Determinant representations of scalar products for the open XXZ chain with non-diagonal boundary terms

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Abstract

With the help of the F-basis provided by the Drinfeld twist or factorizing F-matrix for the open XXZ spin chain with non-diagonal boundary terms, we obtain the determinant representations of the scalar products of Bethe states of the model.

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1 Introduction

The computation of correlation functions (or scalar products of Bethe states) is one of major challenging problems in the theory of quantum integrable models [1, 2]. There are two approaches in the literature for computing the correlation functions of a quantum integrable model. One is the vertex operator method (see e.g. [3, 4, 5, 6, 7, 8]) which works only on an infinite lattice, and another one is based on the detailed analysis of the structure of the Bethe states [9, 10]. As for the second approach which usually works for models with finite size, it is well known that in the framework of quantum inverse scattering method (QISM) [2] Bethe states are obtained by applying pseudo-particle creation operators to reference state (pseudo-vacuum). However, the apparently simple action of creation operators is plagued with non-local effects arising from polarization clouds or compensating exchange terms on the level of local operators. This makes the direct calculation of correlation functions of models with finite size challenging.

Progress has recently been made on the second approach with the help of the Drinfeld twists or factorizing F-matrices [11]. Working in the F-basis provided by the F-matrices, the authors in [12, 13] managed to calculate the form factors and correlation functions of the XXX and XXZ chains with periodic boundary condition (or closed chains) analytically and expressed them in determinant forms. Then the determinant representation of the scalar products and correlation functions of the supersymmetric t-J model [14] and its q-deformed model [15] with periodic boundary condition was obtained within the corresponding F-basis given in [16].

It was noticed [17, 18] that the F-matrices of the closed XXX and XXZ chains also make the pseudo-particle creation operators of the open XXX and XXZ chains with diagonal boundary terms polarization free. This is mainly due to the fact that the closed chain and the corresponding open chain with diagonal boundary terms share the same reference state [19]. However, the story for the open XXZ chain with non-diagonal boundary terms is quite different [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. Firstly, the reference state (all spin up state) of the closed chain is no longer a reference state of the open chain with non-diagonal boundary terms [21, 22, 26]. Secondly, at least two reference states (and thus two sets of Bethe states) are needed [34] for the open XXZ chain with non-diagonal boundary terms in order to obtain its complete spectrum [35, 36]. As a consequence, the F-matrix found in [12] is no longer the *desirable* F-matrix for the open XXZ chain with non-diagonal boundary terms.

Very recently, we have succeeded in obtaining the factorizing F-matrices for the open XXZ chain with integrable boundary conditions given by the non-diagonal K-matrices (2.9) and (2.11) [37]. In this paper, we shall investigate the determinant representations of the scalar products of the Bethe states of the open XXZ chain with non-diagonal boundary terms with the help of the associated F-matrices.

The paper is organized as follows. In section 2, we briefly describe the open XXZ chain with non-diagonal boundary terms, and introduce the pseudo-particle creation operators and the two sets of Bethe states of the model. In section 3, we introduce the face picture of the model and express the scalar products in terms of the operators in the face picture. In section 4, we present the F-matrix of the open XXZ chain in the face picture and give the completely symmetric and polarization free representations of the pseudo-particle creation/annihilation operators in the F-basis. In section 5, with the help of the F-basis, we obtain the determinant representations of the scalar products of Bethe states. In section 6, we summarize our results and give some discussions.

2 The inhomogeneous spin-$\frac{1}{2}$ XXZ open chain

Throughout, V denotes a two-dimensional linear space. The spin-$\frac{1}{2}$ XXZ chain can be constructed from the well-known six-vertex model R-matrix $\overline{R}(u) \in \mathrm{End}(V \otimes V)$ [2] given by

$$\overline{R}(u) = \begin{pmatrix} 1 & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & 1 \end{pmatrix}. \qquad (2.1)$$

The coefficient functions read: $b(u) = \frac{\sin u}{\sin(u+\eta)}$, $c(u) = \frac{\sin \eta}{\sin(u+\eta)}$. Here we assume η is a generic complex number. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE),

$$R_{1,2}(u_1 - u_2)R_{1,3}(u_1 - u_3)R_{2,3}(u_2 - u_3) = R_{2,3}(u_2 - u_3)R_{1,3}(u_1 - u_3)R_{1,2}(u_1 - u_2), \quad (2.2)$$

and the unitarity, crossing-unitarity and quasi-classical properties [26]. We adopt the standard notations: for any matrix $A \in \mathrm{End}(V)$, A_j (or A^j) is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as A on the j-th space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the i-th and j-th ones.

One introduces the "row-to-row" (or one-row) monodromy matrix $T(u)$, which is an 2×2 matrix with elements being operators acting on $V^{\otimes N}$, where $N = 2M$ (M being a positive integer),

$$T_0(u) = \overline{R}_{0,N}(u - z_N)\overline{R}_{0,N-1}(u - z_{N-1}) \cdots \overline{R}_{0,1}(u - z_1). \tag{2.3}$$

Here $\{z_j | j = 1, \cdots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters.

Integrable open chain can be constructed as follows [19]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$\begin{aligned} &\overline{R}_{1,2}(u_1 - u_2)K_1^-(u_1)\overline{R}_{2,1}(u_1 + u_2)K_2^-(u_2) \\ &\quad = K_2^-(u_2)\overline{R}_{1,2}(u_1 + u_2)K_1^-(u_1)\overline{R}_{2,1}(u_1 - u_2), \end{aligned} \tag{2.4}$$

and the latter satisfies the dual RE

$$\begin{aligned} &\overline{R}_{1,2}(u_2 - u_1)K_1^+(u_1)\overline{R}_{2,1}(-u_1 - u_2 - 2\eta)K_2^+(u_2) \\ &\quad = K_2^+(u_2)\overline{R}_{1,2}(-u_1 - u_2 - 2\eta)K_1^+(u_1)\overline{R}_{2,1}(u_2 - u_1). \end{aligned} \tag{2.5}$$

For open spin-chains, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.3), one needs to consider the "double-row" monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u)K^-(u)\hat{T}(u), \quad \hat{T}(u) = T^{-1}(-u). \tag{2.6}$$

Then the double-row transfer matrix of the XXZ chain with open boundary (or the open XXZ chain) is given by

$$\tau(u) = tr(K^+(u)\mathbb{T}(u)). \tag{2.7}$$

The QYBE and (dual) REs lead to that the transfer matrices with different spectral parameters commute with each other [19]: $[\tau(u), \tau(v)] = 0$. This ensures the integrability of the open XXZ chain.

In this paper, we will consider the K-matrix $K^-(u)$ which is a generic solution to the RE (2.4) associated the six-vertex model R-matrix [38, 39]

$$K^-(u) = \begin{pmatrix} k_1^1(u) & k_2^1(u) \\ k_1^2(u) & k_2^2(u) \end{pmatrix} \equiv K(u). \tag{2.8}$$

The coefficient functions are

$$\begin{aligned}
k^1_1(u) &= \frac{\cos(\lambda_1-\lambda_2)-\cos(\lambda_1+\lambda_2+2\xi)e^{-2iu}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)},\\
k^1_2(u) &= \frac{-i\sin(2u)e^{-i(\lambda_1+\lambda_2)}e^{-iu}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)},\\
k^2_1(u) &= \frac{i\sin(2u)e^{i(\lambda_1+\lambda_2)}e^{-iu}}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)},\\
k^2_2(u) &= \frac{\cos(\lambda_1-\lambda_2)e^{-2iu}-\cos(\lambda_1+\lambda_2+2\xi)}{2\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)}.
\end{aligned} \tag{2.9}$$

At the same time, we introduce the corresponding *dual* K-matrix $K^+(u)$ which is a generic solution to the dual reflection equation (2.5) with a particular choice of the free boundary parameters:

$$K^+(u) = \begin{pmatrix} k^{+1}_{\ 1}(u) & k^{+1}_{\ 2}(u) \\ k^{+2}_{\ 1}(u) & k^{+2}_{\ 2}(u) \end{pmatrix} \tag{2.10}$$

with the matrix elements

$$\begin{aligned}
k^{+1}_{\ 1}(u) &= \frac{\cos(\lambda_1-\lambda_2)e^{-i\eta}-\cos(\lambda_1+\lambda_2+2\bar{\xi})e^{2iu+i\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)},\\
k^{+1}_{\ 2}(u) &= \frac{i\sin(2u+2\eta)e^{-i(\lambda_1+\lambda_2)}e^{iu-i\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)},\\
k^{+2}_{\ 1}(u) &= \frac{-i\sin(2u+2\eta)e^{i(\lambda_1+\lambda_2)}e^{iu+i\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)},\\
k^{+2}_{\ 2}(u) &= \frac{\cos(\lambda_1-\lambda_2)e^{2iu+i\eta}-\cos(\lambda_1+\lambda_2+2\bar{\xi})e^{-i\eta}}{2\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}.
\end{aligned} \tag{2.11}$$

The K-matrices depend on four free boundary parameters $\{\lambda_1,\ \lambda_2,\ \xi,\ \bar{\xi}\}$. It is very convenient to introduce a vector $\lambda \in V$ associated with the boundary parameters $\{\lambda_i\}$,

$$\lambda = \sum_{k=1}^{2} \lambda_k \epsilon_k, \tag{2.12}$$

where $\{\epsilon_i,\ i = 1, 2\}$ form the orthonormal basis of V such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$.

2.1 Vertex-face correspondence

Let us briefly review the face-type R-matrix associated with the six-vertex model.

Set

$$\hat{\imath} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{2}\sum_{k=1}^{2}\epsilon_k, \quad i = 1, 2, \qquad \text{then } \sum_{i=1}^{2} \hat{\imath} = 0. \tag{2.13}$$

Let $\mathfrak{h}$ be the Cartan subalgebra of A_1 and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalizable $\mathfrak{h}$-module is a complex finite dimensional vector space W with a weight decomposition $W = \oplus_{\mu\in\mathfrak{h}^*} W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x\,v = \mu(x)\,v$, $(x \in \mathfrak{h},\, v \in W[\mu])$. For example, the non-zero weight spaces of the fundamental representation $V_{\Lambda_1} = \mathbb{C}^2 = V$ are

$$W[\hat{\imath}] = \mathbb{C}\epsilon_i, \;\; i = 1, 2. \tag{2.14}$$

For a generic $m \in V$, define

$$m_i = \langle m, \epsilon_i\rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j\rangle, \quad i, j = 1, 2. \tag{2.15}$$

Let $R(u, m) \in \mathrm{End}(V \otimes V)$ be the R-matrix of the six-vertex SOS model, which is trigonometric limit of the eight-vertex SOS model [40] given by

$$R(u; m)=\sum_{i=1}^{2} R(u; m)^{ii}_{ii} E_{ii}\otimes E_{ii}+\sum_{i\neq j}^{2} \left\{R(u; m)^{ij}_{ij} E_{ii}\otimes E_{jj}+R(u; m)^{ji}_{ij} E_{ji}\otimes E_{ij}\right\}, \tag{2.16}$$

where E_{ij} is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R(u; m)^{ii}_{ii} = 1, \quad R(u; m)^{ij}_{ij} = \frac{\sin u \sin(m_{ij} - \eta)}{\sin(u + \eta) \sin(m_{ij})}, \quad i \neq j, \tag{2.17}$$

$$R(u; m)^{ji}_{ij} = \frac{\sin \eta \sin(u + m_{ij})}{\sin(u + \eta) \sin(m_{ij})}, \quad i \neq j, \tag{2.18}$$

and m_{ij} is defined in (2.15). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation (or the star-triangle relation) [40]

$$\begin{aligned} &R_{1,2}(u_1 - u_2; m - \eta h^{(3)})R_{1,3}(u_1 - u_3; m)R_{2,3}(u_2 - u_3; m - \eta h^{(1)}) \\ &\quad = R_{2,3}(u_2 - u_3; m)R_{1,3}(u_1 - u_3; m - \eta h^{(2)})R_{1,2}(u_1 - u_2; m). \end{aligned} \tag{2.19}$$

