcal{P}^n
$$

Proof. Clearly, the above is a differential complex. We need to show that the associated symbol complex is exact at each point $x \in M$. Let $\xi \in T_x^* - \{0\}$. By an $Sp(2n)$ transformation, we can set $\xi = e_1$ and take the symplectic form to be $\omega = e_{12} + e_{34} + \ldots + e_{2n-1,2n}$ with $e_1, \ldots, e_{2n}$ providing a basis for T_x^*. For $\mu_k \in P\bigwedge^k T_x^*$, an element in the primitive exterior vector space, we can use the decomposition of Lemma 2.34 to write

$$
\mu_k = e_1 \wedge \beta_1 + e_2 \wedge \beta_2 + (e_{12} - \frac{1}{H+1} \sum_{j=2}^{n} e_{2j-1,2j}) \wedge \beta_3 + \beta_4 \qquad (2.34)
$$

where $\beta_1, \beta_2, \beta_3, \beta_4 \in P\bigwedge^* T_x^*$ are primitive exterior products involving only $e_3, e_4, \ldots, e_{2n-1}, e_{2n}$. Note that when $k = n$, then $\beta_4 = 0$ since there are no primitive n-form without either e_1 or e_2.

From Lemma 2.7, the symbol of the differential operators are given by

$$
\sigma(\partial_+)(x, \xi)\,\mu = (1 - LH^{-1}\Lambda)(\xi \wedge \mu) ,
$$

$$
\sigma(\partial_-)(x, \xi)\,\mu = H^{-1}\Lambda(\xi \wedge \mu) ,
$$

$$
\sigma(\partial_+\partial_-)(x, \xi)\,\mu = (H+1)^{-1}[\xi \wedge (\Lambda(\xi \wedge \mu))] .
$$

Letting $\xi = e_1$ and μ taking the form of (2.34), we have that

$$
\operatorname{im} \sigma(\partial_+) = \left\{ (e_{12} - \frac{1}{H+1} \sum_{j=2}^{n} e_{2j-1,2j}) \wedge \beta_2, e_1 \wedge \beta_4 \right\} \qquad (2.35)
$$

$$
\operatorname{im} \sigma(\partial_-) = \{\beta_2, e_1 \wedge \beta_3\} , \qquad (2.36)
$$

$$
\operatorname{im} \sigma(\partial_+\partial_-) = \{e_1 \wedge \beta_2\} , \qquad (2.37)
$$

---

[2]After proving this proposition, we searched the literature for any mention of such a symplectic elliptic complex. We found only that the simple $n = 2$, $d = 4$ case has appeared in [10]. It was presented there as an example of an elliptic complex that does not imply the corresponding local Poincaré lemmas (which we also had found and is described here in Proposition 3.13)

which imply

$$\ker \sigma(\partial_+) = \left\{ e_1 \wedge \beta_1, (e_{12} - \frac{1}{H+1} \sum_{j=2}^{n} e_{2j-1,2j}) \wedge \beta_3 \right\} , \tag{2.38}$$

$$\ker \sigma(\partial_-) = \{ e_1 \wedge \beta_1, \beta_4 \} , \tag{2.39}$$

$$\ker \sigma(\partial_+ \partial_-) = \left\{ e_1 \wedge \beta_1, (e_{12} - \frac{1}{H+1} \sum_{j=2}^{n} e_{2j-1,2j}) \wedge \beta_3, \beta_4 \right\} . \tag{2.40}$$

Comparing (2.38)-(2.40) with (2.35)-(2.37), and noting that for $k = n$, $\beta_4 = 0$, we find that the symbol sequence is exact, i.e. $\ker \sigma(D_i) = \operatorname{im} \sigma(D_{i-1})$, as required. $\qquad \square$

With an elliptic complex, the associated cohomologies are finite-dimensional. The finiteness of

$$PH^n_{d+d^\Lambda}(M) = \frac{\ker \partial_- \cap \mathcal{P}^n(M)}{\operatorname{im} \partial_+ \partial_- \cap \mathcal{P}^n(M)} , \qquad PH^n_{dd^\Lambda}(M) = \frac{\ker \partial_+ \partial_- \cap \mathcal{P}^n(M)}{\operatorname{im} \partial_+ \cap \mathcal{P}^n(M)} ,$$

were proved previously in Paper I [11]. But we have now also shown the finiteness of $PH^k_{\partial_\pm}(M)$.

Corollary 2.9 *The cohomologies $PH^k_{\partial_+}(M)$ and $PH^k_{\partial_-}(M)$ for $0 \le k < n$ are finite-dimensional.*

3   Properties of $PH_{\partial_\pm}(M)$

3.1   Primitive harmonic forms and isomorphism of $PH_{\partial_+}(M)$ and $PH_{\partial_-}(M)$

To analyze the properties of $PH^*_{\partial_\pm}(M)$, we shall make use of a compatible triple (ω, J, g) of symplectic form, almost complex structure, and Riemannian metric, present on all symplectic manifolds. The Riemannian metric g gives us the standard inner product on differential forms

$$(A, A') = \int_M A \wedge *A' = \int_M g(A, A') \, d\mathrm{vol} , \qquad A, A' \in \Omega^k(M) \tag{3.1}$$

With an inner product, we can define the adjoint operators $(\partial_+^*, \partial_-^*)$. They can easily be expressed in terms of d^* and $d^{\Lambda *}$ using Lemma 2.6.

Lemma 3.1 *On a symplectic manifold (M, ω) with a compatible Riemannian metric g, the adjoints $(\partial_+^*, \partial_-^*)$ take the form*

$$\partial_+^* = \left[d^*(H + R + 1) + d^{\Lambda *} \Lambda \right] \frac{1}{H + 2R + 1} \tag{3.2}$$

$$\partial_-^* = - \left[d^{\Lambda *} - d^* L \frac{1}{H + R} \right] \frac{1}{H + 2R + 1} \tag{3.3}$$

With the adjoint operators at hand, we can define the associated harmonic forms for $\partial_\pm$ operators. The natural $\partial_\pm$ Laplacian is the second-order differential operator

$$\Delta_{\partial_\pm} = \partial_\pm (\partial_\pm)^* + (\partial_\pm)^* \partial_\pm \; , \tag{3.4}$$

which leads to the following definition.

Definition 3.2 A primitive differential form $B_k \in \mathcal{P}^k(M)$ for $k < n$ is called $\partial_\pm$-*harmonic* if $\Delta_{\partial_\pm} B = 0$, or equivalently,

$$\partial_\pm B_k = 0 \quad \text{and} \quad (\partial_\pm)^* B_k = 0 \; . \tag{3.5}$$

We denote the space of $\partial_\pm$-harmonic k-forms by $P\mathcal{H}_{\partial_\pm}^k(M)$.

Now the elliptic complex (2.33) implies that $\Delta_{\partial_\pm}$ are elliptic operators. Thus, applying Hodge theory, we immediately have the following theorem.

Theorem 3.3 *Let M be a compact symplectic manifold. For any compatible triple (ω, J, g), we define the standard inner product on $\mathcal{P}^k(M)$ with respect to g. Then, for $k < n$,*

(i) $\dim \mathcal{H}_{\partial_\pm}^k(M) < \infty$;

(ii) *There is an orthogonal decomposition*

$$\mathcal{P}^k = P\mathcal{H}_{\partial_\pm}^k \oplus \partial_\pm \mathcal{P}^{k\pm 1} \oplus (\partial_\pm)^* \mathcal{P}^{k\mp 1} \; ; \tag{3.6}$$

(iii) *There is a canonical isomorphism: $P\mathcal{H}_{\partial_\pm}^k(M) \cong PH_{\partial_\pm}^k(M)$.*

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Having demonstrated the finiteness of $PH_{\partial_\pm}(M)$, let us compare the solution space of $\partial_\pm$-harmonic forms. We will need to make use of the almost complex structure J and the relation between the Hodge star operator and the symplectic star operator [11] given by

$$* = \mathcal{J} *_s \tag{3.7}$$

where

$$\mathcal{J} = \sum_{p,q} (\sqrt{-1})^{p-q}\, \Pi^{p,q}$$

projects a k-form onto its (p,q) parts times the multiplicative factor $(\sqrt{-1})^{p-q}$. Interestingly, we find that $(\partial_+, \partial_+^*)$ is $\mathcal{J}$-conjugate to $(\partial_-^*, \partial_-)$ up to a non-zero constant.

Lemma 3.4 *For a compatible triple* (ω, J, g) *on a symplectic manifold,*

$$\mathcal{J}\,\partial_+\,\mathcal{J}^{-1} = \partial_-^*\,(H+R)\ , \tag{3.8}$$

$$\mathcal{J}\,\partial_+^*\,\mathcal{J}^{-1} = (H+R)\,\partial_-\ . \tag{3.9}$$

Proof. Acting on a k-form, we have

$$
\begin{aligned}
\mathcal{J}\,\partial_+\,\mathcal{J}^{-1} &= \mathcal{J}\,\frac{1}{H+2R+1}\left[(H+R+1)d + Ld^\Lambda\right]\mathcal{J}^{-1}\\
&= \frac{(-1)^{k+1}}{H+2R+1}\left[(H+R+1)\mathcal{J}*_s d^\Lambda *_s \mathcal{J}^{-1} + \mathcal{J}*_s \Lambda d *_s \mathcal{J}^{-1}\right]\\
&= \frac{-1}{H+2R+1}\left[(H+R+1)*d^\Lambda* + *\Lambda d*\right]\\
&= \frac{1}{H+2R+1}\left[-(H+R+1)d^{\Lambda*} + Ld^*\right]\\
&= \frac{1}{H+2R+1}\left[-(H+R)d^{\Lambda*} + d^*L\right]\\
&= (H+R+1)\partial_-^* = \partial_-^*(H+R)
\end{aligned}
$$

where we have used the expressions for ∂_+ and ∂_-^* in Lemma 2.6 and Lemma 3.1, and various relations involving $*$ and $*_s$. In particular, we applied $d = (-1)^{k+1}(*_s d^\Lambda *_s)$ and $L = *_s\Lambda*_s$ in line two, (3.7) and $*_s\mathcal{J}^{-1} = *(-1)^k$ in line three, $L = (-1)^k * \Lambda*$ in line four, and the relation $d^{\Lambda*} = [L, d^*]$ in line five. As for the second equation, it can be derived similarly, or interpreted simply as the Hodge adjoint of the first equation. $\square$

Thus, by Lemma 3.4, $B_k \in P^k(M)$ is ∂_+-harmonic if and only if $\mathcal{J}B_k$ which is also primitive is ∂_--harmonic. This implies that the two harmonic spaces are isomorphic, and moreover, by Theorem 3.3(iii), the two respective primitive cohomologies are also isomorphic.