Here we have adopted

$$R_{1,2}(u, m - \eta h^{(3)})\, v_1 \otimes v_2 \otimes v_3 = (R(u, m - \eta\mu) \otimes \mathrm{id})\, v_1 \otimes v_2 \otimes v_3, \quad \text{if } v_3 \in W[\mu]. \tag{2.20}$$

Moreover, one may check that the R-matrix satisfies weight conservation condition,

$$\left[h^{(1)} + h^{(2)},\, R_{1,2}(u; m)\right] = 0, \tag{2.21}$$

unitary condition,

$$R_{1,2}(u; m)\, R_{2,1}(-u; m) = \mathrm{id} \otimes \mathrm{id}, \tag{2.22}$$

and crossing relation

$$R(u;m)_{ij}^{kl} = \varepsilon_l\,\varepsilon_j \frac{\sin(u)\sin((m-\eta\hat{\imath})_{21})}{\sin(u+\eta)\sin(m_{21})} R(-u-\eta; m-\eta\hat{\imath})_{\bar{l}\,i}^{\bar{j}\,k}, \tag{2.23}$$

where

$$\varepsilon_1 = 1,\, \varepsilon_2 = -1, \quad \text{and } \bar{1} = 2,\, \bar{2} = 1. \tag{2.24}$$

Define the following functions: $\theta^{(1)}(u) = e^{-iu}$, $\theta^{(2)}(u) = 1$. Let us introduce two intertwiners which are 2-component column vectors $\phi_{m,m-\eta\hat{\jmath}}(u)$ labelled by $\hat{1}$, $\hat{2}$. The k-th element of $\phi_{m,m-\eta\hat{\jmath}}(u)$ is given by

$$\phi^{(k)}_{m,m-\eta\hat{\jmath}}(u) = \theta^{(k)}(u + 2m_j). \tag{2.25}$$

Explicitly,

$$\phi_{m,m-\eta\hat{1}}(u) = \begin{pmatrix} e^{-i(u+2m_1)} \\ 1 \end{pmatrix}, \qquad \phi_{m,m-\eta\hat{2}}(u) = \begin{pmatrix} e^{-i(u+2m_2)} \\ 1 \end{pmatrix}. \tag{2.26}$$

Obviously, the two intertwiner vectors $\phi_{m,m-\eta\hat{\imath}}(u)$ are linearly *independent* for a generic $m \in V$.

Using the intertwiner vectors, one can derive the following face-vertex correspondence relation [21]

$$\begin{aligned} &\overline{R}_{1,2}(u_1-u_2)\phi^1_{m,m-\eta\hat{\imath}}(u_1)\phi^2_{m-\eta\hat{\imath},m-\eta(\hat{\imath}+\hat{\jmath})}(u_2) \\ &\qquad = \sum_{k,l} R(u_1-u_2;m)_{ij}^{kl}\phi^1_{m-\eta\hat{l},m-\eta(\hat{l}+\hat{k})}(u_1)\phi^2_{m,m-\eta\hat{l}}(u_2). \end{aligned} \tag{2.27}$$

Then the QYBE (2.2) of the vertex-type R-matrix $\overline{R}(u)$ is equivalent to the dynamical Yang-Baxter equation (2.19) of the SOS R-matrix $R(u,m)$. For a generic m, we can introduce other types of intertwiners $\bar{\phi}$, $\tilde{\phi}$ which are both row vectors and satisfy the following conditions,

$$\bar{\phi}_{m,m-\eta\hat{\mu}}(u)\,\phi_{m,m-\eta\hat{\nu}}(u) = \delta_{\mu\nu}, \quad \tilde{\phi}_{m+\eta\hat{\mu},m}(u)\,\phi_{m+\eta\hat{\nu},m}(u) = \delta_{\mu\nu}, \tag{2.28}$$

from which one can derive the relations,

$$\sum_{\mu=1}^{2} \phi_{m,m-\eta\hat{\mu}}(u)\,\bar{\phi}_{m,m-\eta\hat{\mu}}(u) = \mathrm{id}, \tag{2.29}$$

$$\sum_{\mu=1}^{2} \phi_{m+\eta\hat{\mu},m}(u)\,\tilde{\phi}_{m+\eta\hat{\mu},m}(u) = \mathrm{id}. \tag{2.30}$$

One may verify that the K-matrices $K^{\pm}(u)$ given by (2.8) and (2.10) can be expressed in terms of the intertwiners and *diagonal* matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ as follows

$$K^{-}(u)^{s}_{t}=\sum_{i,j}\phi^{(s)}_{\lambda-\eta(\hat{\imath}-\hat{\jmath}),\ \lambda-\eta\hat{\imath}}(u)\mathcal{K}(\lambda|u)^{j}_{i}\bar{\phi}^{(t)}_{\lambda,\ \lambda-\eta\hat{\imath}}(-u), \tag{2.31}$$

$$K^{+}(u)^{s}_{t}=\sum_{i,j}\phi^{(s)}_{\lambda,\ \lambda-\eta\hat{\jmath}}(-u)\tilde{\mathcal{K}}(\lambda|u)^{j}_{i}\tilde{\phi}^{(t)}_{\lambda-\eta(\hat{\jmath}-\hat{\imath}),\ \lambda-\eta\hat{\jmath}}(u). \tag{2.32}$$

Here the two *diagonal* matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ are given by

$$\mathcal{K}(\lambda|u)\equiv \mathrm{Diag}(k(\lambda|u)_1,\ k(\lambda|u)_2)=\mathrm{Diag}(\frac{\sin(\lambda_1+\xi-u)}{\sin(\lambda_1+\xi+u)},\ \frac{\sin(\lambda_2+\xi-u)}{\sin(\lambda_2+\xi+u)}), \tag{2.33}$$

$$\begin{aligned}\tilde{\mathcal{K}}(\lambda|u)&\equiv \mathrm{Diag}(\tilde{k}(\lambda|u)_1,\ \tilde{k}(\lambda|u)_2)\\ &=\mathrm{Diag}(\frac{\sin(\lambda_{12}-\eta)\sin(\lambda_1+\bar{\xi}+u+\eta)}{\sin\lambda_{12}\sin(\lambda_1+\bar{\xi}-u-\eta)},\ \frac{\sin(\lambda_{12}+\eta)\sin(\lambda_2+\bar{\xi}+u+\eta)}{\sin\lambda_{12}\sin(\lambda_2+\bar{\xi}-u-\eta)}).\end{aligned} \tag{2.34}$$

Although the vertex type K-matrices $K^{\pm}(u)$ given by (2.8) and (2.10) are generally non-diagonal, after the face-vertex transformations (2.31) and (2.32), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ become *simultaneously* diagonal. This fact enabled the authors to apply the generalized algebraic Bethe ansatz method developed in [22] for SOS type integrable models to diagonalize the transfer matrices $\tau(u)$ (2.7) [26, 34].

2.2 Two sets of eigenstates

In order to construct the Bethe states of the open XXZ model with non-diagonal boundary terms specified by the K-matrices (2.9) and (2.11), we need to introduce the new double-row monodromy matrices $\mathcal{T}^{\pm}(m|u)$ [37]:

$$\mathcal{T}^{-}(m|u)^{\nu}_{\mu} = \tilde{\phi}^{0}_{m-\eta(\hat{\mu}-\hat{\nu}),m-\eta\hat{\mu}}(u)\ \mathbb{T}_0(u)\phi^{0}_{m,m-\eta\hat{\mu}}(-u), \tag{2.35}$$

$$\mathcal{T}^{+}(m|u)^{j}_{i} = \prod_{k\neq j}\frac{\sin(m_{jk})}{\sin(m_{jk}-\eta)}\ \phi^{t_0}_{m-\eta(\hat{\jmath}-\hat{\imath}),m-\eta\hat{\jmath}}(u)\left(\mathbb{T}^{+}(u)\right)^{t_0}\bar{\phi}^{t_0}_{m,m-\eta\hat{\jmath}}(-u), \tag{2.36}$$

where t_0 denotes transposition in the 0-th space (i.e. auxiliary space) and $\mathbb{T}^{+}(u)$ is given by

$$\left(\mathbb{T}^{+}(u)\right)^{t_0} = T^{t_0}(u)\left(K^{+}(u)\right)^{t_0}\hat{T}^{t_0}(u). \tag{2.37}$$

These double-row monodromy matrices, in the face picture, can be expressed in terms of the face type R-matrix $R(u;m)$ (2.16) and K-matrices $\mathcal{K}(\lambda|u)$ (2.33) and $\tilde{\mathcal{K}}(\lambda|u)$ (2.34) (for the details see Appendix A).

So far only two sets of Bethe states (i.e. eigenstates) of the transfer matrix for the models with non-diagonal boundary terms have been found [34]. These two sets of states are [37]

$$|\{v_i^{(1)}\}\rangle^{(I)} = \mathcal{T}^{+}(\lambda+2\eta\hat{1}|v_1^{(1)})_2^1\cdots\mathcal{T}^{+}(\lambda+2M\eta\hat{1}|v_M^{(1)})_2^1|\Omega^{(I)}(\lambda)\rangle, \tag{2.38}$$

$$|\{v_i^{(2)}\}\rangle^{(II)} = \mathcal{T}^{-}(\lambda-2\eta\hat{2}|v_1^{(2)})_1^2\cdots\mathcal{T}^{-}(\lambda-2M\eta\hat{2}|v_M^{(2)})_1^2|\Omega^{(II)}(\lambda)\rangle, \tag{2.39}$$

where the vector λ is related to the boundary parameters (2.12). The associated reference states $|\Omega^{(I)}(\lambda)\rangle$ and $|\Omega^{(II)}(\lambda)\rangle$ are

$$|\Omega^{(I)}(\lambda)\rangle = \phi^1_{\lambda+N\eta\hat{1},\lambda+(N-1)\eta\hat{1}}(z_1)\phi^2_{\lambda+(N-1)\eta\hat{1},\lambda+(N-2)\eta\hat{1}}(z_2)\cdots\phi^N_{\lambda+\eta\hat{1},\lambda}(z_N), \tag{2.40}$$

$$|\Omega^{(II)}(\lambda)\rangle = \phi^1_{\lambda,\lambda-\eta\hat{2}}(z_1)\phi^2_{\lambda-\eta\hat{2},\lambda-2\eta\hat{2}}(z_2)\cdots\phi^N_{\lambda-(N-1)\eta\hat{2},\lambda-N\eta\hat{2}}(z_N). \tag{2.41}$$

It is remarked that $\phi^k = \mathrm{id}\otimes\mathrm{id}\cdots\otimes\overset{k-th}{\phi}\otimes\mathrm{id}\cdots$.

If the parameters $\{v_k^{(1)}\}$ satisfy the first set of Bethe ansatz equations given by

$$\begin{aligned}&\frac{\sin(\lambda_2+\xi+v_\alpha^{(1)})\sin(\lambda_2+\bar{\xi}-v_\alpha^{(1)})\sin(\lambda_1+\bar{\xi}+v_\alpha^{(1)})\sin(\lambda_1+\xi-v_\alpha^{(1)})}{\sin(\lambda_2+\bar{\xi}+v_\alpha^{(1)}+\eta)\sin(\lambda_2+\xi-v_\alpha^{(1)}-\eta)\sin(\lambda_1+\xi+v_\alpha^{(1)}+\eta)\sin(\lambda_1+\bar{\xi}-v_\alpha^{(1)}-\eta)}\\
&\quad=\prod_{k\neq\alpha}^{M}\frac{\sin(v_\alpha^{(1)}+v_k^{(1)}+2\eta)\sin(v_\alpha^{(1)}-v_k^{(1)}+\eta)}{\sin(v_\alpha^{(1)}+v_k^{(1)})\sin(v_\alpha^{(1)}-v_k^{(1)}-\eta)}\\
&\quad\quad\times\prod_{k=1}^{2M}\frac{\sin(v_\alpha^{(1)}+z_k)\sin(v_\alpha^{(1)}-z_k)}{\sin(v_\alpha^{(1)}+z_k+\eta)\sin(v_\alpha^{(1)}-z_k+\eta)},\quad \alpha=1,\cdots,M,\end{aligned} \tag{2.42}$$

the Bethe state $|v_1^{(I)},\cdots,v_M^{(1)}\rangle^{(1)}$ becomes the eigenstate of the transfer matrix with eigenvalue $\Lambda^{(1)}(u)$ given by [37]

$$\begin{aligned}\Lambda^{(1)}(u) &= \frac{\sin(\lambda_2+\bar{\xi}-u)\sin(\lambda_1+\bar{\xi}+u)\sin(\lambda_1+\xi-u)\sin(2u+2\eta)}{\sin(\lambda_2+\bar{\xi}-u-\eta)\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_1+\xi+u)\sin(2u+\eta)}\\
&\quad\quad\times\prod_{k=1}^{M}\frac{\sin(u+v_k^{(1)})\sin(u-v_k^{(1)}-\eta)}{\sin(u+v_k^{(1)}+\eta)\sin(u-v_k^{(1)})}\\
&\quad+\frac{\sin(\lambda_2+\bar{\xi}+u+\eta)\sin(\lambda_1+\xi+u+\eta)\sin(\lambda_2+\xi-u-\eta)\sin 2u}{\sin(\lambda_2+\bar{\xi}-u-\eta)\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)\sin(2u+\eta)}\\
&\quad\quad\times\prod_{k=1}^{M}\frac{\sin(u+v_k^{(1)}+2\eta)\sin(u-v_k^{(1)}+\eta)}{\sin(u+v_k^{(1)}+\eta)\sin(u-v_k^{(1)})}\\
&\quad\quad\times\prod_{k=1}^{2M}\frac{\sin(u+z_k)\sin(u-z_k)}{\sin(u+z_k+\eta)\sin(u-z_k+\eta)}.\end{aligned} \tag{2.43}$$

If the parameters $\{v_k^{(2)}\}$ satisfy the second Bethe Ansatz equations

$$\frac{\sin(\lambda_1+\xi+v_\alpha^{(2)})\sin(\lambda_1+\bar{\xi}-v_\alpha^{(2)})\sin(\lambda_2+\bar{\xi}+v_\alpha^{(2)})\sin(\lambda_2+\xi-v_\alpha^{(2)})}{\sin(\lambda_1+\bar{\xi}+v_\alpha^{(2)}+\eta)\sin(\lambda_1+\xi-v_\alpha^{(2)}-\eta)\sin(\lambda_2+\xi+v_\alpha^{(2)}+\eta)\sin(\lambda_2+\bar{\xi}-v_\alpha^{(2)}-\eta)}$$
$$=\prod_{k\neq\alpha}^{M}\frac{\sin(v_\alpha^{(2)}+v_k^{(2)}+2\eta)\sin(v_\alpha^{(2)}-v_k^{(2)}+\eta)}{\sin(v_\alpha^{(2)}+v_k^{(2)})\sin(v_\alpha^{(2)}-v_k^{(2)}-\eta)}$$
$$\times\prod_{k=1}^{2M}\frac{\sin(v_\alpha^{(2)}+z_k)\sin(v_\alpha^{(2)}-z_k)}{\sin(v_\alpha^{(2)}+z_k+\eta)\sin(v_\alpha^{(2)}-z_k+\eta)},\quad \alpha=1,\cdots,M, \tag{2.44}$$

the Bethe states $|v_1^{(2)},\cdots,v_M^{(2)}\rangle^{(II)}$ yield the second set of the eigenstates of the transfer matrix with the eigenvalues [34],

$$\Lambda^{(2)}(u)=\frac{\sin(2u+2\eta)\sin(\lambda_1+\bar{\xi}-u)\sin(\lambda_2+\bar{\xi}+u)\sin(\lambda_2+\xi-u)}{\sin(2u+\eta)\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)\sin(\lambda_2+\xi+u)}$$
$$\times\prod_{k=1}^{M}\frac{\sin(u+v_k^{(2)})\sin(u-v_k^{(2)}-\eta)}{\sin(u+v_k^{(2)}+\eta)\sin(u-v_k^{(2)})}$$
$$+\frac{\sin(2u)\sin(\lambda_1+\bar{\xi}+u+\eta)\sin(\lambda_2+\xi+u+\eta)\sin(\lambda_1+\xi-u-\eta)}{\sin(2u+\eta)\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\xi+u)\sin(\lambda_1+\xi+u)}$$
$$\times\prod_{k=1}^{M}\frac{\sin(u+v_k^{(2)}+2\eta)\sin(u-v_k^{(2)}+\eta)}{\sin(u+v_k^{(2)}+\eta)\sin(u-v_k^{(2)})}$$
$$\times\prod_{k=1}^{2M}\frac{\sin(u+z_k)\sin(u-z_k)}{\sin(u+z_k+\eta)\sin(u-z_k+\eta)}. \tag{2.45}$$

3 Scalar products

It was shown that in order to compute correlation functions of the closed XXZ chain [2] and the open XXZ chain with diagonal boundary terms [17, 18], one suffices to calculate the scalar products of an on-shell Bethe state and a general state (an off-shell Bethe state). The aim of this paper is to give the explicit expressions of the following scalar products of the open XXZ chain with non-diagonal boundary terms:

$$S^{I,II}(\{u_\alpha\};\{v_i^{(2)}\})={}^{(I)}\langle\{u_\alpha\}|\{v_i^{(2)}\}\rangle^{(II)},\quad S^{II,I}(\{u_\alpha\};\{v_i^{(1)}\})={}^{(II)}\langle\{u_\alpha\}|\{v_i^{(1)}\}\rangle^{(I)}, \tag{3.1}$$
$$S^{I,I}(\{u_\alpha\};\{v_i^{(1)}\})={}^{(I)}\langle\{u_\alpha\}|\{v_i^{(1)}\}\rangle^{(I)},\quad S^{II,II}(\{u_\alpha\};\{v_i^{(2)}\})={}^{(II)}\langle\{u_\alpha\}|\{v_i^{(2)}\}\rangle^{(II)}, \tag{3.2}$$

where the dual states ${}^{(I)}\langle\{u_\alpha\}|$ and ${}^{(II)}\langle\{u_\alpha\}|$ are given by

$${}^{(I)}\langle\{u_\alpha\}|=\langle\Omega^{(I)}(\lambda)|\mathcal{T}^-(\lambda-2(M-1)\eta\hat{1}|u_M)_1^2\ldots\mathcal{T}^-(\lambda|u_1)_1^2, \tag{3.3}$$
$${}^{(II)}\langle\{u_\alpha\}|=\langle\Omega^{(II)}(\lambda)|\mathcal{T}^+(\lambda+2(M-1)\eta\hat{2}|u_M)_2^1\ldots\mathcal{T}^+(\lambda|u_1)_2^1, \tag{3.4}$$

and $\langle\Omega^{(I)}(\lambda)|$, $\langle\Omega^{(II)}(\lambda)|$ are

$$\langle\Omega^{(I)}(\lambda)| = \tilde{\phi}^1_{\lambda,\lambda-\eta\hat{1}}(z_1)\ldots\tilde{\phi}^N_{\lambda-(2M-1)\eta\hat{1},\lambda-2M\eta\hat{1}}(z_N), \tag{3.5}$$

$$\langle\Omega^{(II)}(\lambda)| = \tilde{\phi}^1_{\lambda+2M\eta\hat{2},\lambda+(2M-1)\eta\hat{2}}(z_1)\ldots\tilde{\phi}^N_{\lambda+\eta\hat{1},\lambda}(z_N). \tag{3.6}$$

Some remarks are in order. The parameters $\{u_\alpha\}$ in (3.1)-(3.2) are free parameters, namely, they do not need to satisfy the Bethe ansatz equations. However the parameters $\{v_i^{(1)}\}$ and $\{v_i^{(2)}\}$ need to satisfy the Bethe ansatz equations (2.42) and (2.44) respectively.

The K-matrices $K^\pm(u)$ given by (2.8) and (2.10) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (2.31) and (2.32), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (2.33) and (2.34) *simultaneously* become diagonal. This fact suggests that it would be much simpler if one performs all calculations in the face picture.

3.1 Face picture

Let us introduce the face type one-row monodromy matrix (c.f (2.3))

$$\begin{aligned} T_F(l|u) &\equiv T^F_{0,1\ldots N}(l|u) \\ &= R_{0,N}(u-z_N;l-\eta\sum_{i=1}^{N-1}h^{(i)})\ldots R_{0,2}(u-z_2;l-\eta h^{(1)})R_{0,1}(u-z_1;l), \\ &= \begin{pmatrix} T_F(l|u)^1_1 & T_F(l|u)^1_2 \\ T_F(l|u)^2_1 & T_F(l|u)^2_2 \end{pmatrix} \end{aligned} \tag{3.7}$$

where l is a generic vector in V. The monodromy matrix satisfies the face type quadratic exchange relation [41, 42]. Applying $T_F(l|u)^i_j$ to an arbitrary vector $|i_1,\ldots,i_N\rangle$ in the N-tensor product space $V^{\otimes N}$ given by

$$|i_1,\ldots,i_N\rangle = \epsilon^1_{i_1}\ldots\epsilon^N_{i_N}, \tag{3.8}$$

we have

$$\begin{aligned} T_F(l|u)^i_j|i_1,\ldots,i_N\rangle &\equiv T_F(m;l|u)^i_j|i_1,\ldots,i_N\rangle \\ &= \sum_{\alpha_{N-1}\ldots\alpha_1}\sum_{i'_N\ldots i'_1} R(u-z_N;l-\eta\sum_{k=1}^{N-1}\hat{i}'_k)^{i\ \ i'_N}_{\alpha_{N-1}\ i_N}\ldots \\ &\qquad \times R(u-z_2;l-\eta\hat{i}'_1)^{\alpha_2\ i'_2}_{\alpha_1\ i_2}R(u-z_1;l)^{\alpha_1\ i'_1}_{j\ \ i_1}\ |i'_1,\ldots,i'_N\rangle, \end{aligned} \tag{3.9}$$

where $m = l-\eta\sum_{k=1}^N\hat{i}_k$. With the help of the crossing relation (2.23), the face-vertex correspondence relation (2.27) and the relations (2.28), following the method developed in