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Proposition 3.5 *Let (M,ω) be a compact symplectic manifold and let $0 \leq k < n$. Then $PH_{\partial_+}^k(M) \cong PH_{\partial_-}^k(M)$ and $PH_{\partial_+}^k(M) \cong PH_{\partial_-}^k(M)$.*

Coupled with the isomorphism of $PH_{d+d^\Lambda}^n(M) \cong PH_{dd^\Lambda}^n(M)$, we find that the analytical index of the elliptic complex (2.33) is trivial.

Corollary 3.6 *The index of the elliptic complex of (2.33) is zero.*

Let us note further that the isomorphism between $PH_{\partial_+}^k(M)$ and $PH_{\partial_-}^k(M)$ leads to a natural pairing between the two cohomologies, similar to that for $PH_{d+d^\Lambda}^n(M)$ and $PH_{dd^\Lambda}^n(M)$ described in Paper I [11, Prop. 3.26],

Proposition 3.7 *On a compact symplectic manifold (M,ω), there is a natural pairing*

$$PH_{\partial_+}^k(M) \otimes PH_{\partial_-}^k(M) \longrightarrow \mathbb{R} \ ,$$

defined by

$$B_k \otimes B_k' \longrightarrow \int_M \frac{1}{(n-k)!} \, \omega^{n-k} \wedge B_k \wedge B_k' \ ,$$

which is non-degenerate.

Proof. Let us first interpret the integral. Combining (2.10) and (3.7), we obtain the well-known relation (see e.g. [5]),

$$* \frac{1}{r!} L^r B_k = (-1)^{\frac{k(k+1)}{2}} \frac{1}{(n-k-r)!} \, L^{n-k-r} \mathcal{J}(B_k) \ .$$

Hence, the integral can be re-written as

$$\int_M \frac{1}{(n-k)!} \, \omega^{n-k} \wedge B_k \wedge B_k' = (-1)^{\frac{k(k+1)}{2}} \int_M B_k \wedge *(\mathcal{J}^{-1} B_k) \ .$$

In this form and noting Lemma 3.4, it is clear that the pairing is well-defined since the integral is independent of the choice of the representatives of the two cohomology classes. Now to show non-degeneracy, we can choose B_k and B_k' to be the respective harmonic representatives. In particular, let $B_k \in PH_{\partial_+}^k(M)$ and $B_k' = \mathcal{J} B_k \in PH_{\partial_-}^k(M)$. We thus have for $B_k \neq 0$

$$B_k \otimes *B_k' \longrightarrow \int_M \frac{1}{(n-k)!} \, \omega^{n-k} \wedge B_k \wedge B_k' = (-1)^{\frac{k(k+1)}{2}} \|B_k\|^2 \neq 0 \ .$$

$\square$

3.2 Local primitive Poincaré lemmas

We now consider local Poincaré lemmas for the various cohomologies we have studied. Except for cohomologies of degree zero forms and the cohomology $PH^1_{\partial_+}$, all other local primitive cohomologies turn out to be trivial. At the end of this subsection, we shall use the ∂_--Poincaré lemma to demonstrate the equivalence of $PH_{\partial_-}(M)$ with the Čech cohomology of $\mathcal{P}'^n$, the space of ∂_--closed primitive n-forms.

On a open unit disk, the Poincaré lemma states that only $H^0_d(U)$ is non-empty. By the symplectic star operation, there is also the d^Λ-Poincaré lemma

Lemma 3.8 *(d^Λ-Poincaré lemma)Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. If $A_k \in \Omega^k(U)$ is d^Λ-closed and $k < 2n$, then there exists a $A'_{k+1} \in \Omega^{k+1}(U)$ such that $A_k = d^\Lambda A'_{k+1}$.*

Proof. Let $\tilde{A}_{2n-k} = *_s A_k$. Then $d^\Lambda A_k = (-1)^{k+1} *_s d *_s A_k = (-1)^{k+1} *_s d\tilde{A}_{2n-k} = 0$. By the Poincaré lemma, we can write $\tilde{A}_{2n-k} = (-1)^k d\tilde{A}'_{2n-k-1}$, where the additional $(-1)^k$ factor is inserted for convenience. Then, letting $A'_{k+1} = *_s \tilde{A}'_{2n-k-1}$, we have

$$A_k = *_s \tilde{A}_{2n-k} = (-1)^k *_s d\tilde{A}'_{2n-k-1} = (-1)^k *_s d *_s A'_{k+1} = d^\Lambda A'_{k+1} \ .$$

$\square$

Proposition 3.9 *(Primitive Poincaré lemma) Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. If $B_k \in P^k(U)$ is d-closed and $0 < k \leq n$, then there exists a form $B'_{k-1} \in P'^{k-1}(U)$ such that $B_k = dB'_{k-1}$.*

Proof. By the Poincaré lemma, there exists a $(k-1)$-form with the property $B_k = dA_{k-1}$. We give a standard construction of A_{k-1} (see for example [8, Appendix 5]) and show that A_{k-1} turns out to be primitive and ∂_--closed.