[22, 37], we find that the scalar products (3.1)-(3.2) can be expressed in terms of the face-type double-row monodromy operators as follows:

$$
\begin{aligned}
S^{I,II}(\{u_\alpha\};\{v_i^{(2)}\}) &= \langle 1,\ldots,1|\mathcal{T}_F^-(\lambda-2(M-1)\eta\hat{1},\lambda|u_M)_1^2\ldots\mathcal{T}_F^-(\lambda,\lambda|u_1)_1^2 \\
&\quad\times\mathcal{T}_F^-(\lambda+2\eta\hat{1},\lambda|v_1^{(2)})_1^2\ldots\mathcal{T}_F^-(\lambda+2M\eta\hat{1},\lambda|v_M^{(2)})_1^2|2,\ldots,2\rangle, \qquad (3.10)\\
S^{II,I}(\{u_\alpha\};\{v_i^{(1)}\}) &= \langle 2,\ldots,2|\mathcal{T}_F^+(\lambda,\lambda+2(M-1)\eta\hat{2}|u_M)_2^1\ldots\mathcal{T}_F^+(\lambda,\lambda|u_1)_2^1 \\
&\quad\times\mathcal{T}_F^+(\lambda,\lambda-2\eta\hat{2}|v_1^{(1)})_2^1\ldots\mathcal{T}_F^+(\lambda,\lambda-2M\eta\hat{2}|v_M^{(1)})_2^1|1,\ldots,1\rangle, \qquad (3.11)\\
S^{I,I}(\{u_\alpha\};\{v_i^{(1)}\}) &= \langle 1,\ldots,1|\mathcal{T}_F^-(\lambda-2(M-1)\eta\hat{1},\lambda|u_M)_1^2\ldots\mathcal{T}_F^-(\lambda,\lambda|u_1)_1^2 \\
&\quad\times\mathcal{T}_F^+(\lambda,\lambda+2\eta\hat{1}|v_1^{(1)})_2^1\ldots\mathcal{T}_F^+(\lambda,\lambda+2M\eta\hat{1}|v_M^{(1)})_2^1|1,\ldots,1\rangle, \qquad (3.12)\\
S^{II,II}(\{u_\alpha\};\{v_i^{(2)}\}) &= \langle 2,\ldots,2|\mathcal{T}_F^+(\lambda,\lambda+2(M-1)\eta\hat{2}|u_M)_2^1\ldots\mathcal{T}_F^+(\lambda,\lambda|u_1)_2^1 \\
&\quad\times\mathcal{T}_F^-(\lambda-2\eta\hat{2},\lambda|v_1^{(2)})_1^2\ldots\mathcal{T}_F^-(\lambda-2M\eta\hat{2},\lambda|v_M^{(2)})_1^2|2,\ldots,2\rangle. \qquad (3.13)
\end{aligned}
$$

The above double-row monodromy matrix operators $\mathcal{T}_F^-(m,\lambda|u)_1^2$ and $\mathcal{T}_F^+(\lambda,m|u)_2^1$ are given in terms of the one-row monodromy matrix operator $T_F(m;l|u)_j^i$ [37]

$$
\begin{aligned}
\mathcal{T}_F^-(m,\lambda|u)_1^2 &= \frac{\sin(m_{21})}{\sin(\lambda_{21})}\prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)} \\
&\times\left\{\frac{\sin(\lambda_1+\xi-u)}{\sin(\lambda_1+\xi+u)}T_F(m,\lambda|u)_1^2T_F(m+\eta\hat{2},\lambda+\eta\hat{2}|-u-\eta)_2^2\right. \\
&\left.-\frac{\sin(\lambda_2+\xi-u)}{\sin(\lambda_2+\xi+u)}T_F(m+2\eta\hat{2},\lambda|u)_2^2T_F(m+\eta\hat{1},\lambda+\eta\hat{1}|-u-\eta)_1^2\right\}, \qquad (3.14)\\
\mathcal{T}_F^+(\lambda,m|u)_2^1 &= \prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)} \\
&\times\left\{\frac{\sin(\lambda_{12}-\eta)\sin(\lambda_1+\bar{\xi}+u+\eta)}{\sin(m_{12}-\eta)\sin(\lambda_1+\bar{\xi}-u-\eta)}T_F(\lambda+2\eta\hat{2},m+2\eta\hat{2}|u)_2^1T_F(\lambda+\eta\hat{2},m+\eta\hat{2}|-u-\eta)_2^2\right. \\
&\left.-\frac{\sin(\lambda_{21}-\eta)\sin(\lambda_2+\bar{\xi}+u+\eta)}{\sin(m_{21}+\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}T_F(\lambda,m+2\eta\hat{2}|u)_2^2T_F(\lambda+\eta\hat{2},m+\eta\hat{2}|-u-\eta)_2^1\right\}.
\end{aligned}
$$

(3.15)

In the next section we shall construct the Drinfeld twist (or factorizing F-matrix) in the face picture for the open XXZ chain with non-diagonal boundary terms. In the resulting F-basis, the two sets of pseudo-particle creation/annihilation operators $\mathcal{T}_F^{\pm}$ given by (3.14) and (3.15) take completely symmetric and polarization free forms simultaneously. These polarization free forms allow us to construct the explicit expressions of the scalar products (3.10)- (3.13).

4 F-basis

In this section, we construct the Drinfeld twist [11] (factorizing F-matrix) on the N-fold tensor product space $V^{\otimes N}$ (i.e. the quantum space of the open XXZ chain) and the associated representations of the pseudo-particle creation/annihilation operators in this basis.

4.1 Factorizing Drinfeld twist F

Let $\mathcal{S}_N$ be the permutation group over indices $1, \ldots, N$ and $\{\sigma_i | i = 1, \ldots, N-1\}$ be the set of elementary permutations in $\mathcal{S}_N$. For each elementary permutation σ_i, we introduce the associated operator $R^{\sigma_i}_{1\ldots N}$ on the quantum space

$$R^{\sigma_i}_{1\ldots N}(l) \equiv R^{\sigma_i}(l) = R_{i,i+1}(z_i - z_{i+1}|l - \eta \sum_{k=1}^{i-1} h^{(k)}), \tag{4.1}$$

where l is a generic vector in V. For any σ, $\sigma' \in \mathcal{S}_N$, operator $R^{\sigma\sigma'}_{1\ldots N}$ associated with $\sigma\sigma'$ satisfies the following composition law [37](and references therein):

$$R^{\sigma\sigma'}_{1\ldots N}(l) = R^{\sigma'}_{\sigma(1\ldots N)}(l)\, R^{\sigma}_{1\ldots N}(l). \tag{4.2}$$

Let σ be decomposed in a minimal way in terms of elementary permutations,

$$\sigma = \sigma_{\beta_1} \ldots \sigma_{\beta_p}, \tag{4.3}$$

where $\beta_i = 1, \ldots, N-1$ and the positive integer p is the length of σ. The composition law (4.2) enables one to obtain operator $R^{\sigma}_{1\ldots N}$ associated with each $\sigma \in \mathcal{S}_N$. The dynamical quantum Yang-Baxter equation (2.19), weight conservation condition (2.21) and unitary condition (2.22) guarantee the uniqueness of $R^{\sigma}_{1\ldots N}$. Moreover, one may check that $R^{\sigma}_{1\ldots N}$ satisfies the following exchange relation with the face type one-row monodromy matrix (3.7)

$$R^{\sigma}_{1\ldots N}(l) T^F_{0,1\ldots N}(l|u) = T^F_{0,\sigma(1\ldots N)}(l|u) R^{\sigma}_{1\ldots N}(l - \eta h^{(0)}), \qquad \forall \sigma \in \mathcal{S}_N. \tag{4.4}$$

Now, we construct the face-type Drinfeld twist $F_{1\ldots N}(l) \equiv F_{1\ldots N}(l; z_1, \ldots, z_N)$ [1] on the N-fold tensor product space $V^{\otimes N}$, which satisfies the following three properties:

$$\text{I. lower} - \text{triangularity}; \tag{4.5}$$

$$\text{II. non} - \text{degeneracy}; \tag{4.6}$$

$$\text{III. factorizing property}: \ R^{\sigma}_{1\ldots N}(l) = F^{-1}_{\sigma(1\ldots N)}(l) F_{1\ldots N}(l), \ \forall \sigma \in \mathcal{S}_N. \tag{4.7}$$

[1]In this paper, we adopt the convention: $F_{\sigma(1\ldots N)}(l) \equiv F_{\sigma(1\ldots N)}(l; z_{\sigma(1)}, \ldots, z_{\sigma(N)})$.

Substituting (4.7) into the exchange relation (4.4) yields the following relation

$$F^{-1}_{\sigma(1\ldots N)}(l)F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u) = T^F_{0,\sigma(1\ldots N)}(l|u)F^{-1}_{\sigma(1\ldots N)}(l-\eta h^{(0)})F_{1\ldots N}(l-\eta h^{(0)}). \tag{4.8}$$

Equivalently,

$$F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u)F^{-1}_{1\ldots N}(l-\eta h^{(0)}) = F_{\sigma(1\ldots N)}(l)T^F_{0,\sigma(1\ldots N)}(l|u)F^{-1}_{\sigma(1\ldots N)}(l-\eta h^{(0)}). \tag{4.9}$$

Let us introduce the twisted monodromy matrix $\tilde{T}^F_{0,1\ldots N}(l|u)$ by

$$\begin{aligned}\tilde{T}^F_{0,1\ldots N}(l|u) &= F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u)F^{-1}_{1\ldots N}(l-\eta h^{(0)}) \\ &= \begin{pmatrix} \tilde{T}_F(l|u)^1_1 & \tilde{T}_F(l|u)^1_2 \\ \tilde{T}_F(l|u)^2_1 & \tilde{T}_F(l|u)^2_2 \end{pmatrix}.\end{aligned} \tag{4.10}$$

Then (4.9) implies that the twisted monodromy matrix is symmetric under $\mathcal{S}_N$, namely,

$$\tilde{T}^F_{0,1\ldots N}(l|u) = \tilde{T}^F_{0,\sigma(1\ldots N)}(l|u), \quad \forall \sigma \in \mathcal{S}_N. \tag{4.11}$$

Define the F-matrix:

$$F_{1\ldots N}(l) = \sum_{\sigma\in\mathcal{S}_N} \sum_{\{\alpha_j\}=1}^{2}{}^{*} \prod_{j=1}^{N} P^{\sigma(j)}_{\alpha_{\sigma(j)}}\, R^{\sigma}_{1\ldots N}(l), \tag{4.12}$$

where P^i_α is the embedding of the project operator P_α in the $i^{\rm th}$ space with matric elements $(P_\alpha)_{kl} = \delta_{kl}\delta_{k\alpha}$. The sum $\sum^*$ in (4.12) is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

$$\begin{aligned}\alpha_{\sigma(i+1)} &\geq \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) > \sigma(i), \\ \alpha_{\sigma(i+1)} &> \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) < \sigma(i).\end{aligned} \tag{4.13}$$

From (4.13), $F_{1\ldots N}(l)$ obviously is a lower-triangular matrix. Moreover, the F-matrix is non-degenerate because all its diagonal elements are non-zero. It was shown [37] that the F-matrix also satisfies the factorizing property (4.7). Hence, the F-matrix $F_{1\ldots N}(l)$ given by (4.12) is the desirable Drinfeld twist.

4.2 Completely symmetric representations

Direct calculation shows [37] that the twisted operators $\tilde{T}_F(l|u)^j_i$ defined by (4.10) indeed simultaneously have the following polarization free forms

$$\tilde{T}_F(l|u)^2_2 = \frac{\sin(l_{21}-\eta)}{\sin\left(l_{21}-\eta+\eta\langle H,\epsilon_1\rangle\right)} \otimes_i \begin{pmatrix} \frac{\sin(u-z_i)}{\sin(u-z_i+\eta)} & \\ & 1 \end{pmatrix}_{(i)}, \tag{4.14}$$

$$\tilde{T}_F(l|u)^2_1 = \sum_{i=1}^{N} \frac{\sin\eta \sin(u-z_i+l_{12})}{\sin(u-z_i+\eta)\sin l_{12}} E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-z_j)\sin(z_i-z_j+\eta)}{\sin(u-z_j+\eta)\sin(z_i-z_j)} & \\ & 1 \end{pmatrix}_{(j)}, \quad (4.15)$$

$$\tilde{T}_F(l|u)^1_2 = \frac{\sin(l_{21}-\eta)}{\sin(l_{21}+\eta\langle H, \epsilon_1-\epsilon_2\rangle)} \sum_{i=1}^{N} \frac{\sin\eta \sin(u-z_i+l_{21}+\eta+\eta\langle H, \epsilon_1-\epsilon_2\rangle)}{\sin(u-z_i+\eta)\sin(l_{21}+\eta+\eta\langle H, \epsilon_1-\epsilon_2\rangle)}$$

$$\times E^i_{21} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-z_j)}{\sin(u-z_j+\eta)} & \\ & \frac{\sin(z_j-z_i+\eta)}{\sin(z_j-z_i)} \end{pmatrix}_{(j)}, \quad (4.16)$$

where $H = \sum_{k=1}^{N} h^{(k)}$. Applying the above operators to the arbitrary state $|i_1, \ldots, i_N\rangle$ given by (3.8) leads to

$$\tilde{T}_F(m, l|u)^2_2 = \frac{\sin(l_{21}-\eta)}{\sin(l_2 - m_1 - \eta)} \otimes_i \begin{pmatrix} \frac{\sin(u-z_i)}{\sin(u-z_i+\eta)} & \\ & 1 \end{pmatrix}_{(i)}, \quad (4.17)$$

$$\tilde{T}_F(m, l|u)^2_1 = \sum_{i=1}^{N} \frac{\sin\eta \sin(u-z_i+l_{12})}{\sin(u-z_i+\eta)\sin l_{12}}$$

$$\times E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-z_j)\sin(z_i-z_j+\eta)}{\sin(u-z_j+\eta)\sin(z_i-z_j)} & \\ & 1 \end{pmatrix}_{(j)}, \quad (4.18)$$

$$\tilde{T}_F(m, l|u)^1_2 = \frac{\sin(l_{21}-\eta)}{\sin(m_{21}-2\eta)} \sum_{i=1}^{N} \frac{\sin\eta \sin(u-z_i+m_{21}-\eta)}{\sin(u-z_i+\eta)\sin(m_{21}-\eta)}$$

$$\times E^i_{21} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-z_j)}{\sin(u-z_j+\eta)} & \\ & \frac{\sin(z_j-z_i+\eta)}{\sin(z_j-z_i)} \end{pmatrix}_{(j)}. \quad (4.19)$$