Start with the radial vector field $V = x^i \partial_i$. Such a vector fields only scales differential forms. For instance, $\mathcal{L}_V \omega = 2\omega$. Hence, a primitive differential form remains primitive under a diffeomorphism generated by V. Acting on a primitive d-closed form, we have

$$\mathcal{L}_V B_k = d\, i_V B_k \ .$$

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Note that $i_V B_k$ is also primitive. Moreover, since $\mathcal{L}_V B_k$ remains primitive, this implies that $i_V B_k \in P'^{k-1}(U)$.

We introduce the operator $T : \Omega^k \to \Omega^k$ which is inverse to the Lie derivative $\mathcal{L}_V$ and commutes with d,

$$T\,\mathcal{L}_V = id\ , \qquad\qquad dT = T\,d\ .$$

It can be checked [8, p. 385] that such a T is given by

$$T\left(\frac{1}{k!}\,A_{i_1\ldots i_k}\,dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) = \frac{1}{k!}\left(\int_0^1 t^{k-1} A_{i_1\ldots i_k}(tx)\,dt\right) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

With these properties, we can write

$$B_k = (T\,\mathcal{L}_V)B_k = T\,d(i_V B_k) = d(T\,i_V B_k)$$

As mentioned $i_V B_k$ is a primitive $(k-1)$-form, that is $L^{n-k+2}(i_V B_k) = 0$. Clearly, we also have $L^{n-k+2} T(i_V B_k) = 0$ and so $B'_{k-1} := T(i_V B_k)$ must also be primitive. Lastly, since $dB'_{k-1} \in P^k(U)$, this implies $B'_{k-1} \in P'^{k-1}(U)$. $\qquad\square$

Proposition 3.10 (*Primitive dd^Λ-Poincaré lemma*) *Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. If $B_k \in \mathcal{P}^k(U)$ is d-closed and $0 < k \leq n$, then there exists a $B''_k \in \mathcal{P}^k(U)$ such that $B_k = dd^\Lambda B''_k$.*

Proof. By Proposition 3.9, since B_k is d-closed, we can write $B_k = dB'_{k-1}$ for some $B'_{k-1} \in P'^{k-1}(U)$. But since $d^\Lambda B'_{k-1} = 0$, by the d^Λ-Poincaré lemma, there exists $A''_k \in \Omega^k(U)$ such that $B'_{k-1} = d^\Lambda A''_k$ and hence $B_k = dd^\Lambda A''_k$. But this implies by Lemma 3.9 of [11] that there exists a primitive k-form B''_k such that $B_k = dd^\Lambda B''_k$. $\qquad\square$

Lefschetz decomposition and the commutativity of dd^Λ with the $sl(2)$ representation (L, Λ, H) [11] then implies that dd^Λ-Poincaré lemma holds for all differential forms.

Corollary 3.11 (*Local dd^Λ lemma*) *Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. If $A_k \in \Omega^k(U)$ is $d + d^\Lambda$-closed and $k > 0$, then there exists a $A'_k \in \Omega^k(U)$ such that $A_k = dd^\Lambda A'_k$.*

Proposition 3.12 *(Primitive $d + d^\Lambda$-Poincaré lemma) Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. If $B_k \in P^k(U)$ is dd^Λ-closed and $0 < k \le n$, then there exists primitive forms $\widehat{B}'_{k-1}$ and $\widehat{B}''_{k+1}$ such that $B_k = \partial_+ \widehat{B}'_{k-1} + \partial_- \widehat{B}''_{k+1}$ for $k < n$ and $B_k = \partial_+ \widehat{B}'_{k-1}$ for $k = n$.*

Proof. Let $B_{k-1} = d^\Lambda B_k$. Since $dB_{k-1} = 0$, dd^Λ-lemma implies we can write

$$B_{k-1} = dd^\Lambda B'_{k-1} = -d^\Lambda(dB'_{k-1})$$

Let $B''_k = B_k + dB'_{k-1}$ which is clearly d^Λ-closed. By the d^Λ-Poincaré lemma $B''_k = B_k + dB'_{k-1} = d^\Lambda A''_{k+1}$ or $B_k = dB'_{k-1} + d^\Lambda A''_{k+1}$. But then by Lemma 3.20 of [11], we can write $B_k = \partial_+ \widehat{B}'_{k-1} + \partial_- \widehat{B}''_{k+1}$. $\qquad\square$

Proposition 3.13 *(Primitive ∂_+-Poincaré lemma) Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. Then $\dim PH^0_{\partial_+}(U) = \dim PH^1_{\partial_+}(U) = 1$, $\dim PH^k_{\partial_+}(U) = 0$ for $2 \le k < n$, and $\dim PH^n_{\partial_+}(U) = \infty$.*

Proof. Let $B_k \in \mathcal{P}^k$ be ∂_+-closed. Then either (1) $dB_k = 0$ or (2) $\partial_+ B_k = 0$ but $dB_k = LB^1_{k-1} \ne 0$. In the case of (1), it follows from Proposition 3.9 that there exists a $B_{k-1} \in \mathcal{P}^{k-1}$ such that $B_k = \partial_+ B_{k-1}$. So let us consider case (2) which we will analyze in three parts.