It then follows that the two pseudo-particle creation operators (3.14) and (3.15) in the F-basis simultaneously have the following completely symmetric polarization free forms:

$$\tilde{\mathcal{T}}^-_F(m, \lambda|u)^2_1 = \frac{\sin m_{12}}{\sin(m_1-\lambda_2)} \prod_{k=1}^{N} \frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}$$

$$\times \sum_{i=1}^{N} \frac{\sin(\lambda_1+\xi-z_i)\sin(\lambda_2+\xi+z_i)\sin 2u \sin\eta}{\sin(\lambda_1+\xi+u)\sin(\lambda_2+\xi+u)\sin(u-z_i+\eta)\sin(u+z_i)}$$

$$\times E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-z_j)\sin(u+z_j+\eta)\sin(z_i-z_j+\eta)}{\sin(u-z_j+\eta)\sin(u+z_j)\sin(z_i-z_j)} & \\ & 1 \end{pmatrix}_{(j)}, \quad (4.20)$$

$$\tilde{\mathcal{T}}^+_F(\lambda, m|u)^1_2 = \frac{\sin(m_{21}+\eta)}{\sin(m_2-\lambda_1)} \prod_{k=1}^{N} \frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}$$

$$\times \sum_{i=1}^{N} \frac{\sin(\lambda_2+\bar{\xi}-z_i)\sin(\lambda_1+\bar{\xi}+z_i)\sin(2u+2\eta)\sin\eta}{\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)\sin(u+z_i)\sin(u-z_i+\eta)}$$

$$\times E^i_{21} \otimes_{j\neq i} \begin{pmatrix} \frac{\sin(u-z_j)\sin(u+z_j+\eta)}{\sin(u-z_j+\eta)\sin(u+z_j)} & \\ & \frac{\sin(z_j-z_i+\eta)}{\sin(z_j-z_i)} \end{pmatrix}_{(j)} . \quad (4.21)$$

5 Determinant representations of the scalar products

Due to the fact that the states $|1,\ldots,1\rangle$, $|2,\ldots,2\rangle$ and their dual states $\langle 1,\ldots,1|$, $\langle 2,\ldots,2|$ are invariant under the action of the F-matrix $F_{1\ldots N}(l)$ (4.12), the calculation of the scalar products (3.10)-(3.13) can be performed in the F-basis. Namely,

$$S^{I,II}(\{u_\alpha\};\{v_i^{(2)}\}) = \langle 1,\ldots,1|\tilde{\mathcal{T}}_F^-(\lambda-2(M-1)\eta\hat{1},\lambda|u_M)_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda,\lambda|u_1)_1^2$$
$$\times\tilde{\mathcal{T}}_F^-(\lambda+2\eta\hat{1},\lambda|v_1^{(2)})_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda+2M\eta\hat{1},\lambda|v_M^{(2)})_1^2|2,\ldots,2\rangle, \quad (5.1)$$
$$S^{II,I}(\{u_\alpha\};\{v_i^{(1)}\}) = \langle 2,\ldots,2|\tilde{\mathcal{T}}_F^+(\lambda,\lambda+2(M-1)\eta\hat{2}|u_M)_2^1\ldots\tilde{\mathcal{T}}_F^+(\lambda,\lambda|u_1)_2^1$$
$$\times\tilde{\mathcal{T}}_F^+(\lambda,\lambda-2\eta\hat{2}|v_1^{(1)})_2^1\ldots\tilde{\mathcal{T}}_F^+(\lambda,\lambda-2M\eta\hat{2}|v_M^{(1)})_2^1|1,\ldots,1\rangle, \quad (5.2)$$
$$S^{I,I}(\{u_\alpha\};\{v_i^{(1)}\}) = \langle 1,\ldots,1|\tilde{\mathcal{T}}_F^-(\lambda-2(M-1)\eta\hat{1},\lambda|u_M)_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda,\lambda|u_1)_1^2$$
$$\times\tilde{\mathcal{T}}_F^+(\lambda,\lambda+2\eta\hat{1}|v_1^{(1)})_2^1\ldots\tilde{\mathcal{T}}_F^+(\lambda,\lambda+2M\eta\hat{1}|v_M^{(1)})_2^1|1,\ldots,1\rangle, \quad (5.3)$$
$$S^{II,II}(\{u_\alpha\};\{v_i^{(2)}\}) = \langle 2,\ldots,2|\tilde{\mathcal{T}}_F^+(\lambda,\lambda+2(M-1)\eta\hat{2}|u_M)_2^1\ldots\tilde{\mathcal{T}}_F^+(\lambda,\lambda|u_1)_2^1$$
$$\times\tilde{\mathcal{T}}_F^-(\lambda-2\eta\hat{2},\lambda|v_1^{(2)})_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda-2M\eta\hat{2},\lambda|v_M^{(2)})_1^2|2,\ldots,2\rangle. \quad (5.4)$$

In the above equations, we have used the identity: $\hat{1}=-\hat{2}$. Thanks to the polarization free representations (4.20) and (4.21) of the pseudo-particle creation/annihilation operators, we can obtain the determinant representations of the scalar products.

5.1 The scalar products $S^{I,II}$ and $S^{II,I}$

It was shown [44] that the scalar product $S^{I,II}(\{u_\alpha\};\{v_i^{(2)}\})$ (resp. $S^{II,I}(\{u_\alpha\};\{v_i^{(1)}\})$) can be expressed in terms of some determinant no matter the parameters $\{v_i^{(2)}\}$ (resp.$\{v_i^{(1)}\}$) satisfy the associated Bethe ansatz equations or not. In this subsection we do not require these parameters being the roots of the Bethe ansatz equations. Let us introduce two functions

$$\mathcal{Z}_N^{(I)}(\{\bar{u}_J\})\equiv S^{I,II}(\{u_\alpha\};\{v_i\})$$
$$=\langle 1,\ldots,1|\tilde{\mathcal{T}}_F^-(\lambda-2(M-1)\eta\hat{1},\lambda|\bar{u}_N)_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda+2M\eta\hat{1},\lambda|\bar{u}_1)_1^2|2,\ldots,2\rangle, \quad (5.5)$$

$$\mathcal{Z}_N^{(II)}(\{\bar{u}_J\}) \equiv S^{II,I}(\{u_\alpha\};\{v_i\})$$
$$= \langle 2,\ldots,2|\tilde{\mathcal{T}}_F^{+}(\lambda,\lambda+2(M-1)\eta\hat{2}|\bar{u}_N)_2^1\ldots\tilde{\mathcal{T}}_F^{+}(\lambda,\lambda-2M\eta\hat{2}|\bar{u}_1)_2^1|1,\ldots,1\rangle, \quad (5.6)$$

where N free parameters $\{\bar{u}_J|J=1,\ldots N\}$ are given by

$$\bar{u}_i = u_i \text{ for } i=1,\ldots M, \qquad \text{and} \qquad \bar{u}_{M+i} = v_i \text{ for } i=1,\ldots M. \quad (5.7)$$

Note that these functions $\mathcal{Z}_N^{(I)}(\{\bar{u}_J\})$ and $\mathcal{Z}_N^{(II)}(\{\bar{u}_J\})$ correspond to the partition functions of the six-vertex model with domain wall boundary conditions and one reflecting end [43] specified by the non-diagonal K-matrices (2.8) and (2.10) respectively [44].

The polarization free representations (4.20) and (4.21) of the pseudo-particle creation/annihilation operators allowed ones [44] to express the above functions in terms of the determinants representations of some $N\times N$ matrices as follows:

$$\mathcal{Z}_N^{(I)}(\{\bar{u}_J\}) = \prod_{k=1}^{M}\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{12}-2k\eta+\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{12}-k\eta+\eta)}\prod_{l=1}^{N}\prod_{i=1}^{N}\frac{\sin(\bar{u}_i+z_l)}{\sin(\bar{u}_i+z_l+\eta)}$$
$$\times\frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(\bar{u}_\alpha-z_i)\sin(\bar{u}_\alpha+z_i+\eta)\det\mathcal{N}^{(I)}(\{\bar{u}_\alpha\};\{z_i\})}{\prod_{\alpha>\beta}\sin(\bar{u}_\alpha-\bar{u}_\beta)\sin(\bar{u}_\alpha+\bar{u}_\beta+\eta)\prod_{k<l}\sin(z_k-z_l)\sin(z_k+z_l)}, \quad (5.8)$$

$$\mathcal{Z}_N^{(II)}(\{\bar{u}_J\}) = \prod_{k=1}^{M}\frac{\sin(\lambda_{21}+\eta-2k\eta)\sin(\lambda_{21}-\eta+2k\eta)}{\sin(\lambda_{21}-k\eta)\sin(\lambda_{21}+k\eta-\eta)}\prod_{l=1}^{N}\prod_{i=1}^{N}\frac{\sin(\bar{u}_i+z_l)}{\sin(\bar{u}_i+z_l+\eta)}$$
$$\times\frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\sin(\bar{u}_\alpha+z_i)\sin(\bar{u}_\alpha-z_i+\eta)\det\mathcal{N}^{(II)}(\{\bar{u}_\alpha\};\{z_i\})}{\prod_{\alpha>\beta}\sin(\bar{u}_\alpha-\bar{u}_\beta)\sin(\bar{u}_\alpha+\bar{u}_\beta+\eta)\prod_{k<l}\sin(z_l-z_k)\sin(z_l+z_k)}, \quad (5.9)$$

where the $N\times N$ matrices $\mathcal{N}^{(I)}(\{\bar{u}_\alpha\};\{z_i\})$ and $\mathcal{N}^{(II)}(\{\bar{u}_\alpha\};\{z_i\})$ are given by

$$\mathcal{N}^{(I)}(\{\bar{u}_\alpha\};\{z_i\})_{\alpha,j} = \frac{\sin\eta\sin(\lambda_1+\xi-z_j)}{\sin(\bar{u}_\alpha-z_j)\sin(\bar{u}_\alpha+z_j+\eta)\sin(\lambda_1+\xi+\bar{u}_\alpha)}$$
$$\times\frac{\sin(\lambda_2+\xi+z_j)\sin(2\bar{u}_\alpha)}{\sin(\lambda_2+\xi+\bar{u}_\alpha)\sin(\bar{u}_\alpha-z_j+\eta)\sin(\bar{u}_\alpha+z_j)}, \quad (5.10)$$

$$\mathcal{N}^{(II)}(\{\bar{u}_\alpha\};\{z_i\})_{\alpha,j} = \frac{\sin\eta\sin(\lambda_2+\bar{\xi}-z_j)}{\sin(\bar{u}_\alpha-z_j)\sin(\bar{u}_\alpha+z_j+\eta)\sin(\lambda_2+\bar{\xi}-\bar{u}_\alpha-\eta)}$$
$$\times\frac{\sin(\lambda_1+\bar{\xi}+z_j)\sin(2\bar{u}_\alpha+2\eta)}{\sin(\lambda_1+\bar{\xi}-\bar{u}_\alpha-\eta)\sin(\bar{u}_\alpha-z_j+\eta)\sin(\bar{u}_\alpha+z_j)}. \quad (5.11)$$

The above determinant representations are crucial to construct the determinant representations of the remaining scalar products $S^{I,I}$ and $S^{II,II}$ in the next subsection.