(2a) Let $2 \le k < n$. Since $dB_k = LB^1_{k-1}$, we have

$$d^2 B_k = LdB^1_{k-1} = LB^{10}_k = 0 .$$

Since LB^{10}_k can not be identically zero unless $k = n$, we find that $dB^1_{k-1} = 0$. Now by the primitive Poincaré lemma, $B^1_{k-1} = d\widehat{B}_{k-2}$. Thus, $dB_k = LB^1_{k-1}$ implies

$$d(B_k - L\widehat{B}_{k-2}) = 0 \qquad \Longrightarrow \qquad B_k - L\widehat{B}_{k-2} = d\tilde{A}_{k-1}$$

Lefschetz decomposing $\tilde{A}_{k-1} = \tilde{B}_{k-1} + L\tilde{B}_{k-3} + \dots$, it is clear that $B_k = \partial_+ \tilde{B}_{k-1}$.

(2b) Let $k = 1$. If $dB_1 = LB^1_0$, then clearly $B_1 \ne dB_0 = \partial_+ B_0$. But with $d^2 B_1 = LdB^1_0 = 0$, which implies $dB^1_0 = 0$, i.e. B^1_0 is a constant. This gives us at most a one-parameter space for $PH^1_{\partial_+}(U)$. For if both $dB^A_1 = dB^B_1 = LB^1_0$, it follows from the d-Poincaré lemma that $B^A_1 = B^B_1 + dB^{AB}_0 = B^B_1 + \partial_+ B^{AB}_0$. Thus, B^A_1 and B^B_1 are in the same class in $PH^1_{\partial_+}(U)$. And the representative of $PH^1_{\partial_+}(U)$ is simply the tautological one-form.

(2c) Let $k = n$. $d^2 B_n = L\, dB_{n-1}^1 = L\, \partial_+ B_{n-1}^1$ is identically zero and gives no condition on B_{n-1}^1. If B_{n-1}^1 is d-closed as in case (1), then certainly $B_n = \partial_+ \widehat{B}_{n-1}$. But in general B_{n-1}^1 is arbitrary and so $\dim PH_{\partial_+}^n(U) = \infty$. $\qquad\square$

Proposition 3.14 *(Primitive ∂_--Poincaré lemma) Let U be an open unit disk in $\mathbb{R}^{2n}$ and $\omega = \sum dx^i \wedge dx^{i+n}$, the canonical symplectic form. Then $\dim PH_{\partial_-}^k(U) = 0$ for $0 \le k < n$.*

Proof. Let $B_k \in \mathcal{P}^k$ be ∂_--closed. Then either (1) $dB_k = 0$ or (2) $\partial_- B_k = 0$ but $dB_k = B_{k+1}^0 \ne 0$ for $k < n$. In case (1), for $k > 0$, it follows from Proposition 3.10 that there exists a $B_{k+1} \in \mathcal{P}^{k+1}$ such that $B_k = \partial_- B_{k+1}$. For $k = 0$, B_0 being d-closed implies that it must be a constant. However, if c is a nonzero constant, then we can have $B_0 = c = c\,\partial_- B_1$ where B_1 is the tautological one-form, i.e. $dB_1 = \omega$.

As for case (2). With $k < n$ and $dB_k = B_{k+1}^0$, we have that B_{k+1}^0 must be d-closed. Hence, by the primitive dd^Λ-lemma we can write $dB_k = B_{k+1}^0 = dd^\Lambda \tilde{B}_{k+1}$. This means that $d(B_k - d^\Lambda \tilde{B}_{k+1}) = 0$ and apply the primitive dd^Λ-lemma again to $B_k - d^\Lambda \tilde{B}_{k+1}$, we find $B_k - d^\Lambda \tilde{B}_{k+1} = dd^\Lambda \tilde{B}_k'$ or

$$B_k = d^\Lambda \tilde{B}_{k+1} + d^\Lambda \tilde{B}_{k+1}' = d^\Lambda (\tilde{B}_{k+1} + \tilde{B}_{k+1}') \ .$$

$\qquad\square$

Let $\mathcal{P}'(M)$ be the space of primitive forms that are ∂_--closed. Then the ∂_- - Poincaré lemma implies the exactness of the following sequence of primitive sheaves $\mathcal{P}$

$$0 \longrightarrow \mathcal{P}'^n \xrightarrow{\ i\ } \mathcal{P}^n \xrightarrow{\ \partial_-\ } \mathcal{P}^{n-1} \xrightarrow{\ \partial_-\ } \ \cdots\ \xrightarrow{\ \partial_-\ } \mathcal{P}^1 \xrightarrow{\ \partial_-\ } \mathcal{P}^0 \xrightarrow{\ \partial_-\ } 0 \qquad (3.10)$$

Since the $\mathcal{P}^k$ allows for partition of unity, they are fine sheaves, and thus, the Čech cohomology $\check{H}^k(M, \mathcal{P}^k) = 0$ for $k > 0$. Then, by standard argument, we have the following

Theorem 3.15 *For (M^{2n}, ω) a compact symplectic manifold,*

$$PH_{\partial_-}^k(M) \cong \check{H}^{n-k}(M, \mathcal{P}'^n) \qquad\qquad \text{for } 0 \le k < n \ .$$

3.3 Comparing $PH_{\partial_\pm}(M)$ with $H_d(M)$ and $H_{d^\Lambda}(M)$ and the $\partial_+\partial_-$-lemma

Let us note that all zero-forms and one-forms are primitive forms. Therefore, we may expect that the $PH^k_{\partial_\pm}(M)$ may become equivalent to one of the standard cohomology at low degree. Indeed, this is the case as the following proposition shows.

Proposition 3.16 *On a compact symplectic manifold (M, ω), we have the following equivalence*

$$PH^k_{\partial_+}(M) = H^k_d(M) \ , \quad PH^k_{\partial_-}(M) = H^k_{d^\Lambda}(M) \ , \qquad \text{for } k = 0, 1 \ .$$

where $H^k_d(M)$ is the de Rham cohomology and $H^k_{d^\Lambda}(M) = (\ker d^\Lambda \cap \Omega^k(M))/(\operatorname{im} d^\Lambda \cap \Omega^k(M))$.

Proof. Note first that the action of ∂_- on zero- and one-forms is identical to that of d^Λ modulo a non-zero constant (i.e. $-1/H$). For ∂_+, the action on zero-forms is identical to d. So the equivalence at degree $k = 0$ is trivial.

For $PH^1_{\partial_+}(M)$, note first that one forms that are ∂_+-exact are also d-exact. So the question is whether there are any ∂_--closed one-forms that are not d-closed. Now, if $B_1 \in \mathcal{P}^1(M)$ is ∂_--closed, then $dB_1 = LB^1_0$. Furthermore, $d^2 = 0$ implies $dB^1_0 = 0$, which means $B^1_0 = c$, a constant. But $c = 0$ since otherwise the symplectic form would be trivial in de Rham cohomology. Thus, we find that $PH^1_{\partial_+}(M) = H^1_d(M)$ having used the compactness of M.

For $PH^1_{\partial_-}(M)$, as mentioned ∂_--closed one-forms are also d^Λ-closed. Moreover, ∂_--exact one-forms are also trivially d^Λ-exact. We shall now show that any d^Λ-exact one-forms are in fact also ∂_--exact. Let B_1 be d^Λ-exact; that is

$$B_1 = d^\Lambda A_2 = d^\Lambda(B_2 + LB_0)$$
$$= -(n-1)\partial_- B_2 + dB_0$$

We therefore need to show that there exists a $B'_2 \in \mathcal{P}^2(M)$ such that $dB_0 = \partial_- B'_2$. To do this, we can assume $\int_M B_0 = 0$, (since if necessary, we can always subtract a constant factor from B_0 without affecting dB_0). With B_0 integrating to zero and trivially d^Λ-close, B_0 must be d^Λ-exact, i.e. $B_0 = d^\Lambda B'_1$. Clearly then, we now have

$$B_1 = -(n-1)\partial_- B_2 + d(d^\Lambda B'_1)$$
$$= \partial_- \left[-(n-1)B_2 + n\, \partial_+ B'_1 \right]$$

where we have used the relation $dd^\Lambda = (H + R + 1)\partial_- \partial_+ = \partial_-(H + R + 2)\partial_+$. Therefore, we have shown that a d^Λ-exact one-form is also ∂_--exact and this completes the proof. $\qquad\square$

So at degree $k = 0, 1$, we have that $PH^k_{\partial_+}(M)$ is equivalent to the de Rham cohomolgy and $PH^k_{\partial_-}(M)$ to the d^Λ-cohomology. At higher degree, the equivalence generally does not hold no longer. To maintain some kind of equivalence, we can assume additional conditions on M. A useful condition is the dd^Λ-lemma. Recall that we say that the dd^Λ-lemma holds on a symplectic manifold (M, ω) if it satisfies the following condition: Let $A \in \Omega^*(M)$ be a differential form that is both d- and d^Λ-closed, then either it is not exact or else it must be $d-$, d^Λ-, and dd^Λ-exact. Now since we are dealing with only primitive forms, it is not to difficult to show that the dd^Λ-lemma for $\Omega(M)$ is equivalent to the $\partial_+\partial_-$-lemma defined below for $\mathcal{P}(M)$.

Definition 3.17 [$\partial_+\partial_-$-lemma] On a symplectic manifold (M, ω), let $B_k \in \mathcal{P}^k(M)$ be d-closed. We then say that the $\partial_+\partial_-$-lemma holds if the following properties are equivalent:

  (i) B_k is ∂_+-exact;

 (ii) B_k is ∂_--exact if $k < n$;

(iii) B_k is $\partial_+\partial_-$-exact.

Interestingly, it has been shown by Merkulov [7] and Guillemin [4] (see also Cavalcanti [2]) that a symplectic manifold exhibits the dd^Λ-lemma (or equivalently the $\partial_+\partial_-$-lemma) if and only if the the strong Lefschetz property holds. Here, strong Lefschetz is the property that the map in de Rham cohomology $\varphi : H^k_d(M) \to H^{2n-k}_d(M)$ given by $A_k \to [\omega]^{n-k} \wedge A_k$ is an isomorphism for all $k \leq n$. Imposing the $\partial_+\partial_-$-lemma or the strong Lefschetz, we have the following property for $PH^k_{\partial_\pm}(M)$.