5.2 The scalar products $S^{I,I}$ and $S^{II,II}$

Let us introduce two sets of functions $\{H_j^{(I)}(u;\{z_i\},\{v_i\})|j=1,\ldots,M\}$ and $\{H_j^{(II)}(u;\{z_i\},\{v_i\})|j=1,\ldots,M\}$

$$H_j^{(I)}(u;\{z_i\},\{v_i\}) = F_1(u)\prod_{l=1}^{N}\frac{\sin(u+z_l)}{\sin(u+z_l+\eta)}\frac{\prod_{k\neq j}\sin(u+v_k+2\eta)\sin(u-v_k+\eta)}{\sin(u-v_j)\sin(u+v_j+\eta)\sin(2u+\eta)}$$
$$-F_2(u)\prod_{l=1}^{N}\frac{\sin(u-z_l+\eta)}{\sin(u-z_l)}\frac{\prod_{k\neq j}\sin(u+v_k)\sin(u-v_k-\eta)}{\sin(u-v_j)\sin(u+v_j+\eta)\sin(2u+\eta)}, \quad (5.12)$$
$$H_j^{(II)}(u;\{z_i\},\{v_i\}) = F_3(u)\prod_{l=1}^{N}\frac{\sin(u-z_l)}{\sin(u-z_l+\eta)}\frac{\prod_{k\neq j}\sin(v_k+u+2\eta)\sin(v_k-u-\eta)}{\sin(u+v_j+\eta)\sin(u-v_j)\sin(2u+\eta)}$$
$$-F_4(u)\prod_{l=1}^{N}\frac{\sin(u+z_l+\eta)}{\sin(u+z_l)}\frac{\prod_{k\neq j}\sin(v_k+u)\sin(v_k-u+\eta)}{\sin(u+v_j+\eta)\sin(u-v_j)\sin(2u+\eta)}, \quad (5.13)$$

where the coefficients $\{F_i(u)|i=1,2,3,4\}$ are

$$F_1(u) = \sin(\lambda_2+\bar{\xi}+u+\eta)\sin(\lambda_2+\xi-u-\eta)\sin(\lambda_1+\bar{\xi}-u-\eta)\sin(\lambda_1+\xi+u+\eta), \quad (5.14)$$
$$F_2(u) = \sin(\lambda_2+\bar{\xi}-u)\sin(\lambda_2+\xi+u)\sin(\lambda_1+\bar{\xi}+u)\sin(\lambda_1+\xi-u), \quad (5.15)$$
$$F_3(u) = \sin(\lambda_2+\bar{\xi}-u-\eta)\sin(\lambda_2+\xi+u+\eta)\sin(\lambda_1+\bar{\xi}+u+\eta)\sin(\lambda_1+\xi-u-\eta), \quad (5.16)$$
$$F_4(u) = \sin(\lambda_2+\bar{\xi}+u)\sin(\lambda_2+\xi-u)\sin(\lambda_1+\bar{\xi}-u)\sin(\lambda_1+\xi+u). \quad (5.17)$$

Let us consider the scalar product $S^{I,I}(\{u_\alpha\};\{v_i^{(1)}\})$ defined by (3.2). The expression (5.3) of $S^{I,I}(\{u_\alpha\};\{v_i^{(1)}\})$ under the F-basis and the polarization free representations (4.20) and (4.21) of the pseudo-particle creation/annihilation operators allow us to compute the scalar product following the similar procedure as that in [13] for the bulk case as follows. In front of each operators $\tilde{\mathcal{T}}_F^-$ in (5.3), we insert a sum over the complete set of spin states $|j_1,\ldots,j_i\gg$, where $|j_1,\ldots,j_i\gg$ is the state with i spins being ϵ_2 in the sites $j_1,\ldots,j_i$ and $2M-i$ spins being ϵ_1 in the other sites. We are thus led to consider some intermediate functions of the form

$$G^{(i)}(u_1,\ldots,u_i|j_{i+1},\ldots,j_M;\{v_i^{(1)}\})=\ll j_{i+1},\ldots,j_M|\tilde{\mathcal{T}}_F^-(\lambda-2(i-1)\eta\hat{1},\lambda|u_i)_1^2\ldots\tilde{\mathcal{T}}_F^-(\lambda,\lambda|u_1)_1^2$$
$$\times\tilde{\mathcal{T}}_F^+(\lambda,\lambda+2\eta\hat{1}|v_1^{(1)})_2^1\ldots\tilde{\mathcal{T}}_F^+(\lambda,\lambda+2M\eta\hat{1}|v_M^{(1)})_2^1|1,\ldots,1\rangle,$$
$$i=0,1,\ldots,M, \quad (5.18)$$

which satisfy the following recursive relation:

$$G^{(i)}(u_1,\ldots,u_i|j_{i+1},\ldots,j_M;\{v_i^{(1)}\})$$

$$= \sum_{j\neq j_{i+1},\ldots,j_M} \ll j_{i+1},\ldots,j_M|\tilde{\mathcal{T}}^-_F(\lambda-2(i-1)\eta\hat{1},\lambda|u_i)^2_1|j,j_{i+1},\ldots,j_M\gg$$

$$\times G^{(i-1)}(u_1,\ldots,u_{i-1}|j,j_{i+1},\ldots,j_M;\{v^{(1)}_i\}),\quad i=1,\ldots,M. \quad (5.19)$$

Note that the last of these functions $\{G^{(i)}|i=0,\ldots,M\}$ is precisely the scalar product $S^{I,I}(\{u_\alpha\};\{v^{(1)}_i\})$, namely,

$$G^{(M)}(u_1,\ldots,u_M;\{v^{(1)}_i\}) = S^{I,I}(\{u_\alpha\};\{v^{(1)}_i\}), \quad (5.20)$$

whereas the first one,

$$G^{(0)}(j_1,\ldots,j_M;\{v^{(1)}_i\}) = \ll j_1,\ldots,j_M|\tilde{\mathcal{T}}^+_F(\lambda,\lambda+2\eta\hat{1}|v^{(1)}_1)^1_2\ldots\tilde{\mathcal{T}}^+_F(\lambda,\lambda+2M\eta\hat{1}|v^{(1)}_M)^1_2|1,\ldots,1\rangle,$$

is closely related to the partition function computed in [44]. Solving the recursive relations (5.19), we find that if the parameters $\{v^{(1)}_k\}$ satisfy the first set of Bethe ansatz equations (2.42) the scalar product $S^{I,I}(\{u_\alpha\};\{v^{(1)}_i\})$ has the following determinant representation

$$S^{I,I}(\{u_\alpha\};\{v^{(1)}_i\})=\prod_{k=1}^{M}\left\{\frac{\sin(\lambda_{12}+2\eta-2k\eta)\sin(\lambda_{12}-\eta+2k\eta)}{\sin(\lambda_{12}-(k-1)\eta)\sin(\lambda_{12}+k\eta)}\prod_{l=1}^{N}\frac{\sin(u_k-z_l)\sin(v^{(1)}_k-z_l)}{\sin(u_k-z_l+\eta)\sin(v^{(1)}_k-z_l+\eta)}\right\}$$

$$\times\frac{\det\bar{\mathcal{N}}^{(I)}(\{u_\alpha\};\{v^{(1)}_i\})}{\prod_{\alpha<\beta}\sin(u_\alpha-u_\beta)\sin(u_\alpha+u_\beta+\eta)\prod_{k>l}\sin(v^{(1)}_k-v^{(1)}_l)\sin(v^{(1)}_k+v^{(1)}_l+\eta)}, \quad (5.21)$$

where the $M\times M$ matrix $\bar{\mathcal{N}}^{(I)}(\{u_\alpha\};\{v^{(1)}_i\})$ is given by

$$\bar{\mathcal{N}}^{(I)}(\{u_\alpha\};\{v^{(1)}_i\})_{\alpha,j}=\frac{\sin\eta\sin(2u_\alpha)\sin(2v^{(1)}_j+2\eta)H^{(I)}_j(u_\alpha;\{z_i\},\{v^{(1)}_i\})}{\sin(\lambda_1+\xi+u_\alpha)\sin(\lambda_2+\xi+u_\alpha)\sin(\lambda_2+\bar{\xi}-v^{(1)}_j-\eta)\sin(\lambda_1+\bar{\xi}-v^{(1)}_j-\eta)}. \quad (5.22)$$

Using the similar method as above, we have that the scalar product $S^{II,II}(\{u_\alpha\};\{v^{(2)}_i\})$ has the following determinant representation provided that the parameters $\{v^{(2)}_k\}$ satisfy the second set of Bethe ansatz equations (2.44)

$$S^{II,II}(\{u_\alpha\};\{v^{(2)}_i\})=\prod_{k=1}^{M}\left\{\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{21}-\eta+2k\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{21}+(k-1)\eta)}\prod_{l=1}^{N}\frac{\sin(u_k+z_l)\sin(v^{(2)}_k+z_l)}{\sin(u_k+z_l+\eta)\sin(v^{(2)}_k+z_l+\eta)}\right\}$$

$$\times\frac{\det\bar{\mathcal{N}}^{(II)}(\{u_\alpha\};\{v^{(2)}_i\})}{\prod_{\alpha<\beta}\sin(u_\alpha-u_\beta)\sin(u_\alpha+u_\beta+\eta)\prod_{k>l}\sin(v^{(2)}_k-v^{(2)}_l)\sin(v^{(2)}_k+v^{(2)}_l+\eta)}, \quad (5.23)$$

where the $M \times M$ matrix $\bar{\mathcal{N}}^{(II)}(\{u_\alpha\};\{v_i^{(2)}\})$ is given by

$$\bar{\mathcal{N}}^{(II)}(\{u_\alpha\};\{v_i^{(2)}\})_{\alpha,j}=\frac{\sin\eta\sin(2u_\alpha+2\eta)\sin(2v_j^{(2)})H_j^{(II)}(u_\alpha;\{z_i\},\{v_i^{(2)}\})}{\sin(\lambda_2+\bar{\xi}-u_\alpha-\eta)\sin(\lambda_1+\bar{\xi}-u_\alpha-\eta)\sin(\lambda_2+\xi+v_j^{(2)})\sin(\lambda_1+\xi+v_j^{(2)})}. \tag{5.24}$$