Proposition 3.18 *On a symplectic manifold (M, ω), if the $\partial_+\partial_-$-lemma holds, or equivalently the strong Lefschetz property holds, then for $1 < k < n$, we have*

$$PH^k_{\partial_+}(M) = H^k_d(M) \cap \mathcal{P}^k(M) := \frac{\ker d \cap \mathcal{P}^k(M)}{d\Omega^{k-1} \cap \mathcal{P}^k(M)} \ , \tag{3.11}$$

$$PH^k_{\partial_-}(M) = H^k_{d^\Lambda}(M) \cap \mathcal{P}^k(M) := \frac{\ker d^\Lambda \cap \mathcal{P}^k(M)}{d^\Lambda\Omega^{k+1} \cap \mathcal{P}^k(M)} \ . \tag{3.12}$$

Proof. Consider first $PH_{\partial_+}^k(M)$. If B_k is ∂_+-closed, then in general, we have $dB_k = LB_{k-1}^1$. Now since $dB_{k-1}^1 = 0$ and $B_{k-1}^1 = \partial_- B_k$, we can use the $\partial_+\partial_-$-lemma to write $B_{k-1}^1 = \partial_+\partial_- B_{k-1}'$ for some $B_{k-1}' \in \mathcal{P}^{k-1}(M)$. Therefore, we have

$$d(B_k + \partial_+ B_{k-1}') = L(B_{k-1}^1 + \partial_-\partial_+ B_{k-1}') = 0 \ ,$$

implying that in every cohomology class of $PH_{\partial_+}^k(M)$, there must exist a representative which is also d-closed, having assumed of course that the $\partial_+\partial_-$-lemma holds.

We now only need to show that for a primitive d-closed form B_k, if $B_k = \partial_+ B_{k-1}$, then there exists a $A_{k-1} \in \Omega^{k-1}(M)$ such that $B_k = dA_{k-1}$. But since B_k is d-closed and ∂_+-exact, it must also be $\partial_+\partial_-$-exact, or equivalently dd^Λ-exact. Therefore, we must have $B_k = d(d^\Lambda B_k')$ for some $B_k' \in \mathcal{P}^k(M)$, which completes the proof of (3.11).

Consider now $PH_{\partial_-}^k(M)$. Acting on primitive forms, $d^\Lambda : \mathcal{P}^k(M) \to \mathcal{P}^{k-1}(M)$. Therefore, ∂_--closedness is in fact equivalent to d^Λ-closedness. Moreover, a ∂_--exact form is also d^Λ-exact, but the converse is generally not true. Thus we have to show that a d^Λ-exact form can also be expressed as a ∂_--exact form if the $\partial_+\partial_-$-lemma holds.

Let $B_k \in \mathcal{P}^k(M)$ be d^Λ-exact (i.e. $B_k = d^\Lambda A_{k+1}$). Then, since B_k is d^Λ-closed (and equivalently ∂_--closed), we must either have (i) $dB_k = 0$, or (ii) $dB_k = B_{k+1}^0$. For case (i), B_k satisfies the criteria for the dd^Λ-lemma and so we can write immediately, $B_k = \partial_-(\partial_+ B_k')$, for some $B_k' \in \mathcal{P}^k(M)$, noting again that $dd^\Lambda \sim \partial_+\partial_-$. For case (ii), clearly $dB_{k+1}^0 = 0$; therefore we can apply the $\partial_+\partial_-$-lemma to B_{k+1}^0 and write $B_{k+1}^0 = \partial_+\partial_- B_{k+1}'$, for some $B_{k+1}' \in \mathcal{P}^{k+1}(M)$. We thus obtain

$$d(B_k - \partial_- B_{k+1}') = 0 \ .$$

which reduces the problem to case (i). Thus applying the $\partial_+\partial_-$-lemma again, we find that $B_k = \partial_- B_{k+1}' + \partial_-(\partial_+ B_k')$, for some $B_k' \in \mathcal{P}^k(M)$. □

As mentioned in the above proof, $\ker \partial_- = \ker d^\Lambda \cap \mathcal{P}$, but in general $\operatorname{im} \partial_- \subset \operatorname{im} d^\Lambda \cap \mathcal{P}$. This thus give a lower bound on the dimension of the $PH_{\partial_\pm}^k(M)$.

Proposition 3.19 *On a compact symplectic manifold (M, ω),*

$$\dim PH_{\partial_+}^k(M) = \dim PH_{\partial_-}^k(M) \geq \dim \left(H_{d^\Lambda}^k(M) \cap \mathcal{P}^k(M) \right) \ .$$

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4   Example: A symplectic nilmanifold

We can explicitly calculate and compare the different primitive cohomologies on a six-dimensional compact symplectic nilmanifold. Let $M = M^6$ be the nilmanifold of type $(0, 0, 0, 12, 14, 15 + 23 + 24)$. This means that there exists a basis of one-forms $e_1, e_2, \ldots, e_6$ on M with the following alegbra,

$$
\begin{aligned}
de_1 &= 0 & de_4 &= e_1 \wedge e_2 \\
de_2 &= 0 & de_5 &= e_1 \wedge e_4 \\
de_3 &= 0 & de_6 &= e_1 \wedge e_5 + e_2 \wedge e_3 + e_2 \wedge e_4
\end{aligned} \tag{4.1}
$$

This nilmanifold has the following Betti numbers $(b_1, b_2, b_3) = (3, 5, 6)$ [9]. For our calculation, let us take the symplectic form to be

$$
\omega = e_1 \wedge e_6 + e_2 \wedge e_5 - e_3 \wedge e_4 \ . \tag{4.2}
$$

It can be easily checked that ω of (4.2) is both d-closed and non-degenerate as required. In Table 2, we give the basis elements for the cohomologies $H_d(M)$, $H_{d^\Lambda}(M)$, $PH_{\partial_\pm}(M)$, $PH_{d+d^\Lambda}(M)$, and $PH_{d \cap d^\Lambda}(M)$.

Clearly, the $\partial_+ \partial_-$-lemma generally does not hold for this nilmanifold. Take for instance, e_{12}. It is primitive, d-closed, and ∂_+-exact, i.e. $e_{12} = \partial_+ e_4$. Moreover, it is also ∂_--exact, since $e_{12} = \partial_-(e_{416} - e_{425})$. However, it is not $\partial_+ \partial_-$-exact.

Notice for $k = 2$, $\dim PH^2_{\partial_+}(M) = \dim[H^2_{dR} \cap \mathcal{P}^2(M)] + 1$. The difference is due to the presence of the two form $(e_{35} - e_{45})$ which is ∂_+-closed but not d-closed. Explicitly, we have

$$
d(e_{35} - e_{45}) = e_{134} - e_{125} = -e_1 \wedge \omega \ ,
$$

hence we see that the map $\varphi : H^1(M) \to H^3(M)$ given by $[\omega] \wedge$ is not injective for this nilmanifold with ω of (4.2). Similarly, $\dim PH^2_{\partial_-}(M) = \dim[H^2_{d^\Lambda} \cap \mathcal{P}^2(M)] + 1$. This is due to the fact that $e_{24} \in \operatorname{im} d^\Lambda$ but not in the image of ∂_-. Specifically, we have

$$
d^\Lambda[(e_{625} + e_{634}) + \omega \wedge e_6] = 2 \, e_{24}
$$

	$k=0$	$k=1$	$k=2$	$k=3$
H_d^k	1	e_1, e_2, e_3	$\omega, e_{13}, (e_{23} - e_{24}),$ $(e_{15} - e_{23}), (e_{26} - e_{45})$	$\omega \wedge e_2, \omega \wedge e_3, (e_{315} + e_{415}), e_{425},$ $(e_{534} + e_{623}), (e_{516} + e_{534} + 2\,e_{263} + e_{624})$
$H_{d^\Lambda}^k$	1	e_4, e_5, e_6	$\omega, e_{46}, (e_{15} - e_{23}),$ $(e_{26} - e_{45}), (e_{35} + e_{45})$	$\omega \wedge e_2, \omega \wedge e_3, (e_{315} + e_{415}), e_{425},$ $(e_{534} + e_{623}), (e_{516} + e_{534} + 2\,e_{263} + e_{624})$
$PH_{\partial_+}^k$	1	e_1, e_2, e_3	$e_{13}, (e_{23} - e_{24}), (e_{15} - e_{23}),$ $(e_{26} - e_{45}), (e_{35} - e_{45})$	
$PH_{\partial_-}^k$	1	e_4, e_5, e_6	$e_{24}, e_{46}, (e_{15} - e_{23}),$ $(e_{26} - e_{45}), (e_{35} + e_{45})$	
$PH_{d+d^\Lambda}^k$	1	e_1, e_2, e_3	$e_{12}, e_{13}, e_{14}, e_{24},$ $(e_{15} - e_{23}), (e_{26} - e_{45}),$ $(e_{15} + e_{23} + e_{24})$	$e_{315}, e_{415}, (e_{125} + e_{134}), (e_{216} - e_{234}),$ $(e_{314} - e_{325}), (e_{316} - e_{325} + e_{416} - e_{425}),$ $(e_{516} + e_{534} + 2\,e_{263} + e_{624})$
$PH_{d\cap d^\Lambda}^k$	1	$\emptyset$	$(e_{15} - e_{23}), (e_{26} - e_{45})$	$(e_{315} + e_{415}), (e_{516} + e_{534} + 2\,e_{263} + e_{624})$

Table 2: Bases for H_d, H_{d^Λ}, PH_{∂_+}, PH_{∂_-}, PH_{d+d^Λ}, and $PH_{d\cap d^\Lambda}$ of the six dimensional nilmanifold in terms of exterior products of the one-forms e_i (4.1) and symplectic form ω (4.2).

where the presence of the non-primitive term $d^\Lambda(\omega \wedge e_6) = \partial_+ e_6 = e_{15} + e_{23} + e_{24}$ is essential. Effectively, we have a primitive d-closed two-form $B_2 = e_{15} + e_{23} + e_{24} = \partial_+ e_6 \neq \partial_- B_3$.

Now, we could have chosen a different symplectic form. For instance, consider the same nilmanifold but with the symplectic form given by

$$\omega' = e_1 \wedge e_3 + e_2 \wedge e_6 - e_4 \wedge e_5 \ . \tag{4.3}$$

In this case, it is easy to show that the map $\varphi : H^1(M) \to H^3(M)$ now using $[\omega']\wedge$ is injective. Furthermore, any closed primitive two-form, if ∂_+-exact is also ∂_--exact. In this case, we have $PH^2_{\partial_+}(M, \omega') = H^2_d \cap \mathcal{P}^2(M, \omega')$ and $PH^2_{\partial_-}(M, \omega') = H^2_{d^\Lambda} \cap \mathcal{P}^2(M, \omega')$. And moreover, we have

$$\dim PH^2_{\partial_\pm}(M, \omega) = \dim PH^2_{\partial_\pm}(M, \omega') + 1 \ ,$$

which shows that $PH^2_{\partial_\pm}(M)$ can vary as the de Rham class of the symplectic form is varied.

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