Now we are in position to compute the norms of the Bethe states which can be obtained by taking the limit $u_\alpha \to v_\alpha^{(i)}$, $\alpha = 1, \ldots M$. The norm of the first set of Bethe state (2.38) is

$$\begin{aligned}\mathbb{N}^{I,I}(\{v_\alpha^{(1)}\}) &= \lim_{u_\alpha\to v_\alpha^{(1)}} S^{I,I}(\{u_\alpha\};\{v_i^{(1)}\})\\ &= \prod_{k=1}^{M}\left\{\frac{\sin(\lambda_{12}+2\eta-2k\eta)\sin(\lambda_{12}-\eta+2k\eta)}{\sin(\lambda_{12}-(k-1)\eta)\sin(\lambda_{12}+k\eta)}\prod_{l=1}^{N}\frac{\sin^2(v_k^{(1)}-z_l)}{\sin^2(v_k^{(1)}-z_l+\eta)}\right\}\\ &\times\prod_{\alpha\neq\beta}\frac{\sin(v_\alpha^{(1)}+v_\beta^{(1)})\sin(v_\alpha^{(1)}-v_\beta^{(1)}-\eta)}{\sin(v_\alpha^{(1)}-v_\beta^{(1)})\sin(v_\alpha^{(1)}+v_\beta^{(1)}+\eta)}\det\Phi^{(I)}(\{v_\alpha^{(1)}\}),\end{aligned} \tag{5.25}$$

where the matrix elements of $M \times M$ matrix $\Phi^{(I)}(\{v_\alpha\})$ are given by

$$\begin{aligned}\Phi_{\alpha,j}^{(I)}(\{v_\alpha\}) &= \frac{\sin\eta\sin(\lambda_2+\bar{\xi}-v_\alpha)\sin(\lambda_1+\bar{\xi}+v_\alpha)\sin(\lambda_1+\xi-v_\alpha)}{\sin(\lambda_1+\xi+v_\alpha)\sin(\lambda_2+\bar{\xi}-v_j-\eta)\sin(\lambda_1+\bar{\xi}-v_j-\eta)}\\ &\times\frac{\sin 2v_\alpha\sin(2v_j+2\eta)}{\sin(2v_\alpha+\eta)\sin(2v_j+\eta)}\prod_{l=1}^{N}\frac{\sin(v_\alpha-z_l+\eta)}{\sin(v_\alpha-z_l)}\\ &\times\frac{\partial}{\partial v_\alpha}\ln\left\{\frac{\sin(\lambda_2+\bar{\xi}+v_j+\eta)\sin(\lambda_2+\xi-v_j-\eta)\sin(\lambda_1+\bar{\xi}-v_j-\eta)\sin(\lambda_1+\xi+v_j+\eta)}{\sin(\lambda_2+\bar{\xi}-v_j)\sin(\lambda_2+\xi+v_j)\sin(\lambda_1+\bar{\xi}+v_j)\sin(\lambda_1+\xi-v_j)}\right.\\ &\left.\times\prod_{l=1}^{N}\frac{\sin(v_j+z_l)\sin(v_j-z_l)}{\sin(v_j+z_l+\eta)\sin(v_j-z_l+\eta)}\prod_{k\neq j}\frac{\sin(v_j+v_k+2\eta)\sin(v_j-v_k+\eta)}{\sin(v_j+v_k)\sin(v_j-v_k-\eta)}\right\},\end{aligned} \tag{5.26}$$

the norm of the second set of Bethe state (2.39) is given by

$$\begin{aligned}\mathbb{N}^{II,II}(\{v_\alpha^{(2)}\}) &= \lim_{u_\alpha\to v_\alpha^{(2)}} S^{II,II}(\{u_\alpha\};\{v_i^{(2)}\})\\ &= \prod_{k=1}^{M}\left\{\frac{\sin(\lambda_{12}+2k\eta)\sin(\lambda_{21}-\eta+2k\eta)}{\sin(\lambda_{12}+k\eta)\sin(\lambda_{21}+(k-1)\eta)}\prod_{l=1}^{N}\frac{\sin^2(v_k^{(2)}+z_l)}{\sin^2(v_k^{(2)}+z_l+\eta)}\right\}\\ &\times\prod_{\alpha\neq\beta}\frac{\sin(v_\alpha^{(2)}+v_\beta^{(2)}+2\eta)\sin(v_\alpha^{(2)}-v_\beta^{(2)}-\eta)}{\sin(v_\alpha^{(2)}-v_\beta^{(2)})\sin(v_\alpha^{(2)}+v_\beta^{(2)}+\eta)}\det\Phi^{(II)}(\{v_\alpha^{(2)}\}),\end{aligned} \tag{5.27}$$

where the matrix elements of $M \times M$ matrix $\Phi^{(II)}(\{v_\alpha\})$ are given by

$$\Phi^{(II)}_{\alpha,j}(\{v_\alpha\}) = \frac{\sin\eta\sin(\lambda_2+\xi+v_\alpha+\eta)\sin(\lambda_1+\bar{\xi}+v_\alpha+\eta)\sin(\lambda_1+\xi-v_\alpha-\eta)}{\sin(\lambda_1+\bar{\xi}-v_\alpha-\eta)\sin(\lambda_2+\xi+v_j)\sin(\lambda_1+\xi+v_j)}$$
$$\times\frac{\sin(2v_\alpha+2\eta)\sin 2v_j}{\sin(2v_\alpha+\eta)\sin(2v_j+\eta)}\prod_{l=1}^{N}\frac{\sin(v_\alpha-z_l)}{\sin(v_\alpha-z_l+\eta)}$$
$$\times\frac{\partial}{\partial v_\alpha}\ln\Bigg\{\frac{\sin(\lambda_2+\bar{\xi}-v_j-\eta)\sin(\lambda_2+\xi+v_j+\eta)\sin(\lambda_1+\bar{\xi}+v_j+\eta)\sin(\lambda_1+\xi-v_j-\eta)}{\sin(\lambda_2+\bar{\xi}+v_j)\sin(\lambda_2+\xi-v_j)\sin(\lambda_1+\bar{\xi}-v_j)\sin(\lambda_1+\xi+v_j)}$$
$$\times\prod_{l=1}^{N}\frac{\sin(v_j+z_l)\sin(v_j-z_l)}{\sin(v_j+z_l+\eta)\sin(v_j-z_l+\eta)}\prod_{k\neq j}\frac{\sin(v_j+v_k+2\eta)\sin(v_j-v_k+\eta)}{\sin(v_j+v_k)\sin(v_j-v_k-\eta)}\Bigg\}. \quad (5.28)$$

Moreover, one may check that if the parameters $\{u_\alpha\}$ satisfy the Bethe ansatz equations (i.e. on shell) but different from $\{v^{(i)}_\alpha\}$ the corresponding scalar products $S^{I,I}(\{u_\alpha\};\{v^{(1)}_i\})$ or $S^{II,II}(\{u_\alpha\};\{v^{(2)}_i\})$ vanishes, namely, the corresponding Bethe states are orthogonal.

6 Conclusions

We have studied scalar products between an on-shell Bethe state and a general state (or an off-shell Bethe state) of the open XXZ chain with non-diagonal boundary terms, where the non-diagonal K-matrices $K^{\pm}(u)$ are given by (2.9) and (2.11). In our calculation the factorizing F-matrix (4.12) in the face picture of the open XXZ chain, which leads to the polarization free representations (4.20) and (4.21) of the associated pseudo-particle creation/annihilation operators, has played an important role. It is found that the scalar products can be expressed in terms of the determinants (5.8), (5.9), (5.21) and (5.23). By taking the on shell limit, we obtain the determinant representations (or Gaudin formula) (5.25)-(5.26) and (5.27)-(5.28) of the norms of the Bethe states.

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Appendix A: $\mathcal{T}^{\pm}(m|u)$ in the face picture

The K-matrices $K^{\pm}(u)$ given by (2.8) and (2.10) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (2.31) and (2.32), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (2.33) and (2.34) *simultaneously* become diagonal. This fact suggests that it would be much simpler if one performs all calculations in the face picture.

Associated with the vertex type monodromy matrices $T(u)$ (2.3) and $\hat{T}(u)$ (2.6), we introduce the following operators

$$T(m,l|u)^j_\mu = \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\,T_0(u)\,\phi^0_{l+\eta\hat{\mu},l}(u), \tag{A.1}$$

$$S(m,l|u)^\mu_i = \bar{\phi}^0_{l,l-\eta\hat{\mu}}(-u)\,\hat{T}_0(u)\,\phi^0_{m,m-\eta\hat{\imath}}(-u). \tag{A.2}$$

Moreover, for the case of $m = l - \eta\sum_{k=1}^N \hat{\imath}_k$, we introduce a generic state in the quantum space from the intertwiner vector (2.25)

$$|i_1,\ldots,i_N\rangle^m_l = \phi^1_{l,l-\eta\hat{\imath}_1}(z_1)\phi^2_{l-\eta\hat{\imath}_1,l-\eta(\hat{\imath}_1+\hat{\imath}_2)}(z_2)\ldots\phi^N_{l-\eta\sum_{k=1}^{N-1}\hat{\imath}_k,l-\eta\sum_{k=1}^N\hat{\imath}_k}(z_N). \tag{A.3}$$

We can evaluate the action of the operator $T(m,l|u)$ on the state $|i_1,\ldots,i_N\rangle^m_l$ from the face-vertex correspondence relation (2.27)

$$\begin{aligned}
T(m,l|u)^j_\mu|i_1,\ldots,i_N\rangle^m_l &= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\,T_0(u)\,\phi^0_{l+\eta\hat{\mu},l}(u)|i_1,\ldots,i_N\rangle^m_l\\
&= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\overline{R}_{0,N}(u-z_N)\ldots\overline{R}_{0,1}(u-z_1)\phi^0_{l+\eta\hat{\mu},l}(u)\phi^1_{l,l-\eta\hat{\imath}_1}(z_1)\ldots\\
&= \sum_{\alpha_1,i'_1} R(u-z_1;l+\eta\hat{\mu})^{\alpha_1 i'_1}_{\mu\ i_1}\phi^1_{l+\eta\hat{\mu},l+\eta\hat{\mu}-\eta\hat{\imath}'_1}(z_1)\tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\overline{R}_{0,N}(u-z_N)\ldots\\
&\qquad\times\overline{R}_{0,2}(u-z_2)\phi^0_{l+\eta\hat{\mu}-\eta\hat{\imath}'_1,l-\eta\hat{\imath}_1}(u)\phi^2_{l-\eta\hat{\imath}_1,l-\eta(\hat{\imath}_1+\hat{\imath}_2)}(z_2)\ldots\\
&\ \ \vdots\\
&= \sum_{\alpha_1\ldots\alpha_{N-1}}\sum_{i'_1\ldots i'_N} R(u-z_N;l+\eta\hat{\mu}-\eta\sum_{k=1}^{N-1}\hat{\imath}'_k)^{j\ \ \ \ i'_N}_{\alpha_{N-1}i_N}\ldots\\
&\qquad\times R(u-z_1;l+\eta\hat{\mu})^{\alpha_1 i'_1}_{\mu\ i_1}|i'_1,\ldots,i'_N\rangle^{l+\eta\hat{\mu}-\eta\sum_{k=1}^N\hat{\imath}'_k}_{l+\eta\hat{\mu}}.
\end{aligned} \tag{A.4}$$

Comparing with (3.9), we have the following correspondence

$$T(m,l|u)^j_\mu|i_1,\ldots,i_N\rangle^m_l \longleftrightarrow T_F(m+\eta\hat{\mu};l+\eta\hat{\mu}|u)^j_\mu|i_1,\ldots,i_N\rangle, \tag{A.5}$$

where vector $|i_1,\ldots,i_N\rangle$ is given by (3.8). Hereafter, we will use O_F to denote the face version of operator O in the face picture.

Noting that

$$\hat{T}_0(u) = \overline{R}_{1,0}(u+z_1)\ldots\overline{R}_{N,0}(u+z_N),$$

we obtain the action of $S(m,l|u)_i^{\mu}$ on the state $|i_1,\ldots,i_N\rangle_l^m$

$$S(m,l|u)_i^{\mu}|i_1,\ldots,i_N\rangle_l^m = \sum_{\alpha_1\ldots\alpha_{N-1}}\sum_{i'_1\ldots i'_N} R(u+z_1;l)_{i_1\alpha_{N-1}}^{i'_1\,\mu} R(u+z_2;l-\eta\hat{\imath}_1)_{i_2\alpha_{N-2}}^{i'_2\,\alpha_{N-1}} \\ \times\ldots R(u+z_N;l-\eta\sum_{k=1}^{N-1}\hat{\imath}_k)_{i_N\,i}^{i'_N\,\alpha_1}|i'_1,\ldots,i'_N\rangle_{l-\eta\hat{\mu}}^{l-\eta\hat{\mu}-\eta\sum_{k=1}^{N}\hat{\imath}'_k}. \tag{A.6}$$

Then the crossing relation of the R-matrix (2.23) enables us to establish the following relation:

$$S(m,l|u)_i^{\mu} = \varepsilon_{\bar{i}}\varepsilon_{\bar{\mu}}\frac{\sin(m_{21})}{\sin(l_{21})}\prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}T(m,l|-u-\eta)_{\bar{\mu}}^{\bar{i}}, \tag{A.7}$$

where the parities are defined in (2.24) and m_{21} (or l_{21}) is defined in (2.15).

Now we are in the position to express $\mathcal{T}^{\pm}$ (2.35) and (2.36) in terms of $T(m,l)_j^i$ and $S(l,m)_j^i$ which both can be expressed in terms of the face type R-matrix (2.16). By (2.29) and (2.30), we have

$$\begin{aligned}\mathcal{T}^{-}(m|u)_i^j &= \tilde{\phi}^0_{m-\eta(\hat{\imath}-\hat{\jmath}),m-\eta\hat{\imath}}(u)\ \mathbb{T}(u)\ \phi^0_{m,m-\eta\hat{\imath}}(-u)\\ &= \tilde{\phi}^0_{m-\eta(\hat{\imath}-\hat{\jmath}),m-\eta\hat{\imath}}(u)T_0(u)K_0^{-}(u)\hat{T}_0(u)\phi^0_{m,m-\eta\hat{\imath}}(-u)\\ &= \sum_{\mu,\nu}\tilde{\phi}^0_{m-\eta(\hat{\imath}-\hat{\jmath}),m-\eta\hat{\imath}}(u)T_0(u)\phi^0_{l-\eta(\hat{\nu}-\hat{\mu}),l-\eta\hat{\nu}}(u)\tilde{\phi}^0_{l-\eta(\hat{\nu}-\hat{\mu}),l-\eta\hat{\nu}}(u)\\ &\qquad\times K_0^{-}(u)\phi^0_{l,l-\eta\hat{\nu}}(-u)\bar{\phi}^0_{l,l-\eta\hat{\nu}}(-u)\hat{T}_0(u)\phi^0_{m,m-\eta\hat{\imath}}(-u)\\ &= \sum_{\mu,\nu}T(m-\eta\hat{\imath},l-\eta\hat{\nu}|u)_{\mu}^{j}\mathcal{K}(l|u)_{\nu}^{\mu}S(m,l|u)_i^{\nu}\\ &\stackrel{\text{def}}{=}\ \mathcal{T}^{-}(m,l|u)_i^j,\end{aligned} \tag{A.8}$$

where the face-type K-matrix $\mathcal{K}(l|u)_{\nu}^{\mu}$ is given by

$$\mathcal{K}(l|u)_{\nu}^{\mu} = \tilde{\phi}^0_{l-\eta(\hat{\nu}-\hat{\mu}),l-\eta\hat{\nu}}(u)K_0^{-}(u)\phi^0_{l,l-\eta\hat{\nu}}(-u). \tag{A.9}$$

Similarly, we have

$$\begin{aligned}\mathcal{T}^{+}(m|u)_i^j &= \prod_{k\neq j}\frac{\sin m_{jk}}{\sin(m_{jk}-\eta)}\sum_{\mu,\nu}T(l-\eta\hat{\mu},m-\eta\hat{\jmath}|u)_i^{\nu}\tilde{\mathcal{K}}(l|u)_{\nu}^{\mu}S(l,m|u)_{\mu}^{j}\\ &\stackrel{\text{def}}{=}\ \mathcal{T}^{+}(l,m|u)_i^j\end{aligned} \tag{A.10}$$

with

$$\tilde{\mathcal{K}}(l|u)^{\mu}_{\nu}=\bar{\phi}^{0}_{l,l-\eta\hat{\mu}}(-u)K^{+}_{0}(u)\phi^{0}_{l-\eta(\hat{\mu}-\hat{\nu}),l-\eta\hat{\mu}}(u). \tag{A.11}$$

Thanks to the fact that when $l=\lambda$ the corresponding face-type K-matrices $\mathcal{K}(\lambda|u)$ (A.9) and $\tilde{\mathcal{K}}(\lambda|u)$ (A.11) become diagonal ones (2.33) and (2.34), we have

$$\mathcal{T}^{-}(m,\lambda|u)^{j}_{i}=\sum_{\mu}T(m-\eta\hat{\imath},\lambda-\eta\hat{\mu}|u)^{j}_{\mu}k(\lambda|u)_{\mu}S(m,\lambda|u)^{\mu}_{i}, \tag{A.12}$$

$$\mathcal{T}^{+}(\lambda,m|u)^{j}_{i}=\prod_{k\neq j}\frac{\sin m_{jk}}{\sin(m_{jk}-\eta)}\sum_{\mu}T(\lambda-\eta\hat{\mu},m-\eta\hat{\jmath}|u)^{\mu}_{i}\tilde{k}(\lambda|u)_{\mu}S(\lambda,m|u)^{j}_{\mu}, \tag{A.13}$$

where the functions $k(\lambda|u)_{\mu}$ and $\tilde{k}(\lambda|u)_{\mu}$ are given by (2.33) and (2.34) respectively. The relation (A.7) implies that one can further express $\mathcal{T}^{\pm}(m|u)^{j}_{i}$ in terms of only $T(m,l|u)^{j}_{i}$. Here we present the results for the pseudo-particle creation operators $\mathcal{T}^{-}(m|u)^{2}_{1}$ in (2.39) and $\mathcal{T}^{+}(m|u)^{1}_{2}$ in (2.38):

$$\begin{aligned}\mathcal{T}^{-}(m|u)^{2}_{1}=\mathcal{T}^{-}(m,\lambda|u)^{2}_{1}&=\frac{\sin(m_{21})}{\sin(\lambda_{21})}\prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}\\&\times\Bigg\{\frac{\sin(\lambda_1+\xi-u)}{\sin(\lambda_1+\xi+u)}T(m+\eta\hat{2},\lambda+\eta\hat{2}|u)^{2}_{1}T(m,\lambda|-u-\eta)^{2}_{2}\\&-\frac{\sin(\lambda_2+\xi-u)}{\sin(\lambda_2+\xi+u)}T(m+\eta\hat{2},\lambda+\eta\hat{1}|u)^{2}_{2}T(m,\lambda|-u-\eta)^{2}_{1}\Bigg\},\end{aligned} \tag{A.14}$$

$$\begin{aligned}\mathcal{T}^{+}(m|u)^{1}_{2}=\mathcal{T}^{+}(\lambda,m|u)^{1}_{2}&=\prod_{k=1}^{N}\frac{\sin(u+z_k)}{\sin(u+z_k+\eta)}\\&\times\Bigg\{\frac{\sin(\lambda_{12}-\eta)\sin(\lambda_1+\bar{\xi}+u+\eta)}{\sin(m_{12}-\eta)\sin(\lambda_1+\bar{\xi}-u-\eta)}T(\lambda+\eta\hat{2},m+\eta\hat{2}|u)^{1}_{2}T(\lambda,m|-u-\eta)^{2}_{2}\\&-\frac{\sin(\lambda_{21}-\eta)\sin(\lambda_2+\bar{\xi}+u+\eta)}{\sin(m_{21}+\eta)\sin(\lambda_2+\bar{\xi}-u-\eta)}T(\lambda+\eta\hat{1},m+\eta\hat{2}|u)^{2}_{2}T(\lambda,m|-u-\eta)^{1}_{2}\Bigg\}.\end{aligned} \tag{A.15}$$

Similar to (A.5), we have the correspondence,

$$\mathcal{T}^{-}(m,l|u)^{2}_{1}|i_1,\ldots,i_N\rangle^{m}_{l}\longleftrightarrow\mathcal{T}^{-}_{F}(m,l|u)^{2}_{1}|i_1,\ldots,i_N\rangle, \tag{A.16}$$

$$\mathcal{T}^{+}(m,l|u)^{1}_{2}|i_1,\ldots,i_N\rangle^{m}_{l}\longleftrightarrow\mathcal{T}^{+}_{F}(m,l|u)^{1}_{2}|i_1,\ldots,i_N\rangle. \tag{A.17}$$

This gives rise the expressions of the operators $\mathcal{T}^{\pm}_{F}(m|u)$ given by (3.14) and (3.15).

Some remarks are in order. It follows from (A.4) that the action of the operator $T(m,l|u)$ on the state $|i_1,\ldots,i_N\rangle^{m}_{l}$ can be expressed in terms of the face type R-matrix (2.16). This implies that the corresponding actions of $\mathcal{T}^{\pm}(m|u)$ can also be expressed in terms of the R-matrix and the K-matrices (2.33) and (2.34). Moreover, the transfer matrix $t(u)$ (2.7) can be given as a linear combination of either $\mathcal{T}^{-}(m|u)^i_i$:

$$\begin{aligned}\tau(u) &= tr(K^+(u)\mathbb{T}(u)) = \sum_{\mu,\nu} tr\left(K^+(u)\phi_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\tilde{\phi}_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u)\right.\\ &\qquad\qquad \left.\times\, \mathbb{T}(u)\phi_{\lambda,\lambda-\eta\hat{\mu}}(-u)\bar{\phi}_{\lambda,\lambda-\eta\hat{\mu}}(-u)\right)\\ &= \sum_{\mu,\nu}\tilde{\mathcal{K}}(\lambda|u)^{\mu}_{\nu}\mathcal{T}^{-}(\lambda|u)^{\nu}_{\mu} = \sum_{\mu}\tilde{k}(\lambda|u)_{\mu}\mathcal{T}^{-}(\lambda|u)^{\mu}_{\mu},\end{aligned} \tag{A.18}$$

or $\mathcal{T}^{+}(m|u)^i_i$:

$$\begin{aligned}\tau(u) &= tr(K^+(u)\mathbb{T}(u)) = tr\left((\mathbb{T}^+(u))^{t_0}(K^-(u))^{t_0}\right)\\ &= \sum_{\mu,\nu} tr\left((\mathbb{T}^+(u))^{t_0}(\bar{\phi}^0_{\lambda,\lambda-\eta\hat{\mu}}(-u))^{t_0}(\phi^0_{\lambda,\lambda-\eta\hat{\mu}}(-u))^{t_0}(K^-(u))^{t_0}\right.\\ &\qquad\qquad \left.\times\,(\tilde{\phi}^0_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u))^{t_0}(\phi^0_{\lambda-\eta(\hat{\mu}-\hat{\nu}),\lambda-\eta\hat{\mu}}(u))^{t_0}\right)\\ &= \sum_{\mu,\nu}\prod_{k\neq\mu}\frac{\sin(\lambda_{\mu k}-\eta)}{\sin(\lambda_{\mu k})}\mathcal{K}(\lambda|u)^{\nu}_{\mu}\mathcal{T}^{+}(\lambda|u)^{\mu}_{\nu}\\ &= \sum_{\mu}\prod_{k\neq\mu}\frac{\sin(\lambda_{\mu k}-\eta)}{\sin(\lambda_{\mu k})}k(\lambda|u)_{\mu}\mathcal{T}^{+}(\lambda|u)^{\mu}_{\mu}.\end{aligned} \tag{A.19}$$

It was shown that in Ref.[37] the first set of Bethe states given by (2.38) generated by $\mathcal{T}^{+}(m|u)$ are the eigenstates of our transfer matrix (2.7) with the eigenvalue (2.43) if the parameters $\{v_i^{(1)}\}$ satisfy the Bethe ansatz equation (2.42), while in Ref.[34] the second ones are the eigenstates with the eigenvalue (2.45) provided that the corresponding parameters satisfy (2.44).

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